SERRE-TATE LOCAL MODULI

par N. KATZ

INTRODUCTION. It is now some sixteen years since Serre-Tate [13] discovered that over a ring in which a prime number p is nilpotent, the infinitesimal deformation theory of abelian varieties is completely controlled by, and is indeed equivalent to, the infinitesimal deformation theory of their p-divisible groups.

In the special case of a g-dimensional ordinary abelian variety over an algebraically closed field k of characteristic $p > 0$, they deduced from this general theorem a remarkable and unexpected structure of group on the corresponding formal moduli space $\hat{\mathfrak{M}}$; this structure identifies $\hat{\mathfrak{M}}$ with a g^2-fold product of the formal multiplicative group $\hat{\mathbb{G}}_m$ with itself. The most striking consequence of the existence of a group structure on $\hat{\mathfrak{M}}$ is that it singles out a particular lifting (to some fixed artin local ring) as being "better" than any other, namely the lifting corresponding to the origin in $\hat{\mathfrak{M}}$. The theory of this "canonical lifting" is by now fairly well understood (though by no means completely understood ; for example, when is the canonical lifting of a jacobian again a jacobian ?).

A second consequence is the existence of g^2 canonical coordinates on $\hat{\mathcal{M}}$, corresponding to viewing $\hat{\mathcal{M}}$ as $(\hat{\mathbb{G}}_m)^{g^2}$. It is natural to ask whether the traditional structures associated with deformation theory, e.g. the Kodaira-Spencer mapping, the Gauss-Manin connection on the de Rham cohomology of the universal deformation,... have a particularly simple description when expressed in terms of these coordinates. We will show that this is so. In the late 1960's, Dwork (cf. [3], [4], [6]) showed how a direct study of the F-crystal structure on the de Rham cohomology of the universal formal deformation of an ordinary elliptic curve allowed one to define a "divided-power" function "τ" on $\hat{\mathcal{M}}$ such that $\exp(\tau)$ existed as a "true" function on $\hat{\mathcal{M}}$, and such that this function $\exp(\tau)$ defined an isomorphism of functors $\hat{\mathcal{M}} \xrightarrow{\sim} \hat{\mathbb{G}}_m$. Messing in 1975 announced a proof that Dwork's function $\exp(\tau)$ coincided with the Serre-Tate canonical coordinate on $\hat{\mathcal{M}}$. Unfortunately he never published his proof.

In the case of a g-dimensional ordinary abelian variety, Illusie [5] has used similar F-crystal techniques to define g^2 divided-power functions τ_{ij} on $\hat{\mathcal{M}}$, and to show that their exponentials $\exp(\tau_{ij})$ define an isomorphism of functors $\hat{\mathcal{M}} \xrightarrow{\sim} (\hat{\mathbb{G}}_m)^{g^2}$.

In [8], we used a "uniqueness of group structure" argument to show that the Serre-Tate approach and the Dwork-Illusie approach both impose the <u>same group structure</u> on $\hat{\mathcal{M}}$. Here, we will be concerned with showing that the <u>actual parameters</u> provided by the two approaches coincide. This amounts to explicitly computing the Gauss-Manin connection on H^1_{DR} of the universal deformation in terms of the Serre-Tate parameters. This problem in turn reduces to that of computing the Serre-Tate parameters of <u>square-zero deformations</u> of a canonical lifting in terms of the customary deformation-theoretic description of square-zero deformations, via their Kodaira-Spencer class. The main results are 3.7.1-2-3, 4.3.1-2, 4.5.3, 6.0.1-2

For the sake of completeness, we have included a remarkably simple proof, due to Drinfeld [2], of the "general" Serre-Tate theorem.

TABLE OF CONTENTS

1. DRINFELD'S PROOF OF THE SERRE-TATE THEOREM

1.1. Consider a ring R, an integer $N \geq 1$ such that N kills R, and an ideal $I \subset R$ which is nilpotent, say $I^{\nu+1} = 0$. Let us denote by R_0 the ring R/I. For any functor G on the category of R-algebras, we denote by G_I the subfunctor

$$G_I(A) = \mathrm{Ker}(G(A) \to G(A/IA)) ,$$

and by $\hat{G}$ the subfunctor

$$\hat{G}(A) = \mathrm{Ker}(G(A) \to G(A^{\mathrm{red}})) .$$

LEMMA 1.1.1. _If_ G _is a commutative formal Lie group over_ R, _then the sub-group functor_ G_I _is killed by_ N^{ν}.

PROOF. In terms of coordinates $X_1,\ldots,X_n$ for G, we have

$$([N](X))_i = NX_i + (\deg \geq 2 \quad \text{in} \quad X_1,\ldots,X_n) ;$$

as a point of $G_I(A)$ has coordinates in IA, and N kills R, hence A, we see that

$$[N](G_I) \subset G_{I^2}$$

and more generally that

$$[N](G_{I^a}) \subset G_{I^{2a}} \subset G_{I^{a+1}}$$

for every integer $a \geq 1$. As $I^{\nu+1} = 0$, the assertion is clear. Q.E.D.

LEMMA 1.1.2. _If_ G _is an f.p.p.f. abelian sheaf over_ R (i.e. _on the category of_ R-_algebras_) _such that_ $\hat{G}$ _is locally representable by a formal Lie group, then_ N^{ν} _kills_ G_I.

PROOF. Because I is nilpotent, we have $G_I \subset \hat{G}$, and hence $G_I = (\hat{G})_I$. The result now follows from 1.1.1. Q.E.D.

LEMMA 1.1.3. _Let_ G _and_ H _be f.p.p.f. abelian sheaves over_ R. _Suppose that_

1) G is N-divisible

2) $\hat{H}$ is locally representable by a formal Lie group

3) H is formally smooth.

Let G_o, H_o denote the inverse images of G, H on $R_o = R/I$.

Then

1) the groups $\mathrm{Hom}_{R\text{-gp}}(G,H)$ and $\mathrm{Hom}_{R_o\text{-gp}}(G_o,H_o)$ have no N-torsion

2) the natural map "reduction mod I"

$$\mathrm{Hom}(G,H) \to \mathrm{Hom}(G_o,H_o)$$

is injective

3) for any homomorphism $f_o : G_o \to H_o$, there exists a unique homomorphism "$N^\nu f$" $: G \to H$ which lifts $N^\nu f_o$

4) In order that a homomorphism $f_o : G_o \to H_o$ lift to a (necessarily unique) homomorphism $f : G \to H$, it is necessary and sufficient that the homomorphism "$N^\nu f$" $: G \to H$ annihilate the sub-group $G[N^\nu] = \mathrm{Ker}(G \xrightarrow{N^\nu} G)$ of G.

PROOF. The first assertion 1) results from the fact that G, and so G_o, are N-divisible. For the second assertion, notice that the kernel of the map involved is $\mathrm{Hom}(G,H_I)$, which vanishes because G is N-divisible while, by 1.1.2, H_I is killed by N^ν. For the third assertion, we will simply write down a canonical lifting of $N^\nu f_o$ (it's unicity results from part 2) above). The construction is, for any R-algebra A, the following :

$$\begin{tikzcd}
G(A) \arrow[rr, dashed, "\text{"}N^\nu f\text{"}"] \arrow[dr, "\text{mod } I"'] & & H(A) \\
& G(A/IA) \arrow[r, "f_o"] & H(A/IA) \arrow[u, "N^\nu \times (\text{any lifting})"']
\end{tikzcd}$$

the final oblique homomorphism

$$H(A/IA) \xrightarrow{N^\nu \times (\text{any lifting})} H(A)$$

is defined (because by assumption $H(A) \twoheadrightarrow H(A/IA)$) and well-defined (because the indeterminacy in a lifting lies in $H_I(A)$, a group which by 1.1.2 is killed by N^ν). For 4), notice that if f_o lifts to f, then by unicity of liftings we must have $N^\nu f = "N^\nu f"$ (because both lift $N^\nu f_o$). Therefore $"N^\nu f"$ will certainly annihilate $G[N^\nu]$. Conversely, suppose that $"N^\nu f"$ annihilates $G[N^\nu]$. Because G is N-divisible, we have an exact sequence

$$0 \longrightarrow G[N^\nu] \longrightarrow G \xrightarrow{N^\nu} G \longrightarrow 0 ,$$

froù which we deduce that $"N^\nu f"$ is of the form $N^\nu F$ for <u>some</u> homomorphism $F : G \to H$.

To see that F lifts f_o, notice that the reduction mod I, F_o, of F satisfies $N^\nu F_o = N^\nu f_o$; because $\mathrm{Hom}(G_o, H_o)$ has no N-torsion, we conclude that $F_o = f_o$, as required. Q.E.D.

1.2. We now "specialize" to the case in which N is a power of a prime number p, say $N = p^n$.

Let us denote by $\mathcal{A}(R)$ the category of abelian schemes over R, and by $\mathrm{Def}(R, R_o)$ the category of triples

$$(A_o, G, \varepsilon)$$

consisting of an abelian scheme A_o over R_o, a p-divisible (= Barsotti-Tate) group G over R, and an isomorphism of p-divisible groups over R_o

$$\varepsilon : G_o \xrightarrow{\sim} A_o[p^\infty] .$$

THEOREM 1.2.1 (Serre-Tate). <u>Let</u> R <u>be a ring in which a prime</u> p <u>is nilpotent</u>, $I \subset R$ <u>as nilpotent ideal</u>, $R_o = R/I$. <u>Then the functor</u>

$$\mathcal{A}(R) \to \mathrm{Def}(R, R_o)$$

$$A \mapsto (A_o, A[p^\infty], \text{natural } \varepsilon)$$

<u>is an equivalence of categories</u>.

PROOF. We begin with full-faithfulness. Let A, B be abelian schemes over R. We suppose given a homomorphism

$$f[p^\infty] : A[p^\infty] \to B[p^\infty]$$

of p-divisible groups over R, and a homomorphism

$$f_0 : A_0 \to B_0$$

of abelian schemes over R_0 such that $f_0[p^\infty]$ coincides with $(f[p^\infty])_0$. We must show there exists a unique homomorphism

$$f : A \to B$$

which induces both $f[p^\infty]$ and f_0.

Because both abelian schemes and p-divisible groups satisfy all the hypotheses of 1.1.3, we may make use of its various conclusions. The unicity of f, if it exists, follows from the injectivity of

$$\mathrm{Hom}(A,B) \to \mathrm{Hom}(A_0,B_0) .$$

For existence, consider the canonical lifting "$N^\nu f$" of $N^\nu f_0$:

$$"N^\nu f" : A \to B .$$

We must show that "$N^\nu f$" kills $A[N^\nu]$. But because "$N^\nu f$" lifts $N^\nu f_0$, its associated map "$N^\nu f$"$[p^\infty]$ on p-divisible groups lifts $N^\nu(f_0[p^\infty])$. By unicity, we must have

$$"N^\nu f"[p^\infty] = N^\nu(f[p^\infty]) .$$

Therefore "$N^\nu f$" kills $A[N^\nu]$, and we find "$N^\nu f$" $= N^\nu F$, with F a lifting of f_0. Therefore $F[p^\infty]$ lifts $f_0[p^\infty]$, so again by unicity we find $F[p^\infty] = f[p^\infty]$.

It remains to prove essential surjectivity. We suppose given a triple (A_0,G,ε). We must produce an abelian scheme A over R which gives rise to this triple. Because R is a nilpotent thickening of R_0, we can find an abelian scheme B over R which lifts A_0. The isomorphism of abelian schemes over R_o

$$B_o \xrightarrow[\sim]{\alpha_o} A_o$$

induces an isomorphism of p-divisible groups over R_o,

$$B_o[p^\infty] \xrightarrow{\alpha_o[p^\infty]} A_o[p^\infty] ,$$

and $N^\nu \alpha_o[p^\infty]$ has a unique lifting to a morphism of p-divisible groups over R

$$B[p^\infty] \xrightarrow{"N^\nu \alpha[p^\infty]"} G .$$

This morphism is an isogeny, for an "inverse up to isogeny" is provided by the canonical lifting of $N^\nu \times (\alpha_o[p^\infty])^{-1}$; the composition in either direction

$$\begin{tikzcd} B[p^\infty] \arrow[r, bend left, "\text{"}N^\nu \alpha[p^\infty]\text{"}"] & G \arrow[l, bend left, "\text{"}N^\nu(\alpha[p^\infty])^{-1}\text{"}"] \end{tikzcd}$$

is the endomorphism $N^{2\nu}$ (again by unicity). Therefore we have a short exact sequence

$$0 \to K \to B[p^\infty] \to G \to 0 ,$$

with $K \subset B[N^{2\nu}]$. Applying the criterion of flatness "fibre by fibre" - (permissible because the formal completion of a p-divisible group over R along any section is a finite-dimensional formal Lie variety over R, so in particular flat over R) - we conclude that the morphism "$N^\nu \alpha[p^\infty]$" is flat, because its reduction mod I, which is (multiplication by N^ν) $\times$ (an isomorphism), is flat.

Therefore K is a finite flat subgroup of $B[p^{2n\nu}]$; and so we may form the quotient abelian scheme of B by K :

$$A = B/K .$$

Because K lifts $B_o[N^\nu]$, this quotient A lifts $B_o/B_o[N^\nu] \xrightarrow{\sim} B_o \simeq A_o$, and the exact sequence

$$0 \to K \to B[p^{\infty}] \to G \to 0$$

induces a compatible isomorphism

$$A[p^{\infty}] \simeq B[p^{\infty}]/K \xrightarrow{\sim} G \text{ .} \quad \text{Q.E.D.}$$

1.3. REMARK. Let us return to the general situation of a ring R killed by an integer $N \geqslant 1$, and a nilpotent ideal $I \subset R$, say with $I^{\nu+1} = 0$. Let G be an f.p.p.f. abelian sheaf over R, which is formally smooth and for which $\hat{G}$ is locally representable by a formal Lie group. The fundamental construction underlying Drinfeld's proof is the canonical homomorphism

$$"N^{\nu}" : G(A/IA) \xrightarrow{N^{\nu} \times (\text{any lifting})} G(A)$$

for any R-algebra A. This homomorphism is functorial in A. It is also functorial in G in the sense that if G' is another such, and $f : G \to G'$ is any homomorphism, we have a commutative diagram

$$\begin{array}{ccc} G(A/IA) & \xrightarrow{"N^{\nu}"} & G(A) \\ \downarrow f & & \downarrow f \\ G'(A/IA) & \xrightarrow{"N^{\nu}"} & G'(A) \end{array}$$

for any R-algebra A.

There is in fact a much wider class of abelian-group valued functors on the category of R-algebras to which we can extend the construction of this canonical homomorphism. Rougkly speaking, any abelian-group-valued functor formed out of "cohomology with coefficients in G", where G is as above, will do. Rather than develop a general theory, we will give the most striking examples.

EXAMPLE 1.3.1. Let F be any abelian-group-valued functor on R-algebras, and G as above, for instance G a smooth commutative group-scheme over R. Let $D_G(F)$ denote the "G-dual" of F, i.e. the functor on R-algebras defined, for an arbitrary R-algebra A, by

$$D_G(F)(A) = \varprojlim_{B \text{ an } A\text{-alg}} \mathrm{Hom}_{gp}(F(B), G(B)) \ .$$

We define

$$"N^{\vee}" : D_G(F)(A/IA) \to D_G(F)(A)$$

as follows : given $\varphi \in D_G(F)(A/IA)$, $"N^{\vee}"\varphi \in D_G(F)(A)$ is the inverse limit, over A-algebras B , of the homomorphisms

$$\begin{tikzcd} F(B) \arrow[dr] & & G(B) \\ & F(B/IB) \arrow[r, "\varphi"] & G(B/IB) \arrow[u, "\text{"}N^{\vee}\text{"}"'] \end{tikzcd} .$$

If we take F to be a finite flat commutative group scheme over R , and $G = \mathbb{G}_m$, then $D_G(F)$ is just the Cartier dual $F^{\vee}$ of F . Since F is itself of this form (being $(F^{\vee})^{\vee}$), we conclude the existence of a canonical homomorphism

$$"N^{\vee}" : F(A/IA) \to F(A)$$

functorial in variable R-algebras A and in variable finite flat commutative group-schemes over R . This example is due to Drinfeld [2].

EXAMPLE 1.3.2. Let X be any R-scheme, and G any smooth commutative group scheme over R , or any finite flat commutative group-scheme over R . Let $i \geqslant 0$ be an integer, and consider the functor on R-algebras $\Phi^i_X(G)$ defined as

$$\Phi^i_X(G)(A) = H^i_{f.p.p.f.}(X \otimes_R A, G) \ .$$

Using the $"N^{\vee}"$-homomorphism already constructed for G , we deduce by functoriality the required homomorphism

$$"N^{\vee}" : \Phi^i_X(G)(A/I) \to \Phi^i_X(G)(A) \ ,$$

functorial in variable A , G , and X in an obvious sense.

If we take $G = \mathbb{G}_m$, we have $\Phi^1_X(G)(A) = \mathrm{Pic}(X \otimes_R A)$, $\Phi^2_X(G)(A) = \mathrm{Br}(X \otimes_R A), \dots$.

2. SERRE-TATE MODULI FOR ORDINARY ABELIAN VARIETIES

2.0. Fix an algebraically closed field k of characteristic $p > 0$. We will be concerned with the infinitesimal deformation theory of an ordinary abelian variety A over k. Let A^t be the dual abelian variety ; it too is ordinary, because it is isogenous to A.

We denote by $T_pA(k)$, $T_pA^t(k)$ the "physical" Tate modules of A and A^t respectively. Because A and A^t are ordinary, these Tate modules are free $\mathbb{Z}_p$-modules of rank $g = \dim A = \dim A^t$.

Consider now an artin local ring R with residue field k, and an abelian scheme $\mathbb{A}$ over R which lifts A/k (i.e. we are given an isomorphism $\mathbb{A} \otimes_R k \xrightarrow{\sim} A$). Following a construction due do Serre-Tate, we attach to such a lifting a $\mathbb{Z}_p$-bilinear form $q(\mathbb{A}/R;-,-)$

$$q(\mathbb{A}/R;-,-) : T_pA(k) \times T_pA^t(k) \to \hat{\mathbb{G}}_m(R) = 1+\mathfrak{m} .$$

This bilinear form, which if expressed in terms of $\mathbb{Z}_p$-bases of $T_pA(k)$ and of $T_pA^t(k)$ would amount to specifying g^2 principal units in R, is the complete invariant of $\mathbb{A}/R$, up to isomorphism, as a lifting of A/k. The precise theorem of Serre-Tate is the following, in the case of ordinary abelian varieties.

THEOREM 2.1. Let A be an ordinary abelian variety over an algebraically closed field k of characteristic $p > 0$, and R an artin local ring with residue field k.

1) The construction

$$\mathbb{A}/R \mapsto q(\mathbb{A}/R;-,-) \in \mathrm{Hom}_{\mathbb{Z}_p}(T_pA(k) \otimes T_pA^t(k), \hat{\mathbb{G}}_m(R))$$

establishes a bijection between the set of isomorphism classes of liftings of A/k to R and the group $\mathrm{Hom}_{\mathbb{Z}_p}(T_pA(k) \otimes T_pA^t(k), \hat{\mathbb{G}}_m(R))$.

2) If we denote by $\hat{\mathcal{M}}_{A/k}$ the formal moduli space of A/k, the above construction for variable artin local rings R with residue field k defines an isomorphism of functors

$$\hat{\mathcal{M}}_{A/k} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(T_pA(k) \otimes T_pA^t(k), \hat{\mathbb{G}}_m) \ .$$

3) <u>Given a lifting</u> $\mathbb{A}/R$ <u>of</u> A/k , <u>denote by</u> $\mathbb{A}^t/R$ <u>the dual abelian scheme, which is a lifting of</u> A^t/k . <u>With the canonical identification of</u> A <u>with</u> A^{tt} , <u>we have the symmetry formula</u>

$$q(\mathbb{A}/R; \alpha, \alpha_t) = q(\mathbb{A}^t/R; \alpha_t, \alpha)$$

<u>for any</u> $\alpha \in T_pA(k)$, $\alpha_t \in T_pA^t(k)$.

4) <u>Suppose we are given two ordinary abelian varieties</u> A , B <u>over</u> k , <u>and liftings</u> $\mathbb{A}/R$, $\mathbb{B}/R$. <u>Let</u> $f : A \to B$ <u>be a homomorphism, and</u> $f^t : B^t \to A^t$ <u>the dual homomorphism. The necessary and sufficient condition</u> that f lift to a homomorphism $\mathbb{f} : \mathbb{A} \to \mathbb{B}$ is that

$$q(\mathbb{A}/R; \alpha, f^t(\beta_t)) = q(\mathbb{B}/R; f(\alpha), \beta_t)$$

<u>for every</u> $\alpha \in T_pA(k)$ <u>and every</u> $\beta_t \in T_pB^t(k)$ (N.B. <u>If the lifting</u> $\mathbb{f}$ <u>exists, it is unique</u>).

CONSTRUCTION-PROOF. By the "general" Serre-Tate theorem, the functor

$$\left\{\begin{matrix}\text{abelian schemes}\\ \text{over } R\end{matrix}\right\} \to \left\{\begin{matrix}\text{abelian schemes over } k \text{ together}\\ \text{with liftings of their p-divisible}\\ \text{groups to } R\end{matrix}\right\}$$

$$\mathbb{A}/R \mapsto (\mathbb{A} \underset{R}{\otimes} k, \mathbb{A}[p^\infty])$$

is an equivalence of categories.

Thus if we are given A/k , it is equivalent to "know" $\mathbb{A}/R$ as a lifting of A/k or to know its p-divisible group $\mathbb{A}[p^\infty]$ as a lifting of $A[p^\infty]$. Because A/k is ordinary, its p-divisible group is canonically a product

$$A[p^\infty] = \hat{A} \times T_pA(k) \underset{\mathbb{Z}_p}{\otimes} (\mathbb{Q}_p/\mathbb{Z}_p)$$

of its toroidal formal group and its constant etale quotient. Similarly for A^t . The e_{p^n}-pairings (cf. chapter 5 for a detailed discussion)

$$e_{p^n} : A[p^n] \times A^t[p^n] \to \mu_{p^n}$$

restrict to give pairings

$$e_{p^n} : \hat{A}[p^n] \times A^t(k)[p^n] \to \mu_{p^n}$$

which define isomorphisms of k-group-schemes

$$\hat{A}[p^n] \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(A^t(k)[p^n], \mu_{p^n}) \ ,$$

and, by passage to the limit, an isomorphism of formal groups over k

$$\hat{A} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k), \hat{\mathbb{G}}_m) \ .$$

We denote by

$$E_A : \hat{A} \times T_pA^t(k) \to \hat{\mathbb{G}}_m$$

the corresponding pairing.

Because R is artinian, the p-divisible group of $\mathbb{A}$ has a canonical structure of extension

$$0 \longrightarrow \hat{\mathbb{A}} \longrightarrow \mathbb{A}[p^\infty] \longrightarrow T_pA(k) \otimes (\mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow 0$$

of the constant p-divisible group $T_pA(k) \otimes (\mathbb{Q}_p/\mathbb{Z}_p)$ by $\hat{\mathbb{A}}$, which is the unique toroidal formal group over R lifting $\hat{A}$. Because $\hat{\mathbb{A}}$ and the $\hat{\mathbb{A}}[p^n]$'s are toroidal, the isomorphisms of k-groups

$$\begin{cases} \hat{A}[p^n] \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(A^t(k)[p^n], \mu_{p^n}) \\ \quad \hat{A} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k), \hat{\mathbb{G}}_m) \end{cases}$$

extend uniquely to isomorphisms of R-groups

$$\begin{cases} \hat{\mathbb{A}}[p^n] \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(A^t(k)[p^n], \mu_{p^n}) \\ \quad \hat{\mathbb{A}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k), \hat{\mathbb{G}}_m) \ . \end{cases}$$

We denote by

$$\begin{cases} E_{p^n:\mathbb{A}} : \hat{\mathbb{A}}[p^n] \times A^t(k)[p^n] \to \mu_{p^n} \\ \quad E_{\mathbb{A}} : \hat{\mathbb{A}} \times T_pA^t(k) \to \hat{\mathbb{G}}_m \end{cases}$$

the corresponding pairings.

A straightforward Ext calculation (cf. [9], Appendix) shows that our extension

$$0 \to \hat{\mathbb{A}} \to \mathbb{A}[p^\infty] \to T_pA(k) \otimes (\mathbb{Q}_p/\mathbb{Z}_p) \to 0$$

is obtained from the "basic" extension

$$0 \to T_pA(k) \to T_pA(k) \otimes \mathbb{Q}_p \to T_pA(k) \otimes (\mathbb{Q}_p/\mathbb{Z}_p) \to 0$$

by "pushing out" along a unique homomorphism

$$\begin{array}{c} T_pA(k) \\ \big\downarrow \varphi_{\mathbb{A}/R} \\ \hat{\mathbb{A}}(R) \end{array} \quad .$$

This homomorphism may be recovered from the extension

$$0 \to \hat{\mathbb{A}} \to \mathbb{A}[p^\infty] \to T_pA(k) \otimes (\mathbb{Q}_p/\mathbb{Z}_p) \to 0$$

as follows. Pick an integer n sufficiently large that the maximal ideal $\mathfrak{m}$ of R satisfies

$$\mathfrak{m}^{n+1} = 0 \ .$$

Because $p \in \mathfrak{m}$, and $\hat{\mathbb{A}}$ is a formal Lie group over R, every element of $\hat{\mathbb{A}}(R)$ is killed by p^n. Therefore we can define a group homomorphism

$$"p^n" : A(k) \to \mathbb{A}(R)$$

by decreeing

$$x \in A(k) \to p^n\tilde{x} \quad \text{for any} \quad \tilde{x} \in \mathbb{A}(R) \quad \text{lifting} \quad x \ .$$

If we restrict this homomorphism to $A(k)[p^n]$, we fall into $\hat{\mathbb{A}}(R)$:

$$"p^n" : A(k)[p^n] \to \hat{\mathbb{A}}(R) \ .$$

For variable n, we have an obvious commutative diagram

$$\begin{tikzcd} A(k)[p^{n+1}] \arrow[d, "p"] \arrow[dr, "\text{"}p^{n+1}\text{"}"] & \\ A(k)[p^n] \arrow[r, "\text{"}p^n\text{"}"'] & \hat{\mathbb{A}}(R) \end{tikzcd} ,$$

so in fact we obtain a single homomorphism

$$T_pA(k) \to \hat{\mathbb{A}}(R)$$

as the composite

$$T_pA(k) \twoheadrightarrow A(k)[p^n] \xrightarrow{\text{"}p^n\text{"}} \hat{\mathbb{A}}(R)$$

for any $n \gg 0$. This homomorphism is the required $\varphi_{\mathbb{A}/R}$.

We are now ready to define $q(\mathbb{A}/R;-,-)$. We simply view $\varphi_{\mathbb{A}/R}$ as a homomorphism

$$\begin{tikzcd} T_pA(k) \arrow[r] \arrow[dr, dashed] & \hat{\mathbb{A}}(R) \arrow[d, "\wr", "\text{the pairing } E_{\mathbb{A}}"'] \\ & \mathrm{Hom}(T_pA^t(k),\hat{\mathbb{G}}_m(R)) \end{tikzcd} ,$$

or, what is the same, as the bilinear form

$$q(\mathbb{A}/R;\alpha,\alpha_t) \overset{\text{dfn}}{=\!=} E_{\mathbb{A}}(\varphi_{\mathbb{A}/R}(\alpha);\alpha_t) .$$

We summarize the preceding constructions in a diagram :

$$\begin{Bmatrix}\text{isomorphism classes of} \\ \mathbb{A}/R \text{ lifting } A/k\end{Bmatrix} \xrightarrow[\sim]{\text{Serre-Tate}} \begin{Bmatrix}\text{isomorphism classes of} \\ \mathbb{A}[p^\infty]/R \text{ lifting } A[p^\infty]/k\end{Bmatrix}$$

$$\wr\downarrow$$

$$\mathrm{Ext}_{R\text{-gp}}(T_pA(k)\otimes(\mathbb{Q}_p/\mathbb{Z}_p),\mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k),\hat{\mathbb{G}}_m))$$

$$\text{"pushout"} \ \wr\uparrow \ \wr\downarrow \ \text{"}\varphi_{\mathbb{A}/R}\text{"}$$

$$\mathrm{Hom}_{R\text{-gp}}(T_pA(k),\mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k),\hat{\mathbb{G}}_m))$$

$$\wr\downarrow \ \text{"q"}$$

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_pA(k)\underset{\mathbb{Z}_p}{\otimes}T_pA^t(k),\hat{\mathbb{G}}_m(R)) .$$

Thus the truth of part 1), and, by passage to the limit, of part 2), results from the "general" Serre-Tate theorem. To prove part 4), we argue as follows. Given the homomorphism $f : A \to B$, we know by the general Serre-Tate theorem that it lifts to $\mathbb{f} : \mathbb{A} \to \mathbb{B}$ if and only if it lifts to an $\mathbb{f}[p^\infty] : \mathbb{A}[p^\infty] \to \mathbb{B}[p^\infty]$. Such an $\mathbb{f}[p^\infty]$ will necessarily respect the structure of extension of $\mathbb{A}[p^\infty]$ and of $\mathbb{B}[p^\infty]$, so it will necessarily sit in a commutative diagram of p-divisible groups over R :

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k),\hat{\mathbb{G}}_m) & \to & \mathbb{A}[p^\infty] & \to & T_pA(k)\otimes(\mathbb{Q}_p/\mathbb{Z}_p) & \to 0 \\ & \downarrow{\scriptstyle \circ f^t} & & \downarrow{\scriptstyle \mathbb{f}[p^\infty]} & & \downarrow{\scriptstyle f} & \\ 0 \to & \mathrm{Hom}_{\mathbb{Z}_p}(T_pB^t(k),\hat{\mathbb{G}}_m) & \to & \mathbb{B}[p^\infty] & \to & T_pB(k)\otimes(\mathbb{Q}_p/\mathbb{Z}_p) & \to 0 . \end{array}$$

Conversely, the Serre-Tate theorem assures us that we can lift f to an $\mathbb{f}$ if we can fill in this diagram with an $\mathbb{f}[p^\infty]$. But the necessarily and sufficient condition for the existence of an $\mathbb{f}[p^\infty]$ rendering the diagram commutative is that the "push out" of the top extension by the arrow "f^t" be isomorphic to the "pull-back" of the lower extension by the arrow "f".

The "push-out" along f^t of the upper extension is the element of

$$\begin{array}{c} \mathrm{Ext}_{R\text{-}gp}(T_pA(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p, \mathrm{Hom}_{\mathbb{Z}_p}(T_pB^t(k),\hat{\mathbb{G}}_m)) \\ \wr\!\!\downarrow q \\ \mathrm{Hom}_{\mathbb{Z}_p}(T_pA(k)\otimes T_pB^t(k),\hat{\mathbb{G}}_m(R)) \end{array}$$

defined by the bilinear pairing

$$(\alpha,\beta_t) \to q(\mathbb{A}/R;\alpha,f^t(\beta_t)) \ .$$

The pull-back along f of the lower extension is the element of the same Ext group defined by the bilinear pairing

$$(\alpha,\beta_t) \to q(\mathbb{B}/R;f(\alpha),\beta_t) \ .$$

Therefore $\mathbb{f}[p^\infty]$, and with it $\mathbb{f}$, exists if and only if we have

$$q(\mathbb{A}/R;\alpha,f^t(\beta_t)) = q(\mathbb{B}/R;f(\alpha),\beta_t)$$

for every $\alpha \in T_pA(k)$ and every $\beta_t \in T_pB^t(k)$.

It remains to establish the symettry formula 3), i.e. that

$$q(\mathbb{A}/R;\alpha,\alpha_t) = q(\mathbb{A}^t/R;\alpha_t,\alpha) \ .$$

Choose an integer n such that the maximal ideal $\mathfrak{m}$ of R satisfies

$$\mathfrak{m}^{n+1} = 0 \ .$$

Then the groups $\hat{\mathbb{A}}(R)$ and $\hat{\mathbb{A}}^{\tau}(R)$ are both killed by p^n. Let $\alpha(n)$, $\alpha_t(n)$ denote the images of α, α_t under the canonical projections

$$T_pA(k) \twoheadrightarrow A(k)[p^n] \ , \ T_pA^t(k) \twoheadrightarrow A^t(k)[p^n] \ .$$

Then by construction we have

$$\varphi_{\mathbb{A}/R}(\alpha) = \text{"}p^n\text{"}\alpha(n) \quad \text{in} \quad \hat{\mathbb{A}}(R)$$

$$\varphi_{\mathbb{A}^t/R}(\alpha_\tau) = \text{"}p^n\text{"}\alpha_t(n) \quad \text{in} \quad \hat{\mathbb{A}}^t(R) \ ,$$

and therefore we have

$$\begin{aligned} q(\mathbb{A}/R;\alpha,\alpha_t) &= E_{\mathbb{A}}(\varphi_{\mathbb{A}/R}(\alpha),\alpha_t) \\ &= E_{\mathbb{A},p^n}(\varphi_{\mathbb{A}/R}(\alpha),\alpha_t(n)) \\ &= E_{\mathbb{A},p^n}(\text{"}p^n\text{"}\alpha(n),\alpha_t(n)) \ . \end{aligned}$$

Similarly, we have

$$\begin{aligned} q(\mathbb{A}^t/R;\alpha_t,\alpha) &= E_{\mathbb{A}^t}(\varphi_{\mathbb{A}^t/R}(\alpha_t),\alpha) \\ &= E_{\mathbb{A}^t,p^n}(\varphi_{\mathbb{A}^t/R}(\alpha_t),\alpha(n)) \\ &= E_{\mathbb{A}^t,p^n}(\text{"}p^n\text{"}\alpha_t(n),\alpha(n)) \ . \end{aligned}$$

But for any n the pairings $E_{\mathbb{A};p^n}$ are "computable" in terms of the e_{p^n}-pairings on $\mathbb{A}$, as follows.

LEMMA 2.2. <u>Let</u> $n \geqslant 1$, $x \in \hat{\mathbb{A}}(R)[p^n]$ <u>and</u> $y \in A^t(k)[p^n]$. <u>There exists an artin local ring</u> R' <u>which is finite and flat over</u> R, <u>and a point</u> $Y \in \mathbb{A}^t(R')[p^n]$ <u>which lifts</u> $y \in A^t(k)[p^n]$. <u>For any such</u> R' <u>and</u> Y', <u>we have the equality, inside</u> $\hat{\mathbb{G}}_m(R')$,

$$E_{\mathbb{A},p^n}(x,y) = e_{p^n}(x,Y) \ .$$

PROOF OF LEMMA. Given $y \in A^t(k)[p^n]$, we can certainly lift it to a point $Y_1 \in \mathbb{A}^t(R)$, simply because $\mathbb{A}^t(R)$ is smooth over R. The point $p^nY_1 = Y_2$ lies in $\hat{\mathbb{A}}^t(R)$. Because $\hat{\mathbb{A}}^t$ is p-divisible, and R

is artin local, we can find an artin local R' which is finite flat over R and a point Y_3 in $\hat{\mathbb{A}}^t(R')$ such that $Y_2 = p^n Y_3$. Then $Y = Y_1 - Y_3$ lies in $\mathbb{A}^t(R')[p^n]$, and it lifts y.

Fix such a situation R', Y. The restriction of the e_{p^n}-pairing for $\mathbb{A} \underset{R}{\otimes} R'$

$$e_{p^n} : (\mathbb{A} \underset{R}{\otimes} R')[p^n] \times (\mathbb{A}^t \underset{R}{\otimes} R')[p^n] \to \mu_{p^n}$$

to a map

$$(\hat{\mathbb{A}} \underset{R}{\otimes} R')[p^n] \times Y \to \mu_{p^n}$$

is a homomorphism of toroidal groups over R'

$$\hat{\mathbb{A}}[p^n] \underset{R}{\otimes} R' \to \mu_{p^n}$$

whose reduction modulo the maximal ideal of R' is the homomorphism of toroidal groups over k

$$\hat{A}[p^n] \to \mu_{p^n}$$

defined by

$$e_{p^n}(-,y) .$$

But the homomorphism of toroidal groups over R

$$\hat{\mathbb{A}}[p^n] \to \mu_{p^n}$$

defined by

$$E_{\mathbb{A},p^n}(-,y)$$

is another such lifting. By uniqueness of infinitesimal liftings of maps between toroidal groups, we have the asserted equality. Q.E.D.

Now choose liftings

$$\begin{cases} \mathfrak{a}(n) \in \mathbb{A}(R) & \text{lifting } \alpha(n) \in A(k)[p^n] \\ \mathfrak{a}_t(n) \in \mathbb{A}^t(R) & \text{lifting } \alpha_t(n) \in A^t(k)[p^n] . \end{cases}$$

Because n was chosen large enough that p^n kill $\hat{\mathbb{A}}(R)$ and $\hat{\mathbb{A}}^t(R)$, we have a priori inclusions

$$\begin{cases} \mathfrak{a}(n) \in \mathbb{A}(R)[p^{2n}] \\ \mathfrak{a}_t(n) \in \hat{\mathbb{A}}^t(R)[p^{2n}] \end{cases} .$$

KEY FORMULA 2.3. Hypotheses as above, we have the formula

$$\frac{q(\mathbb{A}/R;\alpha,\alpha_t)}{q(\mathbb{A}^t/R;\alpha_t,\alpha)} = e_{p^{2n}}(\mathfrak{a}(n),\mathfrak{a}_t(n)) .$$

PROOF OF KEY FORMULA. By the previous lemma, we can find an artin local ring R' which is finite and flat over R, together with points

$$\begin{cases} B(n) \in \mathbb{A}(R')[p^n] & \text{lifting} \quad \alpha(n) \in A(k)[p^n] \\ B_t(n) \in \mathbb{A}^t(R')[p^n] & \text{lifting} \quad \alpha_t(n) \in A^t(k)[p^n] \end{cases} .$$

We define the "error terms"

$$\begin{cases} \mathfrak{b}(n) = \mathfrak{a}(n) - B(n) & \text{in} \quad \hat{\mathbb{A}}(R')[p^{2n}] \\ \mathfrak{b}_t(n) = \mathfrak{a}_t(n) - B_t(n) & \text{in} \quad \hat{\mathbb{A}}^t(R')[p^{2n}] \end{cases} .$$

In terms of these $\mathfrak{a}$, B, and $\mathfrak{b}$, we have

$$"p^n"\alpha(n) \stackrel{\text{dfn}}{=\!=} p^n\mathfrak{a}(n) = p^n\mathfrak{b}(n)$$

$$"p^n"\alpha_t(n) \stackrel{\text{dfn}}{=\!=} p^n\mathfrak{a}_t(n) = p^n\mathfrak{b}_t(n) .$$

We now calculate

$$\begin{aligned} q(\mathbb{A}/R;\alpha,\alpha_t) &= \mathbb{E}_{\mathbb{A},p^n}("p^n"\alpha(n),\alpha_t(n)) \\ \text{(by the previous lemma)} \quad &= e_{p^n}("p^n"\alpha(n),B_t(n)) \\ &= e_{p^n}(p^n\mathfrak{b}(n),B_t(n)) , \\ &= e_{p^{2n}}(\mathfrak{b}(n),B_t(n)) \end{aligned}$$

and similarly

$$\begin{aligned} q(\mathbb{A}^t/R;\alpha_t,\alpha) &= E_{\mathbb{A}^t,p^n}("p^n"\alpha_t(n),\alpha(n)) \\ &= e_{p^n}("p^n"\alpha_t(n),B(n)) \\ &= e_{p^n}(p^n\mathfrak{b}_t(n),B(n)) \\ &= e_{p^{2n}}(\mathfrak{b}_t(n),B(n)) \\ &= 1/e_{p^{2n}}(B(n),\mathfrak{b}_t(n)) , \end{aligned}$$

this last equality by the skew-symettry of the $e_{p^{2n}}$-pairing.

Therefore the "key formula" is equivalent to the following formula :

$$e_{p^{2n}}(\mathcal{B}(n),B_t(n)).e_{p^{2n}}(B(n),\mathcal{B}_t(n)) = e_{p^{2n}}(\mathcal{C}(n),\mathcal{C}_t(n)) \ .$$

To obtain this last formula, we readily calculate

$$e_{p^{2n}}(\mathcal{C}(n),\mathcal{C}_T(n)) = e_{p^{2n}}(B(n)+\mathcal{B}(n),B_t(n)+\mathcal{B}_t(n))$$

$$= e_{p^{2n}}(B(n),B_t(n)).e_{p^{2n}}(\mathcal{B}(n),\mathcal{B}_t(n)).e_{p^{2n}}(B(n),\mathcal{B}_t(n)).e_{p^{2n}}(\mathcal{B}(n),B_t(n)).$$

The first two terms in the product are identically one ; the first because $B(n)$ and $B_t(n)$ are killed by p^n , so that

$$e_{p^{2n}}(B(n),B_t(n)) = e_{p^n}(p^nB(n),B_t(n)) = e_{p^n}(0,B_t(n)) = 1 \ ;$$

the second because both $\mathcal{B}(n)$ and $\mathcal{B}_t(n)$ lie in their respective formal groups $\hat{\mathbb{A}}(R')[p^{2n}]$ and $\hat{\mathbb{A}}^T(R')[p^{2n}]$, and these groups are toroidal (the $e_{p^{2n}}$-pairing restricted to

$$\hat{\mathbb{A}}[p^{2n}] \times \hat{\mathbb{A}}^t[p^{2n}]$$

must be <u>trivial</u>, since it is equivalent to a homomorphism from a <u>connected</u> group, $\hat{\mathbb{A}}[p^{2n}]$, to an <u>etale</u> group, the Cartier dual of $\hat{\mathbb{A}}^t[p^{2n}]$, and any such homomorphism is necessarily trivial). Thus we have

$$e_{p^{2n}}(\mathcal{B}(n),\mathcal{B}_t(n)) = 1 \ ,$$

and we are left with the required formula. Q.E.D.

In order to complete our proof of the symettry formula, then, we must explain why

$$e_{p^{2n}}(\mathcal{C}(n),\mathcal{C}_t(n)) = 1 \ ,$$

for <u>some</u> choice of liftings $\mathcal{C}(n)$, $\mathcal{C}_t(n)$ of $\alpha(n)$ and $\alpha_t(n)$ to R .

Let us choose liftings

$$\begin{cases} \mathcal{C}(2n) \in \mathbb{A}(R) \ , \text{ lifting } \ \alpha(2n) \in A(k)[p^{2n}] \\ \mathcal{C}_t(2n) \in \mathbb{A}^t(R) \ , \text{ lifting } \ \alpha_t(2n) \in A^t(k)[p^{2n}] \ . \end{cases}$$

Then the points

$$p^n\mathfrak{a}(2n)\ ,\ p^n\mathfrak{a}_t(2n)$$

are liftings to R of $\alpha(n)$ and $\alpha_t(n)$ respectively. Thus it suffices to show that

$$e_{p^{2n}}(p^n\mathfrak{a}(2n), p^n\mathfrak{a}_t(2n)) = 1\ .$$

But in any case we have

$$e_{p^{2n}}(p^n\mathfrak{a}(2n), p^n\mathfrak{a}_t(2n)) = (e_{p^{3n}}(\mathfrak{a}(2n), \mathfrak{a}_t(2n)))^{p^n}\ .$$

The quantity $e_{p^{3n}}(\mathfrak{a}(2n), \mathfrak{a}_t(2n))$ lies in

$$\mu_{p^{3n}}(R) \subset 1+\mathfrak{m} = \hat{\mathbb{G}}_m(R)$$

and our choice of n, large enough that $\mathfrak{m}^{n+1} = 0$, guarantees that $\hat{\mathbb{G}}_m(R)$ is killed by p^n. Q.E.D.

3. FORMULATION OF THE MAIN THEOREM

3.0. Fix an algebraically closed field k of characteristic $p > 0$, and an ordinary abelian variety A over k. The Serre-Tate q-construction defines an isomorphism

$$\hat{\mathcal{M}}_{A/k} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(T_pA(k) \otimes T_pA^t(k), \hat{\mathbb{G}}_m)$$

of functors on the category of artin local rings with residue field k. In particular, it endows $\hat{\mathcal{M}}$ with a canonical structure of toroidal formal Lie group over the Witt vectors $W = W(k)$ of k.

Let $\mathcal{A}/\hat{\mathcal{M}}$ denote the <u>universal</u> formal deformation of A/k. In this section we will state a fundamental compatibility between the group structure on $\hat{\mathcal{M}}$ and the crystal structure on the de Rham cohomology of $\mathcal{A}/\hat{\mathcal{M}}$, as refected in the Kodaira-Spencer mapping of "traditional" deformation theory.

In order to formulate the compatibility in a succinct manor, we must first make certain definitions.

3.1. Let $\mathcal{R}$ denote the coordinate ring of $\hat{\mathcal{M}}$. Given elements $\alpha \in T_pA(k)$, $\alpha_t \in T_pA^t(k)$, we denote by

$$q(\alpha, \alpha_t) \in \mathcal{R}^{\times}$$

the inversible function on $\hat{\mathcal{M}}$ defined by

$$q(\alpha, \alpha_t) = q(\mathcal{A}/\mathcal{R} : \alpha, \alpha_t) .$$

Here are two characterizations of these functions $q(\alpha, \alpha_t)$. The isomorphism

$$\hat{\mathcal{M}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(T_pA(k) \otimes T_pA^t(k), \hat{\mathbb{G}}_m)$$

gives rise to an isomorphism

$$T_pA(k) \otimes T_pA^t(k) \xrightarrow{\sim} \mathrm{Hom}_{W\text{-}gp}(\hat{\mathcal{M}}, \hat{\mathbb{G}}_m) .$$

Under this isomorphism, we have

$$\alpha \otimes \alpha_t \to q(\alpha, \alpha_t) ,$$

i.e. the functions $q(\alpha,\alpha_t)$ are precisely the characters of the formal torus $\hat{\mathfrak{M}}$.

In particular, if we pick a $\mathbb{Z}_p$-basis $\alpha_1,\ldots,\alpha_g$ of $T_pA(k)$ and a $\mathbb{Z}_p$-basis $\alpha_{t,1},\ldots,\alpha_{t,g}$ of $T_pA^t(k)$, then the g^2 quantities

$$T_{ij} = q(\alpha_i,\alpha_{t,j}) - 1 \in \mathfrak{R}$$

define a ring isomorphism

$$W[[T_{i,j}]] \xrightarrow{\sim} \mathfrak{R} .$$

We will not make use of this isomorphism.

Given an artin local ring R with residue field k , and a lifting $\mathbb{A}/R$ of A/k , there is a unique continuous "classifying" homomorphism

$$f_{\mathbb{A}/R} : \mathfrak{R} \to R$$

for which we have an R-isomorphism of liftings

$$\mathbb{A}/R \xrightarrow{\sim} \mathcal{A} \otimes_{\mathfrak{R}} R .$$

The image of $q(\alpha,\alpha_t)$ under this classifying map is given by the formula

$$f_{\mathbb{A}/R}(q(\alpha,\alpha_t)) = q(\mathbb{A}/R;\alpha,\alpha_t) .$$

3.2. For each linear form

$$\ell \in \mathrm{Hom}_{\mathbb{Z}_p}(T_pA(k) \otimes T_pA^t(k), \mathbb{Z}_p) ,$$

we denote by $D(\ell)$ the translation-invariant (for the group structure on $\hat{\mathfrak{M}}$) continuous derivation of $\mathfrak{R}$ into itself given

$$D(\ell)(q(\alpha,\alpha_t)) = \ell(\alpha \otimes \alpha_t).q(\alpha,\alpha_t) .$$

Formation of $D(\ell)$ defines a $\mathbb{Z}_p$-linear map

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_pA(k) \otimes T_pA^t(k), \mathbb{Z}_p) \to \mathrm{Lie}(\hat{\mathfrak{M}}/W) ,$$

whose associated W-linear map is the isomorphism

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_pA(k) \otimes T_pA^t(k), W) \xrightarrow{\sim} \mathrm{Lie}(\hat{\mathfrak{M}}/W)$$

deduced from the inverse of the q-isomorphism of W-groups

$$\hat{\mathcal{M}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(T_pA(k) \otimes T_pA^t(k), \hat{\mathbb{G}}_m)$$

by applying the functor "Lie".

3.3. We next introduce certain invariant one-forms on $\mathcal{A}$

$$\omega(\alpha_t) \in \underline{\omega}_{\mathcal{A}/\mathcal{R}} \ .$$

For each artin local ring R with residue field k , and each lifting $\mathbb{A}/R$ of A/k , we have given a canonical isomorphism of formal groups over R

$$\hat{\mathbb{A}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k), \hat{\mathbb{G}}_m) \ .$$

This isomorphism yields an isomorphism

$$T_pA^t(k) \xrightarrow{\sim} \mathrm{Hom}_{R\text{-gp}}(\hat{\mathbb{A}}, \hat{\mathbb{G}}_m) \ ,$$

say

$$\alpha_t \to \lambda(\alpha_t) \ .$$

If we denote by dT/T the standard invariant one-form on $\mathbb{G}_m$, we can define an invariant one-form

$$\omega(\alpha_t) \in \underline{\omega}_{\mathbb{A}/R} = \underline{\omega}_{\hat{\mathbb{A}}/R}$$

by the formula

$$\omega(\alpha_t) = \lambda(\alpha_t)^*(dT/T) = d\lambda(\alpha_t)/\lambda(\alpha_t) \ .$$

Equivalently, the construction of $\omega(\alpha_t)$ sits in the diagram

$$\begin{tikzcd} T_pA^t(k) \arrow[r, "\sim"] \arrow[ddr, "\alpha_t \mapsto \omega(\alpha_t)"'] & \mathrm{Hom}_{R\text{-gp}}(\hat{\mathbb{A}}, \hat{\mathbb{G}}_m) \arrow[d, "\mathrm{Lie}"] \\ & \mathrm{Hom}_{R\text{-gp}}(\mathrm{Lie}(\mathbb{A}/R), \mathbb{G}_a) \arrow[d, equal] \\ & \underline{\omega}_{\mathbb{A}/R} \end{tikzcd}$$

More functorially, we can introduce the ring $R[\varepsilon] = R+R\varepsilon$, $\varepsilon^2 = 0$, of dual numbers over R . Then the Lie algebra $\mathrm{Lie}(\mathbb{A}/R)$ is the subgroup of $\hat{\mathbb{A}}(R[\varepsilon])$ defined by

$$\mathrm{Lie}(\mathbb{A}/R) = \mathrm{Ker} \quad \text{of} \quad \mathbb{A}(R[\varepsilon]) \xrightarrow{\varepsilon \to 0} \mathbb{A}(R)$$

$$= \mathrm{Ker} \quad \text{of} \quad \hat{\mathbb{A}}(R[\varepsilon]) \xrightarrow{\varepsilon \to 0} \hat{\mathbb{A}}(R)$$

(the second equality because R is an artin local ring). Let us denote by

$$\cdot : \underline{\omega}_{\mathbb{A}/R} \times \mathrm{Lie}(\mathbb{A}/R) \to R$$
$$(\omega, L) \to \omega . L$$

the natural duality pairing of $\underline{\omega}$ and Lie. Then we have the formula, for any $L \in \mathrm{Lie}(\mathbb{A}/R)$,

$$1 + \varepsilon\omega(\alpha_t).L = \lambda(\alpha_t)(L) \in \mathrm{Lie}(\hat{\mathbb{G}}_m/R)$$

or equivalently

$$1 + \varepsilon\omega(\alpha_t).L = E_{\mathbb{A}}(L, \alpha_t) \ .$$

If we choose an integer n large enough that $p^n R = 0$, we will have

$$\mathrm{Lie}(\mathbb{A}/R) \subset \hat{\mathbb{A}}(R[\varepsilon])[p^n] \ ,$$

so we may rewrite this last formula as

$$1 + \varepsilon\omega(\alpha_t).L = E_{\mathbb{A};p^n}(L, \alpha_t(n)) \ .$$

Finally, if we choose an artin local ring R' which is finite and flat over R , and a point

$$Y \in \mathbb{A}^t(R')[p^n] \quad \text{lifting} \quad \alpha_t(n) \in A^t(k)[p^n] \ ,$$

we may, by lemma 2.2, rewrite this last formula in

$$1+\varepsilon\omega(\alpha_t).L = e_{p^n}(L, Y) \ .$$

The construction of $\omega(\alpha_t)$ defines a $\mathbb{Z}_p$-linear homomorphism

$$T_p A^t(k) \to \omega_{\mathbb{A}/R} \ ,$$
$$\alpha_t \mapsto \omega(\alpha_t)$$

which, in view of the isomorphism

$$\hat{\mathbb{A}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(T_p A^t(k), \hat{\mathbb{G}}_m) \ ,$$

induces an R-linear isomorphism

$$T_pA^t(k) \underset{\mathbb{Z}_p}{\otimes} R \xrightarrow{\sim} \underline{\omega}_{\mathbb{A}/R} \ .$$

The evident functoriality of this construction for variable situations $\mathbb{A}/R$, shows that it extends uniquely to the universal formal deformation $\mathcal{A}/\mathfrak{R}$, i.e. to a $\mathbb{Z}_p$-linear homomorphism

$$T_pA^t(k) \to \underline{\omega}_{\mathcal{A}/\mathfrak{R}}$$
$$\alpha_t \mapsto \omega(\alpha_t)$$

which is compatible with the canonical identifications

$$\omega_{\mathcal{A}/\mathfrak{R}} \underset{\mathfrak{R}}{\otimes} R \xrightarrow{\sim} \underline{\omega}_{\mathbb{A}/R}$$

whenever $\mathbb{A}/R$ is a lifting of A/k to an artin local ring R with residue field k , and R is viewed as an $\mathfrak{R}_{univ}$-algebra in the $\otimes$ via the classifying homomorphism $f_{\mathbb{A}/R} : \mathfrak{R} \to R$ of $\mathbb{A}/R$.
The associated $\mathfrak{R}$-linear map is an isomorphism

$$T_pA^t(k) \underset{\mathbb{Z}_p}{\otimes} \mathfrak{R} \xrightarrow{\sim} \underline{\omega}_{\mathcal{A}/\mathfrak{R}} \ .$$

3.4. The R-linear <u>dual</u> of the isomorphism

$$T_pA^t(k) \otimes R \xrightarrow{\sim} \omega_{\mathbb{A}/R}$$

is obtained by applying the functor "Lie" to the isomorphism

$$\hat{\mathbb{A}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k), \hat{\mathbb{G}}_m) \ .$$

Its inverse provides an R-isomorphism

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k), \mathbb{Z}_p) \underset{\mathbb{Z}_p}{\otimes} R \xrightarrow{\sim} \mathrm{Lie}(\mathbb{A}/R) \ ,$$

which yields, upon passing to the limit, an $\mathfrak{R}$-isomorphism

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k), \mathbb{Z}_p) \underset{\mathbb{Z}_p}{\otimes} \mathfrak{R} \xrightarrow{\sim} \mathrm{Lie}(\mathcal{A}/\mathfrak{R}) \ .$$

The "underlying" $\mathbb{Z}_p$-linear homomorphisms

$$\left\{ \begin{array}{ll} \mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k), \mathbb{Z}_p) & \to \mathrm{Lie}(\mathbb{A}/R) \\ \qquad \text{"} & \to \mathrm{Lie}(\mathcal{A}/\mathfrak{R}) \end{array} \right.$$

will be denoted

$$\check{\alpha}_t \to L(\check{\alpha}_t) .$$

It is immediate from the <u>definition</u> of $L(\check{\alpha}_t)$ that for any situation $\mathbb{A}/R$, any $\alpha_t \in T_pA^t(k)$ and any $\check{\alpha}_t \in \mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k),\mathbb{Z}_p)$, we have the formula

$$\omega(\alpha_t).L(\check{\alpha}_t) = \alpha_t.\check{\alpha}_t \quad \text{in} \quad \mathbb{Z}_p .$$

3.5. Let us make explicit the functoriality of the constructions $\omega(\alpha_t)$, $L(\check{\alpha}_t)$ under morphisms. Thus suppose we have two ordinary abelian varieties A , B over k , liftings of them $\mathbb{A}/R$, $\mathbb{B}/R$ to an artin local ring R with residue field k , and an R-homomorphism

$$\mathbb{f} : \mathbb{A} \to \mathbb{B}$$

lifting a k-homomorphism

$$f : A \to B .$$

LEMMA 3.5.1. <u>Under the induced map</u>

$$\mathbb{f}^* : \underline{\omega}_{\mathbb{B}/R} \to \underline{\omega}_{\mathbb{A}/R}$$

<u>we have the formula</u>

$$\mathbb{f}^*(\omega(\beta_t)) = \omega(f^t(\beta_t))$$

<u>for any</u> $\beta_t \in T_pB^t(k)$.

PROOF. This is immediate from the definition of the ω-construction and the commutativity (by rigidity of toroidal groups !) of the diagram

$$\begin{array}{ccc} \hat{\mathbb{A}} & \xrightarrow{\sim} & \mathrm{Hom}(T_pA^t(k),\hat{\mathbb{G}}_m) \\ \downarrow{\scriptstyle \mathbb{f}} & & \downarrow{\scriptstyle \circ f^t} \\ \hat{\mathbb{B}} & \xrightarrow{\sim} & \mathrm{Hom}(T_pB^t(k),\hat{\mathbb{G}}_m) \end{array} \quad . \qquad \text{Q.E.D.}$$

LEMMA 3.5.2. <u>Under the induced map</u>

$$\mathbb{f}_* : \mathrm{Lie}(\mathbb{A}/R) \to \mathrm{Lie}(\mathbb{B}/R) ,$$

we have the formula

$$\mathbb{f}_*(L(\alpha_t^\vee)) = L(\alpha_t^\vee \circ f^t)$$

for any $\alpha_t^\vee \in \mathrm{Hom}(T_pA^t(k), \mathbb{Z}_p)$.

PROOF. The same. Q.E.D.

LEMMA 3.5.3. *Under the induced map*

$$\begin{array}{ccc} \mathbb{f}^* : H^1(\mathbb{B}, \mathcal{O}_{\mathbb{B}}) & \longrightarrow & H^1(\mathbb{A}, \mathcal{O}_{\mathbb{A}}) \\ \wr\downarrow & & \wr\downarrow \\ \mathrm{Lie}(\mathbb{B}^t/R) & \underset{\mathbb{f}_*^t}{\dashrightarrow} & \mathrm{Lie}(\mathbb{A}^t/R) \end{array}$$

we have the formula

$$\mathbb{f}^*(L(\beta^\vee)) = \mathbb{f}_*^t(L(\beta^\vee)) = L(\beta^\vee \circ f)$$

for any $\beta^\vee \in \mathrm{Hom}(T_pB(k), \mathbb{Z}_p)$.

PROOF. This is the concatenation of the previous lemma and the functoriality of the identification of $H^1(\mathbb{A}, \mathcal{O}_{\mathbb{A}})$ with $\mathrm{Lie}(\mathbb{A}^t/R)$. Q.E.D.

3.6. We next recall the definition of the Kodaira-Spencer mapping. First consider a lifting $\mathbb{A}/R$ of A/k to an artin local ring R with residue field k. Such an R has a unique structure of $W = W(k)$-algebra. This W-algebra structure on R allows us to view $\mathbb{A}$ as a W-scheme. Because $\mathbb{A}$ is smooth over R, we have a locally splittable short exact sequence on $\mathbb{A}$

$$0 \longrightarrow \mathcal{O}_{\mathbb{A}} \underset{R}{\otimes} \Omega^1_{R/W} \longrightarrow \Omega^1_{\mathbb{A}/W} \longrightarrow \Omega^1_{\mathbb{A}/R} \longrightarrow 0 .$$

The coboundary map in the long exact sequence of cohomology

$$\begin{tikzcd}
\underline{\omega}_{\mathbb{A}/R} = H^0(\mathbb{A},\Omega^1_{\mathbb{A}/R}) \arrow[r, "\partial"] \arrow[ddr, dashed, "\mathrm{Kod}"'] & H^1(\mathbb{A},\mathcal{O}_{\mathbb{A}} \underset{R}{\otimes} \Omega^1_{R/W}) \\
 & H^1(\mathbb{A},\mathcal{O}_{\mathbb{A}}) \underset{R}{\otimes} \Omega^1_{R/W} \arrow[u, "\wr"', "\text{(base-change for } \mathbb{A}/R)"] \arrow[d, "\wr"] \\
 & \mathrm{Lie}(\mathbb{A}^t/R) \underset{R}{\otimes} \Omega^1_{R/W} ,
\end{tikzcd}$$

defines the Kodaira-Spencer mapping

$$\mathrm{Kod} : \underline{\omega}_{\mathbb{A}/R} \to \mathrm{Lie}(\mathbb{A}^t/R) \underset{R}{\otimes} \Omega^1_{R/W} .$$

By passage to the limit, we obtain the Kodaira-Spencer mapping in the universal case :

$$\mathrm{Kod} : \underline{\omega}_{\mathcal{A}/\mathcal{R}} \to \mathrm{Lie}(\mathcal{A}^t/\mathcal{R}) \otimes \Omega^1_{\mathcal{R}/W}$$

(with the convention that $\Omega^1_{\mathcal{R}/W}$ denotes the <u>continuous</u> one-forms).

3.7. In this section we state three visibly equivalent forms (3.7.1-2-3) of the fundamental compatibility.

MAIN THEOREM 3.7.1. <u>Under the canonical pairing</u>

$$. : \underline{\omega}_{\mathcal{A}^t/\mathcal{R}} \times \mathrm{Lie}(\mathcal{A}^t/\mathcal{R}) \otimes \Omega^1_{\mathcal{R}/W} \longrightarrow \Omega^1_{\mathcal{R}/W} ,$$

<u>we have the formula</u>

$$\omega(\alpha).\mathrm{Kod}(\omega(\alpha_t)) = \mathrm{dlog}(q(\alpha,\alpha_t)) ,$$

for any $\alpha \in T_pA(k)$ (viewed as $T_pA^{tt}(k)$, so that $\omega(\alpha)$ is defined), and any $\alpha_t \in T_pA^t(k)$.

MAIN THEOREM (bis) 3.7.2. <u>Choose a</u> $\mathbb{Z}_p$-<u>basis</u> $\alpha_1,\ldots,\alpha_g \in T_pA(k)$, <u>and denote by</u> $\alpha_1^\vee,\ldots,\alpha_g^\vee$ <u>the dual base of</u> $\mathrm{Hom}(T_pA(k),\mathbb{Z}_p)$, <u>we have the formula</u>

$$\mathrm{Kod}(\omega(\alpha_t)) = \sum_i L(\alpha_i^\vee) \otimes \mathrm{dlog}\, q(\alpha_i,\alpha_t)$$

<u>for any</u> $\alpha_t \in T_pA^t(k)$.

For each continuous derivation D of $\mathfrak{R}$ into itself consider the map $\mathrm{Kod}(D)$ defined by

$$\begin{tikzcd}
\underline{\omega}_{\mathcal{A}/\mathfrak{R}} \arrow[r, "\mathrm{Kod}"] \arrow[dr, "\mathrm{Kod}(D)"'] & \mathrm{Lie}(\mathcal{A}^t/\mathfrak{R}) \otimes \Omega^1_{\mathfrak{R}/W} \arrow[d, "1 \otimes D"] \\
 & \mathrm{Lie}(\mathcal{A}^t/\mathfrak{R})
\end{tikzcd} .$$

For each element

$$\ell \in \mathrm{Hom}(T_pA(k) \otimes T_pA^t(k), \mathbb{Z}_p) ,$$

and each element

$$\alpha_t \in T_pA^t(k) ,$$

we denote by

$$\ell * \alpha_t \in \mathrm{Hom}(T_pA(k), \mathbb{Z}_p)$$

the element defined by

$$(\ell * \alpha_t)(\alpha) = \ell(\alpha \otimes \alpha_t) .$$

MAIN THEOREM (ter) 3.7.3. _We have the formula_

$$\mathrm{Kod}(D(\ell))(\omega(\alpha_t)) = L(\ell * \alpha_t)$$

for any $\alpha_t \in T_pA^t(k)$ _and any_ $\ell \in \mathrm{Hom}(T_pA(k) \otimes T_pA^t(k), \mathbb{Z}_p)$.
Equivalently, for any $\alpha \in T_pA(k)$, _we have the formula_

$$\omega(\alpha).\mathrm{Kod}(D(\ell))(\omega(\alpha_t)) = \ell(\alpha \otimes \alpha_t) .$$

4. THE MAIN THEOREM : EQUIVALENT FORMS AND REDUCTION STEPS

4.0. Our proof falls naturally into two parts. In the first part, we make use of the canonical Frobenius endomorphism Φ of $\hat{\mathcal{M}}$ to transform the Main Theorem into a theorem (4.3.1.2) giving the precise structure of the Gauss-Manin connection on the De Rham cohomology of the universal formal deformation $\mathbb{A}/\mathfrak{R}$. We then make use of the "rigidity" of these various actors in the universal situation to show that the Main Theorem in its Gauss-Manin reformulation follows from an exact formula (4.5.3) for the Serre-Tate q-parameters of square-zero deformations of the canonical lifting.

The second part of the proof, which amounts to verifying 4.5.3, is given in chapters 5 and 6.

4.1. Let σ denote the absolute Frobenius automorphism of $W = W(k)$. For <u>any</u> W-scheme X , we denote by $X^{(\sigma)}$ the W-scheme obtained from X/W by the extension of scalars $W \xrightarrow{\sigma} W$. Thus we have a tautological cartesian diagram of schemes

$$\begin{array}{ccc} X^{(\sigma)} & \xrightarrow[\sim]{\Sigma} & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}(W) & \xrightarrow[\sim]{\operatorname{Spec}(\sigma)} & \operatorname{Spec}(W) \end{array} .$$

LEMMA 4.1.1. <u>We have a natural isomorphism</u>

$$(\hat{\mathfrak{m}}_{A/k})^{(\sigma)} \xrightarrow{\sim} \hat{\mathfrak{m}}_{A^{(\sigma)}/k}$$

<u>under which</u>

$$\Sigma^*(q(\alpha,\alpha_t)) \longleftarrow q(\sigma(\alpha),\sigma(\alpha_t)) \ .$$

PROOF. Let R be an artin local ring with residue field k , and $\mathbb{A}/R$ an abelian scheme lifting A/k . Then $\mathbb{A}^{(\sigma)}/R^{(\sigma)}$ is a lifting of $A^{(\sigma)}/k$. Because σ is an automorphism, this construction defines a bijection

$$\hat{\mathfrak{m}}_{A/k}(R) \xrightarrow{\sim} \hat{\mathfrak{m}}_{A^{(\sigma)}/k}(R^{(\sigma)})$$

which is functorial for variable R . If we apply it to $\mathfrak{R}$, we find a bijection

$$\begin{array}{ccc} \hat{\mathfrak{m}}_{A/k}(\mathbb{R}) & \simeq & \hat{\mathfrak{m}}_{A^{(\sigma)}/k}(\mathbb{R}^{(\sigma)}) \\ \| & & \| \\ \mathrm{Hom}_{fctr}(\hat{\mathfrak{m}}_{A/k},\hat{\mathfrak{m}}_{A/k}) & & \mathrm{Hom}_{fctr}((\hat{\mathfrak{m}}_{A/k})^{(\sigma)},\mathfrak{m}_{A^{(\sigma)}/k}) \\ \Psi & & \\ \mathrm{id} & & \end{array}$$

The element of $\mathrm{Hom}((\hat{\mathfrak{m}}_{A/k})^{(\sigma)},\hat{\mathfrak{m}}_{A^{(\sigma)}/k})$ corresponding to the identity map is the required isomorphism. Alternatively, this isomorphism is the classifying map for the formal deformation of $A^{(\sigma)}/k$ provided by $\mathbb{A}^{(\sigma)}/\mathbb{R}^{(\sigma)}$.

By "transport of structure", we have for every $\mathbb{A}/R$, the formula

$$\Sigma^*(q(\mathbb{A}/R;\alpha,\alpha_t)) = q(\mathbb{A}^{(\sigma)}/R^{(\sigma)};\sigma(\alpha),\sigma(\alpha_t)) \ ,$$

and hence we have

$$\Sigma^*(q(\alpha,\alpha_t)) = q(\sigma(\alpha),\sigma(\alpha_t)) \ . \quad \text{Q.E.D.}$$

LEMMA 4.1.1.1. <u>The behaviour of the constructions</u> $\omega(\alpha_t)$, $L(\check{\alpha}_t)$ <u>under the construction</u>

$$\mathbb{A}/R \longmapsto \mathbb{A}^{(\sigma)}/R^{(\sigma)}$$

<u>is expressed by formulas</u>

$$\begin{cases} \Sigma^*(\omega(\alpha_t)) = \omega(\sigma(\alpha_t)) \\ \Sigma^*(L(\check{\alpha}_t)) = L(\check{\alpha}_t \circ \sigma^{-1}) \ . \end{cases}$$

PROOF. This is obvious by "transport of structure". Q.E.D.

Given $\mathbb{A}/R$, we denote by $\mathbb{A}'/R$ the <u>quotient</u> of $\mathbb{A}$ by the "canonical subgroup" $\hat{\mathbb{A}}[p]$ of $\mathbb{A}$. The morphism "projection onto the quotient"

$$F_{can} : \mathbb{A} \to \mathbb{A}'$$

lifts the absolute Frobenius morphism

$$F : A \to A^{(\sigma)} \ .$$

LEMMA 4.1.2. For $\alpha \in T_pA(k)$ and $\alpha_t \in T_pA^t(k)$, we have the formulas

$$\begin{cases} F(\alpha) = \sigma(\alpha) \quad , \quad V(\sigma(\alpha)) = p\alpha_t \\ q(\mathbb{A}'/R;\sigma(\alpha),\sigma(\alpha_t)) \;=\; (q(\mathbb{A}/R;\alpha,\alpha_t))^p \ . \end{cases}$$

PROOF. Because the morphism F_{can} exists, and lifts F, the lifting criterion yields the formula

$$q(\mathbb{A}/R;\alpha,V(\sigma(\alpha_t))) \;=\; q(\mathbb{A}'/R;F(\alpha),\sigma(\alpha_t)) \ .$$

It is visible that

$$F(\alpha) = \sigma(\alpha) \qquad \text{for} \quad \alpha \in T_pA(k) \ .$$

Applying this to A^t, we have

$$F(\alpha_t) = \sigma(\alpha_t) \quad \text{for} \quad \alpha_t \in T_pA^t(k) \ .$$

Because $VF = P$, we find, upon applying V, the formula

$$p\alpha_t = V(\sigma(\alpha_t)) \ . \quad \text{Q.E.D.}$$

LEMMA 4.1.3. Let $\alpha_t \in T_pA^t(k)$, and $\alpha^{\vee} \in \mathrm{Hom}(T_pA(k),\mathbf{Z}_p)$. Consider the elements

$$\begin{cases} \omega(\alpha_t) \in \underline{\omega}_{\mathbb{A}/R} = H^0(\mathbb{A},\Omega^1_{\mathbb{A}/R}) \ , \\ \omega(\sigma(\alpha_t)) \in \underline{\omega}_{\mathbb{A}'/R} \\ L(\alpha^{\vee}) \in \mathrm{Lie}(\mathbb{A}^t/R) \simeq H^1(\mathbb{A},\mathcal{O}_{\mathbb{A}}) \\ L(\alpha^{\vee}\circ\sigma^{-1}) \in \mathrm{Lie}((\mathbb{A}^t)'/R) \simeq H^1(\mathbb{A}',\mathcal{O}_{\mathbb{A}'}) \ . \end{cases}$$

Under the morphism F^*_{can} induced by

$$F_{can} : \mathbb{A} \to \mathbb{A}' \ ,$$

we have the formulas

$$F^*_{can}(\omega(\sigma(\alpha_t))) \;=\; p\omega(\alpha_t)$$

$$F^*_{can}(L(\alpha^{\vee}\circ\sigma^{-1})) \;=\; L(\alpha^{\vee}) \ .$$

PROOF. By lemma 3.5.1, we have

$$F^*_{can}(\omega(\sigma(\alpha_t))) \;=\; \omega(V\sigma(\alpha_t)) \;=\; \omega(p\alpha_t)$$

while by lemma 3.5.3 we have

$$F_{\mathrm{can}}^{*}(L(\alpha^{\vee}\circ\sigma^{-1})) = L(\alpha^{\vee}\circ\sigma^{-1}\circ F) = L(\alpha^{\vee}) . \quad \text{Q.E.D.}$$

If we apply the construction

$$\mathbb{A}/R \longmapsto \mathbb{A}'/R$$

to the universal formal deformation $\mathcal{A}/\hat{\mathcal{M}}_{A/k}$ of A/k, we obtain a formal deformation $\mathcal{A}'/\hat{\mathcal{M}}_{A/k}$ of $A^{(\sigma)}/k$. It's classifying map is the unique morphism

$$\Phi : \hat{\mathcal{M}}_{A/k} \longrightarrow \hat{\mathcal{M}}_{A^{(\sigma)}/k} \xrightarrow{\sim} (\hat{\mathcal{M}}_{A/k})^{(\sigma)}$$

such that

$$\Phi^{*}(\mathcal{A}^{(\sigma)}) \simeq \mathcal{A}' .$$

The expression of Φ on the coordinate rings is given, by lemma 4.1.1, as

$$\Phi^{*}\Sigma^{*}(q(\alpha,\alpha_{t})) = q(\alpha,\alpha_{t})^{p} .$$

In terms of the structure of toroidal formal Lie group over W imposed upon $\hat{\mathcal{M}}_{A/k}$ by Serre-Tate, the morphism Φ may be characterized as the unique group homomorphism which reduces mod p to the absolute Frobenius.

The isomorphism

$$\Phi^{*}(\mathcal{A}^{(\sigma)}) \simeq \mathcal{A}'$$

allows us to view F_{can} as a morphism of formal abelian schemes over $\hat{\mathcal{M}}_{A/k}$

$$F_{\mathrm{can}} : \mathcal{A} \longrightarrow \Phi^{*}(\mathcal{A}^{(\sigma)}) .$$

LEMMA 4.1.4. *Let* $\alpha_{t} \in T_{p}A^{t}(k)$, $\alpha^{\vee} \in \mathrm{Hom}(T_{p}A(k),\mathbb{Z}_{p})$. *Consider the elements*

$$\begin{cases} \omega(\alpha_{t}) \in \underline{\omega}_{\mathcal{A}/\hat{\mathcal{M}}} \\ L(\alpha^{\vee}) \in \mathrm{Lie}(\mathcal{A}^{t}/\mathcal{R}) \simeq H^{1}(\mathcal{A},\mathcal{O}_{\mathcal{A}}) . \end{cases}$$

Under the morphism F_{can}^{*} *induced by*

$$F_{\mathrm{can}} : \mathcal{A} \to \Phi^{*}(\mathcal{A}^{(\sigma)}) ,$$

<u>we have the formulas</u>

$$\begin{cases} F_{can}^{*}\Phi^{*}\Sigma^{*}(\omega(\alpha_t)) = p\omega(\alpha_t) \\ F_{can}^{*}\Phi^{*}\Sigma^{*}(L(\alpha^{\vee})) = L(\alpha^{\vee}) \ . \end{cases}$$

PROOF. This follows immediately from 4.1.3 and 4.1.1.1.

COROLLARY 4.1.5. <u>The</u> ω <u>and</u> L <u>constructions define isomorphisms</u>

$$T_pA^t(k) \xrightarrow{\sim} \{\omega \in \underline{\omega}_{\mathcal{A}/\mathfrak{R}} \mid F_{can}^{*}\Phi^{*}\Sigma^{*}(\omega) = p\omega\}$$

$$\operatorname{Hom}(T_pA(k),\mathbb{Z}_p) \xrightarrow{\sim} \left\{ \begin{array}{l} L \in \operatorname{Lie}(\mathcal{A}^t/\mathfrak{R}) \simeq H^1(\mathcal{A},\mathcal{O}_{\mathcal{A}}) \\ \text{such that } F_{can}^{*}\Phi^{*}\Sigma^{*}(L) = L \end{array} \right\} .$$

PROOF. Let $\alpha_{1,t},\ldots,\alpha_{g,t}$ be a $\mathbb{Z}_p$-basis of $T_pA^t(k)$. Then $\omega(\alpha_{1,t}),\ldots,\omega(\alpha_{g,t})$ is an $\mathfrak{R}$-basis of $\underline{\omega}_{\mathcal{A}/\mathfrak{R}}$. Given $\omega \in \underline{\omega}$, it has a unique expression

$$\omega = \sum_i f_i \ \omega(\alpha_{t,i}) \quad , \quad f_i \in \mathfrak{R} \ ,$$

whence

$$F_{can}^{*}\Phi^{*}\Sigma^{*}(\omega) = \Sigma \ \Phi^{*}\Sigma^{*}(f_i).p\omega(\alpha_{t,i}) \ .$$

Therefore, as $\mathfrak{R}$ is torsion-free, we see that

$$F_{can}^{*}\Phi^{*}\Sigma^{*}(\omega) = p\omega$$

$$\Longleftrightarrow \Phi^{*}\Sigma^{*}(f_i) = f_i \quad \text{for } i = 1,\ldots,g \ .$$

But it is obvious that a function $f \in \mathfrak{R}$ satisfies $\Phi^{*}\Sigma^{*}(f) = f$ if and only if f is a constant in $\mathbb{Z}_p$.

The proof of the second assertion is entirely analogous.

4.2. Consider the de Rham cohomology of $\mathcal{A}/\mathfrak{R}$, sitting in its Hodge exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\omega}_{\mathcal{A}/\mathfrak{R}} & \longrightarrow & H^1_{DR}(\mathcal{A}/\mathfrak{R}) & \longrightarrow & H^1(\mathcal{A},\mathcal{O}_{\mathcal{A}}) & \longrightarrow & 0 \\ & & \Big\uparrow\wr\, \omega\otimes 1 & & & & \Big\| & & \\ & & & & & & \mathrm{Lie}(\mathcal{A}^t/\mathfrak{R}) & & \\ & & T_pA^t(k) \underset{\mathbb{Z}_p}{\otimes} \mathfrak{R} & & & & \Big\uparrow\wr\, L\otimes 1 & & \\ & & & & & & \mathrm{Hom}(T_pA(k),\mathbb{Z}_p) \underset{\mathbb{Z}_p}{\otimes} \mathfrak{R} \ . & & \end{array}$$

Let us denote by

$$\mathrm{Fix}(H^1_{DR}) \ , \ p\text{-}\mathrm{Fix}(H^1_{DR})$$

the $\mathbb{Z}_p$-submodules of $H^1_{DR}(\mathcal{A}/\mathfrak{R})$ defined as

$$\mathrm{Fix} = \{\xi \in H^1_{DR} \mid F^*_{can}\Phi^*\Sigma^*(\xi) = \xi\}$$

$$p\text{-}\mathrm{Fix} = \{\xi \in H^1_{DR} \mid F^*_{can}\Phi^*\Sigma^*(\xi) = p\xi\} \ .$$

LEMMA 4.2.1. <u>The maps</u> a , b <u>in the Hodge exact sequence</u>

$$0 \longrightarrow T_pA^t(k) \underset{\mathbb{Z}_p}{\otimes} \mathfrak{R} \xrightarrow{a} H^1_{DR}(\mathcal{A}/\mathfrak{R}) \xrightarrow{b} \mathrm{Hom}(T_pA(k),\mathbb{Z}_p) \underset{\mathbb{Z}_p}{\otimes} \mathfrak{R} \longrightarrow 0$$

<u>induce isomorphisms</u>

$$(1) \qquad T_pA^t(k) \xrightarrow[\sim]{a} p\text{-}\mathrm{Fix}(H^1_{DR})$$

$$(2) \qquad \mathrm{Hom}(T_pA(k),\mathbb{Z}_p) \xleftarrow[b]{\sim} \mathrm{Fix}(H^1_{DR}) \ .$$

PROOF. (1) Let $\xi \in p\text{-}\mathrm{Fix}$. By 4.1.5, it suffices to show that ξ lies in $\underline{\omega}_{\mathcal{A}/\mathfrak{R}}$. For this, it suffices to show that the projection of ξ in $H'(\mathcal{A},\mathcal{O}_{\mathcal{A}})$ vanishes. But this projection lies in $p\text{-}\mathrm{Fix}(H'(\mathcal{A},\mathcal{O}_{\mathcal{A}}))$; in terms of a $\mathbb{Z}_p$-basis $\alpha_i^\vee$ of $\mathrm{Hom}(T_pA(k),\mathbb{Z}_p)$, we have

$$\mathrm{proj}(\xi) = \Sigma\, f_i L(\alpha_i^\vee) \ ,$$

$$p\ \mathrm{proj}(\xi) = F^*_{can}\Phi^*\Sigma^*(\mathrm{proj}(\xi)) = \Sigma\ \Phi^*\Sigma^*(f_i)\ L(\alpha_i^\vee) \ ,$$

whence the coefficients $f_i \in \mathfrak{R}$ satisfy

$$\Phi^*\Sigma^*(f_i) = pf_i \ .$$

Because $\mathfrak{R}$ is flat over $\mathbb{Z}_p$ and p-adically separated, $\mathfrak{R}/p\mathfrak{R}$ is reduced; as $\Phi^*\Sigma^*$ reduces mod p to the absolute Frobenius endomorphism of $\mathfrak{R}/p\mathfrak{R}$, we infer that $f_i = 0$.

(2) By 4.1.4, the endomorphism $F^*_{\mathrm{can}}\Phi^*\Sigma^*$ of $\underline{\omega}_{\mathcal{A}/\mathcal{R}}$ is p-adically nilpotent, and therefore we have

$$\mathrm{Fix}(H^1_{DR}) \cap \underline{\omega}_{\mathcal{A}/\mathcal{R}} = 0 .$$

This means that the projection b induces an injective map

$$\begin{tikzcd} \mathrm{Fix}(H^1_{DR}) \arrow[r, "\mathrm{proj}"] \arrow[dr, "b"'] & \mathrm{Fix}(H^1(\mathcal{A},\mathcal{O}_{\mathcal{A}})) \\ & \mathrm{Hom}(T_pA(k),\mathbb{Z}_p)) \arrow[u, "\wr", "(4.1.5)"'] \end{tikzcd} .$$

To see that it is surjective, fix an element $\alpha^\vee \in \mathrm{Hom}(T_pA(k),\mathbb{Z}_p))$, and choose any element $\xi_0 \in H^1_{DR}$ which projects to $L(\alpha^\vee)$. Because $L(\alpha^\vee)$ is fixed by $F^*_{\mathrm{can}}\Phi^*\Sigma^*$, each of the sequence $\xi_0, \xi_1, \ldots$ of elements of H^1_{DR} defined inductively by

$$\xi_{n+1} = F^*_{\mathrm{can}}\Phi^*\Sigma^*(\xi_n)$$

also projects to $L(\alpha^\vee)$. Therefore for every $n \geqslant 0$ we have

$$\xi_n - \xi_0 = \omega_n \in \underline{\omega}_{\mathcal{A}/\mathcal{R}} ;$$

applying the endomorphism $F^*_{\mathrm{can}}\Phi^*\Sigma^*$ m times, we see by 4.1.4 that

$$\xi_{n+m} - \xi_m = (F^*_{\mathrm{can}}\Phi^*\Sigma^*)^m(\omega_n) \in p^m\underline{\omega}_{\mathcal{A}/\mathcal{R}} .$$

Therefore the sequence ξ_n converges, in the p-adic topology on H^1_{DR} , to an element ξ_∞ which projects to $L(\alpha^\vee)$ and which by construction lies in $\mathrm{Fix}(H^1_{DR})$.

For each element $\alpha^\vee \in \mathrm{Hom}(T_pA(k),\mathbb{Z}_p)$, we denote by

$$\mathrm{Fix}(\alpha^\vee) \in \mathrm{Fix}(H^1_{DR})$$

the unique fixed point which projects to $L(\alpha^\vee)$. Formation of $\mathrm{Fix}(\alpha^\vee)$ defines the isomorphism inverse to b :

$$\mathrm{Hom}(T_pA(k),\mathbb{Z}_p) \underset{b}{\overset{\mathrm{Fix}}{\rightleftarrows}} \mathrm{Fix}(H^1_{DR}) .$$

COROLLARY 4.2.2. <u>The construction "Fix" provides the unique</u> $\mathcal{R}$-<u>splitting of the Hodge exact sequence which respects the action of</u> $F^*_{\mathrm{can}}\Phi^*\Sigma^*$:

$$\begin{tikzcd}
0 \arrow[r] & \underline{\omega}_{\mathcal{A}/\mathcal{R}} \arrow[r] & H^1_{DR}(\mathcal{A}/\mathcal{R}) \arrow[r] & H^1(\mathcal{A},\mathcal{O}_{\mathcal{A}}) \arrow[r] & 0 \\
& T_pA^t(k)\otimes\mathcal{R} \arrow[u, "\omega\otimes 1"', "\wr"] & & \mathrm{Lie}(\mathcal{A}^t/\mathcal{R}) \arrow[u, "\wr"] & \\
& & & \mathrm{Hom}(T_pA(k),\mathbb{Z}_p)\otimes\mathcal{R} \arrow[u, "L\otimes 1", "\wr"'] \arrow[uul, bend left, "\mathrm{Fix}\otimes 1"] &
\end{tikzcd}$$

4.3. In this section we will give further equivalent forms of the Main Theorem, this time formulated in terms of the Gauss-Manin connection on $H^1_{DR}(\mathcal{A}/\mathcal{R})$.

MAIN THEOREM (quat) 4.3.1. <u>Let</u> $\alpha_1,\ldots,\alpha_g$ <u>be a</u> $\mathbb{Z}_p$-<u>basis of</u> $T_pA(k)$, $\alpha_1^\vee,\ldots,\alpha_g^\vee$ <u>the dual basis of</u> $\mathrm{Hom}(T_pA(k),\mathbb{Z}_p)$. <u>Under the Gauss-Manin connection</u>

$$\nabla : H^1_{DR}(\mathcal{A}/\mathcal{R}) \to H^1_{DR}(\mathcal{A}/\mathcal{R}) \otimes \Omega^1_{\mathcal{R}/W}$$

<u>we have the formulas</u>

$$\nabla(\omega(\alpha_t)) = \sum_i \mathrm{Fix}(\alpha_i^\vee)\otimes \mathrm{dlog}\, q(\alpha_i,\alpha_t)$$

$$\nabla(\mathrm{Fix}(\alpha^\vee)) = 0$$

<u>for any</u> $\alpha_t \in T_pA^t(k)$, <u>and any</u> $\alpha^\vee \in \mathrm{Hom}(T_pA(k),\mathbb{Z}_p)$.

For each continuous derivation D of $\mathcal{R}$ into itself we denote by $\nabla(D)$ the map defined by

$$\begin{tikzcd}
H^1_{DR}(\mathcal{A}/\mathcal{R}) \arrow[r, "\nabla"] \arrow[dr, "\nabla(D)"'] & H^1_{DR}(\mathcal{A}/\mathcal{R}) \otimes \Omega^1_{\mathcal{R}/W} \arrow[d, "1\otimes D"] \\
& H^1_{DR}(\mathcal{A}/\mathcal{R})
\end{tikzcd}$$

MAIN THEOREM (cinq) 4.3.2. <u>We have the formulas</u>

$$\begin{cases} \nabla(D(\ell))(\omega(\alpha_t)) = \mathrm{Fix}(\ell * \alpha_t) \\ \nabla(D(\ell))(\mathrm{Fix}(\alpha^\vee)) = 0 , \end{cases}$$

<u>for every</u> $\alpha_t \in T_pA^t(k)$, $\alpha^\vee \in \mathrm{Hom}(T_pA(k),\mathbb{Z}_p)$, $\ell \in \mathrm{Hom}(T_pA(k)\otimes T_pA^t(k),\mathbb{Z}_p)$.

Let us explain why 4.3.1-2 are in fact equivalent to 3.7.1-2-3. That 4.3.1 and 4.3.2 are equivalent to each other is obvious. The implication (4.3.1) $\Longrightarrow$ (3.7.2) comes from the fact that the Kodaira-Spencer mapping Kod is the "associated graded", for the Hodge filtration, of the Gauss-Manin connection, i.e. from the commutativity of the diagram

$$\begin{tikzcd} H^1_{DR}(\mathcal{A}/\mathcal{R}) \arrow[r, "\nabla"] & H^1_{DR}(\mathcal{A}/\mathcal{R}) \otimes \Omega^1_{\mathcal{R}/W} \arrow[d, "\mathrm{proj} \otimes 1"] \\ & H^1(\mathcal{A}, \mathcal{O}_{\mathcal{A}}) \otimes \Omega^1_{\mathcal{R}/W} \arrow[d, phantom, "\wr"] \\ \underline{\omega}_{\mathcal{A}/\mathcal{R}} \arrow[uu, hook] \arrow[r, "\mathrm{Kod}"] & \mathrm{Lie}(\mathcal{A}^t/\mathcal{R}) \otimes \Omega^1_{\mathcal{R}/W} \, . \end{tikzcd}$$

It remains to deduce (4.3.1) from (3.7.2). In terms of a $\mathbb{Z}_p$ base $\{\alpha_i\}$ of $T_pA(k)$ and of the dual base $\alpha_i^{\vee}$ of $\mathrm{Hom}(T_pA(k), \mathbb{Z}_p)$, we must show that

$$\begin{cases} \nabla(\omega(\alpha_t)) = \Sigma\ \mathrm{Fix}(\alpha_i^{\vee}) \otimes \mathrm{dlog}\ q(\alpha_i, \alpha_t) \\ \nabla(\mathrm{Fix}(\alpha^{\vee})) = 0\ . \end{cases}$$

To show this, we must exploit the functoriality of the Gauss-Manin connection. Because we have a morphism

$$F_{can} : \mathcal{A} \to \Phi^*(\mathcal{A}^{(\sigma)}) = \mathcal{A} \underset{\mathcal{R} \nearrow \Phi^*\Sigma^*}{\otimes} \mathcal{R}\ ,$$

the induced map on cohomology is a <u>horizontal</u> map

$$F^*_{can} : \Phi^*\Sigma^*(H^1_{DR}(\mathcal{A}/\mathcal{R}), \nabla) \to (H^1_{DR}(\mathcal{A}/\mathcal{R}), \nabla)\ .$$

Concretely, this means that we have a commutative diagram

$$\begin{tikzcd} H^1_{DR}(\mathcal{A}/\mathcal{R}) \arrow[r, "\nabla \text{ for } \mathcal{A}/\mathcal{R}"] \arrow[d, "\Phi^*\Sigma^*"] & H^1_{DR}(\mathcal{A}/\mathcal{R}) \otimes \Omega^1_{\mathcal{R}/W} \arrow[d, "\Phi^*\Sigma^* \otimes \Phi^*\Sigma^*"] \\ H^1_{DR}(\Phi^*(\mathcal{A}^{(\sigma)})/\mathcal{R}) \arrow[r, "\nabla \text{ for } \Phi^*(\mathcal{A}^{(\sigma)})/\mathcal{R}"] \arrow[d, "F^*_{can}"] & H^1_{DR}(\Phi^*(\mathcal{A}^{(\sigma)})/\mathcal{R}) \otimes \Omega^1_{\mathcal{R}/W} \arrow[d, "F^*_{can} \otimes \mathrm{id}"] \\ H^1_{DR}(\mathcal{A}/\mathcal{R}) \arrow[r, "\nabla \text{ for } \mathcal{A}/\mathcal{R}"] & H^1_{DR}(\mathcal{A}/\mathcal{R}) \otimes \Omega^1_{\mathcal{R}/W}\ . \end{tikzcd}$$

LEMMA 4.3.3. <u>For any</u> $\ell \in \mathrm{Hom}(T_pA(k) \otimes T_pA^t(k), \mathbb{Z}_p)$, <u>the action of</u> $D(\ell)$ <u>under the Gauss-Manin connection on</u> $H^1_{DR}(\mathcal{A}/\mathcal{R})$ <u>satisfies the formula</u>

$$\nabla(D(\ell))(F^*_{can}\Phi^*\Sigma^*(\xi)) = pF^*_{can}\Phi^*\Sigma^*(\nabla(D(\ell))(\xi)) ,$$

<u>for any elements</u> $\xi \in H^1_{DR}(\mathcal{A}/\mathcal{R})$.

PROOF. Let $\{\alpha_i\}_i$ and $\{\alpha_{t,j}\}_j$ be $\mathbb{Z}_p$-bases of $T_pA(k)$ and of $T_pA^t(k)$ respectively. Then the one-forms

$$\eta_{ij} = \mathrm{dlog}\; q(\alpha_i, \alpha_{t,j})$$

form an $\mathcal{R}$-base of $\Omega^1_{\mathcal{R}/W}$. The formula

$$\Phi^*\Sigma^*(q(\alpha,\alpha_t)) = q(\alpha,\alpha_t)^p$$

shows that the η_{ij} satisfy

$$\Phi^*\Sigma^*(\eta_{ij}) = p\,\eta_{ij} .$$

Given $\xi \in H^1_{DR}(\mathcal{A}/\mathcal{R})$, we can write

$$\nabla(\xi) = \sum_{i,j} \lambda_{i,j} \underset{\mathcal{R}}{\otimes} \eta_{ij} ;$$

the coefficients $\lambda_{ij} \in H^1_{DR}(\mathcal{A}/\mathcal{R})$ are given by the formula

$$\lambda_{ij} = \nabla(D(\ell_{ij}))(\xi) ,$$

where we denote by $\{\ell_{i,j}\} \in \mathrm{Hom}(T_pA(k) \otimes T_pA^t(k), \mathbb{Z}_p)$ the dual basis to the basis $\{\alpha_i \otimes \alpha_{t,j}\}_{i,j}$ of $T_pA(k) \otimes T_pA^t(k)$.

The commutativity of our diagram gives

$$\begin{aligned}\nabla(F^*_{can}\Phi^*\Sigma^*(\xi)) &= \Sigma\, F^*_{can}\Phi^*\Sigma^*(\lambda_{ij}) \otimes \Phi^*\Sigma^*(\eta_{ij}) \\ &= p\,\Sigma\, F^*_{can}\Phi^*\Sigma^*(\lambda_{ij}) \otimes \eta_{ij} .\end{aligned}$$

Thus we find

$$\begin{aligned}\nabla(D(\ell_{ij}))(F^*_{can}\Phi^*\Sigma^*(\xi)) = p\, &F^*_{can}\Phi^*\Sigma^*(\lambda_{i,j}) \\ &\quad\| \\ p\, &F^*_{can}\Phi^*\Sigma^*(\nabla(D(\ell_{ij}))(\xi)) .\end{aligned}$$

The assertion for any ℓ follows by $\mathbb{Z}_p$-linearity. Q.E.D.

COROLLARY 4.3.4. _If_ $\xi \in H^1_{DR}(\mathcal{A}/\mathcal{R})$ _satisfies_

$$F^*_{can}\Phi^*\Sigma^*(\xi) = \lambda\ \xi \quad \text{with} \quad \lambda \in W\ ,$$

then for any $\ell \in \mathrm{Hom}(T_pA(k) \otimes T_pA^t(k), \mathbb{Z}_p)$, _the element_ $\nabla(D(\ell))(\xi) \in H^1_{DR}(\mathcal{A}/\mathcal{R})$ _satisfies_

$$p\ F^*_{can}\Phi^*\Sigma^*(\nabla(D(\ell))(\xi)) = \lambda\ \nabla(D(\ell))(\xi)\ .$$

In particular, we have the implications

$$\xi \in \mathrm{Fix}(H^1_{DR}) \implies \nabla(D(\ell))(\xi) = 0$$

$$\xi \in p\text{-}\mathrm{Fix}(H^1_{DR}) \implies \nabla(D(\ell))(\xi) \in \mathrm{Fix}(H^1_{DR})\ .$$

PROOF. The first and last assertions are immediate from 4.3.3. If $\xi \in \mathrm{Fix}(H^1_{DR})$, then the element $\xi' = \nabla(D(\ell))(\xi)$ satisfies

$$\begin{aligned} \xi' &= p\ F^*_{can}\Phi^*\Sigma^*(\xi') \\ &\ \vdots \\ &= p^n(F^*_{can}\Phi^*\Sigma^*)^n(\xi') \\ &\ \vdots \\ &= 0\ . \quad \text{Q.E.D.} \end{aligned}$$

Armed with 4.3.4, we can deduce (4.3.1) from (3.7.2). According to 3.7.2, we have

$$\mathrm{Kod}(\omega(\alpha_t)) = \Sigma\ L(\alpha_i^\vee) \otimes \mathrm{dlog}\ q(\alpha_i, \alpha_t)\ .$$

Therefore we have

$$\mathrm{Kod}(D(\ell))(\omega(\alpha_t)) = \Sigma\ \ell(\alpha_i \otimes \alpha_t) L(\alpha_i^\vee)\ .$$

But the element $\mathrm{Kod}(D(\ell))(\omega(\alpha_t)) \in \mathrm{Lie}(\mathcal{A}^t/R)$ is the _projection_ of $\nabla(D(\ell))(\omega(\alpha_t)) \in H^1_{DR}(\mathcal{A}/\mathcal{R})$. Therefore we have a congruence

$$\nabla(D(\ell))(\omega(\alpha_t)) \equiv \Sigma\ \ell(\alpha_i \otimes \alpha_t)\mathrm{Fix}(\alpha_i^\vee) \quad \mathrm{mod}\ \underline{\omega}_{\mathcal{A}/\mathcal{R}}\ .$$

But $\omega(\alpha_t)$ lies in $p\text{-}\mathrm{Fix}(H^1_{DR})$ (by 4.2.1) ; therefore (4.3.4) shows us that $\nabla(D(\ell))(\omega(\alpha_t))$ lies in $\mathrm{Fix}(H^1_{DR})$. Therefore the above congruence is in fact an equality (because $\mathrm{Fix}(H^1_{DR}) \cap \underline{\omega} = 0$) :

$$\nabla(D(\ell))(\omega(\alpha_t)) = \sum_i \ell(\alpha_i \otimes \alpha_t)\mathrm{Fix}(\alpha_i^\vee)$$
$$= \mathrm{Fix}(\sum_i \ell(\alpha_i \otimes \alpha_t).\alpha_i^\vee)$$
$$= \mathrm{Fix}(\ell * \alpha_t) \ . \quad \text{Q.E.D.}$$

4.4. In this section we will conclude the first part of the proof of 3.7.1 as outlined in 4.0. The key is provided by 4.3.4.

THEOREM 4.4.1. Let $\alpha \in T_pA(k)$, $\alpha_t \in T_pA^t(k)$. There exists a (necessarily unique) character $Q(\alpha,\alpha_t)$ of $\hat{\mathfrak{m}}$ such that

$$\omega(\alpha).\mathrm{Kod}(\omega(\alpha_t)) = \mathrm{dlog}\ Q(\alpha,\alpha_t) \ .$$

PROOF. Let $\{\alpha_i\}$ be a $\mathbb{Z}_p$-basis of $T_pA(k)$, $\{\alpha_{t,j}\}$ a $\mathbb{Z}_p$-basis of $T_pA^t(k)$, and $\ell_{i,j}$ the basis of $\mathrm{Hom}(T_pA(k)\otimes T_pA^t(k),\mathbb{Z}_p)$ dual to $\{\alpha_i \otimes \alpha_{i,j}\}$. Then for any element $\xi \in H^1_{DR}(\mathcal{A}/\mathcal{R})$, we have

$$\nabla(\xi) = \sum_{i,j} \nabla(D(\ell_{ij}))(\xi)\otimes \mathrm{dlog}\ q(\alpha_i,\alpha_{t,j}) \ .$$

In particular, for $\xi = \omega(\alpha_t)$ we find

$$\nabla(\omega(\alpha_t)) = \sum_{i,j} \nabla(D(\ell_{ij}))(\omega(\alpha_t))\otimes \mathrm{dlog}\ q(\alpha_i,\alpha_{t,j}) \ .$$

By 4.3.4 and 4.2.1, we have

$$\nabla(D(\ell_{ij}))(\omega(\alpha_t)) \in \mathrm{Fix}(H^1_{DR}) \ ;$$

so for fixed α_t , there exist unique elements

$$\alpha^\vee_{ij} \in \mathrm{Hom}(T_pA(k),\mathbb{Z}_p)$$

such that

$$\nabla(D(\ell_{ij}))(\omega(\alpha_t)) = \mathrm{Fix}(\alpha^\vee_{ij}) \ .$$

Thus we obtain a formula of the form

$$\nabla(\omega(\alpha_t)) = \sum_{i,j} \mathrm{Fix}(\alpha^\vee_{ij})\otimes \mathrm{dlog}\ q(\alpha_i,\alpha_{t,j})$$

with certain elements $\alpha^\vee_{ij} \in \mathrm{Hom}(T_pA(k),\mathbb{Z}_p)$ depending upon α_t .

Passing to the associated graded, we obtain a formula

$$\mathrm{Kod}(\omega(\alpha_t)) = \sum_{i,j} L(\check{\alpha}_{ij}) \otimes \mathrm{dlog}\, q(\alpha_i, \alpha_{t,j}) .$$

Therefore for $\alpha \in T_pA(k)$, we have

$$\begin{aligned}\omega(\alpha).\mathrm{Kod}(\omega(\alpha_t)) &= \sum_{i,j} (\alpha.\check{\alpha}_{ij})\mathrm{dlog}\, q(\alpha_i, \alpha_{t,j}) \\ &= \mathrm{dlog}\Big(\prod_{i,j} (q(\alpha_i, \alpha_{t,j}))^{\alpha.\check{\alpha}_{ij}}\Big) . \quad \text{Q.E.D.}\end{aligned}$$

COROLLARY 4.4.2. <u>For</u> $\alpha \in T_pA(k)$, $\alpha_t \in T_pA^t(k)$, <u>and</u> $\ell \in \mathrm{Hom}(T_pA(k) \otimes T_pA^t(k), \mathbb{Z}_p)$, <u>we have</u>

$$\omega(\alpha).\mathrm{Kod}(D(\ell))(\omega(\alpha_t)) = \text{a constant in } \mathbb{Z}_p .$$

COROLLARY 4.4.3. <u>Suppose for every integer</u> $n \geq 1$ <u>we can find a homomorphism</u>

$$f_n : \mathcal{R} \to W_n = W_n(k)$$

<u>such that we have</u>

$$f_n(\omega(\alpha).\mathrm{Kod}(D(\ell))(\omega(\alpha_t))) = \ell(\alpha \otimes \alpha_t) \quad \text{in } W_n ,$$

<u>for every</u> $\alpha \in T_pA(k)$, $\alpha_t \in T_pA^t(k)$, <u>and</u> $\ell \in \mathrm{Hom}(T_pA(k) \otimes T_pA^t(k), \mathbb{Z}_p)$. <u>Then the Main Theorem</u> 3.7.4 <u>holds</u>, <u>i.e. we have</u>

$$\omega(\alpha).\mathrm{Kod}(D(\ell))(\omega(\alpha_t)) = \ell(\alpha \otimes \alpha_t) \quad \text{in } \mathcal{R} .$$

PROOF. This is obvious from 4.4.2, because the natural map $\mathbb{Z}_p \to \varprojlim W_n$ is injective !

4.5. In this section we will exploit 4.4.3 to give an infinitesimal formulation of the Main Theorem.

Let R be any artin local ring with residue field k (e.g. $R = W_n(k)$). By the Serre-Tate theorem, there is a unique abelian scheme A_{can}/R lifting A/k for which

$$q(A_{can}/R; \alpha, \alpha_t) = 1 \quad \text{for all } \alpha \in T_pA(k) , \ \alpha_t \in T_pA^t(k) .$$

This is the "canonical lifting", to R , of A/k . It's classifying homomorphism

$$f_{can} : \mathfrak{R} \to R$$

is the unique W-linear homomorphism for which

$$f_{can}(q(\alpha,\alpha_t)) = 1 \text{ , for all } \alpha \in T_pA(k) \text{ , } \alpha_t \in T_pA^t(k) \text{ .}$$

Let D be any continuous derivation of $\mathfrak{R}$ into itself. Then we can define a homomorphism

$$f_{can,D} : \mathfrak{R} \to R[\varepsilon] \qquad (\varepsilon^2 = 0)$$

by defining, for, $r \in \mathfrak{R}$,

$$f_{can,D}(r) \overset{dfn}{=\!=} f_{can}(r) + f_{can}(D(r)) \text{ . } \text{ .}$$

The corresponding abelian scheme over $R[\varepsilon]$

$$A_{can,D} \overset{dfn}{=\!=} \mathcal{A} \underset{\mathfrak{R}}{\otimes} R[\varepsilon]$$

is a first order deformation of A_{can}/R .

Consider its associated locally splittable short exact sequence on $A_{can,D}$:

$$0 \to \mathcal{O}_{A_{can,D}} \underset{R[\varepsilon]}{\otimes} \Omega^1_{R[\varepsilon]/R} \to \Omega^1_{A_{can,D}/R} \to \Omega^1_{A_{can,D}/R[\varepsilon]} \to 0 \text{ .}$$

It's reduction modulo ε is a short exact sequence on A_{can} ,

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{A_{can}} \underset{R}{\otimes} d\varepsilon \arrow[r] & \Omega^1_{A_{can,D}/R}\big|A_{can} \arrow[r] & \Omega^1_{A_{can}/R} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{O}_{A_{can}} \arrow[r, "\times d\varepsilon"] \arrow[u, "\wr"] & \Omega^1_{A_{can,D}/R}\big|A_{can} \arrow[r] \arrow[u, equal] & \Omega^1_{A_{can}/R} \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

which sits in a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\mathcal{A}} \underset{\mathfrak{R}}{\otimes} \Omega^1_{\mathfrak{R}/W} \arrow[r] \arrow[d, "1 \otimes D"] & \Omega^1_{\mathcal{A}/W} \arrow[r] \arrow[dd, "f^*_{can,D}"] & \Omega^1_{\mathcal{A}/\mathfrak{R}} \arrow[r] \arrow[dd, "f^*_{can}"] & 0 \\ & \mathcal{O}_{\mathcal{A}} \arrow[d, "f^*_{can}"] & & & \\ 0 \arrow[r] & \mathcal{O}_{A_{can}} \arrow[r, "\times d\varepsilon"] & \Omega^1_{A_{can,D}/R}\big|A_{can} \arrow[r] & \Omega^1_{A_{can}/R} \arrow[r] & 0 \text{ .} \end{tikzcd}$$

Let us denote by ∂ the coboundary map in the associated long exact cohomology sequence

$$\begin{array}{ccc} H^0(A_{can},\Omega^1_{A_{can}/R}) & \longrightarrow & H^1(A_{can},\Theta_{A_{can}}) \\ \wr\| & & \wr\| \\ \underline{\omega}_{A_{can}/R} & \xrightarrow{\partial} & \mathrm{Lie}(A^t_{can}/R) \ . \end{array}$$

From the commutative diagram (4.5.1) above, we see that

LEMMA 4.5.2. For $\alpha \in T_pA(k)$ and $\alpha_t \in T_pA^t(k)$, we have the formulas

$$\begin{cases} f^*_{can}(\mathrm{Kod}(D)(\omega(\alpha_t)) = \partial(f^*_{can}(\omega(\alpha_t))) \\ f_{can}(\omega(\alpha).\mathrm{Kod}(D)(\omega(\alpha_t))) = f^*_{can}(\omega(\alpha)).\partial(f^*_{can}(\omega(\alpha_t)) \ . \end{cases}$$

MAIN THEOREM 4.5.3. Hypotheses and notations as above, the q-parameters of $A_{can,D}/R[\varepsilon]$ are given by the formula

$$q(A_{can,D}/R[\varepsilon];\alpha,\alpha_t) = 1+\varepsilon f^*_{can}(\omega(\alpha)).\partial(f^*_{can}(\omega(\alpha_t))) \ .$$

Let us explain why 4.5.3 is equivalent to 3.7.1-2-3-4 . Suppose first that 3.7.1 holds. Then

$$\omega(\alpha).\mathrm{Kod}(\omega(\alpha_t)) = \mathrm{dlog}(q(\alpha,\alpha_t)) \ .$$

Therefore we have

$$\omega(\alpha).\mathrm{Kod}(D)(\omega(\alpha t)) = \frac{D(q(\alpha,\alpha_t))}{q(\alpha,\alpha_t)} \ .$$

Applying the homomorphism

$$f_{can} : \mathfrak{R} \to R \ ,$$

we obtain

$$f_{can}(\omega(\alpha).\mathrm{Kod}(D)(\omega(\alpha_t)) = \frac{f_{can}(D(q(\alpha,\alpha_t))}{f_{can}(q(\alpha,\alpha_t))}$$
$$\| $$
$$f_{can}(D(q(\alpha,\alpha_t)) \ .$$

Because $A_{can,D}/R[\varepsilon]$ has classifying map $f_{can,D}$, we have

$$\begin{aligned} q(A_{can,D}/R[\varepsilon];\alpha,\alpha_t) &= f_{can,D}(q(\alpha,\alpha_t)) \\ &= f_{can}(q(\alpha,\alpha_t)) + \varepsilon f_{can}(D(q(\alpha,\alpha_t)) \\ &= 1+\varepsilon f_{can}(\omega(\alpha).\mathrm{Kod}(D)(\omega(\alpha_t))) \\ &= 1+\varepsilon f^*_{can}(\omega(\alpha)).\partial(f^*_{can}(\omega(\alpha_t))) \ . \end{aligned}$$

Conversely, suppose that 4.5.3 holds.

Equating coefficients of ε, we obtain

$$\begin{array}{ccc} f_{can}(D(q(\alpha,\alpha_t)) & = & f^*_{can}(\omega(\alpha)).\partial(f^*_{can}(\omega(\alpha_t)) \\ \| & & \| \\ f_{can}(\mathrm{Dlog}\ q(\alpha,\alpha_t)) & = & f_{can}(\omega(\alpha).\mathrm{Kod}(D)(\omega(\alpha_t)) \ . \end{array}$$

Taking for D one of the derivations $D(\ell)$, $\ell \in \mathrm{Hom}(T_pA(k)\otimes T_pA^t(k),\mathbb{Z}_p)$, we obtain an equality

$$f_{can}(\ell(\alpha\otimes\alpha_t)) = f_{can}(\omega(\alpha).\mathrm{Kod}(D(\ell)(\omega(\alpha_t))) \ .$$

Taking for R the rings W_n, we thus fulfill the criteria of 4.4.3.

Q.E.D.

5. INTERLUDE : NORMALIZED COCYCLES AND THE e_N-PAIRING

5.0. Let S be a scheme, and $\pi : X \to S$ a proper and smooth S-scheme with geometrically connected fibres (i.e., $\pi_*\mathcal{O}_X = \mathcal{O}_S$), given together with a marked section $x : S \to X$:

$$\begin{tikzcd} X \arrow[d, "\pi"] \\ S \arrow[u, bend left, "x"] \end{tikzcd} .$$

As explained in ([11]), under these conditions we may view the relative Picard group $\mathrm{Pic}(X/S) \overset{\mathrm{dfn}}{=\!=} \mathrm{Pic}(X)/\mathrm{Pic}(S)$ as the subgroup of $\mathrm{Pic}(X)$ consisting of $\mathrm{Ker}(\mathrm{Pic}(X) \xrightarrow{x^*} \mathrm{Pic}(S))$. Intrinsically, this means that we view $\mathrm{Pic}(X/S)$ as the group of isomorphism classes of pairs $(\mathcal{L},\ell)$ consisting of an invertible $\mathcal{O}_X$-module $\mathcal{L}$ together with an $\mathcal{O}_S$-basis ℓ of the invertible $\mathcal{O}_S$-module $x^*(\mathcal{L})$. In terms of Cech cocycles, it is convenient to introduce the subsheaf $K^\times$ of $(\mathcal{O}_X)^\times$ consisting of "functions which take the value 1 along x" ; it which sits in the tautological exact sequence

$$0 \longrightarrow K^\times \longrightarrow (\mathcal{O}_X)^\times \longrightarrow x_*(\mathcal{O}_S^\times) \longrightarrow 0 .$$

Then we have a natural isomorphism

$$\mathrm{Pic}(X/S) \simeq H^1(X,K^\times) ,$$

while the assumption $\pi_*\mathcal{O}_X = \mathcal{O}_S$ (and consequently $\pi_*(\mathcal{O}_X)^\times = \mathcal{O}_S^\times$) guarentees that

$$H^0(X,K^\times) = \{1\} .$$

This means that if a normalized cocycle (i.e. one with values in $K^\times$),

$$f_{ij} \in \Gamma(\mathcal{U}_i \cap \mathcal{U}_j ; K^\times)$$

represents the zero-element of $\mathrm{Pic}(X/S)$, then there exist <u>unique</u> functions

$$f_i \in \Gamma(\mathcal{U}_i, K^\times)$$

such that $\{f_{ij}\}$ is the boundary of the normalized cochain $\{f_i\}$:

$$f_{ij} = f_i/f_j \ .$$

The functor $\mathrm{Pic}_{X/S}$ on the category of S-schemes is defined by

$$T \longmapsto \mathrm{Pic}(X \underset{S}{\times} T/T) \ .$$

It's Lie algebra

$$\mathrm{Lie}(\mathrm{Pic}_{X/S}) \overset{\mathrm{dfn}}{=\!=} \mathrm{Ker}(\mathrm{Pic}(X[\varepsilon]/S[\varepsilon]) \longrightarrow \mathrm{Pic}(X/S))$$

is easily described in terms of normalized additive cocycles as follows. Let K^+ be the subsheaf of $\mathcal{O}_X$ consisting of "functions which take the value zero along x", which sits in the exact sequences

$$0 \longrightarrow K^+ \longrightarrow \mathcal{O}_X \longrightarrow x_*(\mathcal{O}_S) \longrightarrow 0$$

$$0 \longrightarrow 1+\varepsilon K^+ \longrightarrow K^\times_{X[\varepsilon]/S[\varepsilon]} \longrightarrow K^\times_{X/S} \longrightarrow 0 \ .$$

Just as above we have a natural isomorphism

$$H^1(X, 1+\varepsilon K^+) \simeq \mathrm{Lie}(\mathrm{Pic}_{X/S})$$

while

$$H^0(X, 1+\varepsilon K^+) = \{1\} \ .$$

Although normalized cocycles are extremely convenient for certain calculations, as we shall see, they bring about no essential novelty over a local base.

LEMMA 5.0.1. _If_ $\mathrm{Pic}(S) = 0$ (e.g. _if_ S _is the spectrum of a local ring_) _the inclusion_ $K^\times \subset (\mathcal{O}_X)^\times$ _induces an isomorphism_

$$\mathrm{Pic}(X/S) = H^1(X, K^\times) \xrightarrow{\sim} H^1(X, \mathcal{O}_X^\times) = \mathrm{Pic}(X) \ .$$

_If_ S _is affine_, _the inclusion_ $K^+ \subset \mathcal{O}_X$ _induces an isomorphism_

$$\begin{array}{rcl} \mathrm{Lie}(\mathrm{Pic}_{X/S}) = H^1(X, 1+\varepsilon K^+) & \xrightarrow{\sim} & H^1(X, 1+\varepsilon\mathcal{O}_X) \\ & & \| \\ & & \mathrm{Ker}(\mathrm{Pic}(X[\varepsilon]) \to \mathrm{Pic}(X)) \ . \end{array}$$

Q.E.D.

PROOF. Obvious from the long cohomology sequences.

5.1. Suppose that X/S is an abelian scheme, with marked point $x=0$. The dual abelian scheme X^t/S is the subfunctor $\mathrm{Pic}^0_{X/S}$ of $\mathrm{Pic}_{X/S}$ which classifies those $(\mathcal{L},\ell)$ whose underlying $\mathcal{L}$ becomes algebraically equivalent to zero on each geometric fibre of X/S. Because abelian varieties "have no torsion", the <u>torsion</u> subgroup-functor of $\mathrm{Pic}_{X/S}$ lies in X^t, i.e. for any integer N and any S-scheme T, we have

$$X^t(T)[N] = \mathrm{Pic}_{X/S}(T)[N] .$$

According to a fundamental theorem, for any integer N the two endomorphisms

$$\mathrm{Pic}_{X/S} \xrightarrow{N} \mathrm{Pic}_{X/S}$$
$$\mathrm{Pic}_{X/S} \xrightarrow{[N_{X/S}]^*} \mathrm{Pic}_{X/S}$$

coincide on the subgroup X^t (cf. [12]).

5.2. The e_N-pairing as defined in Oda [13]

$$e_N : X[N] \times X^t[N] \longrightarrow \mu_N$$

may be described simply in terms of normalized cocycles. Thus suppose we are given points

$$Y \in X(S)[N] \quad , \quad \lambda \in \mathrm{Pic}(X/S)[N] .$$

Choose a normalized cocycle representing λ, say

$$f_{ij} \in \Gamma(\mathcal{U}_i \cap \mathcal{U}_j, K^\times)$$

with respect to some open covering $\mathcal{U}_i$ of X. Then as $[N_{X/S}]^*(\lambda)$ is the zero element in $\mathrm{Pic}(X/S)$, the normalized cocycle

$$[N_{X/S}]^*(f_{ij}) \in \Gamma([N]^*(\mathcal{U}_i) \cap [N]^*(\mathcal{U}_j), K^\times)$$

with respect to the covering $\{[N]^{-1}(\mathcal{U}_i)\}$ must be the boundary of a <u>unique</u> normalized cochain

$$f_i \in \Gamma([N]^{-1}(\mathcal{U}_i), K^\times) \ ;$$

thus we have

$$[N]^*(f_{ij}) = f_i/f_j \ .$$

Now view $Y \in X(S)[N]$ as a morphism

$$Y : S \to X \ .$$

The open sets $Y^{-1}([N]^{-1}(\mathcal{U}_i))$ form an open covering of S ; and the sections

$$f_i(Y) = Y^*(f_i) \in \Gamma(Y^{-1}([N]^{-1}(\mathcal{U}_i)), \mathcal{O}_S^\times)$$

patch together to give a <u>global section</u> over S of $\mathcal{O}_S^\times$; (because on overlaps we have

$$\frac{f_i(Y)}{f_j(Y)} = ([N]^*(f_{ij}))(Y) = f_{ij}(NY) = f_{ij}(0) = 1 \ ,$$

as the cocycle f_{ij} is normalized).

Oda's definition of the e_N-pairing (as the effect of translation by Y on a nowhere vanishing section of the <u>inverse</u> of $[N]^*(\mathcal{L})$, $\mathcal{L}$ a line bundle representing λ) means that we have the formula

$$e_N(Y,\lambda) = \text{the global section of } \mathcal{O}_S^\times \text{ given locally by } 1/f_i(Y) \ .$$

(Of course one can verify <u>directly</u> that this global section of $\mathcal{O}_S^\times$ is independent of the original choice of normalized cocycle representing λ , but this "independence of choice" is already a consequence of its interpretation via the e_N-pairing).

5.3. Suppose now that the scheme S is killed by an integer N . Here are <u>two</u> natural homomorphisms

$$\mathrm{Pic}(X/S)[N] \longrightarrow \underline{\omega}_{X/S} \ .$$

The first, which we will denote

$$\lambda \longmapsto \omega_N(\lambda) \in \underline{\omega}_{X/S} \ ,$$

is defined via the e_N-pairing and the observation that, because N

kills S, we have $\mathrm{Lie}(X/S) \subset X(S[\varepsilon])[N]$. We define $\omega(\lambda)$ as a linear form on $\mathrm{Lie}(X/S)$, by requiring

$$e_N(L,\lambda) = 1 + \varepsilon\omega_N(\lambda).L .$$

Given our "explicit formula" for the e_N-pairing, we can translate this in terms of normalized cocycles, as follows.

Begin with a normalized cocycle f_{ij} for λ, and write

$$[N]^*(f_{ij}) = f_i/f_j$$

for a unique normalized $\mathcal{O}$-cochain $\{f_i\}$; then we have

$$\omega_N(\lambda) = -df_i/f_i \quad \text{on} \quad [N]^{-1}(\mathfrak{U}_i) .$$

(One can verify directly that this formula defines a global one-form on X, independently of the choice of normalized cocycle representing λ, but this independence follows from the e_N-interpretation).

The second, which we will denote

$$\lambda \longrightarrow \text{"dlog(N)"}(\lambda)$$

has nothing to do with the fact that X/S is an abelian scheme. Given $\lambda \in \mathrm{Pic}(X/S)[N]$, choose a normalized cocycle

$$f_{ij} \in \Gamma(\mathfrak{U}_i \cap \mathfrak{U}_j ; K^\times)$$

representing it. Then $(f_{ij})^N$ is a normalized cocycle, for the same covering, which represents $N\lambda = 0$ in $\mathrm{Pic}(X/S)$. Therefore there exist unique functions

$$g_i \in \Gamma(\mathfrak{U}_i, K^\times)$$

such that

$$(f_{ij})^N = g_i/g_j .$$

We define

$$\text{"dlog(N)"}(\lambda) = dg_i/g_i \quad \text{on} \quad \mathfrak{U}_i .$$

Choice of a cohomologous normalized cocycle $f'_{ij} = f_{ij}(h_i/h_j)$ would lead (by uniqueness) to functions $g'_i = g_i(h_i)^N$; as N kills S, and hence

X , we have

$$\mathrm{dlog}(g_i)' = \mathrm{dlog}\, g_i + N\,\mathrm{dlog}\, h_i = \mathrm{dlog}\, g_i \ ,$$

so our construction is well-defined.

For any integer $M \geqslant 1$, S will also be killed by NM , and so we have homomorphisms

$$\omega_{NM} \ , \ \text{"dlog(NM)"} : \mathrm{Pic}(X/S)[NM] \longrightarrow \underline{\omega}_{X/S} .$$

From their explicit descriptions via normalized cocycles, it is clear that they sit in a commutative diagram

$$\begin{tikzcd} & \mathrm{Pic}(X/S)[NM] \arrow[dl, "\text{"dlog(NM)"}"'] \arrow[dr, "\omega_{NM}"] \arrow[dd, "M\ =\ {[M]^*}"] & \\ \underline{\omega}_{X/S} & & \underline{\omega}_{X/S} . \\ & \mathrm{Pic}(X/S)[N] \arrow[ul, "\text{"dlog(N)"}"] \arrow[ur, "\omega_N"'] & \end{tikzcd}$$

LEMMA 5.4. <u>If</u> N <u>kills</u> S , <u>then for any</u> $\lambda \in \mathrm{Pic}(X/S)[N^2]$ <u>we have</u>

$$\text{"dlog}(N^2)\text{"}(\lambda) = -\omega_{N^2}(\lambda) \quad \text{in} \quad \underline{\omega}_{X/S} .$$

PROOF. Let us begin with a normalized cocycle f_{ij} representing λ , with respect to some open covering $\{\mathcal{U}_i\}$. Then

$$\begin{cases} [N]^*(f_{ij}) \ \text{ represents } \ [N]^*(\lambda) = N\lambda \ , \text{ on the covering } \ [N]^{-1}(\mathcal{U}_i) \\ \qquad f_{ij}^N \ \text{ represents } \ N\lambda = [N]^*(\lambda) \ , \text{ on the covering } \ \mathcal{U}_i \ . \end{cases}$$

We compute $\text{"dlog}(N^2)\text{"}(\lambda) = \text{"dlog(N)"}(N\lambda) = \text{"dlog(N)"}([N]^*(\lambda))$ by using the normalised cocycle for $[N]^*(\lambda)$ given by

$$[N]^*(f_{ij}) \ \text{ on the covering } \ [N]^{-1}(\mathcal{U}_i) \ .$$

There exist <u>unique</u> functions

$$f_{ij} \in \Gamma([N]^{-1}(\mathcal{U}_i), K^\times)$$

such that

$$([N]^*(f_{ij}))^N = h_i/h_j \ ,$$

and by definition we have

$$\text{"dlog}(N)\text{"}([N]^*(\lambda) = dh_i/h_i \quad \text{on} \quad [N]^{-1}(\mathcal{U}_i) \ .$$

Similarly, we compute $\omega_{N^2}(\lambda) = \omega_N([N]^*(\lambda)) = \omega_N(N\lambda)$ by using the normalized cocycle for $N\lambda$ given by

$$(f_{ij})^N \quad \text{on the covering} \quad \mathcal{U}_i \ .$$

There exist unique functions

$$H_i \in \Gamma([N]^{-1}(\mathcal{U}_i), K^\times)$$

such that

$$[N]^*((f_{ij})^N) = H_i/H_j \ ,$$

and by definition we have

$$\omega_N(N\lambda) = -dH_i/H_i \quad \text{on} \quad [N]^{-1}(\mathcal{U}_i) \ .$$

By uniqueness, we must have $H_i = h_i$, and hence we find

$$\omega_{N^2}(\lambda) = \omega_N(N\lambda) = -\text{"dlog}(N)\text{"}([N]^*(\lambda)) = \text{"dlog}(N^2)\text{"}(\lambda) \ . \text{ Q.E.D.}$$

COROLLARY 5.5. Let k be an algebraically closed field of characteristic $p > 0$, A/k an ordinary abelian variety, R an artin local ring with residue field k , and X/R an abelian scheme lifting A/k . For any n sufficiently large that p^n kills R , we have a commutative diagram

$$\begin{tikzcd} T_pX^t(R) \arrow[r, "\text{reduce mod.}\mathfrak{m}"] \arrow[d] & T_pA^t(k) \arrow[d, "\alpha_t \mapsto \omega(\alpha_t)"] \\ X^t(R)[p^n] \arrow[r, "-\text{"dlog}(p^n)\text{"}"'] & \underline{\omega}_{X/R} \end{tikzcd} .$$

PROOF. From the description (3.3) of the $\alpha_t \mapsto \omega(\alpha_t)$ construction in terms of the e_{p^n}-pairing, it is obvious that the diagram

$$\begin{array}{ccccc}
\{(\lambda(n)\} & & T_pX^t(R) & \longrightarrow & T_pA(k) \\
\downarrow & & \downarrow & & \downarrow \omega \\
\lambda(n) & & X^t(R)[p^n] & \xrightarrow[\omega_{p^n}]{} & \underline{\omega}_{X/R}
\end{array}$$

is commutative. By the previous lemma, we have

$$\omega_{p^n}(\lambda(n)) = \omega_{p^{2n}}(\lambda(2n)) = -\text{"dlog}(p^{2n})\text{"}(\lambda(2n)) = -\text{"dlog}(p^n)\text{"}(\lambda(n)) \ .$$

Q.E.D.

6. THE END OF THE PROOF

6.0. Let k be an algebraically closed field of characteristic $p > 0$, and A/k an ordinary abelian variety over k. We <u>fix</u> an artin local ring R with residue field k. Having fixed R, we denote by X/R the <u>canonical</u> lifting of A/k to R.

We denote by

$$\begin{cases} \alpha_t \longmapsto \omega(\alpha_t) \in \underline{\omega}_{X/R} \\ \alpha \longmapsto \omega(\alpha) \in \underline{\omega}_{X^t/R} \end{cases}$$

the homomorphisms

$$\begin{cases} T_pA^t(k) \longrightarrow \underline{\omega}_{X/R} \\ T_pA(k) \longrightarrow \underline{\omega}_{X^t/R} \end{cases}.$$

Let $R[\varepsilon]$ denote the dual numbers over R $(\varepsilon^2 = 0)$. We <u>fix</u> an abelian scheme $\tilde{X}/R[\varepsilon]$ which lifts X/R. We denote by

$$\partial : \underline{\omega}_{X/R} \longrightarrow H^1(X, \mathcal{O}_X) = \mathrm{Lie}(X^t/R)$$

the coboundary in the long exact cohomology sequence attached to the short exact sequence of sheaves on X

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{d\varepsilon} \Omega^1_{\tilde{X}/R}|X \longrightarrow \Omega^1_{X/R} \longrightarrow 0 .$$

As explained at the end of chapter 4, our Main Theorem in all it equivalent forms results from the following "intrinsic" form of 4.5.3.

THEOREM 6.0.1. <u>The Serre-Tate q-parameters of</u> $\tilde{X}/R[\varepsilon]$ <u>are given by the formula</u>

$$q(\tilde{X}/R[\varepsilon]; \alpha, \alpha_t) = 1 + \varepsilon\omega(\alpha).\partial(\omega(\alpha_t)) .$$

By the symmetry formula (2.1.4), it is equivalent to prove

THEOREM 6.0.2. <u>The Serre-Tate q-parameters of</u> $(\tilde{X})^t/R[\varepsilon]$ <u>are given by the formula</u>

$$q((\tilde{X})^t/R[\varepsilon];\alpha_t,\alpha) = 1+\varepsilon\omega(\alpha).\partial(\omega(\alpha_t)) .$$

We will deduce 5.0.2 from a sequence of lemmas.

LEMMA 6.1. <u>The natural maps</u> "<u>reduction modulo the maximal ideal</u> $\mathfrak{m}$ <u>of</u> R"

$$\begin{cases} T_pX(R) \longrightarrow T_pX(k) = T_pA(k) \\ T_pX^t(R) \longrightarrow T_pX^t(k) = T_pA^t(k) \end{cases}$$

<u>are bijective</u>.

PROOF. First of all, the maps are injective, for their kernels are the groups $T_p\hat{X}(R)$, $T_p\hat{X}^t(R)$; as the groups $\hat{X}(R)$ and $\hat{X}^t(R)$ are killed by p^n as soon as the maximal ideal $\mathfrak{m}$ of R satisfies $\mathfrak{m}^{n+1}=0$, their T_p's are reduced to zero.

For surjectivity, we must use the fact that X/R is canonical, i.e., has $q(X/R;\alpha,\alpha_t)=1$. This means that for all n sufficiently large, the map

$$\begin{aligned} \varphi_{X/R} : T_pA(k) &\twoheadrightarrow A(k)[p^n] \longrightarrow \hat{X}(R) \\ \alpha &\longrightarrow \alpha(n) \longrightarrow p^n \times (\text{any lifting of } \alpha(n) \text{ in } X(R)) \end{aligned}$$

vanishes, i.e. the "reduction mod $\mathfrak{m}$" map is surjective for $n \gg 0$:

$$X(R)[p^n] \twoheadrightarrow A(k)[p^n] .$$

In fact, this map is surjective for every n , for we have a commutative diagram

$$\begin{array}{ccc} X(R)[p^{n+m}] & \twoheadrightarrow & A(k)[p^{n+m}] \\ \downarrow p^n & & \downarrow p^n \\ X(R)[p^m] & \longrightarrow & A(k)[p^m] \end{array} .$$

Thus we obtain a short exact sequence of projective systems

$$0 \longrightarrow \{\hat{X}(R)[p^n]\}_n \longrightarrow \{X(R)[p^n]\}_n \longrightarrow \{A(k)[p^n]\}_n \longrightarrow 0 ,$$

the first of which is "essentially zero" (because $\hat{X}(R)$ is killed by p^n for $n \gg 0$), so in particular satisfies the Mittag-Leffler condition. Passing to inverse limits, we obtain the required isomorphism

$$T_pX(R) \xrightarrow{\sim} T_pA(k) .$$

For X^t/R, we simply note that by the symmetry formula (1.2.1.4) we have $q(X^t/R;\alpha_t,\alpha) = q(X/R;\alpha,\alpha_t) = 1$; then repeat the argument. Q.E.D.

LEMMA 6.2. <u>The deformation homomorphism</u>

$$\varphi_{(\tilde{X})^t/R[\varepsilon]} : T_pA^t(k) \longrightarrow (\hat{\tilde{X}})^t(R[\varepsilon])$$

<u>takes values in the subgroup</u> $\mathrm{Ker}(\tilde{X}^t(R[\varepsilon]) \longrightarrow X^t(R)) = \mathrm{Ker}(\mathrm{Pic}(\tilde{X}) \longrightarrow \mathrm{Pic}(X))$.

PROOF. Because X^t/R is canonical, i.e. $q(X^t/R;\alpha_t,\alpha) = 1$, by the symmetry formula, the homomorphism $\varphi_{X^t/R} : T_pA^t(k) \longrightarrow \hat{X}^t(R)$ vanishes. The result follows from the commutativity of the diagram

$$\begin{array}{ccc} T_pA^t(k) & \xrightarrow{\varphi_{(\tilde{X})^t/R}} & (\hat{\tilde{X}})^t(R[\varepsilon]) \\ & \searrow^{\varphi_{X^t/R}} & \Big\downarrow \text{reduce mod } \varepsilon \\ & & \hat{X}^t(R) \end{array} \quad \text{Q.E.D.}$$

6.3. The short exact sequence of sheaves on $\tilde{X}$

$$0 \longrightarrow 1+\varepsilon\mathcal{O}_X \longrightarrow (\mathcal{O}_{\tilde{X}})^\times \longrightarrow (\mathcal{O}_X)^\times \longrightarrow 0$$

leads to an isomorphism

$$H^1(X,1+\varepsilon\mathcal{O}_X) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Pic}(\tilde{X}) \longrightarrow \mathrm{Pic}(X)) = \mathrm{Ker}(\tilde{X}^t(R[\varepsilon]) \longrightarrow X^t(R)) .$$

If we replace $\tilde{X}$ by the trivial deformation $X[\varepsilon]$ of X/R, we obtain an isomorphism

$$H^1(X,1+\varepsilon\mathcal{O}_X) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Pic}(X[\varepsilon]) \longrightarrow \mathrm{Pic}(X)) \stackrel{\mathrm{dfn}}{=\!=} \mathrm{Lie}(X^t/R) .$$

LEMMA 6.3.1. <u>Let</u> $L \in H^1(X,1+\varepsilon\mathcal{O}_X)$, <u>and</u> $\alpha \in T_pA(k)$. <u>Under the canonical pairings</u>

$$E_{(\tilde{X})^t} : \widehat{(\tilde{X})^t} \times T_pA(k) \longrightarrow \hat{\mathbb{G}}_m$$

$$E_{X^t} : (\widehat{X^t}) \times T_pA(k) \longrightarrow \hat{\mathbb{G}}_m$$

we have

$$E_{(\tilde{X})^t}(L_1,\alpha) = E_{X^t}(L_2,\alpha) = 1+\varepsilon\omega(\alpha).L_3 ,$$

where

L_1 = "L viewed as lying in $\mathrm{Ker}(\tilde{X}^t(R[\varepsilon]) \longrightarrow X^t(R))$"

L_2 = "L viewed as lying in $\mathrm{Ker}(X^t(R[\varepsilon]) \longrightarrow X^t(R))$"

L_3 = "L viewed as lying in $\mathrm{Lie}(X^t/R)$".

PROOF. The second of the asserted equalities is the definition of $\omega(\alpha)$, cf. 3.3 ; we have restated it "pour memoire". We now turn to the first assertion. Fix an integer n such that $m^n = 0$ in R . Then the maximal ideal (m, ε) of $R[\varepsilon]$ satisfies $(m,\varepsilon)^{n+1} = 0$. Also p^n kills R , hence we have $p^nL = 0$.

Choose a finite flat artin local $R[\varepsilon]$-algebra S , and a point

$$Y \in \tilde{X}(S)[p^n] \text{ lifting } \alpha(n) \text{ in } A(k)[p^n] .$$

Denote by S_o the finite flat artin local R-algebra defined as

$$S_o = S/\varepsilon S ,$$

and denote by $Y_o \in X(S_o)[p^n]$ the image of Y under the "reduction mod ε" map

$$\tilde{X}(S)[p^n] \longrightarrow X(S_o)[p^n]$$

$$Y \longrightarrow Y_o .$$

By lemma (2.2), we have

$$E_{(\tilde{X})^t}(L_1,\alpha) = E_{(\tilde{X}^t);p^n}(L_1,\alpha(n)) = e_{(\tilde{X})^t;p^n}(L_1,Y) ,$$

and similarly

$$E_{X^t}(L_2,\alpha) = e_{X^t;p^n}(L_2,Y_o) .$$

By the skew-symettry of the e_{p^n}-pairing, it suffices to show that

$$e_{\tilde{X};p^n}(Y,L_1) = e_{X;p^n}(Y_o,L_2) \ .$$

In order to show this, we represent L by a normalized cocycle on some affine open covering $\mathfrak{U}_i$ of X :

$$1+\varepsilon f_{ij} \quad ; \quad f_{ij}(0)=0 \quad \text{if} \quad 0 \in \mathfrak{U}_i \cap \mathfrak{U}_j \ .$$

Because $p^n L = 0$, the "autoduality" of multiplication by integers on abelian schemes shows that

$$[p^n]^*_{\tilde{X}}(L_1) = 0 \quad , \quad [p^n]^*_X(L_2) = 0 \ .$$

Therefore the normalized cocycles for the covering $[p^n]^{-1}(\mathfrak{U}_i)$

$$[p^n]^*_{\tilde{X}}(1+\varepsilon f_{ij}) = 1+\varepsilon[p^n]^*_X(f_{ij}) = [p^n]^*_X(1+\varepsilon f_{ij})$$

may be written as the coboundary of a common normalized zero-cochain

$$1+\varepsilon[p^n]^*_X(f_{ij}) = \frac{1+\varepsilon f_i}{1+\varepsilon f_j} \quad , \quad f_i(0)=0 \quad \text{if} \quad 0 \in [p^n]^{-1}(\mathfrak{U}_i) \ .$$

By definition of the e_{p^n}-pairing, we have, for any index i such that $Y \in [p^n]^{-1}(\mathfrak{U}_i)$, the formulas

$$\begin{cases} e_{\tilde{X};p^n}(Y,L_1) = \dfrac{1}{(1+\varepsilon f_i)(Y)} = 1-(\varepsilon f_i)(Y) \\[2ex] e_{X;p^n}(Y_o,L_2) = \dfrac{1}{(1+\varepsilon f_i)(Y_o)} = 1-\varepsilon f_i(Y_o) \ . \end{cases}$$

The fact that Y_o is $Y \bmod \varepsilon$ makes it evident that

$$(\varepsilon f_i)(Y) = \varepsilon f_i(Y_o) \quad \text{in} \quad \varepsilon S \ . \quad \text{Q.E.D.}$$

COROLLARY 6.3.2. _If we interpret the deformation homomorphism as a map_

$$\varphi_{(\tilde{X})^t/R[\varepsilon]} : T_pA^t(k) \longrightarrow H^1(X,1+\varepsilon\mathcal{O}_X) \simeq \mathrm{Lie}(X^t/R) \ ,$$

we have the formula

$$q((\tilde{X})^t/R[\varepsilon];\alpha_t,\alpha) = 1+\varepsilon\omega(\alpha).\varphi_{(\tilde{X})^t/R[\varepsilon]}(\alpha_t) \ .$$

PROOF. This follows immediately from the definition of q in terms of φ and E, and lemmas 5.2 and 5.3.1.

6.4. In this section, we analyze the deformation homomorphism

$$\varphi_{(\tilde{X})^t/R[\varepsilon]} : T_pA^t(k) \longrightarrow H^1(X, 1+\varepsilon\mathcal{O}_X) .$$

Recall that this homomorphism is defined as the composite, for any n sufficiently large that $\mathfrak{m}^r = 0$,

$$\begin{tikzcd} T_pA^t(k) \arrow[r, two heads] & A^t(k)[p^n] \arrow[r, "p^n \times \text{(any lifting)}"] \arrow[r, bend right, "\varphi"'] & (\tilde{X}^t)(R[\varepsilon]) . \end{tikzcd}$$

Because X/R is canonical, we have an isomorphism (4.6.1)

$$T_pX^t(R) \xrightarrow{\simeq} T_pA^t(k) ,$$

and this sits in a commutative diagram

$$\begin{tikzcd} T_pA^t(k) \arrow[r, two heads] \arrow[rr, bend left, "\varphi"] & A^t(k)[p^n] \arrow[r, "p^n \times \text{(any lifting)}"] & (\tilde{X})^t(R[\varepsilon]) & \\ T_pX^t(R) \arrow[u, "\wr"] \arrow[r] & X^t(R)[p^n] \arrow[u, "\text{reduce mod } \mathfrak{m}"'] \arrow[r, "p^n \times \text{(any lifting)}"'] & \mathrm{Ker}((\tilde{X})^t(R[\varepsilon]) \arrow[u, hook] \arrow[r] & X^t(R)) \\ & & H^1(X, 1+\varepsilon\mathcal{O}_X) \arrow[u, "\wr"] \simeq \mathrm{Lie}(X^t/R) . & \end{tikzcd}$$

In order to complete the proof of 6.0.2, it suffices in view of 6.3.2, to prove

THEOREM 6.4.1. <u>For</u> R <u>artin local with algebraically closed residue field</u> k <u>of characteristic</u> $p > 0$, X/R <u>the canonical lifting of an ordinary abelian variety</u> A/k, <u>and</u> $\tilde{X}/R[\varepsilon]$ <u>a deformation of</u> X/R, <u>we have the formula</u>

$$\partial(\omega(\alpha_t)) = \varphi_{(\tilde{X})^t/R[\varepsilon]}(\alpha_t) \qquad \text{in } \mathrm{Lie}(X^t/R)$$

<u>for every</u> $\alpha_t \in T_pA^t(k)$.

According to 5.5, the construction $\alpha_o \longmapsto \omega(\alpha_t)$ sits in a commutative diagram, for any n such that p^n kills R :

$$\begin{tikzcd} T_pA^t(k) \arrow[r, "\omega"] & \underline{\omega}_{X/R} \\ T_pX^t(R) \arrow[u, "\wr"] \arrow[r] & X^t(R)[p^n] \arrow[u, "-\text{"dlog}(p^n)\text{"}"'] \end{tikzcd} .$$

Therefore 6.4.1 would follow from the more precise

THEOREM 6.4.2. _Hypotheses as in_ 6.4.1, _for any_ n _such that_ p^n _kills_ R , _and any element_ $\lambda \in X^t(R)[p^n]$, _we have the identity, in_ $\mathrm{Lie}(X^t/R)$

$$\partial(\text{"dlog}(p^n)\text{"}(\lambda)) = -p^n \times (\text{any lifting of } \lambda \text{ to an invertible sheaf on } \tilde{X}) .$$

6.5. In this section we will prove 6.4.2. Given any ring R killed by any integer N , and any proper smooth R-scheme X/R with geometrically connected fibres and a marked point $x \in X(R)$, there is a natural homomorphism

$$\mathrm{Pic}(X/R)[N] \longrightarrow H^0(X, (\mathcal{O}_X)^\times \underset{\mathbb{Z}}{\otimes} (\mathbb{Z}/N\mathbb{Z}))$$

defined as follows. Given $\lambda \in \mathrm{Pic}(X/R)[N]$, represent it by a normalized cocycle $\{f_{ij}\}$. Then there exists a unique normalized 0-chain $\{f_i\}$ such that

$$(f_{ij})^N = f_i/f_j .$$

A cohomologous normalized cocycle, say $g_{ij} = f_{ij} \times (h_i/h_j)$, leads to

$$(g_{ij})^N = f_i(h_i)^N/f_j(h_j)^N .$$

Therefore the $\{f_i\}$ "are" a well-defined global section of $(\mathcal{O}_X)^\times \underset{\mathbb{Z}}{\otimes} (\mathbb{Z}/N\mathbb{Z})$. This construction

$$\mathrm{Pic}(X/R)[N] \ni \lambda \longmapsto \{f_i\} \in H^0(X, (\mathcal{O}_X)^\times \underset{\mathbb{Z}}{\otimes} (\mathbb{Z}/N\mathbb{Z}))$$

defines our homomorphism.

Suppose we are in addition given a deformation $\tilde{X}/R[\varepsilon]$ of X/R, together with a marked point $\tilde{x} \in \tilde{X}(R[\varepsilon])$ which lifts x. We have an exact sequence of sheaves of units

$$0 \longrightarrow 1 + \varepsilon\mathcal{O}_X \longrightarrow (\mathcal{O}_{\tilde{X}})^\times \longrightarrow (\mathcal{O}_X)^\times \longrightarrow 0 .$$

Because N kills R, it also kills $\mathcal{O}_X$, so kills $1+\varepsilon\mathcal{O}_X$; the serpent lemma, applied to this exact sequence and the endomorphism "N", therefore leads to a short exact sequence of "units mod N" :

$$0 \longrightarrow 1 + \varepsilon\mathcal{O}_X \longrightarrow (\mathcal{O}_{\tilde{X}})^\times \underset{\mathbb{Z}}{\otimes} (\mathbb{Z}/N\mathbb{Z}) \longrightarrow (\mathcal{O}_X)^\times \otimes (\mathbb{Z}/N\mathbb{Z}) \longrightarrow 0 .$$

We will denote by

$$\Delta(N) : H^0(X,(\mathcal{O}_X)^\times \otimes (\mathbb{Z}/N\mathbb{Z})) \longrightarrow H^1(X, 1 + \varepsilon\mathcal{O}_X)$$

the coboundary map in the associated long exact sequence of cohomology.

The "units mod N" exact sequence maps to the Kodaira-Spencer short exact sequence by "dlog", and gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 1 + \varepsilon\mathcal{O}_X \arrow[r] \arrow[d, "\frac{1}{\varepsilon}\log"', "\wr"] & (\mathcal{O}_{\tilde{X}})^\times \otimes (\mathbb{Z}/N\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{dlog}"] & (\mathcal{O}_X)^\times \otimes (\mathbb{Z}/N\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{dlog}"] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, "d\varepsilon"] & \Omega^1_{\tilde{X}/R}|X \arrow[r] & \Omega^1_{X/R} \arrow[r] & 0 .
\end{tikzcd}$$

This diagram in turn gives a commutative diagram of coboundary maps in the long exact sequences of cohomology :

$$\begin{tikzcd}
\mathrm{Pic}(X/R)[N] \arrow[d] \arrow[dd, bend right=60, "\text{"dlog(N)"}"'] & \\
H^0(X,(\mathcal{O}_X)^\times \otimes (\mathbb{Z}/N\mathbb{Z})) \arrow[r, "\Delta(N)"] \arrow[d, "\mathrm{dlog}"] & H^1(X, 1 + \varepsilon\mathcal{O}_X) \arrow[d, "\wr", "\frac{1}{\varepsilon}\log"'] \\
H^0(X,\Omega^1_{X/R}) \arrow[r, "\partial"] & H^1(X,\mathcal{O}_X) .
\end{tikzcd}$$

LEMMA 6.5.1. *Hypotheses as in* 6.5 *above, suppose that every element of* $\mathrm{Pic}(X/R)[N]$ *lifts to an element of* $\mathrm{Pic}(\tilde{X}/R[\varepsilon])$ (*a condition automatically fulfulled if* $\mathrm{Pic}^{\tau}_{\tilde{X}/R[\varepsilon]}$ *is smooth, in particular when* X/R *is an abelian scheme*). *Then the diagram*

$$\begin{array}{ccc} \mathrm{Pic}(X/R)[N] & & \\ \downarrow & \overset{\lambda \mapsto N \times (\text{any lifting of } \lambda \text{ to an invertible sheaf on } \tilde{X})}{\searrow} & \\ H^0(X, (\mathcal{O}_X)^{\times} \underset{\mathbb{Z}}{\otimes} (\mathbb{Z}/N\mathbb{Z})) & \xrightarrow{-\Delta(N)} & H^1(X, 1+\varepsilon\mathcal{O}_X) \end{array}$$

is commutative.

PROOF. Given $\lambda \in \mathrm{Pic}(X/R)[N]$, represent it by a normalized cocycle f_{ij} on some affine open covering $\mathcal{U}_i$ of X ; we may assume f_{ij} to be the reduction modulo ε of a normalized cocycle $\tilde{f}_{ij}$ on $\tilde{X}$ representing a lifting of λ to $\tilde{X}$. Because $\lambda \in \mathrm{Pic}(X/R)[N]$, we have

$$(f_{ij})^N = f_i/f_j$$

for a normalized 0-cochain $\{f_i\}$. Choose liftings

$$\tilde{f}_i \in \Gamma(\mathcal{U}_i, (\mathcal{O}_{\tilde{X}})^{\times})$$

of the functions $f_i \in \Gamma(\mathcal{U}_i, (\mathcal{O}_X)^{\times})$.
Then

$$\Delta_N \ (\text{the section } \{f_i\}) = \begin{cases} \text{the element of } H^1(X, 1+\varepsilon\mathcal{O}_X) \\ \text{represented by the 1-cocycle} \\ \qquad (\tilde{f}_i/\tilde{f}_j)(\tilde{f}_{ij})^{-N}, \end{cases}$$

while

$$N \times (\text{any lifting of } \lambda) = \begin{cases} \text{the element of } H^1(X, 1+\varepsilon\mathcal{O}_X) \\ \text{represented by the 1-cocycle} \\ \qquad (\tilde{f}_{ij})^N \cdot (\tilde{f}_j/\tilde{f}_i) \ . \quad \text{Q.E.D.} \end{cases}$$

If we combine 6.5.1 with the commutative diagram immediately preceding it, we find a commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(X/R)[N] \arrow[d, "\text{"dlog(N)"}"'] \arrow[dr, "-(N\times(\text{any lifting}))"] & \\
& H^1(X, 1+\varepsilon\mathcal{O}_X) \arrow[d, phantom, "\wr"] \\
H^0(X, \Omega^1_{X/R}) \arrow[r, "\partial"] & H^0(X, \mathcal{O}_X) \ .
\end{tikzcd}$$

In particular, this proves 6.4.2, (take $N = p^n$) and with it our "main theorem" in all its forms (3.7.1-2-3, 4.3.1-2, 4.5.3, 6.0.1-2).

REFERENCES

[1] P. DELIGNE.- Variétés Abéliennes Ordinaires Sur un Corps Fini. Inv. Math. 8 (1969), pp. 238-243.

[2] V.G. DRINFEL'D.- Coverings of p-adic symmetric domains (Russian). Funkcional. Anal. i. Prilozen. 10 (1976), n° 2, pp. 29-40.

[3] B. DWORK.- P-adic Cycles. Pub. Math. I.H.E.S., n° 37 (1969), pp. 27-116.

[4] B. DWORK.- Normalized Period Matrices. Annals of Math. 94 (1971), pp. 337-388.

[5] L. ILLUSIE and P. DELIGNE.- Cristaux Ordinaires et Coordonnées Canoniques. Exposé V, this volume.

[6] N. KATZ.- Travaux de Dwork. Séminaire Bourbaki 1971/72. Springer Lecture Notes 317 (1973), pp. 167-200.

[7] N. KATZ.- P-adic L-functions, Serre-Tate Local Moduli, and Ration of Solutions of Differential Equations. Proceedings of the International Congress of Mathematicians, Helsinki, 1978. Academica Scientrarum Fennica, Hungary (1980), vol. I, pp. 365-371.

[8] N. KATZ.- Appendix to Expose Vbis, this volume.

[9] W. MESSING.- The Crystals Associated to Barsotti-Tate Groups : with Applications to Abelian Schemes. Springer Lecture Notes 264 (1972).

[10] W. MESSING.- q Serre-Tate = q Dwork, abstract, AMS Notices, 1976.

[11] D. MUMFORD.- Geometric Invariant Theory. Springer-Verlag, Berlin-Heidelberg-New York (1965).

[12] D. MUMFORD.- Abelian Varieties. Oxford University Press, London (1970).

[13] T. ODA.- The first De Rham cohomology group and Dieudonné modules. Ann. sci. Ec. Norm. Sup, 4e série, t. 2 (1969), pp. 63-135.

[14] J.-P. SERRE and J. TATE.- Numeographed notes from the 1964 A.M.S. Summer Institute in Algebraic Geometry at Woods Hole.

University of Princeton
Dept. Math.
08544 PRINCETON, N.J., U.S.A.