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Free Actions of Finite Groups on Varieties. II

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Introduction

Serre [6] showed that any finite group can be made to act freely on a suitable non-singular complete intersection. Here we consider the converse problem. Given a projective variety X of dimension n together with a projective embedding $X \hookrightarrow \mathbb{P}^{n+r}$, we show that if G is a finite group which acts freely on X and which suitably respects the given projective embedding, then (1) its order must divide the square of the degree of X, (2) any element of G has order dividing the degree of X, and (3) only primes $p \geqq (n+r+1)/r$ can divide the order of G (cf. Sect. 3 for precise statements). In Paper I with this title [1], we proved (1) and (2) over the field $\mathbb{C}$ using topological methods, and allowing G to act on X by continuous orientation-preserving automorphisms. In this paper we prove (1), (2), and (3) over an arbitrary algebraically closed ground field, by using elementary algebraic geometry and linear algebra, and we show by example that (3) need not hold in the topological setting.

I. Generalities

Let k be an algebraically closed field, X a proper k-scheme, and G a finite group which acts freely on X by k-automorphisms (i.e. for $g \in G$, $g \neq \mathrm{id}$, g has no fixed points in X).

We assume that the quotient scheme $Y = X/G$ exists; this is automatically the case if X is projective. Let $\mathscr{F}$ be a coherent G-sheaf on X, i.e., G operates on the pair $(X, \mathscr{F})$ compatibly with its action on X. Then $\mathscr{F}$ descends to yield a coherent sheaf $\mathscr{G}$ on $X/G = Y$. In terms of the projection $\pi: X \to Y$, we have the formulas

$$\mathscr{F} = \pi^*(\mathscr{G}), \qquad \mathscr{G} = (\pi_*(\mathscr{F}))^G.$$

Proposition 1.1. *The Euler characteristic* $\chi(X, \mathscr{F}) = \Sigma(-1)^i \dim_k H^i(X, \mathscr{F})$ *is divisible by* $\#(G)$. *More precisely, one has the formula*

$$\chi(X, \mathscr{F}) = \#(G) \cdot \chi(Y, \mathscr{G}).$$

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Proof (cf. Mumford [4, pp. 70 and 120–121]).

A group-theoretic version of Proposition 1.1 is

Proposition 1.2. *The virtual k-representation of G provided by* $\Sigma(-1)^i H^i(X, \mathscr{F})$ *is an integer multiple of the regular representation. More precisely, one has the formula in the representation ring* $R_k(G)$

$$\Sigma(-1)^i H^i(X, \mathscr{F}) = \chi(Y, \mathscr{G}) \cdot \operatorname{Reg}(G).$$

Proof. By Brauer theory, it is equivalent to prove both of the following statements A) and B) about Brauer traces:

A) For $g \in G$, $g \neq \mathrm{id}$ of order prime to char(k), we have

$$\Sigma(-1)^i \text{ Brauer trace } (g \mid H^i(X, \mathscr{F})) = 0.$$

B) For $g = \mathrm{id}$, we have

$$\chi(X, \mathscr{F}) = \#(G) \cdot \chi(Y, \mathscr{G}).$$

Statement B) is true by Proposition 1. Statement A) follows from the truth of Proposition 1.2 with G replaced by its cyclic subgroups of order prime to char(k), and in this case the result is proven in Ellingsrud-Lønsted [3] (cf. also [0, 2, 5]).

Corollary 1.3. *The arithmetic genus* $\chi(X, \mathcal{O}_X)$ *is divisible by* $\#(G)$, *as are the various Euler characteristics* $\chi(X, \Omega^i_{X/k})$, $i \geqq 1$.

Proof. This is Proposition 1.1, with $\mathscr{F} = \mathcal{O}_X, \Omega^1_{X/k}, \ldots$ QED

Corollary 1.4. *Let X be a projective k-scheme with* $\chi(X, \mathcal{O}_X) = 1$, *e.g. a complete intersection in* $\mathbb{P}^N$ *of multi-degree* $(d_1, \ldots, d_r)$, *with* $\Sigma d_i \leqq N$. *Then no non-trivial finite group can operate freely on X.*

II. Linear Actions on Projective Varieties

Let k be an algebraically closed field, V a finite-dimensional k-vector space, $\mathbb{P}(V)$ the projective space of all hyperplane in V, and $\mathcal{O}_{\mathbb{P}}(1)$ the tautological quotient line bundle on $\mathbb{P}(V)$. For any closed subscheme $X \xrightarrow{i} \mathbb{P}(V)$ we denote by $\mathcal{O}_X(1)$ the inverse image line bundle $i^*(\mathcal{O}_{\mathbb{P}}(1))$ on X, and by $\mathcal{O}_X(j)$ the j-th tensor power of $\mathcal{O}_X(1)$. The dimension of X, $\dim(X)$, is defined to be the maximum of the dimensions of the irreducible components of X. The codimension of X in $\mathbb{P}(V)$, $\operatorname{cdm}(X, \mathbb{P}(V))$, is the minimum of the codimensions of the irreducible components. The degree of X in $\mathbb{P}(V)$ is the number of points, counted with multiplicity, of the intersection of X with a general linear subspace of codimension $= \dim(X)$ in $\mathbb{P}(V)$.

Theorem 2.1. *Suppose that a finite group G operates k-linearly on V, in such a way that under the induced action of G on* $\mathbb{P}(V)$, *the closed subscheme* $X \subset \mathbb{P}(V)$ *is G-stable, and the induced action of G on X is free. Then*

2.1.1. $\#(G)$ *divides* degree(X).

2.1.2. *If a prime p divides* $\#(G)$, *then* $p \geqq \dim(V)/\operatorname{cdm}(X, \mathbb{P}(V))$.

Proof. An action of G on V is equivalent to an action on the pair $(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(1))$. If $X \subset \mathbb{P}(V)$ is a G-stable sub-scheme, then by restriction G acts on the pair $(X, \mathcal{O}_X(1))$, i.e., $\mathcal{O}_X(1)$ is a G-sheaf on X. Because G acts freely on X, $\mathcal{O}_X(1)$ descends to a line-bundle $\mathcal{L}$ on the quotient $Y=X/G$.

Suppose first that X is irreducible and non-singular, say of dimension n. Then the degree of X in $\mathbb{P}(V)$ is simply the n-fold self-intersection number of the first chern class of $\mathcal{O}_X(1)$ with itself in the ring $A^{\cdot}(X)$ of algebraic cycles modulo numerical equivalence on X:

$$\operatorname{degree}(X)=(C_1(\mathcal{O}_X(1)))^n \in A^n(X) \simeq \mathbb{Z}.$$

Because $\pi: X \to Y=X/G$ is finite etale of degree $\#(G)$, the induced map $\pi^*: A^n(Y) \to A^n(X)$ is multiplication by $\#(G)$. Therefore we have:

$$\begin{aligned}\operatorname{degree}(X) &= (C_1(\mathcal{O}_X(1)))^n\\ &=(C_1(\pi^*(\mathcal{L})))^n\\ &=(\pi^*(C_1(\mathcal{L})))^n\\ &=\pi^*((C_1(\mathcal{L}))^n)\\ &=\#(G)\cdot(C_1(\mathcal{L}))^n,\end{aligned}$$

which proves 2.1.1 in this case.

In the general case, we must resort to an artifice. Let $P(T) \in \mathbb{Q}[T]$ denote the Hilbert polynomial of $(X, \mathcal{O}_X(1))$, i.e.,

$$P(j)=\chi(X, \mathcal{O}_X(j)) \quad \text{for all} \quad j \in \mathbb{Z}.$$

One knows that $P(T)$ has degree $n=\dim(X)$, and that it is a $\mathbb{Z}$-linear combination of the binomial functions

$$\binom{T}{r}=\frac{T(T-1)\ldots(T-(r-1))}{r!}$$

say

$$P(T)=\sum_{r=0}^{n} A_r \binom{T}{r},$$

with leading coefficient $A_n=\operatorname{degree}(X)$. Therefore the degree of X is the "n-th difference" of the function $P(T)$:

$$\operatorname{degree}(X)=\sum_{r=0}^{n}(-1)^r\binom{n}{r}\cdot\chi(X, \mathcal{O}_X(n-r))$$

Applying Proposition 1, we may rewrite this as

$$\operatorname{degree}(X)=\#(G)\cdot\sum_{r=0}^{n}(-1)^r\binom{n}{r}\cdot\chi(Y, \mathcal{L}^{\otimes(n-r)}).$$

We now prove 2.1.2. Let $g \in G$ be an element of order p. Then g operates on V by an automorphism $A \in \mathrm{GL}(V)$ with $A^p=1$. A fixed point of g acting on $\mathbb{P}(V)$ is represented by a non-zero eigenvector of A^t operating on the dual space V^t of V. For any eigenvalue ζ of A^t, the linear subspace $\operatorname{Ker}(A^t-\zeta)$ of V^t, whose dimension we denote $m(\zeta)$, projects to an $m(\zeta)-1$ dimensional set of fixed points of g in $\mathbb{P}(V)$. Because g has no fixed points in X, X must be disjoint from this set of fixed points, whence

$$\mathrm{cdm}(X, \mathbb{P}(V)) > m(\zeta) - 1,$$

i.e.

$$\mathrm{cdm}(X, \mathbb{P}(V)) \geqq m(\zeta).$$

To prove 2.1.2, we must show that for at least one eigenvalue ζ, we have

$$m(\zeta) \geqq \dim(V^t)/p.$$

As Steinberg pointed out to me, this follows just from the fact that V^t is annihilated by a polynomial of degree p in A^t [in our case $(A^t)^p - 1$], say

$$(A^t - \zeta_1)(A^t - \zeta_2)\dots(A^t - \zeta_p)(V^t) = 0.$$

For consider the sequence of subspaces W_i of V^t given by

$$W_i = \begin{cases} (A^t - \zeta_i)\dots(A^t - \zeta_p)(V^t) & \text{if } i \leqq p \\ V^t & \text{if } i = p+1. \end{cases}$$

We have

$$0 = W_1 \subset W_2 \subset \dots \subset W_p \subset V^t = W_{p+1},$$

so that

$$\dim(V^t) = \sum_{i=1}^{p} (\dim(W_{i+1}) - \dim(W_i)).$$

By definition of the W_i, we have short exact sequences

$$0 \to W_{i+1} \cap (\mathrm{Ker}(A^t - \zeta_i)) \to W_{i+1} \xrightarrow{A^t - \zeta_i} W_i \to 0$$

whence

$$\begin{aligned} \dim(V^t) &= \sum_{i=1}^{p} \dim(W_{i+1} \cap \mathrm{Ker}(A^t - \zeta_i)) \\ &\leqq \sum_{i=1}^{p} \dim(\mathrm{Ker}(A^t - \zeta_i)) = \sum_{i=1}^{p} m(\zeta_i) \\ &\leqq p \cdot \max_i (m(\zeta_i)). \quad \text{QED} \end{aligned}$$

Example 2.2. Take for X a hypersurface of degree $d \geqq 2$ in $\mathbb{P}^{n+1}$. If a finite group G acts freely on X via a linear action on the ambient $n+2-$ dimensional vector space, then $\#(G)$ divides d, and only primes $p \geqq n+2$ divide $\#(G)$. Let d_1 be the corresponding factor of d, i.e.,

$$d_1 = \prod_{p \geqq n+2} p^{\mathrm{ord}_p(d)};$$

so that $\#(G)$ must divide d_1. We will give an example to show that, at least in characteristic prime to d_1, this bound is attained. Take for G the group $\boldsymbol{\mu}_{d_1}$ of d_1-th roots of unity, acting on the Fermat hypersurface

$$\sum_{i=0}^{n+1} (X_i)^d = 0$$

by the Godeaux action

$$\zeta : (, \ldots, X_i, \ldots) \to (, \ldots, \zeta^i X_i, \ldots).$$

We must verify that this action is free, i.e. that if $\zeta \in \boldsymbol{\mu}_{d_1}$ and $\zeta \neq 1$, then the diagonal matrix

$$\begin{pmatrix} 1 & & & \\ & \zeta & & \\ & & \ddots & \\ & & & \zeta^{n+1} \end{pmatrix}$$

has all its eigenvalues dictinct [for then it's fixed points in $\mathbb{P}^{n+1}$ are the $n+2$ points $(0, \ldots, 0, 1, 0 \ldots 0)$, none of which lies on the Fermat]. But if we have

$$\zeta^i = \zeta^j \quad \text{with} \quad 0 \leqq i \leqq j \leqq n+1,$$

then we would have

$$\zeta^h = 1 \quad \text{with} \quad 1 \leqq h = j - i \leqq n+1.$$

But such an h is relatively prime to d_1, since d_1 is divisible by no primes $\leqq n+1$, and therefore $\zeta^h = 1$ forces $\zeta = 1$, contradiction.

Question 2.3. In the general case, what added information is contained in the fact that $\#(G)$ must divide all of the coefficients A_r of the Hilbert polynomial, i.e. $\#(G)$ must divide $\chi(X, \mathcal{O}_X(j))$ for all j? What is the best bound for $\#(G)$ in the case of a complete intersection of codimension $\geqq 2$?

III. Projective Actions on Projective Varieties

Theorem 3.1. *Let k be an algebraically closed field, V a finite-dimensional k-vector space, and G a finite group which acts projectively on V (i.e., we are given a homomorphism from G to $\mathrm{PGL}(V) = \mathrm{GL}(V)/k^\times$). Suppose that under the induced action of G on $\mathbb{P}(V)$, the closed sub-scheme $X \subset \mathbb{P}(V)$ is G-stable, and the induced action of G on X is free. Then*

3.1.1. $\#(G)$ *divides* (degree $(X))^2$.

3.1.2. *Every element of G has order dividing* degree (X).

3.1.3. *If a prime ℓ divides $\#(G)$, then $\ell \geqq \dim(V)/\mathrm{cdm}(X, \mathbb{P}(V))$.*

3.1.4. *If* $\mathrm{char}(k) = p > 0$, *the order of any p-Sylow subgroup of G divides* degree (X).

Theorem 3.1 bis. *Let k be an algebraically closed field, X a projective k-scheme with $H^0(X, \mathcal{O}_X) = k$, $\mathcal{O}_X(1)$ a very ample invertible sheaf on X, and G a finite group operating freely on X. Suppose that the isomorphism class of $\mathcal{O}_X(1)$ in $\mathrm{Pic}(X)$ is fixed by G. Then* 3.1.1–4 *hold with $V = H^0(X, \mathcal{O}_X(1))$.*

Corollary 1 (of 3.1 bis). *Over an algebraically closed field k, let X be a not-necessarily smooth complete intersection of dimension* $n \geqq 3$ *in* $\mathbb{P}^{n+r}$. *If* $n \geqq r$, *then X admits no fixed-point-free involution. More generally, if p is a prime such that* $n \geqq (p-1)r$, *then X admits no fixed-point-free automorphism of order p.*

Proof. By SGA 2, XII, Corollary 3.7, $\mathrm{Pic}(X) \simeq \mathbb{Z}$ with generator $\mathcal{O}_X(1)$. This generator is distinguished from its inverse $\mathcal{O}_X(-1)$ by the property of having sections. Therefore its isomorphism class is preserved by any automorphism of X. The result now follows, via 3.1 bis, from 3.1.3.

Over $\mathbb{C}$, there is a similar result valid for arbitrary smooth varieties (i.e. not necessarily complete intersections). We are indebted to Arthus Ogus for pointing this out to us.

Corollary 2 (of 3.1 bis). *Over* $\mathbb{C}$, *let* $X \subset \mathbb{P}^{n+r}$ *be a smooth connected variety of dimension n and codimension r. If* $n \geqq r+2$, *then X admits no fixed point-free holomorphic involutions.*

Proof. By results of Barth, Larsen, and Ogus (cf. [8, Corollary 4.10]), such an X has $\mathrm{Pic}(X) = \mathbb{Z}$ with generator $\mathcal{O}_X(1)$, and the argument proceeds as above.

Remark. This last corollary becomes false if we drop the hypothesis "holomorphic." Indeed, if X is any smooth (even-dimensional) algebraic variety defined over the reals, then complex conjugation provides an (orientation-preserving) real-analytic involution of the complex manifold $X(\mathbb{C})$, whose fixed point set is precisely the set $X(\mathbb{R})$ of real points on X. Thus if X *has no real points*, we have a fixed-point-free involution of $X(\mathbb{C})$. For example, take X to be a smooth complete intersection of hypersurfaces, each of which is defined over $\mathbb{R}$, and one of which is a Fermat hypersurface F of some even degree $2k$

$$\Sigma(X_i)^{2k} = 0.$$

Then X has no real points, because visibly F has no real points.

Theorem 3.1 ter. *Let X be an irreducible non-singular projective variety over* $\mathbb{C}$, *for which* $H^1(X^{an}, \mathbb{C}) = 0$ *(e.g., if* $\pi_1(X^{an})$ *is finite). Let* $h \in H^2(X^{an}, \mathbb{Z})$ *denote the cohomological first chern class of a very ample invertible sheaf* $\mathcal{O}_X(1)$. *Suppose that a finite group G acts freely on* X^{an} *by holomorphic automorphisms which fix the element* $h \in H^2(X^{an}, \mathbb{Z})$ *(a condition which is automatically fulfilled if* $H^2(X^{an}, \mathbb{Z}) \simeq \mathbb{Z}$, *e.g. if X is a complete intersection of dimension at least three). Then 3.1.1–3 hold with* $V = H^0(X, \mathcal{O}_X(1))$.

Proof. We first deduce 3.1 ter from 3.1 bis. The hypothesis $H^1(X^{an}, \mathbb{C}) = 0$ guarantees, by Hodge theory, that $H^1(X^{an}, \mathcal{O}_X an) = 0$. The exponential sequence together with GAGA then show that the cohomological first chern class is injective

$$\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(X^{an}) \overset{c_1}{\hookrightarrow} H^2(X^{an}, \mathbb{Z}),$$

whence, by GAGA, 3.1 ter becomes a special case of 3.1 bis.

We next deduce 3.1 bis from 3.1. For each $g \in G$, choose an isomorphism.

$$\psi(g): g^*(\mathcal{O}_X(1)) \xrightarrow{\sim} \mathcal{O}_X(1).$$

Such a $\psi(g)$ is unique up to an element of $H^0(X, \underline{\mathrm{Aut}}(\mathcal{O}_X(1))) \simeq H^0(X, (\mathcal{O}_X)^\times) = (H^0(X, \mathcal{O}_X))^\times = k^\times$. Therefore the action of $g \in G$ on $V = H^0(X, \mathcal{O}_X(1))$ defined as

$$H^0(X, \mathcal{O}_X(1)) \xrightarrow[\sim]{g^*} H^0(X, g^*(\mathcal{O}_X(1))) \xrightarrow[\sim]{\psi(g)} H^0(X, \mathcal{O}_X(1))$$

provides a projective representation of G on V, which induces the given action of G on $X \subset \mathbb{P}(V)$. Thus 3.1 bis follows 3.1.

It remains to prove 3.1. To prove 3.1.2 and 3.1.3, it suffices to remark that they follow from 2.1.1 and 2.1.2, together with fact that a projective representation of a *cyclic* group on a vector space over an algebraically closed field can always be *linearized*. Similarly, 3.1.4 follows from 2.1.1 and the corresponding fact for projective representations of p-groups over perfect fields of characteristic p [because $k^\times$ is then uniquely p-divisible, so that the obstruction to linearization lies in an abelian group $H^2(p\text{-group}, k^\times)$ which is simultaneous killed by a power of p and on which "p" is an automorphism].

It remains to prove 3.1.1. Because this is a divisibility assertion, we may prove it prime by prime, i.e., replace G by its various Sylow subgroups. This reduces us to the case when G is a nilpotent group. In this case, 3.1.1 follows from 2.1.1 and the following theorem in group cohomology, applied to the obstruction, in $H^2(G, k^\times)$, to linearizing the given projective representation.

Theorem 3.2. *Let G be a finite nilpotent group, and k an algebraically closed field. View $k^\times$ as a trivial G-module. Then given any element $\xi \in H^2(G, k^\times)$, there exists a subgroup $K(\xi) \subset G$ such that ξ dies in $H^2(K(\xi), k^\times)$, and such that*

$$\#(G) \quad \textit{divides} \quad (\#(K(\xi)))^2.$$

Proof. We proceed by induction on $\#(G)$. Let us choose a central extension E of G by $k^\times$ which represents ξ:

$$1 \longrightarrow k^\times \longrightarrow E \xrightarrow{\pi} G \longrightarrow 1.$$

Let H be a non-trivial subgroup of the center $Z(G)$ over which this extension splits (e.g. H cyclic). Choose a group-theoretic splitting s: $H \to E$, and then extend s arbitrarily to a set-theoretic splitting $S: G \to E$.

$$\begin{tikzcd} 1 \arrow[r] & k^\times \arrow[r] & E \arrow[r, "\pi"] & G \arrow[r] \arrow[l, bend right, "S"'] & 1 \\ & & & H \arrow[u, hook] \arrow[ul, bend left, "s"] & \end{tikzcd}$$

Now denote by $E_1 \subset E$ the *normalizer* of $s(H)$. Then E_1 contains $s(H)$ (because $s: H \to E$ is a group homomorphism) and E_1 contains $k^\times$ (because $k^\times$ lies in the center of E). Therefore E_1 is the complete inverse image of a subgroup G_1 in G, and $G_1 \subset H$. Because $\mathscr{s}(H)$ is normal in E_1, we have a commutative diagram of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & k^\times & \longrightarrow & E_1 & \longrightarrow & G_1 & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & k^\times & \longrightarrow & E_1/\mathscr{s}(H) & \rightarrow & G_1/H & \longrightarrow & 1. \end{array}$$

By induction, the extension $E_1/\mathscr{s}(H)$ of G_1/H splits when restricted to a subgroup $K_2 \subset G_1/H$, with

$$\#(G_1/H) \quad \textit{divides} \quad (\#(K_2))^2.$$

The commutative diagram above shows that the extension E_1 of G_1 is the pull-back by $G_1 \twoheadrightarrow G_1/H$ of the extension $E_1/\mathscr{s}(H)$.

Therefore E_1 splits over the subgroup $K(\xi) \subset G_1$ defined by

$$K(\xi) \overset{dfn}{=\!=} \text{ the inverse image by } \quad G_1 \twoheadrightarrow G_1/H \text{ of } K_2.$$

Because E_1 is simply the restriction to $G_1 \subset G$ of E, our original extension E of G splits over $K(\xi)$.

It remains to verify that

$$\#(G) \quad \textit{divides} \quad (\#(K(\xi)))^2.$$

By definition of $K(\xi)$, we have

$$(\#(K(\xi)))^2 = (\#(H))^2(\#(K_2))^2.$$

and the inductive hypothesis may be rewritten

$$\#(G_1) \cdot \#(H) \quad \textit{divides} \quad (\#(H))^2(\#(K_2))^2,$$

so that it suffices to verify that

$$\#(G) \quad \textit{divides} \quad \#(G_1)\#(H).$$

Let us denote by $H^\vee$ the group $\operatorname{Hom}(H, k^\times)$. Then we have, simply because H is abelian and k is a field,

$$\#(H^\vee) \quad \textit{divides} \quad \#(H).$$

Therefore, the required divisibility follows from the existence of an exact sequence of groups

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow H^\vee.$$

which we will now construct. Recall that G_1 is defined as

$$G_1 = \{g \in G \mid S(g) \textit{ normalizes } \mathscr{s}(H)\}.$$

Because $H \subset Z(G)$, the element $S(g)\mathscr{s}(h)S(g)^{-1}$ in E projects to h; therefore if this element lies in $\mathscr{s}(H)$, it must be *equal* to $\mathscr{s}(h)$. Thus we may define a pairing

$$\langle\,,\rangle : G \times H \rightarrow k^\times$$

by the formula

$$S(g)\mathscr{A}(h)S(g)^{-1}=\langle g,h\rangle \mathscr{A}(h).$$

For *fixed* $g\in G$, the map

$$\langle g,-\rangle : H\to k^{\times}$$

is visibly a homomorphism, which is trivial if and only if $g\in G_1$. The map

$$G\to H^{\check{}}=\mathrm{Hom}(H,k^{\times})$$

$$g\mapsto\langle g,-\rangle$$

is easily seen to be itself a homomorphism, whence the required exact sequence. QED

Examples 3.3. (1) Take for X an abelian variety of dimension $g\geqq 1$, $\mathscr{L}$ a line bundle on X, and $N\geqq 1$ an integer. The abelian group $X_N(k)$ of points of order N in $X(k)$ operates freely on X, by translations, and this action fixes the isomorphism class of $\mathscr{L}^{\otimes N}$. Suppose that $\mathscr{L}$ defines a principal polarization of X, and that $N\geqq 3$. Then $\mathscr{L}^{\otimes N}$ is very ample, and the degree of X in the corresponding projective embedding is N^g. For N prime to char(k), our group $X_N(k)$ has order N^{2g}, so that our result $\#(G)$ divides $(\deg(X))^2$ is best possible. Notice that if char(k) $=p>0$ and we take $N=p$, then by 3.1.4 we have $\#(X_p(k))$ divides p^g, a familiar fact from the characteristic p theory of abelian varieties.

(2) Fix a prime $p\geqq 3$. Consider, in characteristic $\neq p$, the one-parameter family of hypersurfaces of degree p and dimension $p-2$, defined by the equation

$$\sum_{i=0}^{p-1}(X_i)^p=\mu\prod_{i=0}^{p-1}X_i.$$

It will be convenient to view the ambient p-dimensional vector space V as being

$$k[T]/(T^p-1)$$

via the identification

$$X_i\leftrightarrow T^i.$$

We obtain a *projective* representation of the group $\mathbb{Z}/p\mathbb{Z}\times\boldsymbol{\mu}_p$ on this vector space by the rule

$$(a,\zeta): f(T)\mapsto T^a f(\zeta T).$$

This action respects each of our hypersurfaces [the $\mathbb{Z}/p\mathbb{Z}$ acts by cyclic permutation of the p coordinates X_i, and the $\boldsymbol{\mu}_p$ acts by the Godeaux action $\zeta:(,\ldots,X_i,\ldots)\mapsto(,\ldots,\zeta^i X_i,\ldots)$].

One checks by explicit calculation that at $\mu=0$, and therefore at all but finitely many values of μ, this defines a free action of $\mathbb{Z}/p\mathbb{Z}\times\boldsymbol{\mu}_p$, a group of order p^2, on a hypersurface of degree p. For $p=3$, we recover the $g=1$, $N=3$ case of example (1), for our family of hypersurfaces becomes the universal family of elliptic curves with level three structure, the action of the group $\mathbb{Z}/3\mathbb{Z}\times\boldsymbol{\mu}_3$ is the action by translation of the points of order 3, and the very ample $\mathcal{O}_X(1)$ is $\mathcal{L}^{\otimes 3}$ where $\mathcal{L}$ is the standard principal polarization on an elliptic curve.

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