SÉMINAIRE N. BOURBAKI

NICHOLAS KATZ

Travaux de Dwork

Séminaire N. Bourbaki, 1971-1972, exp. nº 409, p. 167-200.

<http://www.numdam.org/item?id=SB_1971-1972__14__167_0>

TRAVAUX DE DWORK

par Nicholas KATZ

Introduction.

This talk is devoted to a part of Dwork's work on the variation of the zeta function of a variety over a finite field, as the variety moves through a family. Recall that for a single variety $V/\mathbb{F}_q$, its zeta function is the formal series in t

$$\mathrm{Zeta}(V/\mathbb{F}_q;t) = \exp\left(\sum_{n\geq 1} \frac{t^n}{n} (\#\text{ of points on } V \text{ rational over } \mathbb{F}_{q^n})\right).$$

As a power series it has coefficients in $\mathbb{Z}$, and in fact it is a rational function of t [4]. We shall generally view it as a rational function of a p-adic variable.

Suppose now we consider a one parameter family of varieties, i.e. a variety $V/\mathbb{F}_p[\lambda]$. For each integer $n \geq 1$ and each point $\lambda_o \in \mathbb{F}_{p^n}$, the fibre $V(\lambda_o)/\mathbb{F}_{p^n}$ has a zeta function $\mathrm{Zeta}(V(\lambda_o)/\mathbb{F}_{p^n};t)$. We want to understand how this rational function of t varies when we vary λ_o in the algebraic closure of $\mathbb{F}_p$. Ideally, we might wish a "formula", of a p-adic sort, for, say, one of the reciprocal zeroes of Zeta $(V(\lambda_o)/\mathbb{F}_{p^n};t)$. A natural sort of "formula" would be a p-adic power series $a(x) = \sum a_n x$ with coefficients $a_n \in \mathbb{Z}_p$ tending to zero, with the property :

for every $n \geq 1$ and for every $\lambda_o \in \mathbb{F}_{p^n}$, let $X_o \in$ the algebraic closure of Q_p be the unique quantity lying over λ_o which satisfies $X_o = X_o^{p^n}$. Then

$$a(X_o)a(X_o^p)\ldots a(X_o^{p^{n-1}})$$

is a reciprocal zero of $\mathrm{Zeta}(V(\lambda_o)/\mathbb{F}_{p^n};t)$, i.e., the numerator of $\mathrm{Zeta}(V(\lambda_o)/\mathbb{F}_{p^n};t)$ is divisible by $(1 - a(X_o)a(X_o^p)\ldots a(X_o^{p^{n-1}})t)$.

Now it is unreasonable to expect such a formula unless we can at least describe a priori which reciprocal zero it's a formula for ! If, for example, we knew a priori that one and only one of the reciprocal zeroes were a p-adic unit, then we might reasonably hope for a formula for it. If, on the other hand, we knew a priori that precisely $\nu \geq 2$ of the reciprocal zeroes were p-adic units, we oughtn't hope to single one out ; we could expect at best that we could describe the polynomial of degree ν which has those ν as its reciprocal zeroes. For instance, we might hope for a $\nu \times \nu$ matrix $A(X)$ with entries in $\mathbb{Z}_p[[X]]$, their coefficients tending to zero, so that for each $\lambda_o \in \mathbb{F}_{p^n}$, the characteristic polynomial

$$\det(I - t\,A(X_o)A(X_o^p)\ldots A(X_o^{p^{n-1}}))$$

is the above polynomial.

In another optic, zeta functions come from cohomology, and to study their variation we should study the variation of cohomology. As Dwork discovered in 1961-63 in his study of families of hypersurfaces, their cohomology is quite rigid p-adically, forming a sort of structure on the base now called an F-crystal. Thanks to crystalline cohomology, we now know that this is a general phenomenon (cf. pt. 7 for a more precise statement). The relation with the "formula" viewpoint is this : a formula $a(X)$ for one root is sub-F-crystal of rank 1, a formula $A(X)$ for the ν roots "at once" is a sub-F-crystal of rank ν .

So in fact this exposé is about some of Dwork's recent work on variation of F-crystals, from the point of view of p-adic analysis. Due to space limitations, we have systematically suppressed the Monsky-Washnitzer "over-convergent" point of view in favor of the simpler but less rich "Krasner-analytic" or "rigid analytic" one (but cf. [16]). Among the casualties are Dwork's work on "excellent Liftings of Frobenius", and on the p-adic use of the Picard-Lefschetz formula, both of which are entirely omitted.

1. F-crystals ([1],[2]).

In down-to-earth terms, an F-crystal is a differential equation on which a "Frobenius" operates. Let us make this precise.

(1.0) Let k be a perfect field of characteristic $p > 0$, $W(k)$ its Witt vectors, and $S = \operatorname{Spec}(A)$ a smooth affine $W(k)$-scheme. For each $n \geq 0$, we put $S_n = \operatorname{Spec}(A/p^{n+1}A)$, an affine smooth $W_n(k)$-scheme, and for $n = \infty$ we put S^∞ = the p-adic completion of $S = \operatorname{Spec}(\varprojlim A/p^{n+1}A)$. (Function theoretically, $A^\infty = \varprojlim A/p^{n+1}A$ is the ring of those rigid analytic functions of norm ≤ 1 on the rigid analytic space underlying S which are defined over $W(k)$). For any affine $W(k)$-scheme T and any k-morphism $f_o : T_o \longrightarrow S_o$, there exists a compatible system of $W_n(k)$-morphisms $f_n : T_n \longrightarrow S_n$ with f_{n+1} lifting f_n (because T is affine and S smooth), or, equivalently, a $W(k)$-morphism $f : T^\infty \longrightarrow S^\infty$ lifting f_o. Of course, there is in general no unicity in the lifting f.

In particular, noting by σ the Frobenius automorphism of $W(k)$, there exists a σ-linear endomorphism φ of S^∞ which lifts the p'th power endomorphism of S_o. The interplay between S_o, S, S^∞ and φ is given by :

Lemma 1.1. (Tate-Monsky [24],[27]). *Denote by* $\mathbb{C}$ *the completion of the algebraic closure of the fraction field* K *of* $W(k)$, *and by* $\mathcal{O}_C$ *its ring of integers.*

1.1.1. *The successive inclusions between the sets below are all bijections*

a) *the* $\mathbb{C}$-*valued points of* S (*as* $W(k)$-*scheme*)

b) *the continuous* $W(k)$-*homomorphisms* $A^\infty \longrightarrow \mathcal{O}_C$

c) " " $A^\infty \longrightarrow \mathbb{C}$

d) *the closed points of* $S^\infty \otimes \mathbb{C}$.

1.1.2. *Every k-valued point e_o of S_o lifts uniquely to a W(k)-valued point e of S^∞ which verifies $\varphi \circ e = e \circ \sigma$. In fact, for any isometric extension $\bar{\sigma}$ of σ to C, e is the unique $\mathbb{C}$-valued point of S^∞ which lifts e_o and verifies $\varphi \circ e = e \circ \bar{\sigma}$. The point e is called the φ-Teichmuller representative of e_o. The Teichmuller points of S^∞ ($\mathbb{C}$-valued points e satisfying $\varphi \circ e = e \circ \bar{\sigma}$) are in bijective correspondence with the points of S_o with values in the algebraic closure $\bar{k}$ of k, and all take values in $W(\bar{k})$.*

(1.2) Let H be a locally free S^∞-module of finite rank, with an integrable connection ∇ (for the *continuous* derivations of $S^\infty/W(k)$) which is nilpotent. This means that for any continuous derivation D of $S^\infty/W(k)$ which is p-adically topologically nilpotent as additive endomorphism of A^∞, the additive endomorphism $\nabla(D)$ of H is also p-adically topologically nilpotent. For any affine W(k)-scheme T which is p-adically complete, any pair of maps

$$\begin{tikzcd} T \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & S^\infty \end{tikzcd}$$

which are congruent modulo a divided-power ideal of T ((p), for example), the connection ∇ provides an isomorphism

$$\chi(f,g) : f^*H \xrightarrow{\sim} g^*H \quad .$$

This isomorphism satisfies

(i) $\chi(g,h)\ \chi(f,g) = \chi(f,h)$ if
$$\begin{tikzcd} T \arrow[r, shift left=2, "f"] \arrow[r, "g"] \arrow[r, shift right=2, "h"'] & S^\infty \end{tikzcd}$$

(ii) $\chi(fk,gk) = k^*\chi(f,g)$ if
$$\begin{tikzcd} R \arrow[r, "k"] & T \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & S^\infty \end{tikzcd}$$

(iii) $\chi(id,id) = id$.

The universal example of such a situation
$$\begin{tikzcd} T \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & S^\infty \end{tikzcd}$$
is provided by

the "closed divided power neighborhood of the diagonal" P.D.-$\Delta(S^{\infty})$, with its two projections to S^{∞}. When, for examples, S is etale over $\mathbb{A}^{n}_{W(k)}$, P.D.-$\Delta(S^{\infty})$ is the spectrum of the ring of convergent divided power series over A^{∞} in n indeterminates, the formal expressions

$$\sum a_{i_1,\dots,i_n} \frac{t_1^{i_1}}{i_1!} \cdots \frac{t_n^{i_n}}{i_n!}$$

whose coefficients $a_{i_1,\dots,i_n}$ are elements of A^{∞} which tend to zero (in the p-adic topology of A^{∞}).

Any situation $T \underset{g}{\overset{f}{\rightrightarrows}} S^{\infty}$ of the type envisioned above can be factored uniquely

$$\begin{tikzcd} T \arrow[r, "f \times g"] & \text{P.D.-}\Delta(S^{\infty}) \arrow[r, shift left, "pr_2"] \arrow[r, shift right, "pr_2"'] & S^{\infty} \end{tikzcd} ,$$

and we have

$$\chi(f,g) = (f \times g)^{*} \chi(pr_1, pr_2) .$$

In fact, giving the isomorphism $\chi(pr_1, pr_2)$, subject to a cocycle condition, is equivalent to giving the nilpotent integrable connection ∇.

(1.3) We may now define an F-crystal $\underline{H} = (H, \nabla, F)$ as consisting of :

(1) a "differential equation" (H, ∇) as above

(2) for every lifting $\varphi : S^{\infty} \longrightarrow S^{\infty}$ of Frobenius, a horizontal morphism

$$F(\varphi) : \varphi^{*} H \longrightarrow H$$

which becomes an isomorphism upon tensoring with Q.

For different liftings φ_1, φ_2, we require the commutativity of the diagram below. (compare [11], section 5 and [12], section 2)

(1.3.1)

$$\begin{tikzcd}
\varphi_1^*H \arrow[r, "F(\varphi_1)"] \arrow[d, "\chi(\varphi_1,\varphi_2)"', "\wr"] & H \\
\varphi_2^*H \arrow[ur, "F(\varphi_2)"'] &
\end{tikzcd}$$

(1.4) Given a k-valued point e_o of S_o, let φ_1 and φ_2 be two liftings of Frobenius, and e_1 and e_2 the corresponding Teichmuller representatives. By inverse image, we obtain two F-crystals on W(k), $(e_1^*H, e_1^*(F(\varphi_1))$ and $(e_2^*H, e_2^*F(\varphi_2))$ which are explicitly isomorphic

$$\begin{tikzcd}
(e_1^*H)^{(\sigma)} \arrow[r, "e_1^*(F(\varphi_1))"] \arrow[d, "\sigma^*\chi(e_1,e_2)"', "\wr"] & e_1^*H \arrow[d, "\chi(e_1,e_2)", "\wr"'] \\
(e_2^*H)^{(\sigma)} \arrow[r, "e_2^*(F(\varphi_2))"] & e_2^*H
\end{tikzcd}$$

We thus obtain an F-crystal on W(k) (a free W(k)-module of finite rank together with a σ-linear endomorphism which is an isomorphism over K) which depends *only* on the point e_o of S_o. In case k is a finite field $\mathbb{F}_{p^n}$, then for every multiple, m, of n, the m-th iterate of the σ-linear endomorphism is *linear* over $W(\mathbb{F}_{p^m})$. Its characteristic polynomial $\det(1 - t F^m)$ is denoted

$$P(\underline{H}; e_o, \mathbb{F}_{p^m}, t) .$$

2. *F-crystals over* W(k) *and their Newton polygons* [19].

Theorem 2.(Manin-Dieudonné). *Let* (H,F) *be an* F-*crystal over* h/(k), *and suppose* k *algebraically closed*.

2.1. H *admits an increasing finite filtration of* F-*stable sub-modules*

$$0 \subset H_o \subset H_1 \subset \dots$$

whose associated graded is free, with the following property. There exists a sequence of rational numbers in "lowest terms"

$$0 \leq \frac{a_o}{n_o} < \frac{a_1}{n_1} < \frac{a_2}{n_2} < \dots$$

(if $a_o = 0$, $n_o = 1$; $n_i \geq 1$, $a_i \geq 0$, and $(a_i, n_i) = 1$ if $a_i \neq 0$)

such that

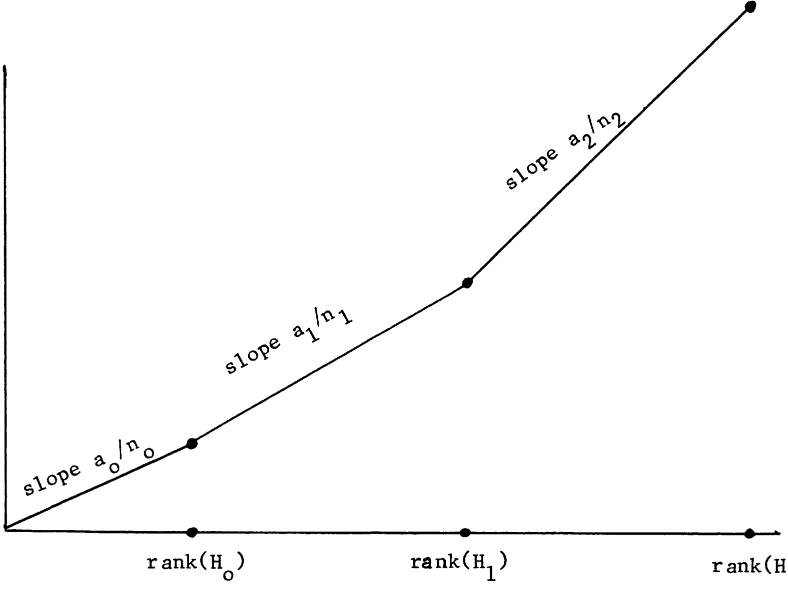
2.1.1. $(H_i/H_{i-1}) \otimes K$ admits a K-base of vectors x which satisfy $F^{n_i}(x) = p^{a_i}x$, and its dimension is a multiple of n_i .

2.1.2. If $a_o/n_o = 0$, then H_o itself admits a $W(k)$ base of elements x satisfying $Fx = x$, F is topologically nilpotent on H/H_o , and the rank of H_o is equal to the stable rank of the p-linear endomorphism of the k-space H/pH induced by F ; H_o is then called the "unit root part" of H , or the "slope zero" part.

2.1.3. If (H,F) is deduced by extension of scalars from an F-crystal $(\mathbb{H},\mathbb{F})$ over $W(k_o)$, k_o a perfect subfield of k , then the filtration descends to an $\mathbb{F}$-stable filtration of $\mathbb{H}$. In case k_o is a finite field $\mathbb{F}_{p^n}$, the eigenvalues of $\mathbb{F}^n$ on the i'th associated graded have p-adic ordinal na_i/n_i .

2.2. The rational numbers a_i/n_i are called the slopes of the F-crystal, and the ranks of H_i/H_{i-1} are called the multiplicities of the slopes. The slopes and their multiplicities characterize the F-crystal up to isogeny.

It is convenient to assemble the slopes and their multiplicities in the Newton polygon



When (H,F) comes by extension of scalars from $(\mathbb{H},\mathbb{F})$ over $W(\mathbb{F}_{p^n})$, this Newton polygon is the "usual" Newton polygon of the characteristic polynomial $P(\underline{\mathbb{H}};e_0,\mathbb{F}_{p^n},t)$, calculated with the ordinal function normalized by $\mathrm{ord}(p^n) = 1$.

3. Local Results ; F-crystals on $W(k)[[t_1,\ldots t_n]]$.

(3.0) The completion of S^∞ along a k-valued point e_0 of S_0 is (non-canonically) isomorphic to the spectrum of $W(k)[[t_1,\ldots,t_n]]$. In this optic, the set of W(k)-valued points of S^∞ lying over e_0 becomes the n-fold product of pW(k) , and the set of $\mathcal{O}_{\mathbb{C}}$-valued points of S^∞ lying over e_0 becomes the n-fold product of the maximal ideal of $\mathcal{O}_{\mathbb{C}}$ (namely, the values of $t_1,\ldots,t_n$).

By inverse image, any F-crystal on S^∞ gives an F-crystal on $W(k)[[t_1,\ldots,t_n]]$.

Proposition 3.1. *Let* (H,∇,F) *be an* F-*crystal over* $W(k)[[t_1,\dots,t_n]]$.

3.1.1. *Let* $W(k)\langle\langle t_1,t_n\rangle\rangle$ *denote the ring of convergent divided power series over* W(k) (cf. 1.2). *Then* $H \otimes W(k) \langle\langle t_1,\dots,t_n\rangle\rangle$ *admets a basis of horizontal* (*for* ∇) *sections*.

3.1.2. *Let* $K\{\{t_1,\dots,t_n\}\}$ *denote the ring of power series over* K *which are convergent in the open polydisc of radius one* (i.e. *series* $\Sigma\, a_{i_1\dots i_n} t_1^{i_1}\dots t_n^{i_n}$ *such that for every real number* $0 \leq r < 1$, $|a_{i_1\dots i_n}| r^{i_1+\dots+i_n}$ *tends to zero*). *Then* $H \otimes K\{\{t_1,\dots,t_n\}\}$ *admits a basis of horizontal sections*.

3.1.3. *Every horizontal section of* $H \otimes W(k)\langle\langle t_1,\dots,t_n\rangle\rangle$ *fixed by* F *"extends" to a horizontal section of* H (i.e. *over all of* $W(k)[[t_1,\dots,t_n]]$).

Proof: 3.1.1. is completely formal : the two homomorphisms $f,g : W(k)[[t_1,\dots,t_n]] \longrightarrow W(k)\langle\langle t_1,\dots,t_n\rangle\rangle$ given by f = natural inclusion, g = evaluation e at (0,...,0), followed by the inclusion of W(k) in $W(k)\langle\langle t_1,\dots,t_n\rangle\rangle$, are congruent modulo the divided power ideal $(t_1,\dots t_n)$ of the p-adically complete ring $W(k)\langle\langle t_1,\dots,t_n\rangle\rangle$. Thus $\chi(f,g)$ is an isomorphism between $H \otimes W(k)\langle\langle t_1,\dots,t_n\rangle\rangle$ with its induced connection and the "constant" module $H(0,\dots,0) \otimes_{W(k)} W(k)\langle\langle t_1,\dots,t_n\rangle\rangle$ with connection $1 \otimes d$.

3.1.2. is more subtle. Let's choose a particularly simple φ (as we may using 1.3.1), the one which sends $t_i \longrightarrow t_1^p$, i=1,...,n , and is σ-linear. Choose a *basis* of the free $W(k)[[t_1,\dots,t_n]]$ module H , and let A_φ denote the matrix of

$F(\varphi) : \varphi^*H \longrightarrow H$. Denote by Y the matrix with entries in $W(k)\langle\langle t_1,\dots,t_n\rangle\rangle$ whose columns are a basis of horizontal sections of $H \otimes W(k)\langle\langle t_1,\dots,t_n\rangle\rangle$ (a "fundamental solution matrix") ; in the notation of (2) above, it's the matrix of $\chi(g,f)$. Because $F(\varphi)$ is <u>horizontal</u>, we have the matricial relation

$$A_\varphi \cdot \varphi(Y) = Y \cdot A_\varphi(0,\dots,0) \quad .$$

We must deduce that Y converges in the open unit polydisc. We know this is true of A_φ , as it even has coefficients in $W(k)[[t_1,\dots,t_n]]$. Since $A_\varphi(0,\dots,0)$ is invertible over K by definition of an F-crystal, we conclude that for any real number $0 \leq r < 1$, we have the implication

> $\varphi(Y)$ converges in the polydisc of radius $r \Longrightarrow Y$ converges in the polydisc of radius r .

On the other hand, writing $Y = \Sigma\, Y_{i_1,\dots,i_n} t_1^{i_1} \dots t_n^{i_n}$, we have $\varphi(Y) = \Sigma\ \sigma(Y_{i},\dots,i_n)\, t_{i_1}^{pi_1} \dots t_{i_n}^{pi_n}$, whence for any real $r \geq 0$, we have the implication

> Y converges in the polydisc of radius $r \Longrightarrow \varphi(Y)$ converges in the polydisc of radius $r^{1/p}$.

Since Y has entries in $W(k)\langle\langle t_1,\dots,t_n\rangle\rangle$, it converges in the polydisc of radius $r_o = |p|^{1/p-1}$, hence, iterating our two implications, in the polydisc of radius r_o^{1/p^n} for every n ; as $\lim(r_o)^{1/p^n} = 1$, we are done.

3.1.3. is similar to 3.1.2, only easier. If y is a column vector with entries in $W(k)\langle\langle t_1,\dots,t_n\rangle\rangle$ satisfying

$$A_\varphi \cdot \varphi(y) = y$$

then for every integer $m \geq 1$ we have

$$A_\varphi . \varphi(A_\varphi) . \varphi^2(A_\varphi) \ldots \varphi^{m-1}(A_\varphi) . \varphi^m(y) = y$$

Since $\varphi^m(y)$ is congruent to $\sigma^m(y(0,\ldots,0)$ modulo $(t_1^{pm},\ldots,t_n^{pm})$, we have a $(t_1,\ldots,t_n)$-adic limit formula for y

$$y = \lim_{n\to\infty} A_\varphi . \varphi(A_\varphi) \ldots \varphi^{n-1}(A_\varphi) \sigma^n(\varphi(0,\ldots,0))$$

which shows that y has entries in $W(k)[[t_1,\ldots,t_n]]$.

Q.E.D.

Remark 3.2. 3.1.2 shows that "most" differential equations do not admit any structure of F-crystal. For example, the differential equation for $\exp(t^{p^n})$ is nilpotent provided $n \geq 1$, but its local solutions around any point $\alpha \in \mathcal{O}_{\mathbb{C}}$ converge only in the disc of radius $|p|^{1/p^n(p-1)}$:

The meaning of 3.1.2 is this : for any two points e_1, e_2 of S^∞ with values in $\mathcal{O}_{\mathbb{C}}$ which are sufficiently near (congruent modulo $p^{1/p-1}$), the connection provides an explicit isomorphism of the two $\mathcal{O}_{\mathbb{C}}$-modules $e_1^*(H)$ and $e_2^*(H)$. If the two points are further apart, but still congruent modulo the maximal ideal of $\mathcal{O}_{\mathbb{C}}$, 3.1.2 says the connection still gives an explicit isomorphism of the $\mathbb{C}$-vector spaces $e_1^*(H) \otimes \mathbb{C}$ and $e_2^*(H) \otimes \mathbb{C}$.

4. *Global results* : *gluing together the "unit root" parts* ([11], thm 4.1)

(4.0) Given an F-crystal $\underline{H} = (H,\nabla,F)$ and an integer $n \geq 0$, we denote by $\underline{H}(-n)$ the F-crystal (H,∇,p^nF). An F-crystal of the form $\underline{H}(-n)$ necessarily has all its slopes $\geq n$, though the converse need not be true.

Theorem 4.1. *Suppose* k *algebraically closed, and* $\underline{H}$ *an* F-*crystal on* S^∞ *such that at every* k-*valued point of* S_o, *its Newton polygon begins with a side of slope zero* , *always of the same length* $\nu \geq 1$ (i.e., *point by point, the unit root part has rank* ν). *Suppose further that there exists a locally free submodule* $\mathrm{Fil} \subset H$ *such that* H/Fil *is locally free of rank* ν , *and such that for every lifting* φ *of Frobenius, we have*

$$F(\varphi)\ (\varphi^*\mathrm{Fil}) \subset p\,H \quad .$$

Then there exists a sub-crystal $\underline{U} \subset \underline{H}$, *of rank* ν , *whose underlying module* U *is transversal to* Fil $(H = U \oplus \mathrm{Fil})$ *such that*

4.1.1. F *is an isomorphism on* $\underline{U}$.

4.1.2. *The connection* ∇ *on* $\underline{U}$ *prolongs to a stratification.*

4.1.3. *The quotient* F-*crystal* $\underline{H}/\underline{U}$ *is of the form* $\underline{V}(-1)$.

4.1.4. *The extension of* F-*crystals* $0 \to \underline{U} \to \underline{H} \to \underline{H}/\underline{U} \to 0$ *splits when pulled back to* $W(k)$ *along any* $W(k)$-*valued point of* S^∞ .

4.1.5. *If the situation* $(\underline{H}, \mathrm{Fil})$ *on* $S^\infty/W(k)$ *comes by extension of scalars from a situation* $(\mathbb{H} , \mathbb{F}\mathrm{il})$ *on* $\mathbb{S}^\infty/W(k_o)$, k_o *a perfect subfield of* k , *the* F-*crystal* $\underline{U}$ *descends to an* F-*crystal* $\underline{\mathbb{U}}$ *on* $\mathbb{S}^\infty/W(k_o)$.

Proof. We may assume Fil, H and H/Fil are free, say of ranks $r-\nu$, r and ν. In terms of a basis of H adopted to the filtration Fil $\subset$ H, the matrix of $F(\varphi)$ for some fixed choice of φ is of the form

$$\begin{array}{c|cc} & r-\nu & \nu \\ \hline r-\nu & pA & C \\ \nu & pB & D \end{array} \quad .$$

The hypothesis that there be ν unit root point by point means D is invertible. Let's begin by finding for a free submodule $U \subset H$ which is transversal to Fil and stable by $F(\varphi).\varphi^*$. This means finding an $r-\nu \times \nu$ matrix η, such that the submodule of H spanned by the colums of

$$\begin{pmatrix} \eta \\ I \end{pmatrix}$$

(I denoting the $\nu \times \nu$ identity matrix) is stable under $F(\varphi) \circ \varphi^*$. But

$$F(\varphi)\varphi^*\begin{pmatrix} \eta \\ I \end{pmatrix} = \begin{pmatrix} pA & C \\ pB & D \end{pmatrix}\begin{pmatrix} \varphi^*(\eta) \\ I \end{pmatrix} = \begin{pmatrix} pA\varphi^*(\eta)+C \\ pB\varphi^*(\eta)+D \end{pmatrix} ,$$

so that F-stability of $\begin{pmatrix} \eta \\ I \end{pmatrix}$ is equivalent to having

$$\begin{pmatrix} pA\varphi^*(\eta)+C \\ pB\varphi^*(\eta)+D \end{pmatrix} = \begin{pmatrix} \eta(pB\varphi^*(\eta)+D) \\ I(pB\varphi^*(\eta)+D \end{pmatrix} ,$$

or equivalently (D being invertible) that η satisfy

4.1.6 $$\eta = (pA\varphi^*(\eta) + C)(I + pD^{-1}B\varphi^*(\eta))^{-1}\cdot D^{-1} \ .$$

Because the endomorphism of $r-\nu\times\nu$ matrices given by

(4.1.7) $$\eta \longrightarrow (pA\ \varphi^*(\eta) + C)(I+pD^{-1}B\varphi^*(\eta))^{-1}\cdot D^{-1}$$

is a <u>contraction mapping</u> in the p-adic topology of A^{∞} , it has a unique fixed point.

In order to prove that U is horizontal, it suffices to do so over the completion of S^{∞} along any closed point e_o of S_o . Let e be the φ-Teichmuller point of S^{∞} with values in $W(k)$ lying over e_o . By hypothesis, $e^*(H)$ contains ν fixed points of $e^*(F(\varphi))$ which span a direct factor of $e^*(H)$, which is necessarily transverse to $e^*(\mathrm{Fil})$. By 3.1.3, these fixed points extend to horizontal sections over $H \otimes W(k)[[t_1,\dots,t_n]] \overset{\mathrm{dfn}}{=\!=} \hat{H}(e)$, which span a direct factor of $\hat{H}(e)$, still transversal to $\mathrm{Fil}(e)$. Write these sections as column vectors :

$$\begin{array}{rl} \begin{array}{c} r-\nu \\[1em] \nu \end{array} & \begin{pmatrix} S_2 \\[1em] S_1 \end{pmatrix} \\ & \quad \nu \end{array} \in M_{r,\nu}(W(k)[[t_1,\dots,t_n]]) \ .$$

By transversality we have S_1 invertible. The fixed-point property is

$$\begin{pmatrix} pA & C \\ pB & D \end{pmatrix} \begin{pmatrix} \varphi^*(S_2) \\ \varphi^*(S_1) \end{pmatrix} = \begin{pmatrix} S_2 \\ S_1 \end{pmatrix}$$

or equivalently

$$\begin{pmatrix} pA & C \\ pB & D \end{pmatrix} \begin{pmatrix} \varphi^*(S_2 S_1^{-1}) \\ I \end{pmatrix} = \begin{pmatrix} S_2 S_1^{-1} \cdot S_1 \varphi^*(S_1^{-1}) \\ S_1 \varphi^*(S_1^{-1}) \end{pmatrix} .$$

Let's put $\mu = S_2 \cdot S_1^{-1}$; we have

$$\begin{cases} pA\varphi^*(\mu) + C = \mu\, S_1 \varphi^*(S_1^{-1}) \\ pB\varphi^*(\mu) + D = S_1 \varphi^*(S_1^{-1}) \end{cases}$$

so μ satisfies $\mu = (pA\varphi^*(\mu) + C)\cdot(1+pD^{-1}B\varphi^*(\mu))^{-1}D^{-1}$.

Since the endomorphism of $M_{r-\nu,\nu}(W(k)[[t_1,\ldots,t_n]])$ defined by 4.1.7 is still a contraction mapping in its p-adic topology, it follows that μ is its unique fixed point, and hence that μ is the power series expansion of our global fixed point η near e_o . This proves that

4.1.8. <u>the inverse image</u> $\hat{U}(e)$ <u>of</u> U <u>over</u> $W(k)[[t_1,\ldots,t_n]]$ <u>is the module spanned by the horizontal fixed points of</u> $F(\varphi)\cdot\varphi^*$ <u>in</u> $\hat{H}(e)$. <u>Hence</u> $\hat{U}(e)$ <u>is horizontal, and stratified, which proves</u> 4.1.2.

4.1.9. <u>The matrices</u> $\mu = S_2 S_1^{-1}$ <u>and</u> $S_1\varphi^*(S_1^{-1})$ <u>with entries in</u> $W(k)[[t_1,\ldots,t_n]]$ <u>are the local expansion of the global matrices</u> η <u>and</u> $pB\varphi^*(\eta) + D$ <u>respectively</u>. <u>This is an example of analytic continuation par excellence</u>.

To see that U is F-stable, notice that once we know it's horizontal, it suffices for it to be $F(\varphi)$-stable for <u>one</u> choice of φ (as it is), thanks to 1.3.1. In terms of the new base of H , adopted to $H = \text{Fil} \oplus U$, the matrix of $F(\varphi)$ is

$$\begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} pA & C \\ pB & D \end{pmatrix} \begin{pmatrix} 1 & \varphi^*(\eta) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} pA - p\eta B & 0 \\ pB & D+pB\varphi^*(\eta) \end{pmatrix}$$

which proves 4.1.1 and 4.1.3. That 4.1.5 holds is clear from the "rational" way η was determined.

It remains to prove 4.1.4. The matrix of F in $M_r(W(k))$ looks like

$$\begin{array}{c|cc} r-\nu & pa & 0 \\ \nu & pb & d \\ \hline & r-\nu & \nu \end{array}$$

in a base adopted to $H = \mathrm{Fil} \oplus U$, with d invertible. It's again a fixed point problem, this time to find a matrix $E \in M_{\nu,r-\nu}(W(k))$ so that the span of the column vectors $\begin{pmatrix} I \\ pE \end{pmatrix}$ is F-stable. But

$$\begin{pmatrix} pa & 0 \\ pb & d \end{pmatrix} \begin{pmatrix} I \\ \sigma p(E) \end{pmatrix} = \begin{pmatrix} pa \\ pb + \cdot pd\sigma(E) \end{pmatrix} \quad ,$$

so F-stability is equivalent to the equation

$$\begin{pmatrix} pa \\ pb + pd\sigma(E) \end{pmatrix} = \begin{pmatrix} pa \\ pE.pa \end{pmatrix} \quad .$$

Thus E must be a fixed point of $E \longrightarrow \sigma^{-1}(-d^{-1}b+pd^{-1}Ea)$, which is again a contraction of $M_{\nu,r-\nu}(W(k))$. Q.E.D.

5. Hodge F-crystals ([20])

5.0. A Hodge F-crystal is an F-crystal (H,∇, F) together with a finite decreasing "Hodge filtration" $H = \mathrm{Fil}^0 \supset \mathrm{Fil}^1 \supset \ldots$ by locally free sub-modules with locally free quotients, subject to the transversality condition

5.0.1 $$\nabla \mathrm{Fil}^i \subset \mathrm{Fil}^{i-1} \otimes \Omega^1$$

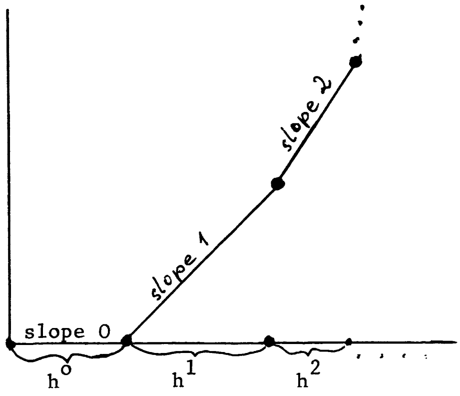
Its Hodge numbers are the integers $h^i = \mathrm{rank}\ (\mathrm{Fil}^i/\mathrm{Fil}^{i+1})$.

A Hodge F-crystal is called divisible if for some lifting φ of Frobenius, we have

5.0.2 $$F(\varphi)\ (\varphi^*(\mathrm{Fil}^i)) \subset p^i H \qquad \text{for } i = 0, 1,\ldots$$

It is rather striking that if p is sufficiently large that $\mathrm{Fil}^p = 0$, then 5.0.2 will hold for every choice of φ if it holds for one. [To see this, one uses the explicit formula (1.3.1) for the variation of $F(\varphi)$ with φ , transversality (5.0.1), and the fact that the function $f(n) = \mathrm{ord}(p^n/n!)$ satisfies $f(n) \geq \inf(n, p-1)$ for $n \geq 1$.]

The Hodge polygon assosciated to the Hodge numbers $h^0, h^1,\ldots$ is the polygon which has slope ν with multiplicity h^ν :



By looking at the first slopes of all exterior powers, one sees:

Lemma 5.1. *The Newton polygon of a divisible Hodge* F-*crystal is always above* (*in the* (x, y) *plane*) *its Hodge polygon.*

5.2. A Hodge F-crystal is called autodual of weight N if H is given a horizontal autoduality $< , > : H \otimes H \longrightarrow \mathcal{O}_{S^\infty}$ such that

5.2.1 *the Hodge filtration is self-dual, meaning* $\perp(\mathrm{Fil}^i) = \mathrm{Fil}^{N+1-i}$.

5.2.2 F *is* p^N-*symplectic*, *meaning that for* x, y ∈ H , and any *lifting* φ , *we have* $< F(\varphi)(\varphi^*x), F(\varphi)(\varphi^*y)> = p^N\varphi^*(< x, y>)$.

The Newton polygon of an autodual Hodge F-crystal of weight N is symmetric, in the sense that its slopes are rational numbers in $[0, N]$ such that the slopes α and $N-\alpha$ occur with the same multiplicity.

As an immediate corollary of 4.1, we get

Corollary 5.3. *Let* $(H, \nabla, F, \mathrm{Fil}, < , >)$ *be an autodual divisible Hodge F-crystal, whose Newton polygon over every closed point of* S_o *has slope zero with multiplicity* h^o . *Then* *H* *admits a three-step* *filtration*

$$\underline{U} \subset \perp(\underline{U}) \subset \underline{H}$$

with:

5.3.1. $\underline{U}$ *the "unit root" part of* $\underline{H}$, *from* 4.1.

5.3.2. $\underline{H}/\perp(\underline{U})$ *is of the form* $\underline{V}_N(-N)$, *where* $\underline{V}_N$ *is a unit-root* F-*crystal* (*its* F *is an isomorphism*).

5.3.3. $\perp(\underline{U})/\underline{U}$ *is of the form* $\underline{H}_1(-1)$, *where* $\underline{H}_1$ *is an autodual divisible Hodge F-crystal of weight* N-2 .

Similarly, we have

Corollary 5.4. *Suppose* $(H, \nabla, F, \mathrm{Fil})$ *is a Hodge* F-*crystal whose Newton polygon coincides with its Hodge polygon over every closed point of* S_o . *Then* $\underline{H}$ *admits a finite increasing filtration*

$$0 \subset \underline{U}_0 \subset \underline{U}_1 \subset \ldots$$

such that

5.4.1. $\underline{U}_i/\underline{U}_{i+1}$ *is of the form* $\underline{V}_i(-i)$, *with* $\underline{V}_i$ *a unit-root* F-*crystal* (F *an isomorphism*)

5.4.2. *the filtration is transverse to the Hodge filtration:*

$H = \mathrm{Fil}^i \oplus U_{i-1}$.

5.4.3. *if* $(H, \nabla, F, \mathrm{Fil})$ *admits an autoduality of weight* N , *the filtration by the* U_i *is autodual:* $\perp(U_i) = U_{N-1-i}$.

Remark 5.5. F-crystals and p-adic representations.

The category of "unit-root" F-crystals on S^∞ (F an isomorphism), such as the V_i occurring in 5.4, is equivalent to the category of continuous representations of the fundamental group $\pi_1(S_o)$ on free $\mathbb{Z}_p$-modules of finite rank (i.e., to the category of "constant tordu" étale p-adic sheaves on S_o).

[Given $\underline{H}$ and a choice of φ , one shows successively that for each $n \geq 0$, there exists a finite étale covering T_n of S_n over which $H/p^{n+1}H$ admits a basis of fixed points of $F(\varphi)\cdot\varphi^*$. The fixed points form a free $\mathbb{Z}/p^{n+1}\mathbb{Z}$ module of rank = rank (H) , on which $\mathrm{Aut}(T_n/S_n)$, hence $\pi_1(S_n) = \pi_1(S_o)$ acts. For n variable, these representations fit together to give the desired p-adic representation of $\pi_1(S_o)$. This construction is inverse to the natural functor from constant tordu p-adic étale sheaves on S_o to F-crystals on S^∞ with F invertible].

6. A conjecture on the L-function of an F-crystal.

6.0. Suppose $\underline{H}$ is an F-crystal on $S^\infty/W(\mathbb{F}_q)$. Denote by Δ_n the points of S_o with values in $\mathbb{F}_{q^n}$ which are of degree precisely n over $\mathbb{F}_q$. The L-function of $\underline{H}$ is the formal power series in $1 + tW(\mathbb{F}_q)[[t]]$ defined by the infinite product (cf. [13], [26])

$$L(\underline{H};\, t) = \prod_{n\geq 1}\ \prod_{e_o\in\Delta_n}\left[P(\underline{H};\, e_o,\, \mathbb{F}_{q^n},\, t^n)\right]^{-1/n}$$

When $\underline{H}$ is a unit root F-crystal, its L-function is the L-function associated to the corresponding étale p-adic sheaf (cf. [13], [26]).

Conjecture 6.1. (cf. [8], [13])

6.1.1. $L(\underline{H};\ t)$ is p-adically meromorphic.

6.1.2. if $\underline{H}$ is a unit root F-crystal, denote by M the corresponding p-adic étale sheaf on S_o, and by $H^i_c(M)$ the étale cohomology groups with compact supports of the geometric fibre $\bar{S}_o = S_o \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ with coefficients in M. These are $\mathbb{Z}_p$-modules of finite rank, zero for $i > \dim S_o$, on which $\mathrm{Gal}(\bar{\mathbb{F}}_q / \mathbb{F}_q)$ acts. Let $f \in \mathrm{Gal}(\bar{\mathbb{F}}_q / \mathbb{F}_q)$ denote the inverse of the automorphism $x \longrightarrow x^q$. Then the function

$$L(\underline{H};\ t) \cdot \prod_{i=0}^{\dim S_o} \det (1 - tf | H^i_c(M))^{(-1)^i}$$

has neither zero nor pole on the circle $|t| = 1$.

Remarks 6.1.1. is (only) known in cases where the F-crystal $\underline{H}$ on S^∞ "extends" to the Washnitzer-Monsky weak completion S^+ of S ([23]), in which case it follows from the Dwork-Reich-Monsky fixed point formula ([4], [25], [24]). Unfortunately, such cases are as yet relatively rare (but cf. [10] for a non-obvious example). It is known ([12a]) that when $S_o = \mathbb{A}^n$, then $L(\underline{H};\ t)$ is meromorphic in the closed disc $|t| \leq 1$. The extension to general S_o of this result should be possible by the methods of ([25]); it would at least make the second part 6.1.2 of the conjecture meaningful. As for 6.1.2 itself, it doesn't seem to be known for any non-constant M. Even for $M = \mathbb{Z}_p$, when L = zeta of S_o, 6.1.2 has only been checked for curves and abelian varieties.

7. F-crystals from geometry ([1], [2])

Let $f : X \longrightarrow S^{\infty}$ be a proper and smooth morphism, with geometrically connected fibres, whose de Rham cohomology is locally free (to avoid derived categories!). Crystalline cohomology tells us that for each integer $i \geq 0$, the de Rham cohomology $H^i = \mathbb{R}^i f_*(\Omega^{\bullet}_{X/S^{\infty}})$ with its Gauss-Manin connection ∇ is the underlying differential equation of an F-crystal $\underline{H}^i$ on S^{∞}. When k is finite, say $\mathbb{F}_q$, then for every point e_o of S_o with values in $\mathbb{F}_{q^n}$, the inverse image X_{e_o} of X over e_o is a variety over $\mathbb{F}_{q^n}$, and its zeta function is given by (cf. 1.4)

$$\mathrm{Zeta}(X_{e_o} / \mathbb{F}_{q^n} ; t) = \prod_{i=0}^{2\dim X_{e_o}} P(\underline{H}^i ; e_o, \mathbb{F}_{q^n}, t)^{(-1)^{i+1}}$$

If in addition we suppose that the Hodge cohomology of X/S^{∞} is locally free, and that X/S^{∞} is projective, then according to Mazur [20], the Hodge F-crystal $\underline{H}^i$ is divisible, provided that $p > i$.

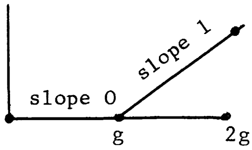
For every p and i we have $F(\varphi)\varphi^*(\mathrm{Fil}^1) \subset p\, H^i$, and the p-linear endomorphism of $H^i/pH^i + \mathrm{Fil}^1 \simeq R^i f_*(\mathcal{O}_X)/pR^i f_*(\mathcal{O}_X) = R^i f_{o*}(\mathcal{O}_{X_o})$ ($f_o : X_o \longrightarrow S_o$ denoting the "reduction modulo p" of $f : X \longrightarrow S^{\infty}$) is the classical Hasse-Witt operation, deduced from the p'th power endomorphism of $\mathcal{O}_{X_o}$. Thus if Hasse-Witt is invertible, we may apply 4.1 to the situation $\underline{H}^i$, $H^i \supset \mathrm{Fil}^1$.

When X/S^{∞} is a smooth hypersurface in $\mathbb{P}^{N+1}_{S^{\infty}}$ of degree prime to p which satisfies a mild technical hypothesis of being "in general position", Dwork gives ([5], [7]) an a priori description of an

F-crystal on S^{∞} whose underlying differential equation is (the primitive part of $H^{N}_{DR}(X/S^{\infty})$ with its Gauss-Manin connection, and whose characteristic polynomial is the "interesting factor" in the zeta function ([14]). The identification of Dwork's F with the crystalline F follows from [14] and (as yet unpublished) work of Berthelot and Meredith (c.f. the Introduction to [2]) relating the crystalline and Monsky-Washnitzer theories ([23], [24]). Dwork's F-crystal is <u>isogenous</u> to a divisible one for <u>every</u> prime p ([7], lemma 7.2).

8. Local study of ordinary curves : Dwork's period matrix T ([11])

7.0. Let $f : X \longrightarrow \mathrm{Spec}(W(k)[[t_1,\dots,t_n]])$ be a proper smooth curve of genus $g \geq 1$. It's crystalline $\underline{H}^1$ is an autodual (cup-product) divisible Hodge F-crystal of weight 1. We assume that it is ordinary, in the sense that modulo p its Hasse-Witt matrix is invertible, or equivalently that its Newton polygon is



(this means geometrically that the jacobian of the special fibre has p^g points of order p). Let's also suppose k algebraically closed, and denote by e the homomorphisme "evaluation at $(0,\dots,0)$": $W(k)[[t_1,\dots,t_n]] \longrightarrow W(k)$. By 2.1.2 and 4.1.4 , $e^*(H^1)$ admits a symplectic base of F-eigenvectors

$$\alpha_1,\dots,\alpha_g \ , \ \beta_1,\dots,\beta_g$$

satisfying

7.0.1
$$\begin{cases} e^*(F)(\alpha_i) = \alpha_i, \ e^*(F)(\beta_i) = p\beta_i \\ < \alpha_i,\alpha_j> = < \beta_i,\beta_j> = 0 \ , \\ < \alpha_i,\beta_j> = - < \beta_j,\alpha_i> = \delta_{ij} \end{cases} .$$

By 3.1.2, this base is the value at $(0,\dots,0)$ of a horizontal base of $H^1 \otimes K\{\{t_1,\dots,t_n\}\}$, which we continue to note $\alpha_1,\dots,\alpha_g$, $\beta_1,\dots,\beta_g$. For each choice of lifting φ , we have

7.0.2 $$\begin{cases} F(\varphi)(\varphi^*(\alpha_i)) = \alpha_i \\ \\ F(\varphi)(\varphi^*(\beta_i)) = p\beta_i \end{cases} .$$

According to 3.1.3, the sections $\alpha_1,\ldots,\alpha_g$ extend to horizontal sections over "all" of H^1, where they span the submodule U of 4.1; in general the β_i do not extend to all of H^1 .

We now wish to express the position of the Hodge filtration $\mathrm{Fil}^1 \subset H$ in terms of the horizontal "frame" provided by the α_i and β_j . Since $H^1 = U \oplus \mathrm{Fil}^1$ is a decomposition of H^1 in submodules isotropic for $< , >$, there is a base $\omega_1,\ldots,\omega_g$ of Fil^1 dual to the base $\alpha_1,\ldots,\alpha_g$ of U .

7.0.3 $$<\omega_i,\omega_j> = 0 \ , \ <\alpha_i, \omega_j> = \delta_{ij} \qquad .$$

In $H \otimes K\{\{t_1,\ldots,t_n\}\}$, the differences $\omega_i - \beta_i$ are orthogonal to U , hence lie in U :

7.0.4 $$\omega_i - \beta_i = \sum_j \tau_{ji}\, \alpha_j \quad ; \quad \tau_{ji} = <\omega_i,\beta_j> \quad .$$

The matrix $T = (\tau_{ij})$ is Dwork's "period matrix" ; it has entries in $W(k) \ll t_1,\ldots,t_n \gg \cap K\{\{t_1,\ldots,t_n\}\}$. Differentiating 7.0.4 via the Gauss-Manin connection, we see :

Lemma 7.1. T *is an indefinite integral of the matrix of the mapping "cup-product with the Kodaira-Spencer class" : for every continuous W(k)-derivation* D *of* $W(k)[[t_1,\ldots,t_n]]$, D(T) *is the matrix of the composite*

7.1.1 $$\mathrm{Fil}^1 \hookrightarrow H^1 \xrightarrow{\nabla(D)} H^1 \xrightarrow{\mathrm{proj}} H/\mathrm{Fil}^1 \simeq U$$

expressed in the dual bases $\omega_1,\ldots,\omega_g$ and $\alpha_1,\ldots,\alpha_g$.

Lemma 7.2. For any lifting φ of Frobenius, we have the following congruences on the τ_{ij} :

7.2.1 $$\varphi^*(\tau_{ij}) - p\,\tau_{ij} \in pW(k)[[t_1,\ldots,t_n]]$$

7.2.2 $$\tau_{ij}(0,\ldots,0) \in pW(k) \quad .$$

Proof. Applying $F(\varphi)\circ\varphi^*$ to the defining equation (7.0.4), we get

$$F(\varphi)(\varphi^*(\omega_i)) - p\,\beta_i = \sum_j \varphi^*(\tau_{ji})\,\alpha_j \quad .$$

Subtracting p times (7.0.4), we are left with

$$F(\varphi)(\varphi^*(\omega_i)) - p\,\omega_i = \sum_j [\varphi^*(\tau_{ji}) - p\tau_{ji}]\,\alpha_j \quad .$$

Since the left side lies in pH^1 , we get

$$\varphi^*(\tau_{ij}) - p\tau_{ij} = \langle F(\varphi)\varphi^*(\omega_i) - p\omega_i, \omega_j\rangle \in pW(k)[[t_1,\ldots,t_n]].$$

To prove that $\tau_{ij}(0,\ldots,0) \in pW(k)$, choose a lifting φ which preserves $(0,\ldots,0)$, for instance, $\varphi(t_i) = t_i^p$ for $i = 1,\ldots,n$, and evaluate (7.2.1) at $(0,\ldots,0)$:

$$\sigma(\tau_{ij}(0,\ldots,0)) - p\tau_{ij}(0,\ldots,0) \in pW(k) \quad .$$

which implies $\tau_{ij}(0,\ldots,0) \in pW(k)$!

QED.

7.3. According to a criterion of Dieudonné and Dwork ([3]), these congruences for $p \neq 2$ imply that the formal series

$$q_{ij} \overset{\text{defn}}{=\!=} \exp(\tau_{ij})$$

lie in $W(k)[[t_1,\ldots,t_n]]$, and have constant terms in $1 + pW(k)$. (When $p = 2$, we cannot define q_{ij} unless τ_{ij} has constant term $\equiv 0$ (4), in which case we would again have the q_{ij} in $W(k)[[t_1,\ldots,t_n]]$).

It is expected that the g^2 principal units q_{ij} in $W(k)[[t_1,\ldots,t_n]]$ are the Serre-Tate parameters of the particular lifting to $W(k)[[t_1,\ldots,t_n]]$ of the jacobian of the special fibre of X given by the jacobian of $X/W(k)[[t_1,\ldots,t_n]]$ (cf. [18], [22]). This seems quite reasonable, because over the ring of <u>ordinary</u> divided power series $W(k) <t_1,\ldots,t_n>$, $p\neq 2$, such liftings are known to the parameterized by the postion of the Hodge filtration, ([21]), which is precisely what (τ_{ij}) <u>is</u>.

<u>Proposition</u> 7.4. <u>The following conditions are equivalent</u>

7.4.1. <u>The Gauss-Manin connection on</u> H^1 <u>extends to a stratification</u> (<u>i.e. horizontal section of</u> $H^1 \otimes W(k) \ll t_1,\ldots,t_n \gg$ <u>extend to horizontal sections of</u> H^1)

7.4.2. <u>Every horizontal section of</u> $H \otimes K\{\{t_1,\ldots,t_n\}\}$ <u>is bounded in the open unit polydisc</u> (<u>i.e. liesin</u> $p^{-m}H^1$ <u>for some</u> m).

7.4.3. <u>The</u> τ_{ij} <u>are all bounded in the open unit polydisc</u> (<u>i.e., lie in</u> $p^{-m}W(k)[[t_1,\ldots,t_n]]$ <u>for some</u> m).

7.4.4. The τ_{ij} all lie in $W(k)[[t_1,\dots,t_n]]$.

7.4.5. The τ_{ij} all lie in $pW(k)[[t_1,\dots t_n]]$.

Proof. Using the congruences 7.2, we get 7.4.3 $\Longleftrightarrow$ 7.4.4 $\Longleftrightarrow$ 7.4.5, by choosing for φ the lifting $\varphi(t_i) = t_i^p$ for $i = 1,\dots,n$. By 7.0.4, 7.4.1 $\Longleftrightarrow$ 7.4.4 and 7.4.2 $\Longleftrightarrow$ 7.4.3.

QED.

Corollary 7.5. Suppose $X/W(k)[[t]]$ is an elliptic curve with ordinary special fibre, and that the induced curve over $k[t]/(t^2)$ is non-constant. Then every horizontal section of H^1 is a $W(k)$-multiple of α_1, the horizontal fixed point of F in H^1.

Proof. The non-constancy modulo (p,t^2) means precisely that the Kodaira-Spencer class in $H^1(X_{\text{special}}, T)$ is non-zero, which for an elliptic curve is equivalent to the non-vanishing modulo (p,t) of the composite mapping :

$$\mathrm{Fil}^1 \hookrightarrow H^1 \xrightarrow{\nabla(\frac{d}{dt})} H^1 \xrightarrow{\text{proj}} H/\mathrm{Fil} \xrightarrow{\sim} U \quad ,$$

whose matrix is $\frac{d\tau}{dt}$. Thus $\frac{d\tau}{dt} \notin (p,t)$, and hence by 7.4 there exists an unbounded horizontal section of $H^1 \otimes K\{\{t\}\}$. Writing it a $a\,\alpha_1 + b\,\beta_1$ $a,b \in K$, we must have $b \neq 0$ because α_1 is bounded. Hence β_1 is unbounded, hence any bounded horizontal section is a K-multiple of α_1, and $H^1 \cap K\alpha_1 = W(k)\alpha_1$.

The interest of this corollary is that it describes the filtration $U \subset H^1$ purely in terms of the differential equation (i.e., without reference to F) as being the span of the horizontal sections of H^1 (the "bounded solutions" of the differential equation). (cf. [9], pt. 4 where this is worked out in great detail for

Legendre's family of elliptic curves]. The general question of when the filtration by slopes can be described in terms of growth conditions to be imposed on the horizontal sections of $H^1 \otimes K\{\{t\}\}$ is not at all understand.

8. <u>An example</u> ([6], [10]). Let's see what all this means in a concrete case : the ordinary part of Legendre's family of elliptic curves. We take $p \neq 2$, $H(\lambda) \in \mathbb{Z}[\lambda]$ the polynomial $\Sigma(-1)^j \binom{\frac{p-1}{2}}{j} \lambda^j$ of degree $p-1/2$, S the smooth $\mathbb{Z}_p$-scheme $\mathrm{Spec}(\mathbb{Z}_p[\lambda][1/\lambda(1-\lambda)\,H(\lambda)])$, and X/S^{∞} the Legendre curve whose affine equation is $y^2 = x(x-1)(X-\lambda)$ (*). The De Rham H^1 is free of rank 2, on ω and ω', where

$$8.0 \qquad \begin{cases} \omega \text{ is the class of the differential of the first kind } dx/y \\ \\ \omega' = \nabla(\frac{d}{d\lambda})(\omega) \end{cases} .$$

The Gauss-Manin connection is specified by the relation

$$8.1 \qquad \lambda(1-\lambda)\ \omega'' + (1-2\lambda)\ \omega' = \frac{1}{4}\omega \quad ; \quad (\omega'' \overset{\text{defn}}{=\!=\!=} (\nabla\ (\frac{d}{d\lambda}))^2(\omega))$$

The Hodge filtration is $H^1 \supset \mathrm{Fil}^1 \subset H^1$ = span of ω . The cup-product is given by $\langle\omega,\omega\rangle = \langle\omega',\omega'\rangle = 0$; $\langle\omega,\omega'\rangle = -\langle\omega',\omega\rangle = -2/\lambda(1-\lambda)$. Horizontal sections are those of the form $\lambda(1-\lambda)f'\omega - \lambda(1-\lambda)f\omega'$, where f is a solution of the ordinary differential equation $(' = \frac{d}{d\lambda})$

$$8.2. \qquad \lambda 1-\lambda f'' + (1-2\lambda)f' = \frac{1}{4} f \qquad .$$

For any point $\alpha \in W(\mathbb{F}_q)$ for which $|H(\alpha).\alpha.(1-\alpha)|=1$ we know by 7.5 and 4.1 that the $W(\overline{\mathbb{F}}_q)$-module of solutions in $W(\overline{\mathbb{F}}_q)[[t-\alpha]]$ of the differential equation 9.2 is free of rank one, and is generated by a solution whose constant term is 1. Denote this solution f_α . According to 4.1.9, the ratis f'_α/f_α is the local expression of a "global" funntion $\eta \in$ the p-adic completion of $\mathbb{Z}_p[\lambda][1/\lambda(1-\lambda)H(\lambda)]$. Now choose a lifting φ of Frobenius, say the one with $\varphi^*(\lambda) = \lambda^p$. For each Teichmuller point α , there exists a unit C_α in $W(\overline{\mathbb{F}}_q)$, such that the function $C_\alpha f_\alpha/\varphi^*(C_\alpha f_\alpha)$ is the local expression of the 1×1 matrix of $F(\varphi)$ on the rank one module U.

(*) $H(\lambda)$ modulo p is the Hasse invariant = 1×1 Hasse-Witt matrix.

This is just the spelling out of 4.1.9, the constant C_α so chosen as to make $C_\alpha f_\alpha$ a fixed point of F. In terms of this matrix, call it $a(\lambda)$, we have a formula for zeta :

For each $\alpha_o \in \mathbb{F}_{q^n}$ such that $y^2 = X(X-1)(X-\alpha_o)$ is the affine equation of an ordinary elliptic curve E_{α_o}, denote by $\alpha \in W(\mathbb{F}_{p^n})$ its Teichmuller representative. The unit root of the numerator of Zeta $(E_{\alpha_o}/\mathbb{F}_{p^n};t)$ is

8.3 $$u_n(\alpha) \overset{\text{defn}}{=\!=} a(\alpha)a(\alpha^p)\ldots a(\alpha^{p^{n-1}})$$

and hence

8.4 $$\text{Zeta}(E_{\alpha_o}/\mathbb{F}_{p^n}\,;\,t) = \frac{(1 - u_n(\alpha)t)(1-(p^n/u(\alpha))t)}{(1-t)(1-p^n t)}$$

This formula, known to Dwork by a completely different approach in 1957, ([6]) was the starting point of his application of p-adic analysis to zeta !

REFERENCES

1 . P. BERTHELOT - A Series of notes in C.R. Acad. Sci. Paris Série A, t. 269, pp. 297-300, 357-360, 397-400, t. 272, pp. 42-45, 141-144, 254-257, 1314-1317, 1397-1400, 1574-1577.

2. P. BERTHELOT - Thèse, Paris VII, 1972.

3. B. DWORK - Norm Residue Symbol in Local Number Fields, Abh. Math. Sem. Univ. Hamburg, 22 (1958), pp. 180-190.

4 B. DWORK - On the Rationality of the Zeta Function of an Algebraic Vairety, Amer. J. Math, 82 (1960), pp.631-648.

5 B. DWORK - On the zeta function a hypersurface, Pub. Math. IHES 12, Paris 1962, pp.5-68.

6 B. DWORK - A deformation theory for the zeta function of a hypersurface, Proc. Int'l Cong. Math. Stockholm, 1962, pp. 247-259.

7 B. DWORK - On the zeta function of a hypersurface II, Annals of Math. 80 (1964), pp. 227-299.

8. B. DWORK - On the rationality of zeta functions and L-series, Proc. of a Conf. on Local Fields, held at Driebergen, 1966. Springer-Verlag, 1967, pp. 40-55.

9. B. DWORK - P-adic Cycles, Publ. Math. IHES 37, Paris, 1969, pp. 27-116

10. B. DWORK - On Hecke Polynomials, Inventiones Math. 12(1971), pp. 249-256.

11. B. DWORK - Normalized Period Matrices. Annals of Math. 94 (1971), pp. 337-388.

12. B. DWORK - Normalized Period Matrices II, to appear in Annals, of Math.

12 a. B . DWORK - p-adic Analysis and Geometry Seminar at IHES, Fall 1971.

13. A. GROTHENDIECK - Formule de Lefschetz et Rationalité des fonctions L , Séminaire Bourbaki, exposé 279. 1964/65.

14. N. KATZ - On the Differential Equations Satisfied by Period Matrices, Pub. Math. IHES 35, Paris, 1968, pp. 71-106.

15. N. KATZ - On the Intersection Matrix of a hypersurface Ann. Sci. E.N.S., 4e série, t.2, fasc.4, 1969, pp. 583-598.

16. N. KATZ - Introduction aux travaux récents de Dwork, IHES multigraph, 1969.

17. N. KATZ - On a theorem of Ax. Amer. J. Math. 93, (1971), pp. 485-499.

18. J. LUBIN - J.P. SERRE, - J. TATE : Elliptic Curves and Formal Groups, Woods Hole Summer Institue, 1964 (miméographed notes).

19. J. MANIN - Theory of Commutative formal groups over fields of finite characteristic. Russian Math. Surveys 18, n°6 (1963).

20. B. MAZUR - "Frobenius and the Hodge filtration" Univ. of Warwick mimeographed notes, 1971.

21. B. MAZUR - and W. MESSING - work in preparation.

22. W. MESSING - The Crystals associated to Barsotti-Tate groups, with application to Abelian schemes. Thesis, Princeton, Univ. 1971

23. P. MONSKY and G. WASHNITZER - Formal Cohomology I. Annals of Math. 88 (1968), pp. 181-217.

24. P. MONSKY - Formal Cohomology II, III ; Annals of Math. 88 (1968), pp. 218-238, and 93 (1971), pp. 315-343?

25. D. REICH - A p-adic fixed point formula, Amer. J. Math 91 (1969), pp. 835-850.

26. J.P. SERRE - Zeta and L functions, in Arithmetic Algebraic Geometry, ed. Schilling, Harper and Row, New York, 1965, pp. 82-92.

27. J. TATE - Rigid Analytic Spaces, Inventiones Math. 12 (1971), pp. 257-289.