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THE REGULARITY THEOREM IN ALGEBRAIC GEOMETRY

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I. Introduction.

A basic finiteness theorem for families of algebraic varieties is that the Picard-Fuchs differential equations have only regular (in the sense of Fuchs) singular points. The theorem was proved analytically by P. A. Griffiths [3], then by P. Deligne, both of whom used Hironaka's resolution of singularities [5] to be able to estimate the growths of solutions.

Just recently, Deligne and the speaker independently found a purely algebro-geometric proof, which makes the theorem a simple corollary of resolution. The method also leads to a direct proof of the monodromy theorem.

II. The notion of regular singular points [1].

Let U be a smooth $\mathbb{C}$-scheme. An algebraic differential equation on U is by definition a pair (M, ∇) consisting of a coherent sheaf M on U with an integrable connection (the existence of ∇ implies that M is, in fact, locally free). We will view ∇ as a homomorphism of abelian sheaves

$$\nabla : M \to \Omega^1_U \otimes_{\mathcal{O}_U} M \tag{2.1}$$

(writing Ω^1_U for $\Omega^1_{U/\mathbb{C}}$) which satisfies the usual product rule and which extends to define a structure of *complex* on $\Omega^{\cdot}_U \otimes_{\mathcal{O}_U} M$, the " absolute de Rham complex " of (M, ∇).

Now let S be a proper and smooth $\mathbb{C}$-scheme, $D = \cup D_i$ a union of connected smooth divisors in S with normal crossings, such that $U \xrightarrow{\sim} S - D$, which we will refer to as a *compactification* of U. Let $Der_D(S/\mathbb{C})$ denote the (locally free) sheaf on S of derivations which preserve the ideal sheaf of each branch D_i of D. The sheaf of differentials on S with logarithmic singularities along D is defined by

$$\begin{aligned} \Omega^1_S(\log D) &\overset{\text{defn}}{=\!=} \mathcal{H}om_{\mathcal{O}_S}(Der_D(S/\mathbb{C}), \mathcal{O}_S) \\ \Omega^p_S(\log D) &= \Lambda^p_{\mathcal{O}_S}\Omega^1_S(\log D) \end{aligned} \tag{2.2}$$

It is immediate that $\Omega^{\cdot}_S(\log D)$ is a subcomplex of $i_*\Omega^{\cdot}_U$ ($i : U \hookrightarrow S$ denoting the inclusion).

Following Fuchs and Deligne, we say that an algebraic differential equation (M, ∇) on U has regular singular points if, for *every* compactification $U = S - D$ as above (by Hironaka [5], such compactifications exist!), there exists a pair $(\overline{M}, \overline{\nabla})$ consisting of a locally free sheaf $\overline{M}$ on S which prolongs M and a homomorphism $\overline{\nabla}$ of abelian sheaves

$$\overline{\nabla}:\quad \overline{M} \rightarrow \Omega^1_S(\log D) \otimes_{\mathcal{O}_S} \overline{M} \tag{2.3}$$

which prolongs ∇.

III. Remarks on the definition.

(3.1) It is rather forbidding in appearance, but is certainly satisfied by ($\mathcal{O}_U$, d = exterior differentiation).

(3.2) A consideration of the local monodromy around D shows that the underlying *analytic* differential equation (M^{an}, ∇^{an}) always admits an *analytic* extension $(\overline{M^{an}}, \overline{\nabla^{an}})$ as above, which, by GAGA, is uniquely *algebrifiable*. Restricting this algebraic data to U, we get a second algebraic differential equation (M', ∇') on U, which depends *only* and *functorially* on (M, ∇), and an isomorphism of (M^{an}, ∇^{an}) with (M'^{an}, ∇'^{an}). The condition that (M, ∇) have regular singular points is that the above isomorphism come from an isomorphism of (M, ∇) and (M', ∇').

(3.3) It follows easily from (3.2) that (M, ∇) has regular singular points if and only if for every morphism $f: V \rightarrow U$ with V a smooth *curve*, the inverse image $f^*(M, \nabla)$ on V has regular singular points.

(3.4) If U is a connected smooth curve, and $U = S - D$ its canonical compactification, (M, ∇) has regular singular points if there exists an extension $(\overline{M}, \overline{\nabla})$ as above with $\overline{M}$ *coherent* ($\overline{M}$/torsion is a locally free extension to which $\overline{\nabla}$ passes over).

(3.5) Combining (3.3) and (3.4), it follows that (M, ∇) has regular singular points if for *one* compactification $U = S - D$ there exists an extension $(\overline{M}, \overline{\nabla})$ as above with $\overline{M}$ coherent

IV. Relative de Rham cohomology [7].

Let $f: U \rightarrow V$ be a proper and smooth morphism of smooth $\mathbb{C}$-schemes, and (M, ∇) an algebraic differential equation on U. Composing ∇ with the projection $\Omega^1_U \otimes_{\mathcal{O}_U} M \rightarrow \Omega^1_{U/V} \otimes_{\mathcal{O}_U} M$, we obtain an integrable V-connection, still noted,

$$\nabla:\quad M \rightarrow \Omega^1_{U/V} \otimes_{\mathcal{O}_U} M \tag{4.1}$$

which extends to provide a structure of complex to $\Omega^{\cdot}_{U/V} \otimes_{\mathcal{O}_U} M$, the " relative de Rham complex of (M, ∇) ". The relative de Rham cohomology sheaves on V of (M, ∇) are defined by

$$H^q_{DR}(U/V, (M, \nabla)) = \mathbb{R}^q f_*(\Omega^{\cdot}_{U/V} \otimes_{\mathcal{O}_U} M) \tag{4.2}$$

These sheaves are coherent, as f is proper, and are endowed with an integrable connection, whose construction we now recall.

Filter the absolute de Rham complex of (M, ∇) by the subcomplexes

$$(4.3)\quad F^i = F^i(\Omega^{\cdot}_U \otimes_{\mathcal{O}_U} M) = \text{image}:\ f^*(\Omega^i_V) \otimes_{\mathcal{O}_U} \Omega^{\cdot - i}_U \otimes_{\mathcal{O}_U} M \to \Omega^{\cdot}_U \otimes_{\mathcal{O}_U} M.$$

The associated graded objects are given by

$$(4.4)\qquad gr^i = F^i/F^{i+1} = f^*(\Omega^i_V) \otimes_{\mathcal{O}_U} (\Omega^{\cdot - i}_{U/V} \otimes_{\mathcal{O}_U} M)$$

The integrable connection sought on $H^q_{DR}(U/V, (M, \nabla))$ is the differential $d_1^{0,q}$ in the spectral sequence of the filtered complex $\Omega^{\cdot}_U \otimes_{\mathcal{O}_U} M$ and the functor $\mathbb{R}^0 f_*$, or, in more down to earth terms, it is the coboundary map δ_q, in the long cohomology sequence of the $\mathbb{R}^q f_*$ arising from the short exact sequence $0 \to gr^1 \to F^0/F^2 \to gr^0 \to 0$. Remember that, by (4.4), we have

$$(4.5)\qquad \begin{cases} \mathbb{R}^q f_*(gr^0) = H^q_{DR}(U/V, (M, \nabla)) \\ \mathbb{R}^{q+1} f_*(gr^1) = \Omega^1_V \otimes_{\mathcal{O}_V} H^q_{DR}(U/V, (M, \nabla)). \end{cases}$$

(4.6) Thus $(H^q_{DR}(U/V, (M, \nabla)), \delta_q)$ is an algebraic differential equation on V. In particular, $H^q_{DR}(U/V, (M, \nabla))$ is locally free; this being so for all q, it follows that *the formation of the $H^q_{DR}(U/V, (M, \nabla))$ is compatible with arbitrary change of base.*

We remark that in the case $(M, \nabla) = (\mathcal{O}_U, d)$, the connection just constructed on $H_{DR}(U/V) \overset{\text{defn}}{=\!=} \mathbb{R} f_*(\Omega^{\cdot}_{U/V})$ is the *Gauss-Manin connection*, and the resulting algebraic differential equation is classically called the *Picard-Fuchs equation.*

V. The regularity theorem.

THEOREM. — *Assumptions as in IV, if (M, ∇) has regular singular points, then the algebraic differential equations $(H^q_{DR}(U/V, (M, \nabla)), \delta_q)$ on V have regular singular points.*

Proof. — Combining (3.4) and (4.6), it suffices to treat the case in which V is a smooth connected curve. Let T be the complete non singular model of the function field of V, so that $V = T - Y$, Y a finite set of points of T, is the canonical compactification of V. By Hironaka [5], we can " compactify " the morphism $f: U \to V$ into a morphism $\pi: S \to T$, so as to have a cartesian diagram

$$(5.1)\qquad \begin{array}{ccc} U & \hookrightarrow & S. \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} \\ V & \hookrightarrow & T \end{array}$$

in which $D \overset{\text{defn}}{=\!=} \{\pi^{-1}(Y)\}^{\text{red.}}$ is a union of connected smooth divisors in S which cross normally, and $U = S - D$ is a *compactification* of U in the sense of II.

Notice that $\pi^*(\Omega^1_T (\log Y))$ is a subsheaf of $\Omega^1_S (\log D))$. We define the (locally free) sheaf of relative differentials with logarithmic singularities along D by

$$(5.2)\qquad \begin{cases} \Omega^1_{S/T} (\log D) \overset{\text{defn}}{=\!=} \Omega^1_S (\log D)/\pi^*(\Omega^1_T (\log Y)) \\ \Omega^p_{S/T} (\log D) \ = \ \Lambda^p_{\mathcal{O}_S} \Omega^1_{S/T} (\log D) \end{cases}$$

The *complex* $\Omega^{\cdot}_{S/T}$ (log D) on S is a prolongation of $\Omega^{\cdot}_{U/V}$, and fits into a short exact sequence of complexes

$$(5.3)\quad 0 \to \pi^*(\Omega^1_T (\log Y) \otimes_{\mathcal{O}_S} \Omega^{\cdot-1}_{S/T} (\log D) \to \Omega^{\cdot}_S (\log D) \to \Omega^{\cdot}_{S/T} (\log D) \to 0$$

Now let $(\overline{M}, \overline{\nabla})$ be an extension of (M, ∇) to S, with $\overline{M}$ locally free and $\overline{\nabla}: \overline{M} \to \Omega^1_S (\log D) \otimes_{\mathcal{O}_S} \overline{M}$ a prolongation of ∇, and consider the complex deduced from $\overline{\nabla}$,

$$(5.4)\qquad \Omega^{\cdot}_S (\log D) \otimes_{\mathcal{O}_S} \overline{M}$$

which is a prolongation of $\Omega^{\cdot}_U \otimes_{\mathcal{O}_U} M$.

Filter $\Omega^{\cdot}_S (\log D) \otimes_{\mathcal{O}_S} \overline{M}$ by the subcomplexes

$$(5.5)\quad F^i = \text{image } \pi^*(\Omega^i_T (\log Y) \otimes_{\mathcal{O}_S} \Omega^{\cdot-i}_S (\log D) \otimes_{\mathcal{O}_S} \overline{M} \to \Omega^{\cdot}_S (\log D) \otimes_{\mathcal{O}_S} \overline{M}$$

The associated graded objects are given by

$$(5.6)\qquad gr^i = F^i/F^{i+1} = \pi^*(\Omega^i_T (\log Y)) \otimes_{\mathcal{O}_S} (\Omega^{\cdot-i}_{S/T} (\log D) \otimes_{\mathcal{O}_S} \overline{M}).$$

In particular, gr^0 is a prolongation of the relative de Rham complex $\Omega_{U/V} \otimes M$ of (M, ∇).

We define the coherent sheaves on T.

$$(5.7)\qquad H^q_{DR}(S/T, (\overline{M}, \overline{\nabla})) \overset{\text{defn}}{=\!=} \mathbb{R}^q\pi_*(\Omega^{\cdot}_{S/T} (\log D) \otimes_{\mathcal{O}_S} M)$$

which are prolongations of the locally free sheaves $H^q_{DR}(U/V, (M, \nabla))$ on V. The extensions of δ_q to homomorphisms of abelian sheaves

$$(5.8)\qquad \overline{\delta}_q:\ H^q(S/T, (\overline{M}, \overline{\nabla})) \to \Omega^1_T (\log Y) \otimes_{\mathcal{O}_T} H^q_{DR}(S/T, (\overline{M}, \overline{\nabla}))$$

are provided by the coboundary maps of the long cohomology sequence of the $\mathbb{R}^q\pi_*$ arising from the short exact sequence $0 \to gr^1 \to F^0/F^2 \to gr^0 \to 0$

Remember that, by (5.6), we have

$$(5.9)\qquad \begin{cases} \mathbb{R}^q\pi_*(gr^0) = H^q_{DR}(S/T, (\overline{M}, \overline{\nabla})) \\ \mathbb{R}^{q+1}\pi_*(gr^1) = \Omega^1_T (\log Y) \otimes_{\mathcal{O}_T} H^q_{DR}(S/T, (\overline{M}, \overline{\nabla})). \end{cases}$$

Thus the $(H^q_{DR}(S/T, (\overline{M}, \overline{\nabla})), \overline{\delta}_q)$ provide the desired extensions of the

$$(H^q_{DR}(U/V, (M, \nabla)), \delta_q).$$

QED.

VI. The exponents.

Notations as in II, let $(\overline{M}, \overline{\nabla})$ be an algebraic differential equation on S with logarithmic singularities along D. For each branch D_i of D, we denote by $\overline{M}(D_i)$ the locally free sheaf $\mathcal{O}_{D_i} \otimes_{\mathcal{O}_S} \overline{M}$ on D_i. Composing $\overline{\nabla}$ with the map " residue along D_i "

$$(6.1)\qquad \Omega^1_S (\log D) \otimes_{\mathcal{O}_S} \overline{M} \xrightarrow{\text{" residue along } D_i\text{"} \otimes 1} \mathcal{O}_{D_i} \otimes_{\mathcal{O}_S} \overline{M} = \overline{M}(D_i)$$

we obtain an ($\mathcal{O}_{D_i}$-linear) endomorphism L_i of $\overline{M}(D_i)$. As D_i is proper, the characteristic polynomial of L_i, $P_i(X) = \det(XI - L_i; \overline{M}(D_i))$ lies in $\mathbb{C}[X]$. Classically, P_i is called the *indicial polynomial* of $(\overline{M}, \overline{\nabla})$ around D_i, and its roots are called *exponents* of $(\overline{M}, \overline{\nabla})$ around D_i. The numbers $\exp(2\pi i\varepsilon)$, ε an exponent, are the proper values of the local monodromy transformation " turning once around D_i " of the space of local holomorphic horizontal sections of $(\overline{M}, \overline{\nabla}) \mid S - D$; thus the exponents, which *depend* on $(\overline{M}, \overline{\nabla})$, are determined modulo $\mathbb{Z}$ by $(\overline{M}, \overline{\nabla}) \mid S - D$.

VII. The Monodromy theorem.

THEOREM. — *Let V be a smooth connected curve, T its canonical compactification, $f: U \to V$ a proper and smooth morphism, and $\pi: S \to T$ a compactification of f as in (5.1). Let $(\overline{M}, \overline{\nabla})$ be an algebraic differential equation on U, and $(\overline{M}, \overline{\nabla})$ an extension to S as in (2.1). Denote by $P_i(X)$ the indicial polynomial of $(\overline{M}, \overline{\nabla})$ around D_i.*

Let $y \in T - V$, and $\pi^{-1}(y) = \Sigma_{i=1}^{r} a_i D_i$ its scheme-theoretic fibre. Then the indicial polynomial at y of $(H^q_{DR}(S/T, (\overline{M}, \overline{\nabla})), \overline{\delta}_q)$ divides a power of

$$\prod_{i=1}^{r} \prod_{j_i=0}^{a_i-1} P_i(a_i X - j_i)$$

Proof. — The question being local around y, let us base-change the entire situation by the inclusion $\operatorname{Spec}(\mathcal{O}_{T,y}) \to T$, but for simplicity keep the same notations (so T henceforth means $\operatorname{Spec}(\mathcal{O}_{T,y})$, etc.). We must now adopt the dual view of the " connection with logarithmic singularities " $\overline{\nabla}$ as an *action* of $Der_D(S/C)$ on $\overline{M}$ satisfying the usual rules [8], and similarly of $\overline{\delta}_q$ as an action of $\mathrm{Der}_y(T/C)$ on $H^q_{DR}(S/T, (\overline{M}, \overline{\nabla}))$. Let t be a uniformizing parameter at y. Then the indicial polynomial at y of $(H^q_{DR}(S/T, (\overline{M}, \overline{\nabla})), \overline{\delta}_q)$ is just the characteristic polynomial of the endomorphism $\overline{\delta}_q\left(t\dfrac{d}{dt}\right)$ of $H^q_{DR}(S/T, (\overline{M}, \overline{\nabla}))(y)$. We will show that

$$\left[\prod_{i=1}^{r} \prod_{j_i=0}^{a_i-1} P_i\left(a_i\overline{\delta}_q\left(t\frac{d}{dt}\right) - j_i\right)\right]^{q+1} [H^q_{DR}(S/T, (\overline{M}, \overline{\nabla}))] \subset tH^q_{DR}(S/T, (\overline{M}, \overline{\nabla})) \tag{7.1}$$

To do this we will use the explicit formulas of [8] for $\overline{\delta}_q\left(t\dfrac{d}{dt}\right)$. Let $\underline{U}$ be a covering of S by affine open sets $U_1, U_2, \ldots$ which is sufficiently fine, in the sense that each U_ν admits coordinates $x_1, \ldots, x_n$, in terms of which D_i is defined by the equation $x_i = 0$ (or by the equation $1 = 0$, if D_i does not meet U_ν), and in terms of which $t = \Pi_{i=1}^{r} x_i^{b_i}$, with $b_i = 0$ or a_i. Let $C^{\cdot\cdot}$ denote the Cech bicomplex of quasi-coherent T-modules $C^{\cdot}(\underline{U}, \Omega^{\cdot}_{S/T}(\log D) \otimes_{\mathcal{O}_S} \overline{M})$, whose (total) cohomology objects are just the

$$H^q_{DR}(S/T, (\overline{M}, \overline{\nabla})).$$

According to [8], we may construct an action σ of $t\dfrac{d}{dt}$ on the underlying sheaf of $C^{\cdot\cdot}$ (i. e., for $h \in \mathcal{O}_{T,y}$ and $c \in C^{\cdot\cdot}$, $\sigma(hc) = t\dfrac{dh}{dt}c + h\sigma(c)$) which commutes with the total coboundary of $C^{\cdot\cdot}$ and induces $\overline{\delta}_q\left(t\dfrac{d}{dt}\right)$ upon passage to cohomology. Indeed, if we choose for each open set U_ν of the covering an element $d_\nu \in \Gamma(U_\nu, \mathrm{Der}_D(S/C))$ which prolongs $t\dfrac{d}{dt}$, there is a σ as above which preserves the filtration F^q of $C^{\cdot\cdot}$ by the first degree, and which on $gr_F^q C^{\cdot\cdot} = C^{q,\cdot}$ is just the Lie derivative $\mathrm{Lie}(\overline{\nabla}(d_{v_0}))$ on

$$\Gamma(U_{v_0} \cap \ldots \cap U_{v_q}, \Omega^{\cdot}_{S/T}(\log D) \otimes_{\mathcal{O}_S} \overline{M}),$$

for $v_0 < \ldots < v_q$.

For each branch D_i of D, we denote by σ_i the action of $t\dfrac{d}{dt}$ on $C^{\cdot\cdot}$ corresponding the choices of liftings of $t\dfrac{d}{dt}$ to an element $d_v^{(i)} \in \Gamma(U_v, Der_D(S/C))$ given by

$$(7.2)\quad d_v^{(i)} = \begin{cases} \dfrac{1}{a_i} x_i \dfrac{\partial}{\partial x_i} & \text{if } D_i \text{ meets } U_v \\ \dfrac{1}{a_j} x_j \dfrac{\partial}{\partial x_j} & \text{if } D_i \text{ does not meet } U_v, \text{ and } j \text{ is the least integer such that } D_j \text{ meets } U_v \end{cases}$$

We define

$$(7.3)\quad \begin{cases} \mathscr{L}_i = \prod_{j_i=0}^{a_i-1} P_i(a_i\sigma_i - j_i) \quad \text{for} \quad i = 1,\ldots, r \\ \mathscr{L} = \mathscr{L}_1 \ldots \mathscr{L}_r \end{cases}$$

The product rule assures that $\mathscr{L}(tF^q) \subset tF^q$, so that to conclude the proof we need only show that $\mathscr{L}(F^q) \subset tF^q + F^{q+1}$, or equivalently, that $\mathscr{L}(gr_F^q C^{\cdot\cdot}) \subset t gr_F^q C^{\cdot\cdot}$. But this last is a " local " statement, namely that over $U_{v_0} \cap \ldots \cap U_{v_q}$, $v_0 < \ldots < v_q$, we have

$$(7.4)\quad \prod_{i=1}^{r} \prod_{j_i=0}^{a_i-1} P_i\,(\mathrm{Lie}\,(a_i\overline{\nabla}(d_{v_0}^{(i)})) - j_i)[\Omega^{\cdot}_{S/T}(\log D) \otimes_{\mathcal{O}_S} \overline{M}] \subset t\Omega^{\cdot}_{S/T}(\log D) \otimes_{\mathcal{O}_S} \overline{M}$$

or, what is equivalent, that over U_{v_0} we have

$$(7.5)\quad \prod_{i=1}^{r} \prod_{j_i=0}^{a_i-1} P_i(a_i\overline{\nabla}(d_{v_0}^{(i)}) - j_i)\overline{M} \subset t\overline{M};$$

Since the various lifting $d_{v_0}^{(i)}$ of $t\dfrac{d}{dt}$ to U_{v_0} were so chosen as to mutually commute, the $\overline{\nabla}(d_{v_0}^{(i)})$ mutually commute (integrability), so we may rearrange the product and " absorb " those P_i corresponding to D_i which do not meet U_{v_0}. Thus we may assume that all the D_i meet U_{v_0}, $t = x_1^{a_1} \ldots x_r^{a_r}$, and $a_i d_{v_0}^{(i)} = x_i \dfrac{\partial}{\partial x_i}$. Since P_i is a polynomial with constant coefficients and $\overline{\nabla}\left(x_i \dfrac{\partial}{\partial x_i}\right)$ is x_j-linear for $j \neq i$, it suffices to show that, for $i = 1,\ldots, r$, we have

$$(7.6)\quad \prod_{j_i=0}^{a_i-1} P_i\left(\overline{\nabla}\left(x_i \frac{\partial}{\partial x_i}\right) - j_i\right)(\overline{M}) \subset x_i^{a_i}\overline{M} \quad \text{over} \quad U_{v_0}.$$

Recalling that the endomorphism L_i of $\overline{M}(D_i)$ is deduced from $\nabla\left(x_i \frac{\partial}{\partial x_i}\right)$ over U_{v_0} by reduction modulo (x_i), we have, by definition of P_i,

$$P_i\left(\nabla\left(x_i \frac{\partial}{\partial x_i}\right)\right)(\overline{M}) \subset x_i \overline{M} \quad \text{over} \quad U_{v_0} \tag{7.7}$$

Combining this with the commutation formula

$$P_i\left(\nabla\left(x_i \frac{\partial}{\partial x_i}\right) - j\right) \circ x_i^j = x_i^j P_i\left(\nabla\left(x_i \frac{\partial}{\partial x^i}\right)\right) \tag{7.8}$$

the desired formula (7.6) (and hence the theorem) follows by induction on a_i. QED.

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