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SLOPE FILTRATION OF F-CRYSTALS

by

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This paper is devoted to the systematic study of the variation of the Hodge and Newton polygons of an F-crystal when that F-crystal moves in a family. As such, it constitutes a natural sequel to my report [6] on Dwork's pioneering investigations of such variation. However, I have tried to make this paper self-contained and accessible to non-specialists.

Some of the results are new, and interesting, even in the "classical" case of F-crystals over perfect fields. I have in mind particularly the "basic" and "sharp" slope estimates (cf 1.4, 1.5) and the "Newton-Hodge" decomposition (cf 1.6). These "pointwise" results are in fact the key to all the "global" results given in 2.3 - 2.7 .

Special thanks are due to Arthur Ogus for suggesting the possible existence of the Newton-Hodge decomposition.

Table of Contents

I. F-Crystals over perfect fields

(1.1) Basic definitions

For any perfect field k of characteristic $p > 0$, we denote by $W(k)$ its ring of Witt vectors, and by

$$\sigma : W(k) \longrightarrow W(k)$$

the absolute Frobenius automorphism. For any integer $a \neq 0$, we have the notion of a σ^a-F-crystal over k, namely a pair (M,F) consisting of a free finitely generated $W(k)$-module M together with a σ^a-linear endomorphism $F:M \longrightarrow M$ which induces an automorphism of $M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. A morphism of σ^a-F-crystals $f:(M,F) \longrightarrow (M,F')$ is a $W(k)$-linear map $f:M \longrightarrow M'$ such that $fF = F'f$. The category of σ^a-F-crystals up to isogeny is obtained from the category of σ^a-F-crystals by keeping the same objects, but tensoring the Hom groups, which are $\mathbb{Z}_p$-modules, over $\mathbb{Z}_p$ with $\mathbb{Q}_p$. An isogeny between σ^a-F-crystals is a morphism of F-crystals which becomes an isomorphism in this new category.

The <u>exterior powers</u> of a σ^a-F-crystal (M,F) are the σ^a-F-crystals $(\Lambda^i M, \Lambda^i(F))$ with underlying module $\Lambda^i_{W(k)}(M)$, and with σ^a-linear endomorphism $\Lambda^i(F)$ defined by

$$\Lambda^i(F)(m_1 \wedge \ldots \wedge m_i) = F(m_1) \wedge \ldots \wedge F(m_i) .$$

For $i = 0$, but $(M,F) \neq 0$, we define $(\Lambda^0 M, \Lambda^0(F))$ to be $(W(k), \sigma^a)$.

The <u>iterates</u> of a σ^a-F-crystal (M,F) are the σ^{an}-F-crystals (M,F^n), $n = 1,2,\ldots$.

(1.2) Hodge polygons

The Hodge numbers $h^0, h^1, h^2, \ldots$ of a σ^a-F-crystal (M,F) are the integers defined as follows (cf [9]). The image $F(M)$ is a $W(k)$-submodule of M of maximal rank, say r, so by the theory of elementary divisors, there exist $W(k)$-bases $\{v_1, \ldots, v_r\}$ and $\{w_1, \ldots, w_r\}$ of M such that

$$F(v_i) = p^{a_i} w_i ,$$

with integers

$$0 \leq a_1 \leq a_2 \leq \cdots \leq a_r .$$

These integers are called the Hodge <u>slopes</u> of (M,F). The Hodge <u>numbers</u> h^i of (M,F) are defined by

$$h^i = \# \text{ of times } i \text{ occurs among } \{a_1 \ldots a_r\} .$$

Thus we have

$$\sum_{i \geq 0} h^i = r \qquad (r = \operatorname{rank}(M))$$

$$M/F(M) \simeq \bigoplus_{i \geq 0} (W(k)/p^i W(k))^{h^i} .$$

Notice that we have the elementary interpretation:

1.2.1
$$h^i = 0 \text{ for } i < A \iff$$
$$F \equiv 0 \bmod p^A \quad \text{i.e.} \quad F(M) \subset p^A M$$

1.2.2
$$h^i = 0 \text{ for } i > B \iff$$
$$M \supset F(M) \supset p^B M \iff$$
$$\exists\, \sigma^{-a}\text{-linear } V : M \longrightarrow M \text{ such that } FV = VF = p^B .$$

According to a marvelous theorem of Mazur [9], these "abstract" Hodge numbers sometimes coincide with more traditional Hodge numbers. Thus let X be a projective smooth $W(k)$-scheme, all of whose Hodge cohomology groups $H^j(X,\Omega^i_{X/W(k)})$ are assumed to be free, finitely generated $W(k)$ modules, whose ranks we denote $h^{i,j}(X)$. Let X_o be the projective smooth k-scheme obtained from X by reduction modulo p. Then for each integer $j \geq 0$, the crystalline cohomology groups $H^j_{cris}(X_o)$ are free finitely generated $W(k)$-modules, given with a σ-linear F

which provides a structure of σ-F-crystal. Mazur's theorem asserts that the abstract Hodge numbers of these σ-F-crystals are given by the formula

$$h^i(H^j_{cris}(X_o);F) = h^{i,j}(X) .$$

Given a σ^a-F-crystal (M,F), whose Hodge slopes are $0 \le a_1 \le \dots \le a_r$, the Hodge slopes of the i^{th} exterior power $(\Lambda^i M, \Lambda^i(F))$, $0 \le i \le r$, are the $\binom{r}{i}$ integers

$$a_{j_1,\dots,j_i} = a_{j_1} + a_{j_2} + \dots + a_{j_i} \qquad 1 \le j_1 < \dots < j_i \le r ,$$

(as follows immediately from computing the matrix of F in the bases $\{v_{j_1} \wedge \dots \wedge v_{j_i}\}$ and $\{w_{j_1} \wedge \dots \wedge w_{j_i}\}$ of $\Lambda^i M$).

The Hodge polygon of (M,F) is the graph of the Hodge function on $[0,r]$ defined on <u>integers</u> $0 \le i \le r$ by

$$\begin{aligned} \mathrm{Hodge}_F(i) &= \text{least Hodge slope of } (\Lambda^i M, \Lambda^i(F)) \\ &= \begin{cases} 0 & \text{if } i = 0 \\ a_1 + \dots + a_i & \text{if } 1 \le i \le r \end{cases} \end{aligned}$$

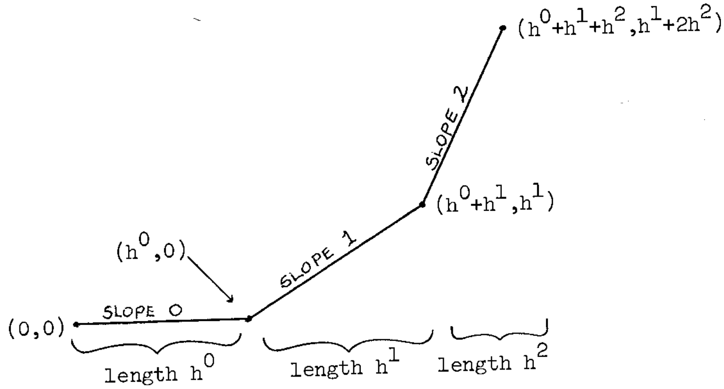
and then extended <u>linearly</u> between successive integers. If we define

$$\begin{aligned} \mathrm{ord}(F) &= \text{greatest integer } A \text{ with } F \equiv 0 \bmod p^A \\ &= \text{least integer } A \text{ with } h^A(M,F) \ne 0 \end{aligned}$$

then we have

$$\mathrm{Hodge}_F(i) = \mathrm{ord}\,(\Lambda^i(F)) .$$

The Hodge polygon thus looks like



The points $(h^0+\ldots h^i,\ h^1+2h^2+\ldots+ih^i)$ at which the Hodge polygon changes slope are called its <u>break-points</u>.

The Hodge polygon is not at all an isogeny invariant, as simple examples show. The only general result I know about its isogeny-behavior is the following trivial "specialization" property.

<u>Lemma 1.2.3.</u>

<u>Suppose we have an exact sequence of</u> σ^a<u>-F-crystals</u>

$$0 \longrightarrow (M_1,F_1) \longrightarrow (M,F) \longrightarrow (M_2,F_2) \longrightarrow 0 .$$

<u>Then the Hodge polygon of the direct sum</u> $(M_1 \oplus M_2\ ,\ F_1 \oplus F_2)$ <u>lies above the Hodge polygon of</u> (M,F) .

<u>Proof</u>. Equivalently, we must show that for $1 \leq i \leq r = \operatorname{rank}(M)$, we have

$$\operatorname{ord}\,(\Lambda^i(F)) \leq \operatorname{ord}\,(\Lambda^i(F_1 \oplus F_2)) .$$

Now we have

$$\Lambda^i(M_1 \oplus M_2) \simeq \bigoplus_{a+b=i} \Lambda^a(M_1) \otimes \Lambda^b(M_2)$$

so that

$$\operatorname{ord}\,(\Lambda^i(F_1 \oplus F_2)) = \min_{a+b=i}\ (\operatorname{ord}\,(\Lambda^a(F_1) \otimes \Lambda^b(F_2)) .$$

But the direct sum $\bigoplus_{a+b=i} \Lambda^a(M_1)\otimes\Lambda^b(M_2)$ is exactly the associated graded of the Koszul filtration (by "how many m_1's") of $\Lambda^i M$. Thus any congruence $\Lambda^i(F) \equiv 0 \bmod P^A$ implies the same congruence for each of the $\Lambda^a(F_1)\otimes\Lambda^b(F_2)$, $a + b = i$, which is to say that we have

$$\mathrm{ord}\,(\Lambda^a(F_1)\otimes\Lambda^b(F_2)) \geq \mathrm{ord}\,(\Lambda^i(F)) \qquad \text{if} \quad a + b = i\ .$$

QED

In fact, Mazur's theorem strongly suggests the desirability of studying F-crystals only up to "Hodge-isogeny", i.e. only regarding as equivalent two F-crystals which have the same Hodge polygon and which are isogenous. We will not pursue that point of view here, except in so far as the "Newton-Hodge" decomposition, which we will discuss further on, may be regarded as a step in that direction.

(1.3) Newton polygons

The Newton slopes of a σ^a-F-crystal (M,F) are the sequence of $r = \mathrm{rank}(M)$ rational numbers

$$0 \leq \lambda_1 \leq \cdots \leq \lambda_r$$

defined in any of the following equivalent ways.

Pick an algebraically closed overfield k' of k, and consider the σ^a-F-crystal over k'

$$(M \otimes_{W(k)} W(k')\ ,\ F\otimes\sigma^a)$$

obtained from (M,F) by "extension of scalars". For each non-negative rational number λ, written in lowest terms N/M, we denote by $E(\lambda)$ the σ^a-F-crystal over k' defined by

$$E(\lambda) = ((\mathbb{Z}_p[T]/(T^M-p^N))\otimes_{\mathbb{Z}_p} W(k')\ ,\ (\text{mult. by } T)\otimes\sigma^a).$$

According to a fundamental theorem of Dieudonné (cf [8]), the category of σ^a-F-crystals up to isogeny over an algebraically closed field k' is semisimple, and the $E(\lambda)$'s give a set of representatives of the simple objects in this category. Thus we can write

$$(M \otimes W(k'), F \otimes \sigma^a) \underset{\text{isog}}{\widetilde{\quad}} \oplus E(N_i/M_i)$$

with a unique finite sequence of rational numbers $N_1/M_1 \leq N_2/M_2 \leq \ldots$, $\Sigma M_i = r$. The Newton slopes of (M,F) are defined to be the sequence of r rational numbers

$$(\lambda_1, \ldots \lambda_r) \overset{\text{dfn}}{=} (N_1/M_1 \text{ repeated } M_1 \text{ times}, N_2/M_2 \text{ repeated } M_2 \text{ times}, \ldots).$$

For each rational number λ, we define

$$\text{mult}(\lambda) = \# \text{ of times } \lambda \text{ occurs among } (\lambda_1, \ldots, \lambda_r).$$

From the above explicit description of the Newton slopes, it is obvious that

1.3.1 $\begin{cases} \sum_{\lambda \in \mathbb{Q}} \text{mult}(\lambda) = r \qquad (r = \text{rank}(M)) \\ \text{for each } \lambda \text{, the product } \lambda\,\text{mult}(\lambda) \text{ lies in } \mathbb{Z} \text{; in} \\ \text{particular the Newton slopes admit } r! \text{ as a common denominator.} \end{cases}$

For the next characterization of the Newton slopes, we choose an auxiliary integer $N \geq 1$ which is divisible by $r!$, $r = \text{rank}(M)$, and consider the discrete valuation ring

$$R = W(k')[X]/(X^N - p) = W(k')[p^{1/N}].$$

We extend σ to an automorphism of R by requiring that $\sigma(X) = X$. For any rational number λ with $N\lambda \in \mathbb{Z}$, we may speak of

$$p^\lambda = \text{the image of } X^{N\lambda} \text{ in } R.$$

Let K denote the fraction field of R. Again by Dieudonné, we know that $M \underset{W(k)}{\otimes} K$ admits a K-basis $e_1, \ldots, e_r$ which transforms under the σ^a-linear endomorphism $F\otimes\sigma^a$ by the formula

$$(F\otimes\sigma^a)(e_i) = p^{\lambda_i} e_i \ .$$

An equivalent, and for us more useful, characterization of the Newton slopes is by the existence of an R-basis $u_1,\ldots,u_r$ of $M \underset{W(k)}{\otimes} R$ with respect to which the "matrix" of $F\otimes\sigma^a$ is upper-triangular, with p^{λ_i} along the diagonal:

$$\begin{pmatrix} p^{\lambda_1} & & \text{entries in } R \\ & \ddots & \\ \bigcirc & & p^{\lambda_r} \end{pmatrix}$$

i.e.

$$F(u_i) \equiv p^{\lambda_i} u_i \bmod \sum_{j<i} Ru_j \ .$$

Either of these last two descriptions makes it obvious that the Newton slopes of the i^{th} exterior power $(\Lambda^i M, \Lambda^i(F))$ of (M,F) are the $\binom{r}{i}$ numbers

$$\lambda_{j_1} + \ldots + \lambda_{j_i} \qquad 1 \leq j_1 < j_2 < \ldots < j_i \leq r \ ,$$

and that the Newton slopes of the n^{th} iterate (M,F^n) of (M,F) are

$$(n\lambda_1,\ldots,n\lambda_r) \ .$$

The last description of the Newton slopes makes clear the elementary interpretations

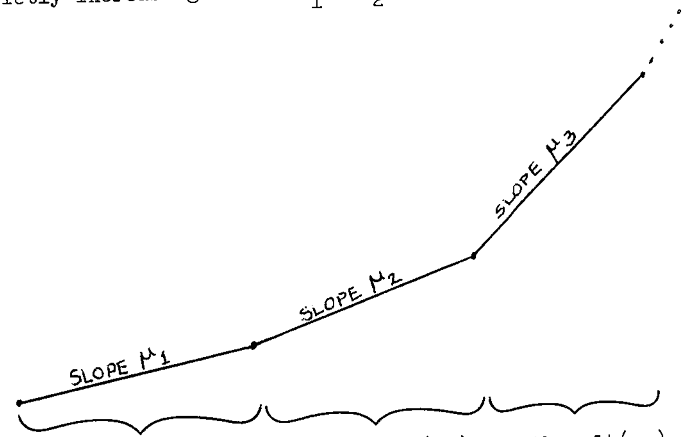
1.3.2 all Newtons slopes λ_i of (M,F) are $= 0$ if and only if F is a σ^a-linear <u>automorphism</u> of M .

1.3.3 all Newton slopes λ_i of (M,F) are > 0 if and only if F is <u>topologically nilpotent</u> on M , i.e. iff and only if $F^r(M) \subset pM$ where $r = \text{rank}(M)$.

The Newton polygon of (M,F) is the graph of the Newton function on $[0,r]$, defined on <u>integers</u> $0 \leq i \leq r$ by

$$\mathrm{Newton}_F(i) = \text{least Newton slope of } (\Lambda^i M, \Lambda^i(F))$$
$$= \begin{cases} 0 & \text{if } i = 0 \\ \lambda_1 + \ldots + \lambda_i & \text{if } 1 \leq i \leq r \end{cases}$$

and then extended linearly between successive integers. In terms of the <u>distinct</u> Newton slopes μ_i of (M,F) together with their multiplicities $\mathrm{mult}(\mu_i)$, arranged in strictly increasing order $\mu_1 < \mu_2 < \ldots$, the Newton polygon looks like



length=mult(μ_1) length=mult(μ_2) length=mult(μ_3) .

The points $(\mathrm{mult}(\mu_1)+\ldots+\mathrm{mult}(\mu_i)\ ,\ \mu_1\ \mathrm{mult}(\mu_1)+\ldots+\mu_i\ \mathrm{mult}(\mu_i))$ at which the Newton polygon changes slope are called its break-points. From the earlier noted fact that the products $\mu_i\ \mathrm{mult}(\mu_i)$ are all <u>integers</u>, it follows that the break-points of the Newton polygon are always <u>lattice-points</u> in $\mathbb{R}^2$, i.e. they have integer coordinates.

By its very construction, the Newton-polygon is an isogeny invariant (indeed over an algebraically closed field it is <u>the</u> isogeny invariant). In contrast to the case of Hodge polygons, we have

Lemma 1.3.4.

Suppose we have an exact sequence of σ^a*-F-crystals*

$$0 \longrightarrow (M_1,F_1) \longrightarrow (M,F) \longrightarrow (M_2,F_2) \longrightarrow 0 .$$

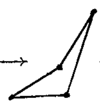
Then the Newton polygon of the direct sum $(M_1 \oplus M_2, F_1 \oplus F_2)$ *coincides with the Newton polygon of* (M,F) .

Proof. Extending scalars, we may suppose k algebraically closed. By Dieudonné's semisimplicity theorem, our exact sequence splits in the "up-to-isogeny" category.

QED

Example. Over $\mathbb{F}_p$, take $M = \mathbb{Z}_p \oplus \mathbb{Z}_p$, $F = \begin{pmatrix} p & 1 \\ 0 & p^2 \end{pmatrix}$.

The Newton and Hodge polygons are



Remarks. In our characterizations of the Newton slopes of a σ^a-F-crystal, we make use of the integer a , (not just of the automorphism σ^a), in order to extend σ^a to an algebraically closed overfield of k . However, in the next section we will give an "internal" characterization (cf 1.4.4) (i.e., one that involves no extension of scalars) of the Newton polygon (in terms of Hodge polygons of iterates). Consequently, the Newton polygon of a given σ^a-F-crystal over k depends only on the automorphism σ^a of k , and not on the auxiliary choice of a . (Of course σ^a as automorphism of k determines a unless k is finite.)

By Manin [8], we know that if σ^a is the identity on k , i.e., if $k \subset \mathbb{F}_{p^a}$, then the Newton slopes of a σ^a-F-crystal (M,F) on k are precisely the p-adic ordinals of the eigenvalues of "F viewed as linear endomorphism of M". In terms of a matrix (F_{ij}) for F , this means that the Newton polygon of the σ^a-F-crystal (M,F) coincides with the Newton polygon of the "reversed" characteristic polynomial $\det(1 - T(F_{ij}))$ of the matrix (F_{ij}) .

However, if $\sigma^a \neq$ id. on k , then the Newton polygon of a σ^a-F-crystal (M,F) need not coincide with the Newton polygon of $\det(1 - T(F_{ij}))$, where

(F_{ij}) is the matrix expressing the action of F on some basis of M. Here is an example, due to B. Gròss. Over $\mathbb{F}_{p^2}$ with $p \equiv 3 \bmod 4$, consider the σ-F-crystal of rank two with matrix

$$\begin{pmatrix} 1-p & (p+1)i \\ (p+1)i & p-1 \end{pmatrix}$$

The eigenvalues of this matrix both have ordinal 1/2 (since trace = 0 , det = p). But this matrix is σ-linearly equivalent, via $\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$, to the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix},$$

and hence our σ-F-crystal has Newton slopes $\{0,1\}$, not $\{1/2,1/2\}$.

(1.4) Newton-Hodge relations; the basic slope estimate

In this section we will discuss various relations between Hodge and Newton polygons.

Theorem 1.4.1 (Mazur)

For any σ^a-F-crystal (M,F), the Newton polygon is above the Hodge polygon. Both polygons have the same initial point (namely $(0,0)$) and the same terminal point (namely $(r, \operatorname{ord}(\det(F)))$.

Proof. For any σ^a-F-crystal of rank one, the Hodge slope and the Newton slope coincide. Applying this remark to $(\Lambda^0 M, \Lambda^0(F)) = (W(k), \sigma^a)$ and to $(\Lambda^r M, \Lambda^r(F))$, we see that the two polygons begin and end together. To show that

$$\mathrm{Newton}_F(i) \geq \mathrm{Hodge}_F(i) \quad \text{for } 1 \leq i \leq r,$$

it suffices to show that for each of the exterior powers of (M,F), we have

$$\text{least Newton slope} \geq \text{least Hodge slope},$$

i.e., we must prove

$$\mathrm{Newton}_F(1) \geq \mathrm{Hodge}_F(1)$$

universally. But in terms of the matrix A_{ij} of $F\otimes\sigma^a$ on $M\otimes R$ with respect to <u>any</u> R-basis $u_1,\ldots,u_r$ of $M\otimes R$, we have

$$\mathrm{Hodge}_F(1) = \mathrm{ord}(F) = \min_{i,j} (\mathrm{ord}_p(A_{ij})) .$$

As we may choose the R-base so that this matrix is

$$\begin{pmatrix} p^{\lambda_1} & & \text{entries in } R \\ & \ddots & \\ 0 & & p^{\lambda_r} \end{pmatrix} ; \qquad \text{Newton slopes } \lambda_1 \leq \cdots \leq \lambda_r$$

we get

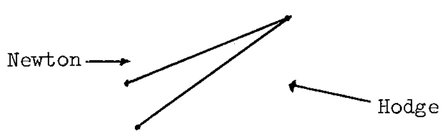
$$\mathrm{Hodge}_F(1) = \min_{i,j} (\mathrm{ord}_p(A_{ij})) \leq \mathrm{ord}(p^{\lambda_1}) = \lambda_1 = \mathrm{Newton}_F(1) .$$

QED

<u>Remarks</u>. If we compare <u>largest</u> rather than <u>smallest</u> slopes, we get

$$\text{greatest Newton slope} \leq \text{greatest Hodge slope} ,$$

simply because the two polygons are both convex, and have the same terminal point, i.e., they end like



Thus denoting by A and B the least and greatest Hodge slopes and by λ_1 and λ_r the least and greatest Newton slopes, we have

1.4.2 $$A \leq \lambda_1 \leq \lambda_r \leq B .$$

As a by-product of this method of proof, we get the

(1.4.3) <u>Basic slope estimate</u>

<u>Let</u> (M,F) <u>be a σ^a-F-crystal of rank</u> r, <u>and let</u> $\lambda \geq 0$ <u>be a rational</u>

number. For any real number x, denote by $\{x\}$ the "next higher integer", i.e., $\{x\} = -[-x]$. Then (M,F) has all Newton slopes $\geq \lambda$ if and only if for all integers $n \geq 1$ we have

$$\operatorname{ord}(F^{n+r-1}) \geq \{n\lambda\}$$

$$\text{i.e.,} \quad F^{n+r-1}(M) \subset p^{\{n\lambda\}}M \ .$$

Proof. We begin with the "if" part. Let λ_1 be the smallest Newton slope of (M,F). Then the smallest Newton slope of (M,F^{n+r-1}) is $(n+r-1)\lambda_1$, while by hypothesis its smallest Hodge slope is $\geq \{n\lambda\}$. By the previous theorem, applied to (M,F^{n+r-1}), we have

$$(n+r-1)\lambda_1 \geq \{n\lambda\} \geq n\lambda \qquad \text{for all} \quad n \geq 1 \ ,$$

whence $\lambda_1 \geq \lambda$ as required.

Conversely, we must show that, still denoting by λ_1 the smallest Newton slope of (M,F), we have

$$\operatorname{ord}(F^{n+r-1}) \geq n\lambda_1 \ .$$

Extending scalars to R (cf 1.3.1ff) it suffices to show

$$(F\otimes\sigma^a)^{n+r-1}(M\otimes R) \subset p^{n\lambda_1}M\otimes R \ .$$

In terms of a suitable R-basis $u_1,\ldots,u_r$ of $M\otimes R$, we have

$$(F\otimes\sigma^a)(u_i) = p^{\lambda_i}u_i + \text{elt. of} \sum_{j<i} Ru_j \ .$$

Iterating, we find, for all $N \geq i$,

$$(F\otimes\sigma^a)^N(u_i) \in p^{N\lambda_1}u_i + p^{(N-1)\lambda_1}Ru_{i-1} + \cdots$$
$$\in p^{(N-i+1)\lambda_1} M\otimes R \ .$$

Since $i \leq r$, and the u_i span $M\otimes R$, we find

$$(F\otimes\sigma^a)^{n+r-1}\,(M\otimes R) \subset p^{n\lambda_1}\,M\otimes R$$

as required.

<u>Corollary 1.4.4</u>

<u>The Newton function of</u> (M,F) <u>is obtained from the Hodge function of the iterates</u> (M,F^n) <u>of</u> F <u>by the (archimedean) limit formula</u>

$$\mathrm{Newton}_F(x) = \lim_{n\to\infty} \frac{1}{n}\,\mathrm{Hodge}_{F^n}(x)\ .$$

<u>valid for</u> $0 \le x \le r = \mathrm{rank}(M)$.

<u>Proof</u>. Since both the Newton and Hodge functions are defined first on integers $0 \le i \le r$, then interpolated linearly between successive integers, it suffices to prove the formula for x = an integer $0 \le i \le r$. The formula, for (M,F) and $x = i$, is equivalent to (indeed term by term identical with) the formula for $(\Lambda^i M, \Lambda^i(F))$ and $x = 1$. Thus it suffices to prove universally that, denoting by λ_1 the smallest Newton slope of (M,F) , we have

$$\lambda_1 = \lim_{n\to\infty} \frac{1}{n}\,\mathrm{ord}(F^n)\ .$$

By the basic slope estimate, we have

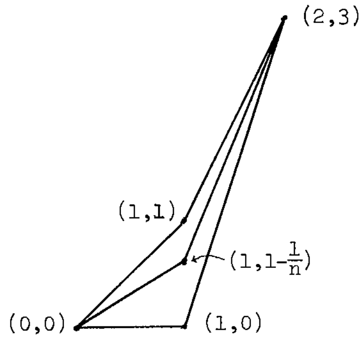
$$(n+r-1)\lambda_1 \ge \mathrm{ord}\,(F^{n+r-1}) \ge \{n\lambda_1\} \ge n\lambda_1\ ,$$

from which the required limit formula is immediate. <u>QED</u>

<u>Examples</u>. Consider the σ-F-crystal over $\mathbb{F}_p$ with $M = \mathbb{Z}_p \oplus \mathbb{Z}_p$ and F given by the matrix

$$F = \begin{pmatrix} p & 1 \\ 0 & p^2 \end{pmatrix}\ .$$

The graphs of the functions Newton_F and $\frac{1}{n}\,\mathrm{Hodge}_{F^n}$ are



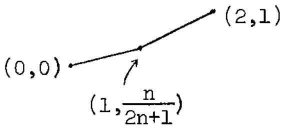
as drawn : the highest is Newton_F , the middle is that of $\frac{1}{n}\,\mathrm{Hodge}_{F^n}$ for some $n \geq 2$, and the lowest is that of Hodge_F .

If instead we take $M = \mathbb{Z}_p \oplus \mathbb{Z}_p$, F given by

$$F = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$$

then the graphs of Newton_F and of all $\frac{1}{2n}\,\mathrm{Hodge}_{F^{2n}}$ coincide:

while the graph of $\frac{1}{2n+1}\,\mathrm{Hodge}_{F^{2n+1}}$ is



The point of these examples is that the Newton polygon may or may not be attained at some finite n , and that the sequence of approximating functions need not be monotone. The common features of the examples, namely that all approximants are convex polygons lying on or below the Newton polygon, but sharing its beginning and terminal points, are indeed common to all examples. This follows from Theorem 1.4.1 applied to (M,F^n) , and the fact that $\frac{1}{n}\,\mathrm{Newton}_{F^n} = \mathrm{Newton}_F$.

(1.5) <u>Sharp slope estimate</u>

In this section we will give a <u>sharpening</u> of the basic slope estimate, in which the "lag" term $r - 1$ is replaced by a sum of certain Hodge numbers.

<u>Sharp Slope Estimate 1.5.1</u>

<u>Let</u> (M,F) <u>be a</u> σ^a<u>-F-crystal, and</u> $\lambda \geq 0$ <u>a rational number. Let</u> $h^0, h^1, \ldots,$ <u>be the Hodge numbers of</u> (M,F). <u>Then all Newton slopes of</u> (M,F) <u>are</u> $\geq \lambda$ <u>if and only if for all integers</u> $n \geq 1$ <u>we have</u>

$$\operatorname{ord}\left(F^{n+\sum_{i<\lambda} h^i}\right) \geq \{n\lambda\} .$$

<u>Proof</u>. The "if" part is proved exactly as for the basic slope estimate. To prove the "only if" part, we first reduce it to a reasonable-sounding assertion about determinants, and then give an unpleasantly computational proof of that assertion.

Suppose then, that λ_1 is the least Newton slope of (M,F) and that we are given a rational λ

$$0 \leq \lambda \leq \lambda_1 .$$

Because the function "ord" assumes only integral values, it suffices to prove that

$$\operatorname{ord}\left(F^{n+\sum_{i<\lambda} h^i}\right) \geq n\lambda .$$

Extending scalars to a suitable R containing p^λ as well as the p^{λ_i}, it suffices to prove that

$$(F \otimes \sigma^a)^{n+\sum_{i<\lambda} h^i} (M \otimes R) \subset p^{n\lambda}\, M \otimes R .$$

At this point, we must observe that for <u>any</u> real $\lambda \geq 0$, we have

$$\operatorname{ord}\left(\Lambda^{n+\sum_{i<\lambda} h^i}(F)\right) \geq n\lambda \qquad \text{for } n \geq 1 ,$$

i.e.,

$$\operatorname{Hodge}_F\left(n + \sum_{i<\lambda} h^i\right) \geq n\lambda .$$

To see that this is true, look at the Hodge polygon. It has a break-point at $\left(\sum_{i<\lambda} h^i, \sum_{i<\lambda} i\,h^i\right)$, and from this point rightwards the slopes are all $\geq \{\lambda\}$, so that we have

$$\mathrm{Hodge}_F\left(n + \sum_{i<\lambda} h^i\right) \geq \sum_{i<\lambda} ih_i + n\{\lambda\} \geq n\lambda$$

for all real $n \geq 0$.

Let us further observe that in terms of a suitable basis $u_1, \ldots, u_r$ of $M \otimes R$, the matrix of $F \otimes \sigma^a$ is of the form

$$\begin{pmatrix} p^{\lambda_1} & & \text{entries in } R \\ & \ddots & \\ 0 & & p^{\lambda_r} \end{pmatrix}$$

an upper triangular matrix over R all of whose diagonal entries lie in the ideal $p^\lambda R$ of R.

Lemma 1.5.2

Let R be any commutative ring with 1, $I \subset R$ an ideal, and $\phi: R \longrightarrow R$ an endomorphism of R such that

$$\text{for any } x \in R, \quad \phi(x) \in xR.$$

Let M be a free R-module of finite rank, and let $F: M \longrightarrow M$ be a ϕ-linear endomorphism of M, whose matrix relative to some R-basis $\{u_i\}$ of M is upper triangular, and has all of its diagonal entries in the ideal I. Suppose that for some integer $k \geq 0$, we have the congruences

$$\Lambda^{k+n}(F) \equiv 0 \bmod I^n \quad \text{for all } n \geq 1.$$

Then we also have

$$F^{k+n} \equiv 0 \bmod I^n \quad \text{for all } n \geq 1.$$

If we apply this to our $R = W(k')[p^{1/N}]$, $\phi = \sigma^a$, $I = p^\lambda R$, $M \otimes R$ with $F \otimes \sigma^a$, we get the "sharp" slope estimate.

To prove the lemma, let us denote by (F_{ij}) the matrix of F relative to the R-basis $\{u_i\}$ of M :

$$F(u_i) = \sum_j F_{ji} u_j \; ; \qquad F_{ij} = 0 \text{ if } i > j \; , \quad F_{i,i} \in I \; .$$

The matrix of F^n is the product matrix

$$(F_{ij})(\phi(F_{ij})) \ldots (\phi^{n-1}(F_{ij})) \; ,$$

each of whose entries is a sum of products of the form

$$F_{i_1,i_2} \cdot \phi(F_{i_2,i_3}) \cdots \phi^{n-1}(F_{i_n,i_{n+1}}) \; .$$

By the hypothesis made on ϕ , each of these products is <u>divisible</u> by the corresponding product "without ϕ "

$$F_{i_1,i_2} \cdot F_{i_2,i_3} \cdots F_{i_n,i_{n+1}} \; .$$

(Of course this product vanishes unless $i_1 \leq i_2 \leq \ldots \leq i_{n+1}$, since (F_{ij}) is upper triangular.) So it suffices to show that each such n-fold product lies in I^{n-k} , for $n \geq k$.

Let $J(n)$ denote the ideal generated by the $n \times n$ minors of $(F_{i,j})$. By hypothesis, we have $J(n) \subset I^{n-k}$ for $n \geq k$, so it suffices to show that

$$F_{i_1,i_2} \cdot \ldots F_{i_n,i_{n+1}} \in J(n) + I \cdot J(n-1) + \ldots + I^{n-1} J(1) + I^n \; ,$$

whenever $1 \leq i_1 \leq \ldots \leq i_{n+1} \leq r$. Since the diagonal entries $F_{i,i}$ lie in I , it suffices to treat the case when $i_1 < i_2 < \ldots < i_{n+1}$. This case follows inductively from the following well-known determinant formula, whose verification is left to the reader.

Formula 1.5.3

Let $(X_{i,j})$ be an $r \times r$ upper-triangular matrix $(X_{i,j}=0$ if $i > j)$ of indeterminates. For each subset $T \subset \{1,\ldots,r\}$ of cardinality $\nu \geq 2$.

$$T = \{t_1 < t_2 \cdots < t_\nu\}$$

we define

$$\det(T) = \det \begin{pmatrix} X_{t_1,t_2} & X_{t_1,t_3} & \cdots & X_{t_1,t_\nu} \\ X_{t_2,t_2} & X_{t_2,t_3} & & X_{t_2,t_\nu} \\ & \ddots & \ddots & \vdots \\ 0 & & X_{t_{\nu-1},t_{\nu-1}} & X_{t_{\nu-1},t_\nu} \end{pmatrix}$$

$$= \text{the } (\nu-1) \times (\nu-1) \text{ minor indexed by}$$
$$(t_1,\ldots,t_{\nu-1}) \times (t_2,\ldots,t_\nu)$$

and we define

$$f(T) = X_{t_1,t_2} X_{t_2,t_3} \cdots X_{t_{\nu-1},t_\nu} .$$

Then we have the formula

$$\det(T) = \sum_{S \subset T-\{t_1,t_\nu\}} (-1)^{\#S} \cdot f(T-S) \cdot \prod_{s\in S} X_{s,s} .$$

(1.6) Newton-Hodge decomposition

In this section we give a fundamental decomposition theorem for σ^a-F-crystals, in terms of the interrelations between their Hodge and Newton polygons. I owe entirely to Ogus the idea that such a decomposition should exist.

Theorem 1.6.1 (Newton–Hodge decomposition)

Let (M,F) be a σ^a-F-crystal of rank r. Let $(A,b) \in \mathbb{Z} \times \mathbb{Z}$ be a break-point of the Newton polygon of (M,F), which also lies on the Hodge polygon of (M,F). Then there exists a unique decomposition of (M,F) as a direct sum

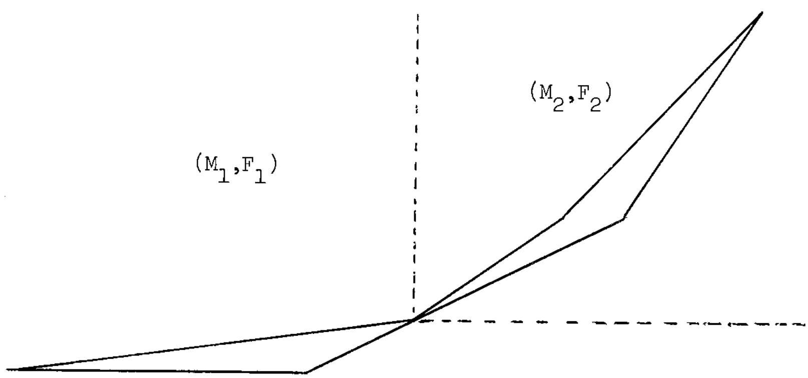
$$(M,F) \simeq (M_1 \oplus M_2 \ , \ F_1 \oplus F_2)$$

of two σ^a-F-crystals (M_1,F_1) and (M_2,F_2), such that

$$1.6.1 \qquad \begin{cases} \text{rank } (M_1) = A \\ \text{Hodge slopes of } (M_1,F_1) = \{\text{first } A \text{ Hodge slopes of } (M,F)\} \\ \text{Newton slopes of } (M_1,F_1) = \{\text{first } A \text{ Newton slopes of } (M,F)\} \end{cases}$$

$$1.6.2 \qquad \begin{cases} \text{rank } (M_2) = r - A \\ \text{Hodge slopes of } (M_2,F_2) = \{\text{last } r-A \text{ Hodge slopes of } (M,F)\} \\ \text{Newton slopes of } (M_2,F_2) = \{\text{last } r-A \text{ Newton slopes of } (M,F)\} \end{cases}$$

In terms of polygons, this means that the Hodge (resp., Newton) polygon of (M,F) is formed by joining end-to-end the Hodge (resp. Newton) polygon of (M_1,F_1) with the translate by (A,b) of the Hodge (resp. Newton) polygon of (M_2,F_2). Pictorially, we have



We give the proof in a series of lemmas.

Lemma 1.6.3

Hypotheses as in Theorem 1.6.1, there exists a unique F-stable W(k)-submodule $M_1 \subset M$ *such that*

$$\begin{cases} M_1 \text{ is free of rank } A\text{, and } M/M_1 \text{ is free of rank } r-A\,. \\ \text{if we put } F_1 = F/M_1\text{, then the Newton slopes of } (M_1,F_1) \\ \text{are the first } A \text{ Newton slopes of } (M,F)\,. \end{cases}$$

Lemma 1.6.4

Hypotheses as in Theorem 1.6.1, and notations as in Lemma 1.6.1 above, put $M_2 = M/M_1$, $F_2 = F/M_2$. *Then we have a short exact sequence of* σ^a*-F-crystals*

$$0 \longrightarrow (M_1,F_1) \longrightarrow (M,F) \longrightarrow (M_2,F_2) \longrightarrow 0$$

in which (M_1,F_1) *and* (M_2,F_2) *satisfy the properties* 1.6.1 *and* 1.6.2 *of the conclusion of Theorem 1.6.1.*

Lemma 1.6.5

The exact sequence of σ^a*-F-crystals in Lemma 1.6.5 above admits a unique splitting*

$$(M,F) \simeq (M_1,F_1) \oplus (M_2,F_2)\ .$$

Proof of Lemma 1.6.3. We first use "Plücker coordinates" to reduce to the case when $A = 1$. The hypothesis that the Newton polygon of (M,F) has a break-point at (A,b) , and that the Hodge polygon of (M,F) goes through (A,b), is *equivalent* to the hypothesis that the Newton polygon of $(\Lambda^A M,\Lambda^A(F))$ has a break-point at $(1,b)$, and that the Hodge polygon of $(\Lambda^A M,\Lambda^A(F))$ goes through $(1,b)$. Admitting temporarily the truth of Lemma 1 for $(\Lambda^A M,\Lambda^A(F))$ and the point $(1,b)$, we obtain a *unique* $\Lambda^A(F)$-stable line L in $\Lambda^A M$, such that $(L,\Lambda^A(F))$ has Newton slope $= b$. Therefore if there exists $M_1 \subset M$ of the sort required in

Lemma 1, we must have $L = \Lambda^A(M_1)$, by <u>uniqueness</u> of L. Conversely, if we can show that L is of the form $\Lambda^A(M_1)$ for some $W(k)$-submodule $M_1 \subset M$ with M/M_1 locally free, then M_1 is uniquely determined by L (Plücker embedding!), M_1 is necessarily F-stable (since L is), and its Newton slopes are necessarily the first a Newton slopes of (M,F) (otherwise the Newton slope of $(L,\Lambda^A(F))$ would be too big).

To verify that L is of this form, it suffices to verify that L satisfies the Plücker equations, and for this we may first make any injective extension of scalars, e.g., from $W(k)$ to the fraction field K of a suitable ring $R = W(k')[p^{1/N}]$ of the sort considered in ~~1.2~~. But over such a K, $M\otimes K$ admits a K-basis $e_1,\ldots,e_r$ with respect to which the matrix of F is

1.3
page 130

$$\begin{pmatrix} p^{\lambda_1} & & \bigcirc \\ & \ddots & \\ \bigcirc & & p^{\lambda_r} \end{pmatrix}$$

where $\lambda_1 \leq \ldots \leq \lambda_A < \lambda_{A+1} \leq \ldots \leq \lambda_r$ are the Newton slopes of (M,F). Inside $\Lambda^A(M)\otimes K$, it is now visible that there is a unique F-stable line of Newton slope $b = \lambda_1 + \ldots + \lambda_A$, namely the K-span of $e_1\wedge\ldots\wedge e_A$. But $L\otimes K$ is also such a line, so by uniqueness, we have $L\otimes K = Ke_1\wedge\ldots\wedge e_A$, whence $L\otimes K$ satisfies the Plücker equations.

It remains to treat the case $(A,b) = (1,b)$. In this case, as the Hodge polygon goes through $(1,b)$, the endomorphism F of M is <u>divisible</u> by p^b. Dividing F by p^b, we are reduced to the case $(A,b) = (1,0)$; i.e., the case in which zero occurs as a Newton slope of (M,F) with multiplicity <u>one</u>. We must find an F-stable line $L \subset M$ on which F induces an automorphism, and show that any such line is unique. For this, it suffices to show that for every integer $n \geq 1$, there is a unique F-stable line L_n in M/p^nM on which F induces an automorphism. Because F has <u>zero</u> as a Newton slope with multiplicity <u>one</u>, all Newton slopes of $\Lambda^2(F)$ are strictly positive, and hence we have

$$\Lambda^2(F^\nu) \equiv 0 \mod p \quad \text{if} \quad \nu \geq \operatorname{rank}(\Lambda^2 M) = \binom{r}{2}$$

i.e.,
$$\Lambda^2(F^\nu) \equiv 0 \mod p^n \quad \text{if} \quad \nu \geq n\binom{r}{2} .$$

But all iterates F^ν of F also have zero as a Newton slope, and hence all iterates F^ν of F have zero as a Hodge slope, i.e.,

$$F^\nu \not\equiv 0 \mod p \quad \text{for} \quad \nu = 1,2,3,\ldots .$$

For any $\nu \geq n\binom{r}{2}$, we thus have $F^\nu \not\equiv 0 \mod p$, but all 2×2 minors of F^ν are $\equiv 0 \mod p^n$. This means exactly that for each $\nu \geq n\binom{r}{2}$, the image of

$$F^\nu : M/p^nM \longrightarrow M/p^nM$$

is a line $L_{n,\nu} \subset M/p^nM$ (in matricial terms, at least one of the columns of the matrix of F^ν is not divisible by p, and all the other columns are congruent mod p^n to $W(k)$-multiples of this column). By the definition of the $L_{n,\nu}$ as images, we have

$$F(L_{n,\nu}) = L_{n,\nu+1} \subset L_{n,\nu} \quad \text{for all} \quad \nu .$$

Since the $L_{n,\nu}$ for $\nu \geq n\binom{r}{2}$ are lines, we must have

$$F(L_{n,\nu}) = L_{n,\nu+1} = L_{n,\nu} \quad \text{for} \quad \nu \geq n\binom{r}{2} .$$

Therefore if we define L_n to be $L_{n,\nu}$ for any $\nu \geq n\binom{r}{2}$, L_n is an F-stable line on M/p^nM on which F induces an automorphism. That L_n is the unique such line results from the fact that

$$L_n = \bigcap_\nu F^\nu(M/p^nM) ,$$

so that L_n must contain any such line, and hence be equal to any such line. This concludes the proof of Lemma 1.6.3 QED .

We now turn to the proof of Lemma 1.6.4. By construction, (M_1,F_1) has as Newton slopes the first A Newton slopes of (M,F), and therefore by (1.3.4)

(M_2,F_2) must have as its Newton slopes the last $r - A$ Newton slopes of (M,F). In terms of a basis of M adapted to the filtration $M_1 \subset M$, the matrix of F looks like

$$\begin{pmatrix} \mathbb{A} & \mathbb{B} \\ 0 & \mathbb{D} \end{pmatrix}$$

with $\mathbb{A}$ the $A \times A$ matrix of F_1 on M_1, and $\mathbb{D}$ the $(r-A) \times (r-A)$ matrix of F_2 on M_2.

Let us begin by showing that this matrix has the same Hodge polygon as does

$$\begin{pmatrix} \mathbb{A} & 0 \\ 0 & \mathbb{D} \end{pmatrix} .$$

For this, it suffices to show that "elementary column operations" allow us to pass from one to the other, i.e., to show that all the columns of $\mathbb{B}$ are $W(k)$-linear combinations of the columns of $\mathbb{A}$. By hypothesis, the Hodge polygon of (M,F) goes through (A,b), and hence

$$\text{all } A \times A \text{ minors of } \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ 0 & \mathbb{D} \end{pmatrix} \text{ are } \equiv 0 \bmod p^b ,$$
$$\text{in particular all } A \times A \text{ minors of } (\mathbb{A},\mathbb{B}) \text{ are } \equiv 0 \bmod p^b .$$

Because the Newton polygon of (M,F) goes through (A,b) and because (M_1,F_1) has as Newton slopes the first a Newton slopes of (M,F), we have

$$\det(\mathbb{A}) = p^b \times \text{unit}.$$

It now follows by Cramer's rule that all columns of $\mathbb{B}$ are $W(k)$-linear combinations of the columns of A. Hence the matrices

$$\begin{pmatrix} \mathbb{A} & \mathbb{B} \\ 0 & \mathbb{D} \end{pmatrix} , \quad \begin{pmatrix} \mathbb{A} & 0 \\ 0 & \mathbb{D} \end{pmatrix}$$

have a common Hodge polygon.

Intrinsically, this means that (M,F) and $(M_1 \oplus M_2, F_1 \oplus F_2)$ have a common Hodge polygon. Therefore we have a partition

$$\{\text{Hodge slopes of } (M,F)\} = \bigcup_{i=1,2} \{\text{Hodge slopes of } (M_i,F_i)\} .$$

So we need only verify that the Hodge slopes of (M_1,F_1) are the A smallest among those of (M,F). But the <u>sum</u> of the smallest A Hodge slopes of (M,F) is b (the Hodge polygon of (M,F) goes through (A,b)). So it suffices to see that the sum of all A of the Hodge slopes of (M_1,F_1) is b. For this just recall that the Newton polygon of (M_1,F_1) <u>ends</u> at (A,b), and hence its Hodge polygon ends there as well. <u>QED</u>

We now turn to the <u>proof of Lemma 1.6.5</u>. In a basis of M adopted to $M_1 \subset M$, the matrix of F is

$$\begin{pmatrix} \mathbb{A} & \mathbb{B} \\ 0 & \mathbb{D} \end{pmatrix} .$$

As we have seen in the proof of Lemma 1.6.4, the columns of $\mathbb{B}$ are all $W(k)$-linear combinations of those of $\mathbb{A}$, so we can write this matrix

$$\begin{pmatrix} \mathbb{A} & \mathbb{A}\mathbb{C} \\ 0 & \mathbb{D} \end{pmatrix}$$

for some integral matrix $\mathbb{C}$.

Let n denote the largest Hodge slope of $\mathbb{A}$, and let m denote the smallest Hodge slope of $\mathbb{D}$. Then $p^n\mathbb{A}^{-1}$ and $p^{-m}\mathbb{D}$ are integral. Since $m \geq n$ by Lemma 1.6.4, $p^{-n}\mathbb{D}$ is integral. Notice that either $p^n\mathbb{A}^{-1}$ is topologically nilpotent (i.e., that the σ^{-a}-F-crystal $(M_1,p^nF_1^{-1})$ has all Newton slopes > 0), or that $p^{-n}\mathbb{D}$ is topologically nilpotent (i.e., $(M_2,p^{-n}F_2)$ has all Newton slopes > 0), or possibly both; this is immediate from the inequalities

$$\begin{cases} \lambda_1 \leq \cdots \leq \lambda_A \leq n \leq m \leq \lambda_{A+1} \leq \cdots \leq \lambda_r \\ \lambda_A < \lambda_{A+1} \end{cases}$$

To split the projection of (M,F) onto (M_2,F_2) is equivalent to finding an $A \times (r-A)$ matrix X, with entries in $W(k)$, so that $\binom{X}{1}$, viewed as the matrix of a $W(k)$-linear cross-section $M_2 \longrightarrow M$ of the projection $M \longrightarrow M_2$, is a morphism of σ^a-F-crystals. Matricially, this means

$$\begin{pmatrix} X \\ 1 \end{pmatrix} \mathbb{D} = \begin{pmatrix} \mathbb{A} & \mathbb{A}\mathbb{C} \\ 0 & \mathbb{D} \end{pmatrix} \begin{pmatrix} X^{\sigma^a} \\ 1 \end{pmatrix}$$

i.e.,

$$X\mathbb{D} = \mathbb{A}X^{\sigma^a} + \mathbb{A}\mathbb{C}$$

i.e.,

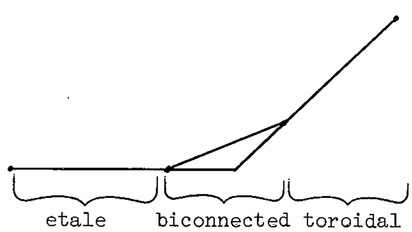
$$p^n\mathbb{A}^{-1}\, X\, p^{-n}\mathbb{D} = X^{\sigma^a} + \mathbb{C}$$

i.e.,

$$X = (p^n\mathbb{A}^{-1}Xp^{-n}\mathbb{D})^{\sigma^{-a}} - \mathbb{C}^{\sigma^{-a}} .$$

Because either $p^n\mathbb{A}^{-1}$ or $p^{-n}\mathbb{D}$ is topologically nilpotent, the method of successive iterations leads to a unique solution of this equation.

Remarks. If we apply this Newton-Hodge decomposition to the contravariant Dieudonné module of a p-divisible group, we recover the cannonical decomposition of such a group over a perfect field into the product of an etale group, a biconnected group, and a toroidal group.



II. F-crystals over $\mathbb{F}_p$-algebras

(2.1) Basic definitions

In this section we recall the basic notions concerning crystals on arbitrary affine schemes in characteristic $p > 0$ (compare [2]). By an absolute test object, we mean a triple (B,I,γ) consisting of a p-adically complete and separated $\mathbb{Z}_p$-algebra B, a closed ideal $I \subset B$ with $p \in I$, and a divided power structure $\gamma = \{\gamma_n\}$ on the ideal I for which $\gamma_n(p)$ = the image, in B, of $p^n/n!$ in $\mathbb{Z}_p$. Given an $\mathbb{F}_p$-algebra A_o, by an A_o-test object we mean a quadruple $(B,I,\gamma;s)$ consisting of an absolute test object (B,I,γ) together with a structure s of A_o-algebra in B/I, i.e., together with a homomorphism of $\mathbb{F}_p$-algebras $s : A_o \longrightarrow B/I$. A map of A_o-test objects $f : (B,I,\gamma;s) \longrightarrow (B',I',\gamma',s')$ is an algebra homomorphism $f : B \longrightarrow C$ which maps I to I', "commutes" with the given divided power structures γ,γ', and induces an A_o-homomorphism $B/I \longrightarrow B'/I'$ (for the given structures s,s').

A <u>crystal</u> M on A_o is rule which assigns to every A_o-test object $(B,I,\gamma;s)$ a p-adically complete and separated B-module, noted $M(B,I,\gamma;s)$, and which assigns to every map $f : (B,I,\gamma;s) \longrightarrow (B',I',\gamma';s')$ of A_o-test objects a B'-isomorphism

$$M(B,I,\gamma;s) \mathop{\hat{\otimes}}_{B} B' \xrightarrow[\sim]{M(f)} M(B',I',\gamma';s')$$

in a way compatible with composition of maps of test objects. A crystal M is said to be locally free of rank r if for all A_o-test objects $(B,I,\gamma;s)$, the B-module $M(B,I,\gamma;s)$ is a locally free B-module of rank r. A <u>morphism</u> of crystals on A_o, $u : M \longrightarrow N$, is a rule which assigns to each A_o-test object $(B,I,\gamma;s)$ a B-module map

$$u(B,I,\gamma;s) : M(B,I,\gamma;s) \longrightarrow N(B,I,\gamma;s)$$

in a way compatible with the isomorphisms $M(f),N(f)$. The category of crystals on A_o up to isogeny is obtained from the category of crystals on A_o by keeping the same objects, but tensoring the Hom groups, which are $\mathbb{Z}_p$-modules, over $\mathbb{Z}_p$ with $\mathbb{Q}_p$. An <u>isogeny</u> between crystals on A_o is a morphism of crystals on A_o which becomes an isomorphism in this new category (explicitly, $u : N \longrightarrow M$ is an isogeny if and only if for some integer $n \geq 0$, there exists $v : M \longrightarrow N$ with $uv = p^n = vu$).

Suppose we are given two $\mathbb{F}_p$-algebras, A_o and B_o, and a homomorphism $\phi : A_o \longrightarrow B_o$. If $(B,I,\gamma;s)$ is a B_o-test object, then $(B,I,\gamma;s\cdot\phi)$ is an A_o-test object. Given a crystal M on A_o, the "inverse image" crystal $M^{(\phi)}$ on B_o is defined by the formula

$$M^{(\phi)}(B,I,\gamma;s) = M(B,I,\gamma;s\phi) .$$

Similarly, given a morphism $u : M \longrightarrow N$ of crystals on A_o, its "inverse image" $u^{(\phi)} : M^{(\phi)} \longrightarrow N^{(\phi)}$ is defined by $u^{(\phi)}(B,I,\gamma;s) = u(B,I,\gamma;s\phi)$.

For any $\mathbb{F}_p$-algebra A_o, we denote by $\sigma : A_o \longrightarrow A_o$ the absolute Frobenius endomorphism $\sigma(x) = x^p$, and by σ^a, $a \geq 1$, its a^{th} iterate. By a σ^a-F-crystal (M,F) on A_o, we mean a locally free (of some rank r) crystal M on A_o together with an <u>isogeny</u> $F:M^{(\sigma^a)} \longrightarrow M$. A morphism of σ^a-F-crystals on A_o, $f:(M,F) \longrightarrow (M',F')$, is a morphism $f:M \longrightarrow M'$ of crystals on A_o such that $F'\cdot f^{(\sigma^a)} = f \cdot F$. The category of σ^a-F-crystals up to isogeny, and the notion of an isogeny between σ^a-F-crystals, are defined in the expected way.

Given a σ^a-F-crystal (M,F) on A_o, and any homomorphism of $\mathbb{F}_p$-algebras $\phi:A_o \longrightarrow B_o$, the inverse image $(M^{(\phi)},F^{(\phi)})$ is a σ^a-F-crystal on B_o (because $\sigma^a \cdot \phi = \phi \cdot \sigma^a$ for any homomorphism ϕ of $\mathbb{F}_p$-algebras).

(2.2) <u>Perfect rings</u>.

When A_o is a perfect $\mathbb{F}_p$-algebra, i.e., when $\sigma:A_o \longrightarrow A_o$ is an automorphism, the ring $W(A_o)$ of Witt vectors of A_o provides an initial object in the category of all A_o-test objects, namely $(W(A_o), (p), \gamma ; s)$. The divided power structure γ on $pW(A_o)$ is uniquely determined by the requirement $\gamma_n(p) = p^n/n!$;

the homomorphism $s:A_o \longrightarrow W(A_o)/(p)$ is the inverse of the isomorphism $W(A_o)/(p) \xrightarrow{\sim} A_o$ obtained by sending a Witt vector to its first component.

<u>Evaluation</u> at this initial object provides an equivalence of categories between the category of crystals on A_o and the category of p-adically complete and separated $W(A_o)$-modules. Given a homomorphism $\phi:A_o \longrightarrow B_o$ of perfect $\mathbb{F}_p$-algebras, the construction $M \longmapsto M^{(\phi)}$ on crystals corresponds to the construction on modules

$$M \longmapsto M \mathop{\hat{\otimes}}_{W(A_o)} W(B_o) \overset{\text{dfn}}{=\!=} \text{the } W(B_o)\text{-module } M^{(\phi)}$$

in which $W(B_o)$ is viewed as a $W(A_o)$-algebra by means of $W(\phi):W(A_o) \longrightarrow W(B_o)$.

If we denote by σ the automorphism $W(\sigma):W(A_o) \xrightarrow{\sim} W(A_o)$, then the category of σ^a-F-crystals on our perfect A_o is equivalent to the category of pairs (M,F) consisting of a locally free (of some rank r) $W(A_o)$-module M together with a σ^a-linear map $F:M \longrightarrow M$ which induces an automorphism of $M\otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

In particular, when A_o is a perfect field k , we recover the more mundane notion of σ^a-F-crystal with which we were concerned in Chapter 1.

(2.3) <u>Grothendieck's specialization theorem</u>.

We now turn to the consideration of a σ^a-F-crystal (M,F) over an arbitrary $\mathbb{F}_p$-algebra A_o . For any homomorphism $\phi:A_o \longrightarrow k$ with k a perfect field, $(M,F)^{(\phi)}$ is a σ^a-F-crystal over k . Its Newton and Hodge polygons depend only on the underlying point $\ker(\phi) \in \mathrm{Spec}(A_o)$, and not on the particular choice of a perfect overfield of the residue field at this point. This allows us to speak of the Newton and Hodge polygons and slopes of (M,F) at the various points of $\mathrm{Spec}(A_o)$. The following theorem and corollary are a slight strengthening of Grothendieck's specialization theorem.

Theorem 2.3.1 (Grothendieck)

Let (M,F) be a σ^a-F-crystal of rank r over an arbitrary $\mathbb{F}_p$-algebra A_o. Let $\lambda \geq 0$ be a real number. The set of points in $\mathrm{Spec}(A_o)$ at which all Hodge (resp. Newton) slopes of (M,F) are $\geq \lambda$ is Zariski closed, and locally on $\mathrm{Spec}(A_o)$ it is the zero-set of a finitely generated ideal.

Corollary 2.3.2

Let P be the graph of any continuous $\mathbb{R}$-valued function on $[0,r]$ which is linear between successive integers. The set of points in $\mathrm{Spec}(A_o)$ at which the Hodge (resp. Newton) polygon of (M,F) lies above P is Zariski closed, and is locally on $\mathrm{Spec}(A_o)$ the zero-set of a finitely generated ideal.

Proof. The Corollary follows by applying the theorem to the various exterior powers of (M,F). The theorem for Newton slopes follows from the theorem for Hodge slopes, applied to a suitable iterate (M,F^n) of (M,F), as follows. Because Hodge slopes are always integers, and Newton slopes are always in $\frac{1}{r!}\mathbb{Z}$ for σ^a-F-crystals of rank r, we may assume that λ lies in $\frac{1}{r!}\mathbb{Z}$. According to the basic slope estimate, we have

$$\begin{array}{ccc} F \text{ has all Newton slopes} \geq \lambda & \Longrightarrow & F^{n+r-1} \text{ has Hodge slopes} \geq n\lambda \\ & & \Downarrow \\ & & F \text{ has all Newton slopes} \geq \dfrac{n\lambda}{n+r-1} \end{array} .$$

Therefore, if we choose n so large that

$$\lambda - \frac{1}{r!} < \frac{n}{n+r-1} \cdot \lambda < \lambda ,$$

then we have

$$F \text{ has all Newton slopes} \geq \lambda \Longleftrightarrow F^{n+r-1} \text{ has all Hodge slopes} \geq n\lambda .$$

So we are reduced to proving the theorem for Hodge slopes.

As replacing A_o by its perfection A_o^{perf} and (M,F) by its inverse image on A_o^{perf} alters neither $\mathrm{Spec}(A_o)$ nor the perfect-field-fibres $(M,F)^{(\phi)}$ of

(M,F) at the points of $\mathrm{Spec}(A_o)$, we may assume that A_o is perfect. As the theorem is local on $\mathrm{Spec}(A_o)$, we may further assume that (M,F) is a free $W(A_o)$-module of rank r with a σ^a-linear endomorphism F. Because Hodge slopes are integers, we may also assume that λ is an integer.

In terms of a basis of M, F is now given by an $r \times r$ matrix $(F_{i,j})$ with entries in $W(A_o)$. For any homomorphism $\phi: A_o \longrightarrow k$ with k a perfect field, $(M,F)^{(\phi)}$ is given by the $r \times r$ matrix $(W(\phi)(F_{i,j}))$ obtained by applying ϕ component-wise to the $F_{i,j}$, individually thought of as Witt vectors. Now $(M,F)^{(\phi)}$ has all Hodge slopes $\geq \lambda$ if and only if all the $W(\phi)(F_{i,j})$ lie in $p^\lambda W(k)$, i.e., if and only if the first λ components of each of the r^2 Witt-vectors $W(\phi)(F_{i,j})$ all vanish, i.e., if and only if ϕ annihilates the ideal in A_o generated by the first λ Witt-vector components of each of the r^2 matrix coefficients $F_{i,j} \in W(A_o)$.

A question. Is there a natural structure of closed subscheme on these Zariski subsets of $\mathrm{Spec}(A_o)$ defined by "slopes $\geq \lambda$"? Given a σ^a-F-crystal over $\mathbb{F}_p[\varepsilon]/(\varepsilon^2)$, does it make sense to ask if its Newton or Hodge slopes are "everywhere" $\geq \lambda$?

(2.4) Newton-Hodge filtration.

In this section we will consider the case in which A_o is an $\mathbb{F}_p$-algebra of one of the following two kinds:

$$\begin{cases} A_o \text{ is smooth over a perfect subring } A_{oo} \text{ of } A_o . \\ A_o \text{ is a formal power series ring in finitely many variables} \\ \qquad \text{over a perfect subring } A_{oo} \text{ of } A_o . \end{cases}$$

In both cases, there exists a p-adically complete and separated $\mathbb{Z}_p$-algebra A_∞ which is flat over $\mathbb{Z}_p$, together with an isomorphism $A_\infty/pA_\infty \xrightarrow{\sim} A_o$, such that for each $n \geq 1$, $A_n \overset{\text{dfn}}{=\!=} A_\infty/p^{n+1}A_\infty$ is formally smooth over $\mathbb{Z}/p^{n+1}\mathbb{Z}$. The algebra A_∞ is naturally a $W(A_{oo})$-algebra; it is unique up to automorphisms which are the identity on $W(A_{oo})$ and which reduce mod p to the identity. The absolute Frobenius map $\sigma : A_o \longrightarrow A_o$ may be lifted, non-uniquely in general, to a ring homomorphism $\Sigma : A_\infty \longrightarrow A_\infty$ which is necessarily σ-linear over $W(A_{oo})$.

The algebra A_∞ provides an A_o-test object, namely $(A_\infty, (p), \gamma; s)$, in which s is the inverse of the given $A_\infty/pA_\infty \xrightarrow{\sim} A_o$. This A_o-test object is "pseudo-initial" in the sense that any A_o-test object receives a map from it, but this map need not be unique. Evaluation at this "pseudo-initial" object provides an equivalence of categories between the category of crystals on A_o and the category of pairs (M, ∇) consisting of a p-adically complete and separated A_∞-module M together with an integrable, nilpotent $W(A_{oo})$-connection.

If we fix a lifting $\Sigma : A_\infty \longrightarrow A_\infty$ of σ, we similarly obtain an equivalence of categories between the category of σ^a-F-crystals on A_o and the category of triples (M, ∇, F_Σ) consisting of a locally free (of some rank r) A_∞-module M together with an integrable, nilpotent $W(A_{oo})$-connection ∇ and a horizontal morphism $F_\Sigma : (M^{(\Sigma^a)}, \nabla^{(\Sigma^a)}) \longrightarrow (M, \nabla)$ which induces an isomorphism after tensoring $M^{(\Sigma)^a}$ and M over $\mathbb{Z}_p$ with $\mathbb{Q}_p$.

Let us denote by A_o^{perf} the perfection of A_o. The method of successive iterations allows us to construct for each choice of Σ, a unique homomorphism $i(\Sigma) : A_\infty \longrightarrow W(A_o^{perf})$ which reduces mod p to the inclusion $A_o \hookrightarrow A_o^{perf}$, and which sits in a commutative diagram

$$\begin{array}{ccc} A_\infty & \xhookrightarrow{i(\Sigma)} & W(A_o^{perf}) \\ \downarrow{\scriptstyle \Sigma} & & \wr\downarrow{\scriptstyle W(\sigma)} \\ A_\infty & \xhookrightarrow{i(\Sigma)} & W(A_o^{perf}) \end{array} .$$

This homomorphism $i(\Sigma)$ should be thought of as the universal "Σ-Teichmuller point" of A_∞. In fact, $i(\Sigma)$ provides a construction of $W(A_o^{perf})$ as the p-adic completion of the "Σ-perfection" $\varinjlim A_\infty$ (in which the successive transition maps $A_\infty \longrightarrow A_\infty$ are all Σ) of A_∞. Notice that

(2.4.1) $$pA_\infty = i(\Sigma)\,(A_\infty) \cap pW(A_o^{perf})$$

simply because $A_\infty/pA_\infty \simeq A_o \subset A_o^{perf} \simeq W(A_o^{perf})\,/\,pW(A_o^{perf})$.

Given a σ^a-F-crystal on A_o , thought of as (M,∇,F_Σ) , its inverse image on A_o^{perf} is the pair $(M,F_\Sigma)^{(i(\Sigma))}$ obtained from (M,F_Σ) on A_∞ by the extension of scalars $i(\Sigma) : A_\infty \longrightarrow W(A_o^{perf})$.

Theorem 2.4.2 (Newton-Hodge filtration)

Let (M,∇,F) be a σ^a-F-crystal over an $\mathbb{F}_p$-algebra A_o of the type discussed above in 2.4. Suppose that $(A,b) \in \mathbb{Z}\times\mathbb{Z}$ is a break point of the Newton polygon of (M,∇,F_Σ) at every point of $\mathrm{Spec}(A_o)$, and that (A,b) lies on the Hodge polygon of (M,∇,F_Σ) at every point of $\mathrm{Spec}(A_o)$. Then there exists a unique F_Σ-stable horizontal A_∞-submodule $M_1 \subset M$, with M_1 locally free of rank a , and $M_2 \overset{dfn}{=} M/M_1$ locally free of rank $r - A$, such that

> *at every point of $\mathrm{Spec}(A_o)$, the Hodge (resp. Newton) slopes of $(M_1,\ \nabla|M_1,\ F_\Sigma|M_1)$ are the A smallest of the Hodge (resp. Newton) slopes of (M,∇,F_Σ) ,*
>
> *at every point of $\mathrm{Spec}(A_o)$, the Hodge (resp. Newton) slopes of $(M_2,\ \nabla|M_2,\ F_\Sigma|M_2)$ are the $r - A$ greatest Hodge (resp. Newton) slopes of (M,∇,F_Σ) .*

Furthermore, when A_o is itself perfect, the exact sequence of σ^a-F-crystals

$$0 \longrightarrow (M_1,\ \nabla|M_1,\ F_\Sigma|M_1) \longrightarrow (M,\nabla,F_\Sigma) \longrightarrow (M_2,\ \nabla|M_2,\ F_\Sigma|M_2) \longrightarrow 0$$

admits a unique splitting.

Proof. Localizing on $\mathrm{Spec}(A_o)$, we may suppose that M is a free A_∞-module of rank r . Consider first the case $(A,b) = (1,b)$. Then the least Hodge slope is b at every point. This means that each matrix coefficient $F_{i,j}$ in A_∞ has $i(\Sigma)(F_{i,j}) \in W(A_o^{perf})$ with its first b Witt-vector components nilpotent, and hence <u>zero</u>, in A_o^{perf}. Therefore $i(\Sigma)(F_{ij})$ lies in $p^b W(A_o^{perf})$, and so F_{ij} lies in $p^b A_\infty$. So dividing F by p^b, we may assume $b = 0$. This means that the matrix coefficients F_{ij} generate the unit ideal in A_∞ (because after extending scalars to $W(A_o^{perf})$, their first Witt-vector components generate the unit ideal in A_o^{perf}; as these first components are just the F_{ij} mod p, in A_o, the F_{ij} mod p generate the unit ideal in A_o and hence the F_{ij} generate the unit ideal in A_∞). For every iterate F^ν of F, its matrix coefficients still generate the unit ideal. But for $\nu \geq n\binom{r}{2}$, all Hodge slopes of $\Lambda^2(F^\nu)$ are $\geq n$, at each point of $\mathrm{Spec}(A_o)$, so that all 2×2 minors of F^ν lie in $p^n A_\infty$ (by the same Witt-vector argument in $W(A_o^{perf})$). So we can construct the required line $L \subset M$ as the "limit" of the images mod p^n of F^ν, for $\nu \geq n\binom{r}{2}$, just as we did in the case of a perfect field. This construction via images of iterates of F makes obvious that L is F-stable and horizontal (since F itself is horizontal). The slope assertions about F on L and on M/L are pointwise, so are already proven.

We do the general case (A,b) by constructing the required line L in $\Lambda^A(M)$. It remains only to see that this line is of the form $\Lambda^A(M_1)$ for some locally free $M_1 \subset M$ of rank A with M/M_1 locally free. [The F-stability and horizontality of M_1 then are consequences of the F-stability and horizontality of the line; the slope assertions about M_1 and M/M_1 are pointwise, so are already proven.] To see that L satisfies the Plücker equations, it suffices to do so after an <u>arbitrary</u> injective extension of scalars. For this purpose we first embed A_∞ in $W(A_o^{perf})$. Then, because A_o^{perf} is reduced we can embed $W(A_o^{perf})$ in the product, indexed by all homomorphisms $\phi : A_o^{perf} \longrightarrow k$ with k a perfect field, of the $W(k)$'s. This reduces us to the case A_o = a perfect field, in which case we have already proven it.

As for the splitting in the case of a perfect A_o, the proof is word-for-word the same as in the case of a perfect field.

Remarks

Let (M,∇,F) be a σ^a-F-crystal over an $\mathbb{F}_p$-algebra A_o of the type discussed above in 2.4. Suppose that at every point of $\operatorname{Spec}(A_o)$, the Newton and Hodge polygons coincide with each other, and that they are constant, i.e., independent of the point. Let us denote by $h^0,h^1,\ldots,$ the Hodge numbers. Then the associated graded pieces of the Newton-Hodge filtration are σ^a-F-crystals (M_i,∇_i,F_i) of rank h^i, such that $F_i = p^i\tilde{F}_i$ with $(M_i,\nabla_i,\tilde{F}_i)$ a "unit-root" (all Newton slopes = 0) σ^a-F-crystal. But a unit-root σ^a-F-crystal of rank h^i is equivalent (cf [7]) to a continuous representation of the fundamental group of $\operatorname{Spec}(A_o)$ in $GL(h^i,W(\mathbb{F}_{p^a}))$. It would be interesting to understand the "meaning" of these p-adic representations, especially when the σ^a-F-crystals in question arise as crystalline cohomology groups of families of varieties.

(2.5) Splitting Theorems.

In this section, we give a splitting theorem up to isogeny for slope filtration of σ^a-F-crystals over perfect rings.

Theorem 2.5.1

Let A_o be a perfect ring, and let

$$0 \longrightarrow (M_1,F_1) \longrightarrow (M,F) \longrightarrow (M_2,F_2) \longrightarrow 0$$

be an exact sequence of σ^a-F-crystals over A_o. Suppose that for some rational number λ, the Newton slopes of (M_1,F_1) at every point of $\operatorname{Spec}(A_o)$ are all $\leq \lambda$, while the Newton slopes of (M_2,F_2) at every point of $\operatorname{Spec}(A_o)$ are all $> \lambda$. Then in the category of σ^a-F-crystals up to isogeny, this exact sequence splits uniquely.

Proof. Localizing on A_o, we may assume that M, M_1, M_2 are free $W(A_o)$-modules of ranks r, r_1, r_2 respectively. Because the Newton slopes of (M,F) at any point of $\operatorname{Spec}(A_o)$ lie in the discrete set $\frac{1}{r!}\mathbb{Z}$, we may in fact choose rational numbers $\lambda_1 < \lambda_2$ such that at all points of $\operatorname{Spec}(A_o)$, (M_1,F_1) has all Newton slopes $\leq \lambda_1$, while (M_2,F_2) has all Newton slopes $\geq \lambda_2$. In terms of a basis of M adopted tò the filtration, the matrix of F has the shape

$$\begin{pmatrix} \mathbb{A} & \mathbb{C} \\ 0 & \mathbb{D} \end{pmatrix} .$$

A splitting of the exact sequence is a morphism $(M_2,F_2) \longrightarrow (M,F)$ which is a cross-section of the projection. In terms of the given bases, the <u>matrix</u> of a splitting is an $r \times r_2$ matrix of the form

$$\begin{pmatrix} X \\ 1 \end{pmatrix}$$

where X is $r_1 \times r_2$, 1 denotes the $r_2 \times r_2$ identity matrix, and where X satisfies the matrix equation

$$\mathbb{A}X^{\sigma^a} + \mathbb{C} = X\mathbb{D}$$

i.e.,

$$X^{\sigma^a} = \mathbb{A}^{-1}X\mathbb{D} - \mathbb{A}^{-1}\mathbb{C}$$

i.e.,

$$X - (\mathbb{A}^{-1}X\mathbb{D})^{\sigma^{-a}} = -(\mathbb{A}^{-1}\mathbb{C})^{\sigma^{-a}} .$$

We must show that this matrix equation has a unique solution matrix X with entries in $W(A_o) \otimes \mathbb{Q}_p$. Let us denote by Mat the space of all $r_1 \times r_2$ matrices with entries in $W(A_o) \otimes \mathbb{Q}_p$, with the linear topology defined by the entry-by-entry congruence modulo $p^nW(A_o)$. Consider the σ^{-a}-linear endomorphism V of Mat defined by

$$X \longmapsto V(X) \overset{\text{dfn}}{=\!=} (\mathbb{A}^{-1}X\mathbb{D})^{\sigma^{-a}}$$

Suppose we can prove that V is topologically nilpotent. Then our matrix equation

$$X - V(X) = -(\mathbb{A}^{-1}\mathbb{C})^{\sigma^{-a}}$$

obviously has the unique solution

$$X = \sum_{n\geq 0} V^n(-(\mathbb{A}^{-1}\mathbb{C})^{\sigma^{-a}}) ,$$

and we are done.

We will deduce the topological nilpotence of V from the basic slope estimate, applied to $\mathbb{A}^{-1}$ and to $\mathbb{D}$. Because (M_2,F_2) has all slopes $\geq \lambda_2$, we have

$$(F_2)^n \equiv 0 \bmod p^{\{(n+1-r_2)\lambda_2\}} .$$

Because (M_1,F_1) has all Newton slopes $\leq \lambda_1$, its <u>determinant</u> has its single Newton=Hodge slope $\leq r_1\lambda_1 \leq \{r_1\lambda_1\}$. So putting $N = \{r_1\lambda_1\}$, the σ^{-a}-linear endomorphism $p^N(F_1)^{-1}$ of $M_1 \otimes \mathbb{Q}_p$ actually maps M_1 to M_1. Hence $(M_1,p^N(F_1)^{-1})$ defines a σ^{-a}-F-crystal over A_o, all of whose slopes are $\geq N - \lambda_1$. So by the basic slope estimate, we have

$$p^{Nn}(F_1)^{-n} \equiv 0 \bmod p^{\{(n+1-r_1)(N-\lambda_1)\}} ,$$

and hence we have

$$p^{\{(n+1-r_2)\lambda_2\}}(F_1)^{-n} \equiv 0 \bmod p^{\{(n+1-r)(\lambda_2-\lambda_1)-N(r-1)\}} .$$

In terms of the matrices $\mathbb{A}^{-1},\mathbb{D}$, these estimates may be rewritten

$$\begin{cases} \mathbb{D}\mathbb{D}^{\sigma^a}\mathbb{D}^{\sigma^{2a}} \cdots \mathbb{D}^{\sigma^{(n-1)a}} \equiv 0 \bmod p^{\{(n+1-r_2)\lambda_2\}} \\ p^{\{(n+1-r_2)\lambda_2\}}\mathbb{A}^{-\sigma^{(n-1)a}} \cdots \mathbb{A}^{-\sigma^a}\mathbb{A}^{-1} \equiv 0 \bmod p^{\{(n+1-r)(\lambda_2-\lambda_1)-N(r-1)\}} \end{cases} ,$$

which together give the estimate for the endomorphism V

$$V^n \equiv 0 \bmod p^{\{(n+1-r)(\lambda_2-\lambda_1)-N(r-1)\}}$$

As $\lambda_2 > \lambda_1$, this estimate establishes the required topological nilpotence of V.

(2.6) Isogeny theorems

In this section, we will give an isogeny theorem for σ^a-F-crystals over curves. Thus let A_o be an $\mathbb{F}_p$-algebra which is either

$$\begin{cases} \text{an integral domain, smooth of dimension} \\ \leq 1 \text{ over a perfect field } k \\ \\ \text{a formal power series ring in one vari-} \\ \text{able } k[[T]] \text{ over a perfect field } k . \end{cases}$$

Theorem 2.6.1

Let (M,∇,F) be a σ^a-F-crystal over an $\mathbb{F}_p$-algebra A_o of the above (2.6) sort. Suppose that λ is a positive real number, such that at every point of $\mathrm{Spec}(A_o)$, all Newton slopes of (M,∇,F) are $\geq \lambda$. Then (M,∇,F) is isogenous to a σ^a-F-crystal (M',∇',F') which is divisible by λ in the sense that

$$\text{for all } n \geq 1 , \ (F')^n \equiv 0 \bmod p^{[n\lambda]}$$

(where $[x]$ denotes the integral part of the real number x).

Proof. In the case when A_o is itself a perfect field, the basic slope estimate gives (r denoting the rank of M)

$$F^{n+r-1} \equiv 0 \bmod p^{\{n\lambda\}} \qquad \text{for all } n \geq 1$$

which in turn implies that if we put $\nu = \{(r-1)\lambda\}$, we have

$$p^{\nu}F^n \equiv 0 \bmod p^{[n\lambda]} \qquad \text{for all } n \geq 0 .$$

Therefore we can define a $W(k)$-module M' with

$$M \subset M' \subset p^{-\nu}M$$

by

$$M' = \sum_{n\geq 0} \text{ image of } \left(\frac{F^n}{p^{[n\lambda]}} : M \longrightarrow p^{-\nu}M \right) .$$

The basic inequality $[x+y] \geq [x] + [y]$ allows one to check that for all $n \geq 0$, the operator $F^n/p^{[n\lambda]}$ maps M' to itself. The fact that M' is "caught" between M and $p^{-\nu}M$ guarentees that M' is a free $W(k)$-module of the correct rank. The inclusion of (M,F) into (M',F) is the required isogeny.

In the general case, the basic slope estimate applied pointwise together with the "Witt-vector-component"-argument already used shows that over A_∞ we still have

$$p^\nu F_\Sigma{}^n \equiv 0 \bmod p^{[n\lambda]} \qquad \text{for all } n \geq 0 .$$

So it is natural to define an A_∞-module M' with

$$M \subset M' \subset p^{-\nu} M$$

by

$$M' = \sum_{n \geq 0} \text{image of} \left(\frac{F_\Sigma{}^n}{p^{[n\lambda]}} : M^{(\Sigma^n)} \longrightarrow M \right) .$$

Clearly M' is horizontal (it's defined in terms of the horizontal maps $F_\Sigma{}^n$), and is stable by all the operators $F_\Sigma{}^n/p^{[n\lambda]}$, now viewed as Σ^{an}-linear endomorphisms of M'. The only problem is that I cannot prove (or disprove!) that M' is a <u>locally free</u> A_∞-module. (Even the fact that M' is finitely generated depends on the fact that, in the case envisioned, A_o is noetherian. Can one give an effective bound on the number of terms needed in the apparently infinite sum of images which defines M' ?)

To circumvent this difficulty, we will define a larger A_∞-module M''

$$M \subset M' \subset M'' \subset p^{-\nu} M \subset M_{\mathbb{Z}_p} \otimes \mathbb{Q}_p$$

which will have <u>all</u> the required properties. Let us denote by K_o the <u>fraction field</u> of A_o, and by $C(K_o)$ the completion of the local ring of A_∞ at the prime ideal pA_∞. (The notation $C(K_o)$ is to remind us that this is a Cohen ring for the field K_o, i.e., a mixed characteristic, complete, discrete, absolutely unramified valuation ring with residue field K_o.) By its construction we see that any derivation of the A_∞ into itself, as well as any endomorphism $\Sigma: A_\infty \longrightarrow A_\infty$ lifting the absolute Frobenius, extends by continuity to this $C(K_o)$. Because $C(K_o)$ is <u>flat</u> over A_∞, we can tensor the chain of inclusions between M and M' to obtain

$$\begin{array}{ccccccc} M & \subset & M' & \subset & p^{-\nu}M & \subset & M\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \\ \cap & & \cap & & \cap & & \cap \\ M\otimes C(K_o) & \subset & M'\otimes C(K_o) & \subset & p^{-\nu}M\otimes C(K_o) & \subset & (M\otimes C(K_o))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \end{array} .$$

We define M'' as the <u>intersection</u> (inside $(M\otimes C(K_o))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$)

$$M'' \overset{\text{dfn}}{=\!=} (M' \underset{A_\infty}{\otimes} C(K_o)) \cap (M\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)$$

This description of M'' shows that it is both horizontal, and stable under all the operators $(F_\Sigma)^n/p^{[n\lambda]}$.

To see that $M'' \subset p^{-\nu}M$, simply notice that

$$(p^{-\nu}M\otimes C(K_O)) \cap (M\otimes_{\mathbb{Z}_p}\mathbb{Q}_p) = p^{-\nu}M$$

(this because M is locally free, and $C(K_o) \cap (A_\infty\otimes\mathbb{Q}_p) = A_\infty$). Thus we have

$$M \subset M'' \subset p^{-\nu}M .$$

Because A_o is noetherian, this shows that M'' is finitely generated. So it remains only to show that M'' is <u>flat</u> over A_∞.

Let us admit for the moment the following assertion about M'':

> for any $f \in A_\infty$ with $f \notin pA_\infty$, M'' has no f-torsion, and M''/fM'' has no p-torsion.

From it, we easily deduce the flatness of M'', as follows. It suffices to show that M'' is flat after we extend scalars from A_∞ to the complete local rings of A_∞ at all closed points of A_o. But such a complete local ring $\hat{A}_\infty$ is of the form $W(k')[[T]]$, where k' is the residue field at the closed point, and T may be chosen as any element in A_∞ whose reduction mod p in A_0 is a uniformizing parameter at the closed point. Because $\hat{A}_\infty$ is flat over A_∞ , we deduce from the admitted assertion that

$$M''\otimes W(k')[[T]] \text{ has no T-torsion and}$$

$$M''\otimes W(k')[[T]]/(T) \text{ has no p-torsion.}$$

Since $M''\otimes W(k')[[T]]$ is finitely generated, its flatness over $W(k')[[T]]$ follows from the local criterion of flatness (SGA 1 , Exposé IV, Thm 5.6).

To prove the assertion, notice first that A_∞ is a domain (being $\mathbb{Z}_p$-flat, p-adically separated, and having A_∞/pA_∞ a domain) , and $f \neq 0$. Since $M'' \subset p^{-\nu}M$, and $p^{-\nu}M$ is locally free and hence flat over A_∞ , there is no f-torsion in M'' . Because $f \notin pA_\infty$, it becomes a unit in $C(K_0)$, and so by the <u>definition</u> of M'' we have

$$p^{-\nu}M/M'' \quad \text{has no f-torsion}$$

Therefore the inclusion $M'' \subset p^{-\nu}M$ gives an inclusion

$$M''/fM'' \subset p^{-\nu}M/fp^{-\nu}M \xrightarrow{\sim} M/fM .$$

So to have M''/fM'' without p-torsion, it suffices if M/fM has no p-torsion. As M is flat over A_∞ , being locally free, it suffices if A_∞/fA_∞ has no p-torsion. This follows from the fact that $f \notin pA_\infty$, while A_∞ is flat over $\mathbb{Z}_p$, p-adically separated, and A_∞/pA_∞ is a domain.

<u>Remark</u>. If we allow A_0 to be a domain which is smooth of arbitrary dimension n over a perfect field k , exactly the same argument shows that M'' will be flat over the complete local rings $\hat{A}_\infty$ of A_∞ at all points of <u>codimension one</u> in $\mathrm{Spec}(A_0)$ (there $\hat{A}_\infty$ will be of the form $C(k')[[T]]$ with k' the no-longer-perfect residue field and $C(k')$ a Cohen ring of k') . In other words, the

isogeny theorem is true outside a closed set of codimension ≥ 2 on <u>any</u> smooth-over-a-perfect field A_o. It would be interesting to know if the isogeny theorem is in fact true, <u>without</u> exceptional sets, in this more general case.

<u>Corollary 2.6.2</u> (Newton filtration)

<u>Let</u> A_o <u>be an</u> $\mathbb{F}_p$<u>-algebra which is either a smooth domain of dimension one over a perfect field</u> k, <u>or is</u> $k[[T]]$. <u>Suppose we are given a</u> σ^a-F-<u>crystal</u> (M,∇,F) <u>over</u> A_o <u>of rank</u> r <u>which at every point of</u> $\mathrm{Spec}(A_o)$ <u>has the same first Newton slope</u> λ, <u>with the same multiplicity</u> A. <u>Then</u> (M,∇,F) <u>is isogenous to a</u> σ^a-F-<u>crystal</u> (M',∇',F') <u>which is divisible by</u> p^λ, <u>and which sits in a short exact sequence of</u> σ^a-F-<u>crystals over</u> A_o

$$0 \longrightarrow (M_1',\nabla',F') \longrightarrow (M',\nabla',F') \longrightarrow (M_2',\nabla',F') \longrightarrow 0$$

<u>in which</u>

$$\left\{\begin{array}{l} (M_1',\nabla',F') \text{ has rank } A \text{, is divisible by} \\ p^\lambda \text{ and at each point of } \mathrm{Spec}(A_o) \text{ all its} \\ \text{Newton slopes are } \lambda \\ \\ (M_2',\nabla',F') \text{ has rank } r-A \text{, is divisible} \\ \text{by } p^\lambda \text{ and at each point of } \mathrm{Spec}(A_o) \text{ all} \\ \text{its Newton slopes are } > \lambda \text{ .} \end{array}\right.$$

<u>Proof</u>. By the isogeny theorem, we may suppose (M,∇,F) itself to be divisible by p^λ. For any integer $n \geq 1$ such that $n\lambda \in \mathbb{Z}$, the n^{th} iterate (M,∇,F^n) is divisible by $p^{n\lambda}$, i.e., all Hodge slopes of (M,∇,F^n) are $\geq n\lambda$, at each point of $\mathrm{Spec}(A_o)$. Since the first Newton slope of (M,∇,F^n) is $n\lambda$, with multiplicity A, at each point of $\mathrm{Spec}(A_o)$, we can apply the Newton-Hodge theorem (cf 2.4) to (M,∇,F^n) and the point $(A,An\lambda)$. This produces a short exact sequence of σ^{an}-F-crystals

$$0 \longrightarrow (M_1,\nabla,F^n) \longrightarrow (M,\nabla,F^n) \longrightarrow (M_2,\nabla,F^n) \longrightarrow 0 \text{ ,}$$

which for $n = 1$ (i.e., the case $\lambda \in \mathbb{Z}$) completes the proof. In general, we need only observe that in terms of a lifting Σ of $\sigma : A_o \longrightarrow A_o$, the submodule $M_1 \subset M$ is simply the intersection

$$\bigcap_{N \geq 1} \text{image of} \left(\frac{F_\Sigma^{nN}}{p^{nN\lambda}} : M^{(\Sigma^{aN})} \longrightarrow M \right),$$

and M_1 is therefore F-stable. Then we can take $(M_1,\nabla,F) \subset (M,\nabla,F)$ as the solution to our problem.

It remains to see why the short exact sequence we have constructed splits uniquely over the perfection A_o^{perf} of A_o. We have proven that, over A_o^{perf}, it splits uniquely in the "up-to-isogeny" category i.e., by an F-compatible map $M_2 \longrightarrow M$ with coefficients in $W(A_o^{perf}) \otimes \mathbb{Q}_p$. We must show this map has coefficients in $W(A_o^{perf})$. But this same map also provides an "up-to-isogeny" splitting of the exact sequence of n^{th} iterates

$$0 \longrightarrow (M_1, F_1{}^n) \longrightarrow (M, F^n) \longrightarrow (M_2, F_2{}^n) \longrightarrow 0 .$$

But for any n with $n\lambda \in \mathbb{Z}$, this is just the Newton-Hodge filtration of (M,F^n) attached to the point $(A, An\lambda)$, which over A_o^{perf} has a unique splitting $(M_2,F_2{}^n) \longrightarrow (M,F^n)$. The underlying map $M_2 \longrightarrow M$ of this splitting, which has coefficients in $W(A_o^{perf})$, must, by uniqueness, coincide with the underlying map of our "up-to-isogeny" splitting.

Corollary 2.6.3

Hypotheses as in the previous Corollary 2.6.2, suppose in addition that the entire Newton polygon of (M,∇,F) is constant, i.e., independent of the point in $\text{Spec}(A_o)$. Let $\lambda_1,\ldots,\lambda_s$ be the distinct Newton slopes, and let $A_1,\ldots,A_s$ be their multiplicities. Then (M,∇,F) is isogenous to a σ^a-F-crystal (M',∇',F') which is divisible by p^{λ_1}, and which admits a filtration

$$0 \subset (M_1',\nabla',F') \subset (M_2',\nabla',F') \subset \ldots \subset (M_s',\nabla',F') = (M',\nabla',F')$$

in which

(M'_i, ∇', F') <u>has rank</u> $A_1 + \ldots + A_i$, <u>and has Newton slopes</u> (λ_1 <u>repeated</u> A_1 <u>times</u>, ..., λ_i <u>repeated</u> A_i <u>times</u>) <u>at each point of</u> $\mathrm{Spec}(A_0)$.

<u>the quotient</u> $(M'/M'_i, \nabla', F')$ <u>has rank</u> $A_{i+1} + \ldots + A_s$, <u>and it is divisible by</u> $p^{\lambda_{i+1}}$.

<u>the associated graded</u> $(M'_i/M'_{i-1}, \nabla', F')$ <u>has rank</u> A_i, <u>is divisible by</u> p^{λ_i}, <u>and has all Newton slopes</u> $= \lambda_i$.

<u>This filtration splits uniquely when we pass to the perfection of</u> A_0.

<u>Proof</u>. We proceed by induction on the number s of distinct Newton slopes. For $s = 1$, the previous Corollary applies. In general, we construct $(M'_1, \nabla', F') \subset (M', \nabla', F')$ as in the previous Corollary. Then we have

$$0 \longrightarrow (M'_1, \nabla', F') \longrightarrow (M', \nabla', F') \longrightarrow (M'/M'_1, \nabla', F') \longrightarrow 0 .$$

By the induction hypothesis applied to $(M'/M'_1, \nabla', F')$, we get an isogeny

$$((M'/M'_1)'', \nabla'', F'') \longrightarrow (M'/M'_1, \nabla', F')$$

whose source satisfies all the conclusions of the Corollary. Taking the "pull-back" by this map of the above extension of $(M'/M'_1, \nabla', F')$ by (M'_1, ∇', F'), we get an extension

$$0 \longrightarrow (M'_1, \nabla', F') \longrightarrow ? \longrightarrow ((M'/M'_1)'', \nabla'', F'') \longrightarrow 0 .$$

The middle term, ?, together with the filtration of it defined first by this exact sequence, then by the inductively given filtration on $((M'/M'_1)'', \nabla'', F'')$, provides a solution to the problem.

The existence of a unique splitting of the filtration when we pass to the perfection of A_0 follows, by induction, from the previous Corollary.

Corollary 2.6.4

Hypotheses as in the previous Corollary, any (M,∇,F) with constant Newton polygon is isogenous to an (M',∇',F') with the property that for any integer $n \geq 1$ which is a common denominator for the Newton slopes, the Hodge and Newton polygons of the σ^{an}-F-crystal $(M',\nabla',(F')^n)$ at each point of $\operatorname{Spec}(A_o)$ coincide.

Proof. Indeed, the (M',∇',F') given by the previous Corollary has the required property.

(2.7) Constancy theorems

Let k be an algebraically closed field of characteristic p, and let A_o be a k-algebra. We say that a σ^a-F-crystal on A_o is constant if it is (isomorphic to) the inverse image of a σ^a-F-crystal on k, by the given algebra map $k \longrightarrow A_o$.

Theorem 2.7.1

Let (M,∇,F) be a σ^a-F-crystal of rank r on $k[[T]]$, with k algebraically closed. Suppose that at the two points of $\operatorname{Spec}(k[[T]])$, the Newton polygons coincide, and that this common Newton polygon has only a single slope, say λ, repeated r times. Then (M,∇,F) is isogenous to a constant σ^a-F-crystal.

Proof. By the isogeny theorem, we may assume that (M,∇,F) is divisible by p^λ, in the sense of 2.5. We will prove it constant. Let N be the denominator of λ. Then F^N is divisible by $p^{N\lambda}$, and all of its Newton slopes, at each point of $\operatorname{Spec}(k[[T]])$ are $N\lambda$. Therefore $(M,\nabla,F^N/p^{N\lambda})$ is a "unit-root" σ^{aN}-F-crystal, so equivalent to a representation of $\pi_1(\operatorname{Spec}(k[[T]])$ in $GL(r,W(\mathbb{F}_{p^{aN}}))$. But $\pi_1(\operatorname{Spec}(k[[T]]) \xrightarrow{\sim} \pi_1(\operatorname{Spec}(k))$ is trivial, because k is algebraically closed. Therefore $(M,\nabla,F^N/p^{N\lambda})$ is trivial as a unit-root σ^{aN}-F-crystal. In particular, (M,∇) is trivial as a crystal, i.e., the $W(k)$-module M^∇ of all horizontal sections of (M,∇) over $A_\infty = W(k)[[T]]$ is free of rank r, and $M^\nabla \otimes_{W(k)} A_\infty \xrightarrow{\sim} M$. Because F is horizontal, it induces a

σ^a-linear endomorphism of M^∇, such that $F^N/p^{N\lambda}$ induces a σ^{aN}-linear automorphism of M^∇. Then (M,∇,F) on $k[[T]]$ is the inverse image of (M^∇,F) on k.

Remarks. 1) The (trivial) representation of $\pi_1(\mathrm{Spec}(k[[T]])$ on a free $W(\mathbb{F}_{p^{aN}})$-module of rank r is provided by the set $(M^\nabla)_{\mathrm{fix}}$ of fixed points of $F^N/p^{N\lambda}$ acting σ^{aN}-linearly on M^∇. In fact, $((M^\nabla)_{\mathrm{fix}},F)$ provides a descent of (M,∇,F) to $\mathbb{F}_{p^a}$.

2) If we omit the words "isogenous to" from the statement of the theorem, it can become false. The simplest geometric counterexample is due to Oort (cf [10]). He begins with a supersingular elliptic curve E_0 over k, and considers the product $E_0(p^\infty) \times E_0(p^\infty)$ of its p-divisible group with itself. In this product, the kernel of F is $\alpha_p \times \alpha_p$. Over the projective line $\mathbb{P}^1$ over k, we get a family of α_p's sitting in $\alpha_p \times \alpha_p$; over a point in $\mathbb{P}^1$ with homogeneous coordinates (μ,ν) sits the image of the closed immersion

$$\begin{cases} \alpha_p \hookrightarrow \alpha_p \times \alpha_p \\ x \longrightarrow (\mu x, \nu x) . \end{cases}$$

If we divide the constant group $E_0(p^\infty) \times E_0(p^\infty)$ over $\mathbb{P}^1$ by this variable α_p, we get a non-constant p-divisible group over $\mathbb{P}^1$. Restricting to the complete local ring at any closed point of $\mathbb{P}^1$, we get a non-constant p-divisible group over $k[[T]]$, whose Dieudonné module provides the required counterexample.

Concretely, this means we begin with the constant σ-F-crystal (M,∇,F) on $k[[T]]$ given by

M: free $W(k)[[T]]$-module with basis e_1,e_2,e_3,e_4

∇: the trivial connection with $\nabla(\frac{d}{dT})(e_i) = 0$ for $i = 1,2,3,4$

F: in terms of the endomorphism Σ of $W(k)[[T]]$ which is σ-linear and maps $T \longrightarrow T^p$, the Σ-linear map $F_\Sigma\colon M \longrightarrow M$ is given by

$$F_\Sigma(e_1) = e_2 \ , \quad F_\Sigma(e_3) = e_4$$

$$F_\Sigma(e_2) = pe_1 \ , \quad F_\Sigma(e_4) = pe_3$$

The $W(k)[[T]]$-submodule $M' \subset M$ spanned by

$$e_1 + T^p e_3 \ , \ e_2 \ , \ pe_3 \ , \ e_4$$

is stable under ∇ and F_Σ, and we have $pM \subset M' \subset M$. But $(M')^\nabla = M^\nabla \cap M'$ is the free $W(k)$-module spanned by e_2, e_4, pe_1 and pe_3, so that $(M')^\nabla \otimes W(k)[[T]] \subsetneqq M'$. Therefore (M',∇,F) is not constant. Alternately, one could observe that the Hodge polygon of (M',∇,F^2) is not constant (its least Hodge slope is 1 at the closed point, 0 at the generic point), and hence (M',∇,F) cannot be constant.

Another very recent counterexample is due to Lubin. He constructs a σ-F-crystal (M,∇,F) over $k[[T]]$ of rank 5, whose Newton slopes are all $2/5$, and whose Hodge numbers are $h^0 = 3$, $h^1 = 2$, at both points of $\mathrm{Spec}(k[[T]])$. In Lubin's example, the Hodge polygon of (M,∇,F^5) is not constant; at the closed point, the least Hodge slope of F^5 is 2, but at the generic point it is 1. Therefore (M,∇,F^5), and à fortiori (M,∇,F), cannot be constant.

Here is the actual example. The module M is free on $e_1,\dots,e_5$ over $W(k)[[T]]$. For the endomorphism Σ of $W(k)[[T]]$ which is σ-linear and sends $T \longmapsto T^p$, F is the linear map $M^{(\Sigma)} \longrightarrow M$ with matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & p \\ p & 0 & 0 & T & 0 \end{pmatrix} .$$

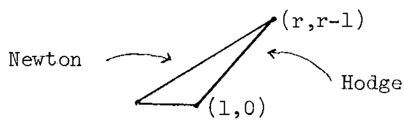
The connection ∇ on M is the unique one for which F is horizontal.

In the positive direction, we do have the following two constancy theorems

Theorem 2.7.2

Let (M,∇,F) *be a* σ^a*-F-crystal of rank* $r \geq 1$ *on* $k[[T]]$, *with* k *algebraically closed. Suppose that at both points of* $\operatorname{Spec}(k[[T]])$, *the Newton slopes are all* $(r-1)/r$, *and the Hodge numbers* h^i *vanish for* $i > 1$. *Then* (M,∇,F) *is constant.*

Proof. The Newton and Hodge polygons must be



at both points of $\operatorname{Spec}(k[[T]])$ (because they start and end together, and the Hodge slopes are 0 and 1). Therefore $h^0 = 1$. Applying the sharp slope estimate (with $\lambda = r-1/r$), we get

$$F^{n+1} \equiv 0 \bmod p^{\{\frac{r-1}{r}\cdot n\}} .$$

In particular, F^r is divisible by p^{r-1} , and hence $(M,\nabla,F^r/p^{r-1})$ is a unit-root F-crystal. Just as in the proof of 2.7.1, this implies that (M,∇,F) is the inverse image of (M^∇,F) on k . QED

Theorem 2.7.3

Let (M,∇,F) *be a* σ^a*-F-crystal of rank* $r \geq 1$ *on* $k[[T]]$, *with* k *algebraically closed. Suppose that at both points of* $\operatorname{Spec}(k[[T]])$, *the Newton slopes are all* $1/r$. *Then* (M,∇,F) *is constant.*

Proof. This time the basic slope estimate shows that $(M,\nabla,F^r/p)$ is a unit-root F-crystal, and we conclude as in 2.7.2.

The theorem 2.7.1 of constancy up to isogeny becomes false as soon as we allow the common Newton polygon to have more than one distinct slope, for there can be highly non-trivial *extensions* of constant F-crystals over $k[[T]]$. The simplest and most important example of such an extension is given by first taking an ordinary elliptic curve E_o over k, then constructing its equicharacteristic universal formal deformation E over $k[[T]]$, and finally taking the p-divisible group $E(p^\infty)$ of E. This p-divisible group over $k[[T]]$ sits an exact sequence

$$0 \longrightarrow \mu_{p^\infty} \longrightarrow E(p^\infty) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0$$

The Dieudonné crystal (M,∇,F) of $E(p^\infty)$, or equivalently the first crystalline cohomology of $E/k[[T]]$, is a σ-F-crystal of rank <u>two</u>, which is an extension of two <u>constant</u> σ-F-crystals of rank <u>one</u>. Even the underlying crystal (M,∇) is highly non-trivial; indeed, M^∇ is free of rank <u>one</u> over $W(k)$. (For a suitable choice of parameter T of $W(k)[[T]]$, the Serre-Tate or the Dwork theory tells us that (M,∇), viewed as a module with connection on $W(k)[[T]]$, admits a basis e_o, e_1 in terms of which the connection is given by

$$\begin{cases} \nabla(\frac{d}{dT})(e_o) = 0 \\ \nabla(\frac{d}{dT})(e_1) = \frac{1}{1+T}\, e_o \,. \end{cases}$$

Because the series $\log(1+T)$ has <u>unbounded</u> coefficients, the module M^∇ of horizontal sections consists only of the $W(k)$ multiples of e_o.)

<u>Theorem 2.7.4</u>

<u>Let</u> (M,∇,F) <u>be a</u> σ^a-<u>F-crystal on</u> $k[[T]]$, <u>with</u> k <u>algebraically closed</u>. <u>Suppose that at the two points of</u> $\mathrm{Spec}(k[[T]])$, <u>the Newton polygons coincide</u>. <u>Then</u> (M,∇,F) <u>is isogenous to a</u> σ^a-<u>F-crystal</u> (M',∇',F') <u>whose inverse image on</u> $(k[[T]])^{\mathrm{perf}}$ <u>is constant</u>.

<u>Proof</u>. This follows by combining Corollary 2.6.2 and Theorem 2.7.1.

<u>Remarks</u>. 1) This gives an alternate proof of Berthelot's Theorem 4.7.1 in [1].

2) B. Gross (cf [4]) attaches to any σ^a-F-crystal over $k((T))^{\mathrm{perf}}$ a representation of $\mathrm{Gal}(k((T))^{\mathrm{alg.cl.}}/k((T))^{\mathrm{perf.}})$ which is trivial if and only if

the σ^a-F-crystal is isogenous to a constant one. Therefore if we begin with a σ^a-F-crystal over $k[[T]]$ with constant Newton polygon, Gross's representation, attached to its inverse image on $k((T))^{\text{perf.}}$, is trivial. Is the converse true?

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