Invent. math. 87, 13–61 (1987)

Inventiones mathematicae

On the calculation of some differential galois groups

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Table of contents

Introduction

This work grew out of another (cf. [Ka-1, Ka-2]) attempt to deal with the following question:

> if one is explicitly given an n'th order differential equation, how can one "tell at a glance" what its differential galois group G_{gal} is?

At one extreme, one wants to recognize when G_{gal} is a finite group, i.e., when the D.E. in question has a full set of algebraic solutions. This problem was solved by Schwarz ("Schwarz's list") in 1872 for the classical hypergeometric equation, but since then there has been remarkably little progress. (Even if one were able to prove Grothendieck's p-curvature conjecture (cf. [Ka-1]), it is not clear whether it would really allow one to "tell at a glance" when G_{gal} is finite.)

This paper is concerned with the opposite extreme, the problem of recognizing when G_{gal}, a priori a Zariski closed subgroup of $\mathscr{GL}(n)$, is "large", in the sense that either it is caught between $\mathscr{SL}(n)$ and $\mathscr{GL}(n)$ or (if n is even), between $\mathscr{Sp}(n)$ and $\mathscr{GSp}(n)$.

Our main result (4.1.4) and its elaborations (4.1.7–8) give an easily checkable sufficient condition for G_{gal} to be "large" in the above sense.

This allows us to exhibit large classes of differential equations whose G_{gal} is large. Fix integers $n \geqq 2$ and $m \geqq 1$ which are *relatively prime*. Then

(A) if $P_n(x)$ and $Q_m(x)$ are polynomials in $\mathbf{C}[x]$ of degrees n and m respectively, G_{gal} is large for

$$P_n\left(\frac{d}{dx}\right) + Q_m(x).$$

(B) if P_n and Q_m as above also satisfy

all roots of P_n are rational numbers with denominator prime to n.

$Q_m(0) = 0$.

then G_{gal} is large for

$$P_n\left(\frac{xd}{dx}\right) + Q_m(x).$$

Once we know that G_{gal} is large, it is *usually* but not always a simple matter to determine its exact value. For example, suppose that the coefficient of x^{n-1} in P_n vanishes. Then

$$G_{\mathrm{gal}} = \begin{cases} \mathscr{SL}(n) & \text{if } n \text{ is odd.} \\ \mathscr{Sp}(n) \text{ or } \mathscr{SL}(n) & \text{if } n \text{ is even.} \end{cases}$$

In the case n even, G_{gal} is $\mathscr{Sp}(n)$ if and only if the $n \times n$ first-order system attached to the operator in question is isomorphic to its dual. By a result of 0. Gabber (1.5.3), a self-adjoint operator gives rise to a self-dual $n \times n$ system, so we have the implication

$$n \text{ even, } P_n(x) = P_n(-x) \Rightarrow G_{\mathrm{gal}} = \mathscr{Sp}(n).$$

In case (A), we expect that the converse implication holds as well, but we can prove this only for $m = 1$. In case (B), the converse implication is false, and only for $m = 1$ do we know how to recognize when $G_{\mathrm{gal}} = \mathscr{Sp}(n)$ (the condition is that the roots of $P_n(x)$ agree modulo $\mathbf{Z}$ with those of $P_n(-x)$).

The problem of recognizing the possible autoduality of the $n \times n$ system attached to an n'th order operator is a special case of recognizing when two different n'th order operators (e.g., the operator and its adjoint) give rise to isomorphic $n \times n$ systems, i.e., of studying the map

n'th order operators $\rightarrow$ isomorphism classes of $n \times n$ systems.

That this study may be non-trivial is shown by the following example: If, $\lambda \neq -1$ the 2×2 systems attached to the operators

$$\begin{cases} \left(\dfrac{d}{dx}\right)^2 - x^2 - \lambda \\ \left(\dfrac{d}{dx}\right)^2 - x^2 - \lambda - 2 \end{cases}$$

are isomorphic (see 4.3).

We now turn to a more detailed discussion of the contents of this work.

Our approach to differential galois group is via the general theory of Tannakian categories; this point of view is much better suited to our purposes than the classical one of looking at differential fields. There are three main ingredients in our approach:

1. a simple global criterion (1.2.5) for G_{gal} to be connected, due to Ofer Gabber.
2. the Tannakian category translation of Levelt's fundamental work [Le-1] on the structure theory of D.E.'s over $\mathbf{C}((x))$ into group and representation-theoretic information about the corresponding "local differential galois group".
3. the classification of "cyclically minuscule" representations of semi-simple Lie groups (this part may be of independent interest).

In working out the second point above, we were very strongly guided by the well-known analogy between D.E.'s on curves over $\mathbf{C}$ and lisse l-adic sheaves, or l-adic representations, on curves over algebraically closed fields of characteristic $p \gg 0, p \neq l$. This analogy is detailed in the Appendix. The interested reader might wish to compare Chapter II of this paper with Chap. I of both [Ka-3] and [Ka-5].

Here is one concrete instance of the above D.E.-lisse sheaf analogy. Let χ be a multiplicative character of a finite field $\mathbf{F}_q$ whose exact order is denoted N. Given $n \geqq 1$ and integers $a_1, \ldots, a_n$, the Kloosterman sheaf (cf. [Ka-4], Chap. 4) $Kl_\psi(\chi^{a_1}, \ldots, \chi^{a_n})$ on $\mathbf{G}_m$ over $\mathbf{F}_q$ is in many ways analogous (cf. [Dw-1, Sp-1]) to the D.E. on $\mathbf{G}_m$ over $\mathbf{C}$

$$\prod_{i=1}^{n} \left(x \frac{d}{dx} - \frac{a_i}{N}\right) - \pi x,$$

for $\pi \in \mathbf{C}^\times$ any non-zero constant. Suppose now that N is prime to n, so that this D.E. has large explicit G_{gal} (see (4.5.4)). It then seems reasonable to hope that the group G_{geom} for the corresponding Kloosterman sheaf is equal to G_{gal} at least in characteristics $p \gg 0$. That it is so for $N=1$ is one of the main results of [Ka-3]. We hope to return to this question in the future.

To conclude this introduction, we would like to point out a curious problem. In [Ka-2], we gave a conjectural description of the Lie algebra of G_{gal} in terms of the p-curvatures of almost all the reductions mod p of the D.E. in question, and we showed that our description was in fact correct for any D.E. on either $\mathbf{A}^1$ or $\mathbf{G}_m$. In this paper, we exhibit extensive classes of D.E.'s on $\mathbf{A}^1$ and on $\mathbf{G}_m$ whose G_{gal} is large. Can one see directly that G_{gal} is large in these cases by using the p-curvature description of its Lie algebra?

I. The global theory: a miscellany

1.1. Generalities

(1.1.1) Let k be a field of characteristic zero, and X/k a smooth, geometrically connected k-scheme which is separated and of finite type over k, and with $X(k)$ non-empty. We denote by D.E. (X/k) the category of all algebraic differential equations on X/k in the sense of [Ka-2]. An object of D.E. (X/k) is a pair (M, ∇) consisting of a locally free $\mathcal{O}_X$-module of finite rank M, together with an integrable connection ∇ on M relative to k; a morphism from (M_1, ∇_1) to (M_2, ∇_2) is a horizonal $\mathcal{O}_X$-linear map from M_1 to M_2. With the obvious notions of tensor product and internal hom, D.E. (X/k) is a "neutral Tannakian category over k"; any rational point $x \in X(k)$ defines a k-valued fibre functor (though not every k-valued fibre functor is of this form).

(1.1.2) Let ω be any k-valued fibre functor on D.E. (X/k). In order to emphasize the analogy with the case of local systems, we denote by

$$\pi_1^{\mathrm{diff}}(X/k, \omega), \tag{1.1.2.1}$$

– read "the differential fundamental group of X/k, with base point ω" – the affine k-groupscheme $\mathcal{A}ut^{\otimes}(\omega)$. By the main theorem on neutral Tannakian categories, ([De-Mi], 2.11) the functor ω defines an equivalence of tensor categories

$$\text{D.E.}\,(X/k) \xrightarrow[\sim]{\omega} \begin{pmatrix} \text{fin. dim. } k\text{-reps.} \\ \text{of } \pi_1^{\mathrm{diff}}(X/k, \omega) \end{pmatrix}; \tag{1.1.2.2}$$

i.e., a D.E. on X/k "is" a finite dimensional k-representation of the pro-algebraic affine k-groupscheme $\pi_1^{\mathrm{diff}}(X/k, \omega)$.

(1.1.3) If k is algebraically closed, any two k-valued fibre functors ω_1 and ω_2 are (non-canonically) isomorphic. In general, any two k-valued fibre functors become isomorphic over $\bar{k}$. Thus the k-groups $\pi_1^{\mathrm{diff}}(X/k, \omega_i)$, $i = 1,2$, become isomorphic over $\bar{k}$, by an isomorphism which is canonical up to composition with an inner automorphism (by a $\bar{k}$-valued point) of either source or target; this is a general fact about "neutral Tannakian categories over k".

(1.1.4) Let V be an object of D.E. (X/k), i.e., V is an (M, ∇) on X, and denote by $\langle V \rangle$ the full subcategory of D.E. (X/k) whose objects are all the sub-quotients of all finite direct sums of the objects $V^{\otimes n} \otimes (V^{\vee})^{\otimes m}$, all $n, m \geqq 0$. Then $\langle V \rangle$ is itself a neutral Tannakian category over k; the restriction to $\langle V \rangle$ of any k-valued fibre functor ω on D.E. (X/k) provides a k-valued fibre functor for $\langle V \rangle$. We denote by

$$G_{\mathrm{gal}}(V, \omega) \subset \mathcal{GL}(\omega(V)), \tag{1.1.4.1}$$

– read "the differential galois group of V, with base point ω" – the Zariski closed subgroup of $\mathcal{GL}(\omega(V))$ which is $\mathcal{A}ut^{\otimes}(\omega | \langle V \rangle)$. Again by ([De-Mi], 2.11), ω defines an equivalence of tensor categories

$$\langle V \rangle \xrightarrow[\sim]{\omega} \begin{pmatrix} \text{fin. dim. } k\text{-reps} \\ \text{of } G_{\mathrm{gal}}(V, \omega) \end{pmatrix}. \tag{1.1.4.2}$$

Dual to the inclusion of $\langle V \rangle$ in D.E.(X/k), we have a homomorphism of k-groupschemes

$$\pi_1^{\mathrm{diff}}(X/k, \omega) \to G_{\mathrm{gal}}(V, \omega),$$

which by ([De-Mi], 2.21) is faithfully flat. The composite

$$\pi_1^{\mathrm{diff}}(X/k, \omega) \to G_{\mathrm{gal}}(V, \omega) \hookrightarrow \mathscr{GL}(\omega(V))$$

is the representation ρ_V of $\pi_1^{\mathrm{diff}}(X/k, \omega)$ which "is" V. In particular, for any "construction of linear algebra" in the sense of [Ka-2], the $G_{\mathrm{gal}}(V, \omega)$-stable subspaces of $\mathrm{Constr}(\omega(V))$ are precisely the ω-fibres of the sub-equations of $\mathrm{Constr}(V)$. By Chevalley's theorem [Chev], $G_{\mathrm{gal}}(V, \omega)$ is the stablizer in $\mathscr{GL}(\omega(V))$ of all such ω-fibres in all such constructions; this was the definition of G_{gal} given in [Ka-2].

(1.1.5) Therefore we may view $G_{\mathrm{gal}}(V, \omega)$ as the *image* in $\mathscr{GL}(\omega(V))$ of $\pi_1^{\mathrm{diff}}(X/k, \omega)$ under ρ_V, and we may interpret $\langle V \rangle$ as equivalent, via ω, to the full subcategory of $\mathscr{R}ep_k(\pi_1^{\mathrm{diff}}(X/k, \omega))$ consisting of those representations which factor through $G_{\mathrm{gal}}(V, \omega)$, viewed as a quotient of π_1^{diff}.

In particular, given an object W of $\langle V \rangle$, corresponding to a representation λ_W of $G_{\mathrm{gal}}(V, \omega)$, we have

(1.1.5.1) $$\langle W \rangle = \langle V \rangle \text{ inside D.E.}(X/k)$$

$$\Leftrightarrow \lambda_W \colon G_{\mathrm{gal}}(V, \omega) \to \mathscr{GL}(\omega(W)) \text{ is faithful (i.e., a closed immersion).}$$

This equivalence will be used later (2.5.9.1) to bound from below the dimensions of faithful representations of $G_{\mathrm{gal}}(V, \omega)$.

1.2. A criterion for connectedness

(1.2.1) In this section, we suppose $k = \mathbf{C}$. We denote by X^{an} the complex manifold "$X(\mathbf{C})$ in its classical topology", and by $\pi_1(X^{\mathrm{an}}, x)$ the classical fundamental group of X^{an} with base point $x \in X^{\mathrm{an}}$. Given $x \in X^{\mathrm{an}}$, the functor $\omega_x =$ "fibre at x" $\xleftarrow{\sim}$ "germs of horizontal sections at x" defines an equivalence of tensor categories

(1.2.1.1) $$\text{D.E.}(X^{\mathrm{an}}) \xrightarrow[\sim]{\omega_x} \begin{pmatrix} \text{fin. dim. } \mathbf{C}\text{-reps.} \\ \text{of } \pi_1(X^{\mathrm{an}}, x) \end{pmatrix}.$$

The corresponding affine pro-algebraic $\mathbf{C}$-group $\mathscr{A}ut^{\otimes}(\omega_x)$ is thus the inverse limit of the Zariski closures of the images of $\pi_1(X^{\mathrm{an}}, x)$ in all its finite-dimensional $\mathbf{C}$-representations.

(1.2.2) For an object V of D.E.(X^{an}), we denote by $\langle V \rangle$ the full subcategory of D.E.(X^{an}) defined just as in the algebraic case. We denote by

(1.2.2.1) $$G_{\mathrm{mono}}(V, x) \hookrightarrow \mathscr{GL}(V(x))$$

– read "the algebraic monodromy group of V with base point x", – the group $\mathscr{A}ut^{\otimes}(\omega_x | \langle V \rangle)$. Concretely, $G_{\mathrm{mono}}(V, x)$ is the Zariski closure in $\mathscr{GL}(V(x))$ of the image of $\pi_1(X^{\mathrm{an}}, x)$ by the monodromy representation of V.

(1.2.3) There is a natural functor

$$\text{(1.2.3.1)} \qquad \mathrm{D.E.}(X/\mathbf{C}) \to \mathrm{D.E.}(X^{\mathrm{an}})$$
$$V \mapsto V^{\mathrm{an}},$$

which for any object V in $\mathrm{D.E.}(X/\mathbf{C})$ maps $\langle V\rangle$ to $\langle V^{\mathrm{an}}\rangle$. The dual map of $\mathbf{C}$-algebraic groups

$$\text{(1.2.3.2)} \qquad G_{\mathrm{mono}}(V^{\mathrm{an}}, x) \to G_{\mathrm{gal}}(V, \omega_x)$$

is a closed immersion (by [De-Mi], 2.21), or cf. ([Ka-2], 5.2)).

(1.2.4) If ρ is a representation of $G_{\mathrm{gal}}(V, \omega_x)$, corresponding to an object W in $\langle V\rangle$, then the restriction of ρ to $G_{\mathrm{mono}}(V^{\mathrm{an}}, x)$ corresponds to the object W^{an} in $\langle V^{\mathrm{an}}\rangle$, and we have a commutative diagram of homomorphisms of $\mathbf{C}$-groups

$$\text{(1.2.4.1)} \qquad \begin{tikzcd} G_{\mathrm{mono}}(V^{\mathrm{an}}, x) \arrow[r, hook] \arrow[d, "\rho|G_{\mathrm{mono}}"] & G_{\mathrm{gal}}(V, \omega_x) \arrow[d, "\rho"] \\ G_{\mathrm{mono}}(W^{\mathrm{an}}, x) \arrow[r, hook] & G_{\mathrm{gal}}(W, \omega_x). \end{tikzcd}$$

(1.2.5) **Proposition.** (O. Gabber). *Notations as above, the natural inclusion* $G_{\mathrm{mono}}(V^{\mathrm{an}}, x) \hookrightarrow G_{\mathrm{gal}}(V, \omega_x)$ *of* $\mathbf{C}$*-groups defines a surjection on groups of connected components*:

$$\text{(1.2.5.1)} \qquad G_{\mathrm{mono}}/(G_{\mathrm{mono}})^0 \twoheadrightarrow G_{\mathrm{gal}}/(G_{\mathrm{gal}})^0 .$$

In particular, we have the implication

$$\text{(1.2.5.2)} \qquad G_{\mathrm{mono}} \text{ connected} \Rightarrow G_{\mathrm{gal}} \text{ connected.}$$

Proof. Pick a faithful representation $\bar{\rho}$ of the finite group $G_{\mathrm{gal}}/(G_{\mathrm{gal}})^0$, and interpret it successively as a representation ρ of G_{gal}, then as an object W of $\langle V\rangle$. By construction, $G_{\mathrm{gal}}(W, \omega_x)$ is finite, and we have a commutative diagram

$$\text{(1.2.5.3)} \qquad \begin{tikzcd} G_{\mathrm{mono}}(V^{\mathrm{an}}, x) \arrow[r, hook] \arrow[d, "\rho|G_{\mathrm{mono}}"] & G_{\mathrm{gal}}(V, \omega_x) \arrow[d, "\rho"] \arrow[r] & G_{\mathrm{gal}}/(G_{\mathrm{gal}})^0 \arrow[dl, hook', "\bar{\rho}"] \\ G_{\mathrm{mono}}(W^{\mathrm{an}}, x) \arrow[r, hook] & G_{\mathrm{gal}}(W, \omega_x). & \end{tikzcd}$$

So it suffices to prove the proposition in the case when $V = W$ has G_{gal} finite. But we have the implications (cf. [Ka-2])

$$\text{(1.2.5.4)} \qquad G_{\mathrm{gal}} \text{ finite} \Rightarrow \text{regular singular points} \Rightarrow G_{\mathrm{mono}} = G_{\mathrm{gal}} . \quad \text{Q.E.D.}$$

1.3. Dependence on the ground field

(1.3.1) Suppose we begin with X/k as in (1.1.1) above, and fix a rational point $x \in X(k)$. Given an extension field L of k, there is an obvious "extension of scalars" functor

$$\text{(1.3.1.1)} \qquad \mathrm{D.E.}(X/k) \to \mathrm{D.E.}\left(X \bigotimes_k L/L\right)$$

$$V \mapsto V_L = V \bigotimes_k L.$$

For a given object V in $\mathrm{D.E.}(X/k)$, this functor maps $\langle V \rangle$ to $\langle V_L \rangle$. In terms of the explicit description of G_{gal} given in [Ka-2], in terms of all sub-equations of "constructions of linear algebra", we see that we have a closed immersion of algebraic groups over L

$$\text{(1.3.1.2)} \qquad G_{\mathrm{gal}}(V_L, \omega_x^L) \hookrightarrow G_{\mathrm{gal}}(V, \omega_x) \bigotimes_k L$$

where ω_x^L denotes the L-valued fibre functor "fibre at $x \in X(k) \in X_L(L)$" on $\mathrm{D.E.}\left(X \bigotimes_k L/L\right)$.

(1.3.2) **Proposition.** (O. Gabber). *The closed immersion* (1.3.1.2) *is an isomorphism, i.e., formation of* G_{gal} *commutes with extension of the ground field.*

Proof. In view of (1.3.1.2), the proposition only becomes "harder" as L grows, so passing to an algebraically closed overfield of L, we may reduce to the case when L is an algebraically closed extension of k. Because k has characteristic zero, the fixed field of $\mathscr{A}ut(L/k)$ is k itself; this is the key point.

By definition, $G_{\mathrm{gal}}(V_L, \omega_x^L)$ is defined inside $\mathscr{GL}(\omega_x(V)) \bigotimes_k L$ as the stabilizer of all (fibres at x of all) sub-equations W of all $M \bigotimes_k L$, for M any "construction of linear algebra" applied to V. But the natural semi-linear action ($\sigma \mapsto id \otimes \sigma$) of $\mathscr{A}ut(L/k)$ on $M \bigotimes_k L$ permutes ($W \mapsto W^{(\sigma)}$) its sub-equations, whence $G_{\mathrm{gal}}(V_L, \omega_x^L)$, viewed inside $\mathscr{GL}(\omega_x(V)) \bigotimes_k L$, is invariant under the semi-linear action ($\sigma \mapsto 1 \otimes \sigma$) of $\mathscr{A}ut(L/k)$ on $\mathscr{GL}(\omega_x(V)) \bigotimes_k L$. Therefore $G_{\mathrm{gal}}(V_L, \omega_x^L)$ is "defined over k", i.e., it is of the form $G \bigotimes_k L$ for a unique Zariski-closed subgroup $G \subset \mathscr{GL}(\omega_x(V))$.

In view of (1.3.1.2), it suffices to show that this k-group G *contains* $G_{\mathrm{gal}}(V, \omega_x)$. By Chevalley's theorem, G is defined inside $\mathscr{GL}(\omega_x(V))$ as the stabilizer of one k-subspace W_x of a "construction of linear algebra" applied to $\omega_x(V)$, i.e., as the stabilizer of a k-subspace W_x of $\omega_x(M)$ for M the "same" construction applied to V itself.

We must show that W_x is stable by $G_{\mathrm{gal}}(V, \omega_x)$, i.e., we must show that W_x is of the form $\omega_x(W)$ for W some sub-equation of M. Because $G \bigotimes_k L = G_{\mathrm{gal}}(V_L, \omega_x^L)$ leaves $W_x \bigotimes_k L$ stable, we know that there exists a sub-equation

$$\mathbf{W} \subset M \bigotimes_k L$$

with $\omega_x^L(\mathbf{W}) = W_x \bigotimes_k L$ inside $\omega_x^L(M) \bigotimes_k L$. For any $\sigma \in \mathscr{A}ut(L/k)$, $\mathbf{W}^{(\sigma)}$ must be equal to $\mathbf{W}$ (inside $M \bigotimes_k L$), because both have the same fibre $W_x \bigotimes_k L$ inside

$\omega_x(M) \otimes_k L = \omega_x^L\left(M \otimes_k L\right)$. Therefore **W** is defined over k, i.e., of the form $W \otimes_k L$ for a unique sub-equation W of M, and necessarily $W_x = \omega_x(W)$, as required. Q.E.D.

1.4. Effect of a covering

(1.4.1) Consider a situation

(1.4.1.1)
$$\begin{array}{c} Y \\ {\scriptstyle \pi}\downarrow) \;\; {\scriptstyle G} \\ X \\ \downarrow \\ \mathrm{Spec}(k) \end{array}$$

where X and Y are k-schemes as in (1.1.1), G is a finite group, and π is a finite etale k-morphism which is galois with group G. We denote by D.E.$(Y/X/k)$ the full subcategory of D.E.(X/k) consisting of those objects V for which $\pi^*(V)$ is trivial. The natural functor "global horizontal sections on Y"

(1.4.1.2)
$$\mathrm{D.E.}(Y/X/k) \to \begin{pmatrix}\text{fin. dim. } k\text{-reps.} \\ \text{of } G\end{pmatrix}$$
$$V \mapsto H^0(Y, \pi^* V)^{\nabla}$$

is easily seen to be an equivalence of tensor categories.

(1.4.3) We have natural functors

(1.4.3.1)
$$\mathrm{D.E.}(Y/k) \xleftarrow{\pi^*} \mathrm{D.E.}(X/k) \xleftarrow{\text{incl.}} \mathrm{D.E.}(Y/X/k).$$

If we fix a k-valued fibre functor ω on D.E.(Y/k) and denote by ω_1 and ω_2 the fibre functors on D.E.(X/k) and on D.E.$(Y/X/k)$ obtained by composition with π^* and with $\pi^* \circ$ incl. respectively, we have dual homomorphisms of k-groupschemes

(1.4.3.2)
$$\pi_1^{\text{diff}}(Y/k, \omega) \xrightarrow{A} \pi_1(X/k, \omega_1) \xrightarrow{B} G.$$

By ([De-Mi], 2.21) the homomorphism A is a closed immersion, (any V in D.E.(Y/k) is a direct factor of $\pi^* \pi_* V \simeq \oplus g(V)$), while the homomorphism B is faithfully flat. Clearly the composite $B \circ A$ is the trivial homomorphism.

(1.4.4) **Proposition.** *The sequence of k-groupschemes*

(1.4.4.1)
$$1 \longrightarrow \pi_1^{\text{diff}}(Y/k, \omega) \xrightarrow{A} \pi_1^{\text{diff}}(X/k, \omega_1) \xrightarrow{B} G \longrightarrow 1$$

is f.p.q.c. exact (e.g., on $\bar{k}$-valued points).

This proposition is the inverse limit of the following more "finite" variant. For a given object V in D.E.(X/k), consider the functors

(1.4.4.2)
$$\langle \pi^* V \rangle \xleftarrow{\pi^*} \langle V \rangle \xleftarrow{\text{incl.}} \langle V \rangle \cap \mathrm{D.E.}(Y/X/k),$$

and denote by G_V the *quotient* of G whose k-representations "are" $\langle V\rangle \cap \mathrm{D.E.}(Y/X/k)$.

(1.4.5) **Proposition.** *The sequence*

$$(1.4.5.1)\qquad 1 \longrightarrow G_{\mathrm{gal}}(\pi^* V, \omega) \xrightarrow{A} G_{\mathrm{gal}}(V, \omega_1) \xrightarrow{B} G_V \longrightarrow 1$$

is f.p.q.c. exact.

Proof. The only non-obvious point is that $\mathrm{Ker}(B) \subset G_{\mathrm{gal}}(\pi^* V, \omega)$.

By definition, $\gamma \in G_{\mathrm{gal}}(\pi^* V, \omega)$ lies in $\mathrm{Ker}(B)$ if and only if it lies in the kernel of all the k-representations ρ_W of $G_{\mathrm{gal}}(V, \omega_1)$ defined by objects W in $\langle V\rangle$ which become trivial on Y, i.e., by objects W in $\langle V\rangle$ such that ρ_W is trivial on $G_{\mathrm{gal}}(\pi^* V, \omega)$. Thus $\mathrm{Ker}(B)$ is the intersection of the kernels of all those finite dimensional k-representations of $G_{\mathrm{gal}}(V, \omega_1)$ which are trivial on the subgroup $G_{\mathrm{gal}}(\pi^* V, \omega)$.

To prove that $\mathrm{Ker}(B) \subset G_{\mathrm{gal}}(\pi^* V, \omega)$, we use a group-analogue of the Lie algebra argument in ([Ka-2], 4.3). Because k has characteristic zero, it suffices to prove this inclusion for field-valued points.

In the notations of ([Ka-2], 4.3), we claim that any field-valued point γ of $G_{\mathrm{gal}}(V, \omega_1)$ *permutes* the lines $\omega(L_1), \ldots, \omega(L_r)$ among themselves. If we grant this, the corresponding permutation representation of $G_{\mathrm{gal}}(V, \omega_1)$ is certainly trivial on $G_{\mathrm{gal}}(\pi^* V, \omega)$, so any element of $\mathrm{Ker}(B)$ maps each $\omega(L_i)$ to itself, as required.

To prove that a given field-valued point γ in $G_{\mathrm{gal}}(V, \omega_1)$ permutes the $\omega(L_i)$, it suffices to prove that for some integer $n \geqq 1$, γ permutes the lines $\omega(L_i)^{\otimes n} = \omega(L_i^{\otimes n})$ in $\omega(\mathrm{Symm}^n(\pi^* W))$.

For any fixed $n \geqq r-1$, the direct sum

$$S(n) = \bigoplus_{1 \leqq i \leqq r} (L_i)^{\otimes n}$$

is a G-stable sub-object of $\mathrm{Symm}^n(\pi^*(W))$ (by ([Ka-2], 4.4)), so of the form $\pi^*(W_n)$. Consider $\mathrm{Symm}^2(S(n)) = \pi^*(\mathrm{Symm}^2(W_n))$; as objects of $\mathrm{D.E.}(Y/k)$, we have

$$\mathrm{Symm}^2(S(n)) = \bigoplus_{1 \leqq i \leqq j \leqq r} (L_i)^{\otimes n} \otimes (L_j)^{\otimes n}.$$

The sub-module

$$S(n; 2) = \bigoplus_{1 \leqq i \leqq r} (L_i)^{\otimes 2n} \subset \mathrm{Symm}^2(S(n))$$

is G-stable, so of the form $\pi^*(W_{n,2})$.

We will exploit the fact that $G_{\mathrm{gal}}(V, \omega_1)$ stabilizes the subspace $\omega(S(n,2))$ of $\omega(\mathrm{Symm}^2(S(n)))$. For each $1 \leqq i \leqq r$, pick a non-zero vector $\ell_i \in \omega(L_i)$. Then

$\ell_1^{\otimes n}, \ldots, \ell_r^{\otimes n}$ is a k-basis of $\omega(S(n))$

$\{\ell_i^{\otimes n} \ell_j^{\otimes n}\}_{1 \leqq i \leqq j \leqq r}$ is a k-basis of $\omega(\mathrm{Symm}^2(S(n)))$

$\ell_1^{\otimes 2n}, \ldots, \ell_r^{\otimes 2n}$ is a k-basis of $\omega(S(n,2))$.

For γ a k'-valued point of $G_{\mathrm{gal}}(V, \omega_1)$, k' an overfield of k, its action on $\omega(S(n))$ is

$$\gamma(\ell_i^{\otimes n}) = \sum A_{ij} \ell_j^{\otimes n} \qquad A_{ij} \in k'.$$

Squaring, we find

$$\gamma(\ell_i^{\otimes 2n}) = \sum (A_{ij})^2 (\ell_j)^{\otimes 2n} + 2 \sum_{j<k} A_{ij} A_{ik} \ell_j^{\otimes n} \ell_k^{\otimes n}.$$

But $\omega(S(n,2))$ is a $G_{\mathrm{gal}}(V,\omega_1)$-stable subspace of $\omega(\mathrm{Symm}^2(S(n)))$, so we must have $A_{ij}A_{ik}=0$ if $j<k$. Consequently, (A_{ij}) has the shape of a permutation matrix, as required. Q.E.D.

(1.4.5.1) **Corollary.** *If* $G_{\mathrm{gal}}(V,\omega_1)$ *is geometrically connected, then* $G_{\mathrm{gal}}(\pi^* V,\omega) \xrightarrow{\sim} G_{\mathrm{gal}}(V,\omega_1)$.

(1.4.6) **Proposition.** *Hypotheses and notations as in (1.4.1)–(1.4.3) above, let W be a non-zero object of* D.E. (Y/k), *and* $V=\pi_* W$ *its direct image in* D.E. (X/k). *Then the representation* ρ_V *of* $\pi_1^{\mathrm{diff}}(X/k,\omega_1)$ *which "is" V is simply the induction of the representation* ρ_W *of* $\pi_1^{\mathrm{diff}}(Y/k,\omega)$ *which "is" W, via the inclusion (1.4.4.1) of* π_1^{diff}*'s.*

Proof. This is just the representation-theoretic translation of the fact that π_* is right adjoint to π^*. Q.E.D.

(1.4.7) – **Note added in proof.** If X and Y are k-schemes as in (1.1.1), and if $\pi\colon Y\to X$ is *any* finite étale k-morphism, then (1.4.4), applied to the "galois closure" of $Y\to X$, shows that $\pi_1^{\mathrm{diff}}(Y/k,\omega)$ is a closed subgroup of finite index in $\pi_1^{\mathrm{diff}}(X/k,\omega_1)$, and the proof of (1.4.6) shows that π_* on D.E.'s corresponds to induction of representations.

1.5. Duality and formal adjoint: a compatibility

(1.5.1) Let R be a commutative ring with unit, $\partial\colon R\to R$ a derivation. By a ∂-module over R we mean a pair (M,D) consisting of a free R-module M of finite rank $n\geqq 1$, and an additive mapping $D\colon M\to M$ which satisfies

(1.5.1.1) $$D(fm) = \partial(f)\,m + fD(m)$$

for $f\in R$ and $m\in M$. The dual $(M^\vee,D)$ of (M,D) is the ∂-module with $M^\vee = \mathrm{Hom}_R(M,R)$ and $D\colon M^\vee\to M^\vee$ defined so that, denoting by

(1.5.1.2) $$(\ ,\ )\colon M^\vee\times M\to R$$

the canonical pairing, we have

(1.5.1.3) $$\partial((m^\vee,m)) = (Dm^\vee,m) + (m^\vee,Dm),$$

for $m\in M$ and $m^\vee\in M^\vee$.

(1.5.2) We say that (M,D) is cyclic if M admits an R-basis $(e_0,\dots,e_{n-1})$ for which $De_i=e_{i+1}$ for $i=0,1,\dots,n-2$; in this case we say that $(e_0,\dots,e_{n-1})$ is a cyclic basis and e_0 a cyclic vector for M. Given a cyclic basis $(e_0,\dots,e_{n-1})$ for M, the expression of $-De_{n-1}$ in this basis gives unique elements $a_0,\dots,a_{n-1}$ in R such that

(1.5.2.1) $$\left(D^n + \sum_{i=0}^{n-1} a_i D^i\right) e_0 = 0.$$

(1.5.3) **Lemma.** (O. Gabber). *Suppose that* (M,D) *is cyclic of rank* $n\geqq 1$ *with cyclic basis* $(e_0,\dots,e_{n-1})$ *and defining relation*

$$\left(D^n + \sum_{i=0}^{n-1} a_i D^i\right) e_0 = 0.$$

Let $(e_0^\vee, \dots, e_{n-1}^\vee)$ *denote the dual basis of* $M^\vee$. *Then* $(M^\vee, D)$ *is cyclic with cyclic vector* $e_{n-1}^\vee$, *and* $e_{n-1}^\vee$ *is annihilated by the formal adjoint*:

$$\left((-D)^n + \sum_{i=0}^{n-1} (-D)^i a_i\right) e_{n-1}^\vee = 0 .$$

Proof. From the defining formulas

$$(De_i^\vee, e_j) + (e_i^\vee, De_j) = \partial(e_i^\vee, e_j) = \partial(\delta_{ij}) = 0 ,$$

we readily calculate

$$\begin{aligned}
&(-D + a_{n-1})\, e_{n-1}^\vee = e_{n-2}^\vee \\
&(-D)\, e_{n-2}^\vee + a_{n-2}\, e_{n-1}^\vee = e_{n-3}^\vee \\
&\quad\vdots \\
&(-D)\, e_i^\vee + a_i\, e_{n-1}^\vee = e_{i-1}^\vee \qquad 1 \leqq i \leqq n-2 \\
&(-D)\, e_0^\vee + a_0\, e_{n-1}^\vee = 0 .
\end{aligned}$$

These relations show that $e_{n-1}^\vee$ is a cyclic vector for $(M^\vee, D)$. Writing D^* in place of $-D$, we rewrite these relations as:

$$\begin{aligned}
&(D^* + a_{n-1})\, e_{n-1}^\vee \equiv e_{n-2}^\vee \\
&(D^*(D^* + a_{n-1}) + a_{n-2})\, e_{n-1}^\vee = e_{n-3}^\vee \\
&(D^*(D^*(D^* + a_{n-1}) + a_{n-2}) + a_{n-3})\, e_{n-1}^\vee = e_{n-4}^\vee \\
&\quad\vdots \\
&(D^*(\cdots(D^*(D^*(D^* + a_{n-1}) + a_{n-2}) + a_{n-3}) + \cdots) + a_0)\, e_{n-1}^\vee = 0 .
\end{aligned}$$

This last relation, multiplied out, is the asserted relation

$$((D^*)^n + (D^*)^{n-1} a_{n-1} + (D^*)^{n-2} a_{n-2} + \cdots + a_0)\, e_{n-1}^\vee = 0 . \quad \text{Q.E.D.}$$

II. The local theory: applications of a result of levelt

2.1. The setting

(2.1.1) Let k be a field of characteristic zero, R a k-algebra which is a complete discrete valuation ring with residue field k, $\mathfrak{m}$ the maximal ideal of R, K the fraction field of R. $\bar{K}$ an algebraic closure of K, and $\bar{k}$ the algebraic closure of k in $\bar{K}$. Given an integer $N \geqq 1$, a finite extension L/K (inside $\bar{K}$) is said to be N-standard if its ramification index is N and if its residue field k' (inside $\bar{k}$) is a galois extension of k which contains the N'th roots of unity. In terms of a uniformizing parameter t in R, we have $k[[t]] \xrightarrow{\sim} R$, $k((t)) \xrightarrow{\sim} K$, and the unique N-standard extension with residue field k' as above is $k'((t^{1/N}))$. Any finite extension of K inside $\bar{K}$ is contained in an N-standard extension for some $N \geqq 1$.

(2.1.2) We denote by $\mathscr{D}$ the ring of all t-adically continuous k-linear differential operators of K to itself. If θ is any non-zero derivation in $\mathscr{D}$, its powers $1, \theta, \theta^2, \dots$ form a K-basis of $\mathscr{D}$ as left K-module.

(2.1.3) We denote by D.E. (K/k) the category of those left $\mathscr{D}$-modules V whose underlying K-vector space is finite-dimensional. In terms of a chosen θ, an object V of D.E. (K/k) is a pair $(M, \nabla(\theta))$ *consisting of a finite-dimensional* K-vector space M together with a k-linear map $\nabla(\theta)$: $M \to M$ satisfying

(2.1.3.1) $$\nabla(\theta)\,(fm) = \theta(f)m + f\nabla(\theta)(m)$$

for all $f \in K$, $m \in M$. In this second description, one sees natural notions of internal hom and tensor product which make D.E. (K/k) into a rigid abelian tensor category (cf. [De-Mi], 1.7) with $\mathrm{End}(\mathbf{1}) = k$ which has a fibre functor with values in K (namely, the functor "underlying K-vector space"). We will see later (2.4.12) that D.E. (K/k) is in fact a neutral Tannakian category over k, by using the fundamental work of Levelt to construct a k-valued fibre functor.

2.2. *Slopes* (Compare [Ince], pp. 424–428, [Ra], pp. 7–12 and [Rob], 1.6).

(2.2.1) Let $N \geqq 1$ be an integer, and L/K an N-standard finite extension field, with residue field k'. We denote by $\mathscr{D}_L$ the ring of all t-adically continuous k-linear differential operators of L to itself. Any operator in $\mathscr{D}_L$ is automatically k'-linear, and every operator D in $\mathscr{D}$ has a unique extension to an element D_L of $\mathscr{D}_L$ which on K coincides with D. [For example, in terms of uniformizing parameter t of K, and the parameter $s = t^{1/N}$ of $L = k'((s))$, the unique extension of $t\dfrac{d}{dt}$ is $\dfrac{1}{N}s\dfrac{d}{ds}$.] Formation of this unique extension defines an injective ring homomorphism $\mathscr{D} \hookrightarrow \mathscr{D}_L$, which gives rise to a canonical isomorphism of right $\mathscr{D}$-modules

(2.2.1.1) $$L \bigotimes_K \mathscr{D} \xrightarrow{\sim} \mathscr{D}_L.$$

There is a natural "extension of scalars" functor

(2.2.1.2) $$\begin{gathered}\mathrm{D.E.}(K/k) \to \mathrm{D.E.}(L/k') \\ V \mapsto V_L = \mathscr{D}_L \bigotimes_{\mathscr{D}} V.\end{gathered}$$

In terms of the $(M, \nabla(\theta))$-description of an object V of D.E. (K/k), V_L is $\left(L \bigotimes_K M, \nabla(\theta_L)\right)$ where θ_L denotes the unique extension of θ to L, and where

(2.2.1.3) $$\nabla(\theta_L)\left(f \bigotimes_K m\right) = \theta_L(f) \bigotimes_K m + f \bigotimes_K \nabla(\theta)(m)$$

for $f \in L$ and $m \in M$.

(2.2.2) According to a fundamental result of Levelt [Le-1], given any non-zero object V in D.E. (K/k), there exists an integer $N \geqq 1$ and an N-standard extension L of K such that V_L is a successive extension of one-dimensional objects of D.E. (L/k').

(2.2.3) If we fix a non-zero derivation θ in $\mathscr{D}$, then any one-dimensional object in D.E. (L/k') is of the form

(2.2.3.1) $$\mathscr{D}_L/\mathscr{D}_L(\theta_L - f)$$

for some $f \in L$, and the group of isomorphism classes of such objects is the group

(2.2.3.2)
(the additive group of L)/the subgroup of elements $\theta_L(g)/g$, for $g \in L^\times$,

via the map which to $\mathscr{D}_L/\mathscr{D}_L(\theta_L - f)$ attaches the image of f in the above quotient group.

(2.2.4) If we write θ in the form

(2.2.4.1) $$\theta = (\text{unit in } k[[t]]) \times t^a \times t \frac{d}{dt}$$

then for any $g \in L^\times$ we have

(2.2.4.2) $$\mathrm{ord}_t(\theta(g)/g) \geqq a \qquad \text{for all } g \in L^\times .$$

Therefore the non-negative rational number

(2.2.4.3) $$\max(0, a - \mathrm{ord}_t(f)) = a - \min(a, \mathrm{ord}_t(f))$$

is a well-defined invariant of the isomorphism class of $\mathscr{D}_L/\mathscr{D}_L(\theta_L - f)$, independent of the auxiliary choice of θ, called its slope (with respect to ord_t).

(2.2.5) Returning to an arbitrary object V in D.E.(K/k) of dimension $n \geqq 1$, its slopes $\lambda_1, \ldots, \lambda_n$ are the n non-negative rational numbers defined as follows. Pick an integer $N \geqq 1$ and an N-standard extension L/K such that V_L is a successive extension of one-dimensional objects, and take for $\lambda_1, \ldots, \lambda_n$ the slopes, as defined above, of the one-dimensional objects (with respect to ord_t). By Jordan-Holder theory, the isomorphism classes of the one-dimensional subquotients of V_L which occur are an intrinsic invariant of V_L, so the slopes $\lambda_1, \ldots, \lambda_n$ are an intrinsic invariant of V_L. Because any two standard extensions (i.e., N-standard for some N) are contained in a third, one sees that the slopes $\lambda_1, \ldots, \lambda_n$ are independent of the auxiliary choice of L as well.

(2.2.6) Similarly, if

(2.2.6.1) $$0 \to V_1 \to V \to V_2 \to 0$$

is a short exact sequence of objects in D.E.(K/k), one has

(2.2.6.2) $$(\text{slopes of } V) = (\text{slopes of } V_1) \cup (\text{slopes of } V_2).$$

(2.2.7) In terms of Levelt's characteristic polynomial $c(X)$ of V (cf. [Le-1]; the characteristic polynomial, unlike the slopes, depends upon the choice of uniformizing parameter t) the slopes may be recovered as follows: if we factor

(2.2.7.1) $$c(X) = \prod_{i=1}^{n} (X - \beta_i), \qquad \beta_i \in \overline{K},$$

then the slopes of V are the numbers

(2.2.7.) $$\max(0, -\mathrm{ord}_t(\beta_i)).$$

Because the characteristic polynomial has coefficients in K, it follows that the slopes have the following integrality property:

(2.2.7.3) each slope λ_i of V occurs with a multiplicity which is a multiple of its exact denominator.

This simple fact, which we view as the differential analogue of the integrality of Swan conductors, i.e., of the Hasse-Arf theorem, immediately yields the following.

(2.2.8) **Irreducibility criterion.** *Let V be an n-dimensional object of* D.E. (K/k) *with $n \geqq 1$, whose slopes satisfy*

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = \begin{pmatrix} \text{a rational number with} \\ \text{exact denominator } n \end{pmatrix}.$$

Then V is a simple object in the category D.E. (K/k).

Proof. If W is a non-zero sub-object of V of dimension r, then its slopes, after renumbering, are $\lambda_1, \ldots, \lambda_r$, whence by (2.2.7.3) r is a multiple of n. Q.E.D.

(2.2.9) In order to make this criterion useful, we need to be able to calculate the slopes when V is given explicitly. Fortunately there is a simple algorithm for doing this when V is presented as a cyclic $\mathscr{D}$-module (in fact one knows (cf. [Ka-4]) that any V is cyclic), and the algorithm itself makes evident the fundamental integrality property (2.2.7.3) of the slopes, independently of the above "characteristic polynomial" argument.

(2.2.10) **Algorithm.** *Let $F(x) \in K[x]$ be a monic polynomial of degree $n \geqq 1$, say*

$$F(x) = \sum_{i=0}^{n} a_i x^i \qquad a_i \in K,\ a_n = 1.$$

Factor it over $\bar{K}$:

$$F(x) = \prod (x - \beta_i).$$

For θ a non-zero derivation in $\mathscr{D}$, we define

$$F(\theta) = \sum_{i=0}^{n} a_i \theta^i \qquad \text{in } \mathscr{D}.$$

If θ is of the form

$$\theta = (\text{unit in } k[[t]]) \times t^a \times t\frac{d}{dt},$$

then the slopes of $V = \mathscr{D}/\mathscr{D}F(\theta)$ are the numbers

$$\max(0, a - \operatorname{ord}_t(\beta_i)), \qquad i = 1, \ldots, n.$$

Proof. Extending scalars to a suitable standard extension L of K, we reduce to the case when V_L is a successive extension of one-dimensional objects of D.E. (L/k'). As explained in ([Rob], §2), this means we have a factorization in $\mathscr{D}$

$$F(\theta) = (\theta - \alpha_1) \cdots (\theta - \alpha_n)$$

with $\alpha_i \in L$, and V_L is a successive extension of the $\mathscr{D}/\mathscr{D}(\theta - \alpha_i)$'s. Therefore the slopes of V are the numbers

$$\max(0, a - \operatorname{ord}_t(\alpha_i)), \qquad i = 1, \ldots, n.$$

That these numbers agree with the

$$\max(0, a-\mathrm{ord}_t(\beta_i)), \qquad i=1,\dots,n$$

when $F(x)=\Pi(x-\beta_i)$ is proven in ([Rob], 1.6). Q.E.D.

(2.2.11) Here are some elementary properties of the slopes, all of which are easily checked by reduction to the one-dimensional case, where they are obvious.

(2.2.11.1) If σ is any continuous automorphism of K which maps k to k, it induces an automorphism, still denoted σ, of $\mathscr{D}$, by the rule

$$\sigma(D)(\sigma(f))=\sigma(D(f))$$

for $D\in\mathscr{D}, f\in K$. [Concretely, σ maps $\frac{d}{dt}$ to $\frac{d}{d(\sigma(t))}$.] Given an object V in D.E. (K/k), we define V^σ to be

$$\mathscr{D}\underset{\sigma\nwarrow\mathscr{D}}{\otimes}V.$$

Then V^σ has the same slopes as V. [Indeed for $V=\mathscr{D}/\mathscr{D}(\theta-f)$, V^σ is $\mathscr{D}/\mathscr{D}(\sigma(\theta)-\sigma(f))$.]

(2.2.11.2) If V has slopes $\lambda_1,\dots,\lambda_n$, then for any $N\geqq 1$, and any N-standard extension L/K, V_L considered as an object of D.E. (L/k') has slopes $N\lambda_1,\dots,N\lambda_n$.

(2.2.11.3) The dual $V^\vee$ of V has the same slopes as V. [Indeed the dual of $\mathscr{D}/\mathscr{D}(\theta-f)$ is $\mathscr{D}/\mathscr{D}(\theta+f)$.]

(2.2.11.4) If V and W are objects of D.E. (K/k) of dimensions $n\geqq 1$ and $m\geqq 1$ respectively, with slopes $\lambda_1,\dots,\lambda_n$ and $\mu_1,\dots,\mu_m$ respectively, then the nm slopes $\nu_{i,j}$ of $V\underset{K}{\otimes}W$ may be indexed by (i,j) in $[1,n]\times[1,m]$ in such a way that

$$\begin{aligned}&0\leqq \nu_{i,j}\leqq \max(\lambda_i,\mu_j)\\ &\nu_{i,j}=\max(\lambda_i,\mu_j) \qquad \text{if } \lambda_i\neq\mu_j.\end{aligned}$$

[Indeed, the tensor product of $\mathscr{D}/\mathscr{D}(\theta-f)$ and $\mathscr{D}/\mathscr{D}(\theta-g)$ is $\mathscr{D}/\mathscr{D}(\theta-f-g)$.]

(2.2.11.5) Fix an isomorphism $K\simeq k((t))$ and an integer $N\geqq 1$, and denote by E the sub-field $k((t^N))$. Because of the identity

$$t^N\frac{d}{d(t^N)}=\frac{1}{N}\,t\,\frac{d}{dt},$$

for any n-dimensional D.E. V on K/k, the underlying E-space of V is an Nn-dimensional D.E. on E/k, denoted $[N]_*(V)$. If V has slopes $\lambda_1,\dots,\lambda_n$, then $[N]_*(V)$ has slopes (λ_1/N repeated N times, $\dots,\lambda_n/N$ repeated N times). [Indeed after extending scalars so that k contains the N'th roots of unity, K/E is an N-standard extension which is galois with group μ_n, and $([N]_*(V))\underset{E}{\otimes}K\xrightarrow{\sim}\bigoplus_{\sigma\in\mathrm{Gal}(K/E)}V^{(\sigma)}$, so the assertion on slopes follows from (2.2.11.2).]

2.3. Irregularity, slope decomposition, and obstructions to descent

(2.3.1) Given an object V in D.E.(K/k) of dimension $n \geqq 1$, with slopes $\lambda_1, \ldots, \lambda_n$, we define its irregularity to be the sum of its slopes:

(2.3.1.1) $$\operatorname{Irr}(V) = \sum_{i=1}^{n} \lambda_i .$$

For the zero-object, we put $\operatorname{Irr}(V)=0$. In view of the fundamental integrality property (1.2.7.3) of the slopes, $\operatorname{Irr}(V)$ is a non-negative *integer*, which vanishes if and only if all slopes of V are zero, i.e., if and only if V has a "regular singular point", cf. ([De-1], II, 6.20).

In view of the algorithm (2.2.10) for the slopes, we have the following simple algorithm for the irregularity:

(2.3.2) **Algorithm.** *Let $F(x) \in K[x]$ be monic of degree $n \geqq 1$;*

$$F(x) = \sum_{i=0}^{n} a_i x^i, \qquad a_i \in K,\ a_n = 1.$$

Let θ be a derivation of the form (unit in $k[[t]]$) $\times\, t\frac{t}{dt}$. Then the irregularity of $V = \mathscr{D}/\mathscr{D}\, F(\theta)$ is given by

$$\operatorname{Irr}(V) = \max\left(0, \max_{0 \leqq i \leqq n-1} (-\operatorname{ord}_t(a_i))\right).$$

(2.3.3) Given an object V in D.E.(K/k), and a rational number $y \geqq 0$, we say that V is "purely of slope y" if (either $V = \{0\}$ or if) all the slopes of V are equal to y.

(2.3.4) **Proposition.** (Compare [Ka-3], 1.1). *Any object V in* D.E.(K/k) *has a unique "slope decomposition" as a direct sum*

$$V = \bigoplus_{y \geqq 0} V(y),$$

indexed by rational numbers $y \geqq 0$, in which $V(y)$ is purely of slope y.

Proof. By galois descent, it suffices to prove the existence and uniqueness of such a decomposition after extension of scalars from K to some standard extension L/K, for by (2.2.11.1) such a decomposition will necessarily be stable by $\operatorname{Gal}(L/K)$. This reduces us to considering the case when V is a successive extension of one-dimensional objects. An induction on $\dim(V)$ reduces us to showing that if V and W are one-dimensional objects in D.E.(K/k) which have different slopes, then $\operatorname{Hom}_{\mathscr{D}}(V, W) = 0$, and any extension of V by W splits uniquely. [Once we know $\operatorname{Hom}_{\mathscr{D}}(V, W) = 0$, the splitting is automatically unique if it exists.] Because V and W have different slopes, they are non-isomorphic, so the required result is a special case of the following lemma.

(2.3.5) **Lemma.** *If V and W are non-isomorphic one-dimensional objects in* D.E.(K/k), *then*

(2.3.5.1) $$\operatorname{Hom}_{\mathscr{D}}(V, W) = 0 = \operatorname{Ext}^1_{\mathscr{D}}(V, W),$$

while for $V=\mathscr{D}/\mathscr{D}(\theta-f)$ itself we have

(2.3.5.2) $$\operatorname{Hom}_{\mathscr{D}}(V,V)=k.$$

(2.3.5.3) $$\operatorname{Ext}^1_{\mathscr{D}}(V,V)$$

is a one-dimensional k-space [with basis the class of $\mathscr{D}/\mathscr{D}(\theta-f)^2$].

Proof. Twisting by $V^\vee$ we reduce to the case when V is trivial, i.e., $V\simeq\mathscr{D}/\mathscr{D}\,\theta$. Then $\mathscr{D}\xrightarrow{\times\theta}\mathscr{D}$ is a $\mathscr{D}$-free resolution of V as left $\mathscr{D}$-module, so we have

$$\operatorname{Ext}^i_{\mathscr{D}}(\mathscr{D}/\mathscr{D}\,\theta, W)=\begin{cases}\ker\theta\colon W\to W & i=0\\ W/\theta(W) & i=1\\ 0 & i\geqq 2.\end{cases}$$

If W has slope $a\geqq 1$, then W is of the form $\mathscr{D}/\mathscr{D}\left(t\dfrac{t}{dt}-f\right)$ where $\operatorname{ord}_t(f)=-a$. Rewriting W as $\mathscr{D}/\mathscr{D}(\theta-1)$ with $\theta=(1/f)\,t\dfrac{d}{dt}$, the derivation θ is topologically nilpotent on K. Identifying W with K via the basis "1" of $W=\mathscr{D}/\mathscr{D}(\theta-1)$, the action of θ on W becomes the action of $1+\theta$ on K. As θ is topologically nilpotent on K, the map $1+\theta$ is bijective on K, i.e., θ is bijective on W.

If W has slope zero, then W is of the form $\mathscr{D}/\mathscr{D}\left(t\dfrac{d}{dt}-f\right)$ where $\operatorname{ord}_t(f)\geqq 0$. Modifying f by adding a logarithmic derivative $t\dfrac{d}{dt}(g)/g$ with g a principal unit in K, we may assume that f is constant, say $f=a\in k$. Then the kernel and cokernel of θ on W become, for $\theta=t\dfrac{d}{dt}$, the kernel and cokernel of $t\dfrac{d}{dt}+a$ on K. If W is non-trivial, then $a\notin\mathbf{Z}$, and $t\dfrac{d}{dt}+a$ is bijective on K.

If W is trivial, then we may take $f=0$, and the ker and coker of $t\dfrac{d}{dt}$ are each k itself, viewed as the constant of $K=k((t))$. *Q.E.D.*

(2.3.6) *Remark.* (Compare [Ka-3], 1.6 and 1.9). In terms of the slope decomposition $V\simeq\bigoplus V(y)$, we have

(2.3.6.1) $$\operatorname{Irr}(V)=\sum_y y\dim(V(y)),$$

and each summand $y\dim(V(y))$ is an integer.

(2.3.7) Fix a uniformizing parameter t. For each $a\in k^\times$, denote by σ_a the k-linear continuous automorphism of K given by $t\mapsto at$.

(2.3.8) **Proposition.** (Compare [Ka-3], 4.1.6). *Let $a\in k^\times$, and V an object of* D.E.(K/k). *Suppose there exists an isomorphism $V^{(\sigma_a)}\simeq V$. Then*

(2.3.8.1) *If $a\in k^\times$ is not a root of unity, then* $\operatorname{Irr}(V)=0$.

(2.3.8.2) *If $a \in k^\times$ is a root of unity of exact order N, then*

$$\mathrm{Irr}(V) \equiv 0 \mod N.$$

Proof. (1). After an extension of scalars from $k((t))$ to a suitable $L = k'((t^{1/N}))$, V_L becomes a successive extension of one-dimensional objects. If we enlarge k' so that it contains $a^{1/N}$, the automorphism σ_a of $k((t))$ extends to a k'-linear continuous automorphism $\sigma_{a^{1/N}}$ of $k'((t^{1/N}))$, given by $t^{1/N} \mapsto a^{1/N} t^{1/N}$. This automorphism $\sigma_{a^{1/N}}$ necessarily permutes the finitely many distinct isomorphism classes of one-dimensional sub-quotients of V_L, so replacing $a^{1/N}$ by a power of itself, we are reduced to proving (1) universally in the case when V is a one-dimensional object in D.E.(K/k). Writing V as $\mathscr{D}/\mathscr{D}\left(t\frac{d}{dt} - f\right)$, the isomorphism class of V is the image of f in

$$(2.3.8.3) \qquad K/t\frac{d}{dt}\log(K^\times) = k((t))/\mathbf{Z} + tk[[t]] \xleftarrow{\sim} k[t^{-1}]/\mathbf{Z},$$

on which σ_a operates by $t \mapsto at$. Thus if

$$f(t) = \sum_{i > -\infty} b_i t^i \qquad b_i \in k$$

then $V \simeq V^{(\sigma_a)}$ if and only if $f(t) \equiv f(at) \mod tk[[t]]$. Because $a^n \neq 1$ for all $n \neq 0$, $f(t) \equiv f(at) \mod tk[[t]]$ if and only if $f \in k[[t]]$, i.e., if and only if $\mathrm{Irr}(V) = 0$.

To prove (2), we may, by extending scalars, assume k to be algebraically closed. Replacing V by its semi-simplication, we may assume V is semi-simple. By decomposing the set of isotypical components of V into orbits under σ_a, we may assume that σ_a cyclically permutes the isotypical components of V. If V has precisely $d \geqq 1$ isotypical components, then $N = dN_1$, each isotypical component is stable by σ_b with $b = a^d$, and by (2.2.11.1) all isotypical components have the same irregularity. Therefore (replacing V by any of its isotypical components, a by a^d and N by N_1) we are reduced to the case when V is isotypical, say $V \simeq (V_1)^k$ with V_1 irreducible. By Jordan-Holder theory, $V^{(\sigma_a)} \xrightarrow{\sim} V$ implies $(V_1)^{(\sigma_a)} \xrightarrow{\sim} V_1$, so we may reduce to the case when V is irreducible.

To prove (2) when V is irreducible and k is algebraically closed we will show that V descends to an object V_0 in D.E.(K_0/k), where K_0 is the subfield $k((t^N))$ of $K = k((t))$. For by (2.2.11.2), we will have

$$\mathrm{Irr}(V) = N\,\mathrm{Irr}(V_0).$$

To perform this descent, view σ_a as a generator of the cyclic galois group of K/K_0, and view the given isomorphism φ: $V^{(\sigma_a)} \simeq V$ as a σ_a-linear automorphism of V. If $\varphi^N = id$, then we can descend. In general, φ^N is a (k-linear) automorphism of V. Let us admit temporarily that $\mathrm{End}_{\mathscr{D}}(V) = k$. Then φ^N is in $k^\times$, say $\varphi^N = \alpha$. Replacing φ by $\alpha^{-1/N}$, we have $\varphi^N = 1$.

To see that $\mathrm{End}_{\mathscr{D}}(V) = k$, we notice that as V is irreducible, $\mathrm{End}_{\mathscr{D}}(V)$ is certainly a division algebra over k. Because k is algebraically closed, any finite-dimensional division algebra over k is necessarily k itself, so it suffices to show that $\mathrm{End}_{\mathscr{D}}(V)$ is finite-dimensional over k. But this is a general fact:

(2.3.9) **Lemma.** *For any two objects V, W in* D.E.(K/k),

$$\dim_k(\mathrm{Hom}_{\mathscr{D}}(V, W)) \leqq \dim_K(V)\,\dim_K(W).$$

Proof. Using the internal hom, we have

$$\mathrm{Hom}_{\mathscr{D}}(V, W) = \mathrm{Hom}_{\mathscr{D}}\left(K, V^{\vee} \bigotimes_K W\right),$$

so we are reduced to the case V trivial. If $\mathrm{Hom}_{\mathscr{D}}(K, W)$ is $\neq 0$, then W contains a one-dimensional sub-object isomorphic to K, so by induction on $\dim W$ we reduce to the case $W = K$. But $\mathrm{Hom}_{\mathscr{D}}(K, K) = k$, because k is the "field of constants" in K. Q.E.D.

2.4. *Canonical extension: construction of a fibre functor* (Compare [Ka-5])

(2.4.1) In this section, we must fix a uniformizing parameter t in R, by means of which we identify $k((t)) \simeq K$. We denote by $\mathbf{G}_m \otimes k$ the multiplicative group over k with coordinate "x":

(2.4.1.1) $$\mathbf{G}_m \otimes k = \mathrm{Spec}(k[x, x^{-1}]).$$

By means of the k-linear embedding

(2.4.1.2) $$\begin{aligned} k[x, x^{-1}] &\hookrightarrow k((t)) \\ x^{-1} &\mapsto t \end{aligned}$$

we view K as the completion at ∞ of the function field of $\mathbf{G}_m \otimes k$. Thus we have a natural inverse image functor

(2.4.1.3) $$\begin{aligned} \mathrm{D.E.}(\mathbf{G}_m \otimes k/k) &\to \mathrm{D.E.}(K/k) \\ V &\mapsto V_\infty. \end{aligned}$$

(2.4.2) We will interpret $\mathrm{D.E.}(\mathbf{G}_m \otimes k/k)$ as the category of $x\dfrac{d}{dx}$-modules over $k[x, x^{-1}]$, cf. (1.5.1). The rank-one objects L which are regular singular at zero are those of the form

(2.4.2.1) $$\left(k[x, x^{-1}],\ D = x\frac{d}{dx} + \sum a_i x^i\right)$$

where $\sum a_i x^i \in k[x]$, and the group of isomorphism classes of such L is the additive group

(2.4.2.2) $$k[x]/\mathbf{Z},$$

via the map $L \mapsto \sum a_i x^i \bmod \mathbf{Z}$. In view of (2.2.3.2) and (2.3.8.3), we see that the inverse image functor

(2.4.2.3) $$\mathrm{D.E.}(\mathbf{G}_m \otimes k/k)_{RS\,\mathrm{at}\,0} \to \mathrm{D.E.}(K/k)$$

induces an *equivalence* between the full subcategories of rank-one objects.

(2.4.3) An object of $\mathrm{D.E.}(\mathbf{G}_m \otimes k/k)$ (resp. of $\mathrm{D.E.}(K/k)$) is called "R.S. unipotent" if it is isomorphic to a successive extension of the trivial object $\left(k[x, x^{-1}], D = x\dfrac{d}{dx}\right)$ $\left(\text{resp. } (K, D) = t\dfrac{d}{dt}\right)$ by itself. We denote by (Nilp End$/k$)

the category of pairs (V,N) consisting of a finite-dimensional k-vector space V together with a k-linear nilpotent endomorphism N of V, with maps the $k[N]$-linear maps. The natural functors

(2.4.3.1)
$$\begin{aligned}(\text{Nilp End}/k) &\to \begin{pmatrix}\text{R.S. unip. objects}\\ \text{in D.E.}(\mathbf{G}_m \otimes k/k)\end{pmatrix}\\ (V,N) &\mapsto \left(k[x,x^{-1}] \underset{k}{\otimes} V,\ D = x\frac{d}{dx}\otimes 1 + 1\otimes N\right),\end{aligned}$$

(2.4.3.2)
$$\begin{aligned}(\text{Nilp End}/k) &\to \begin{pmatrix}\text{R.S. unip. objects}\\ \text{in D.E.}(K/k)\end{pmatrix}\\ (V,N) &\to \left(K \underset{k}{\otimes} V,\ D = t\frac{d}{dt}\otimes 1 - 1\otimes N\right),\end{aligned}$$

are both equivalences, with inverses given by

(2.4.3.3)
$$(M,D) \mapsto \left(\bigcup_{n\geqq 1} \operatorname{Ker}(D^n),\ \pm D\right).$$

Therefore the inverse image functor

(2.4.3.4)
$$\text{D.E.}(\mathbf{G}_m \otimes k/k) \to \text{D.E.}(K/k)$$

induces an equivalence on the full subcategories of R.S. unipotent objects.

(2.4.4) An object of D.E. $(\mathbf{G}_m \otimes k/k)$ is called very special if it is a finite direct sum of objects of the form $L \otimes U$ where

(2.4.4.1) L is of rank-one, regular singular at zero
U is R.S. unipotent.

(2.4.5) An object of D.E. (K/k) is called very special if it is a successive extension of one-dimensional objects.

(2.4.6) **Theorem.** (Levelt). *The inverse image functor*

$$\text{D.E.}(\mathbf{G}_m \otimes k/k) \to \text{D.E.}(K/k)$$

induces an equivalence between the full subcategories of very special objects.

Proof. This is immediate from (2.4.2.3) and the preceding discussion. Q.E.D.

(2.4.7) Given an integer $N \geqq 1$ and a finite galois extension k' of k (inside $\bar{k}$) which contains the N'th roots of unity, an object of D.E. $(\mathbf{G}_m \otimes k/k)$ is called (N,k')-special if its inverse image in D.E. $(\mathbf{G}_m \otimes k'/k')$ by the composite map

(2.4.7.1)
$$\mathbf{G}_m \otimes k' \xrightarrow[(x\mapsto x^N)\otimes \mathrm{id}]{[N]} \mathbf{G}_m \otimes k' \xrightarrow[\mathrm{id}\otimes(\text{extn. of scalars})]{} \mathbf{G}_m \otimes k$$

is very special. An object V of D.E. (K/k) is called (N,k')-special if for the unique N-special extension L/K with residue field k', V_L in D.E. (L/k') is a successive extension of one-dimensional objects.

By combining the previous theorem (2.4.6) with the descent argument of ([Ka-5], 1.4.1), we obtain

(2.4.8) **Theorem.** *For any (N, k') as above, the inverse image functor*

$$\mathrm{D.E.}(\mathbf{G}_m \otimes k) \to \mathrm{D.E.}(K/k)$$

induces an equivalence between the full subcategories of (N, k')-special objects.

(2.4.9) Let us say that an object of $\mathrm{D.E.}(\mathbf{G}_m \otimes k/k)$ (resp. of $\mathrm{D.E.}(K/k)$) is special if it is (N, k')-special for some (N, k') as above. By Levelt's fundamental result ([Le-1]), *every* object of $\mathrm{D.E.}(K/k)$ is special. So, passing to the limit over (N, k'), we obtain

(2.4.10) **Theorem.** *The inverse image functor*

$$\mathrm{D.E.}(\mathbf{G}_m \otimes k/k) \to \mathrm{D.E.}(K/k),$$

when restricted to the full subcategory $\mathrm{S.D.E.}(\mathbf{G}_m \otimes k/k)$ of $\mathrm{D.E.}(\mathbf{G}_m \otimes k/k)$ consisting of the special objects, induces an equivalence of tensor categories

$$\mathrm{S.D.E.}(\mathbf{G}_m \otimes k/k) \xrightarrow{\sim} \mathrm{D.E.}(K/k).$$

(2.4.11) "The" quasi-inverse functor

$$\mathrm{D.E.}(K/k) \xrightarrow{\sim} \mathrm{S.D.E.}(\mathbf{G}_m \otimes k/k)$$

denoted

$$V \mapsto V^{\mathrm{can}},$$

is called the "canonical extension".

(2.4.12) **Corollary.** *For any point $a \in k^\times = (\mathbf{G}_m \otimes k)(k)$, the composite*

$$\begin{tikzcd}
\mathrm{D.E.}(K/k) \arrow[r, "\mathrm{can.}"] \arrow[ddr, dashed] & \mathrm{S.D.E.}(\mathbf{G}_m \otimes k/k) \arrow[d, hook] \\
& \mathrm{D.E.}(\mathbf{G}_m \otimes k/k) \arrow[d, "\text{fibre at } a"] \\
& \begin{pmatrix} \text{fin. dim. } k\text{-vect.} \\ \text{spaces} \end{pmatrix}
\end{tikzcd}$$

is a k-valued fibre functor on $\mathrm{D.E.}(K/k)$.

(2.4.13) For any object V in $\mathrm{D.E.}(K/k)$, and any fibre functor ω on $\mathrm{D.E.}(K/k)$, the "local differential galois group" $G_{\mathrm{loc}}(V, \omega)$ of V as object of $\mathrm{D.E.}(K/k)$ is defined by

$$G_{\mathrm{loc}}(V, \omega) \overset{\mathrm{dfn}}{=} \mathscr{A}ut^{\otimes}(\omega | \langle V \rangle).$$

(2.4.14) **Corollary.** *For any object V in $\mathrm{D.E.}(K/k)$, and any fibre functor ω on $\mathrm{D.E.}(K/k)$ of the form $\omega_a \circ$ can for some $a \in k^\times$ (i.e., $\omega(V) =$ fibre at a of V^{can}), we* *have*

$$G_{\mathrm{loc}}(V, \omega) = G_{\mathrm{gal}}(V^{\mathrm{can}}, \omega_a)$$

inside $\mathscr{GL}(\omega_a(V^{\mathrm{can}}))$.

Proof. It is obvious from the definition of "special" that any sub-object in D.E. $(\mathbf{G}_m \otimes k/k)$ of a special object is itself special. Therefore the category $\langle V^{\mathrm{can}} \rangle$ is the same whether we view V^{can} as lying in S.D.E. or in D.E. Therefore the functors

$$\mathrm{D.E.}(K/k) \xleftarrow{\sim} \mathrm{S.D.E.}(\mathbf{G}_m \otimes k/k) \subset \mathrm{D.E.}(\mathbf{G}_m \otimes k/k)$$

induce equivalences

$$\langle V \rangle \xleftarrow{\sim} \langle V^{\mathrm{can}} \text{ in S.D.E.} \rangle \xrightarrow{\sim} \langle V^{\mathrm{can}} \text{ in D.E.} \rangle. \quad \text{Q.E.D.}$$

(2.4.15) **Corollary.** (O. Gabber). *For L/k any extension field of k, $K_L = L((t))$, V any object of* D.E. (K/k), *V_L its inverse image in* D.E. (K_L/L), *and any $a \in k^\times$, we have*

$$G_{\mathrm{loc}}(V_L, \omega_a^L \circ \mathrm{can}) = G_{\mathrm{loc}}(V, \omega_a \circ \mathrm{can}) \bigotimes_k L$$

inside $\mathscr{GL}(\omega_a(V^{\mathrm{can}})) \bigotimes_k L$.

Proof. Since $(V_L)^{\mathrm{can}} \simeq (V^{\mathrm{can}})_L$, this follows by combining (2.4.14) with (1.3.2). Q.E.D.

(2.4.16) **Corollary** (O. Gabber). *Let V be a* D.E. *on* $\mathbf{G}_m \otimes k/k$, *and L/k an extension field of k. Then V is special on $\mathbf{G}_m \otimes k/k$ if and only if V_L is special on $\mathbf{G}_m \otimes L/L$.*

Proof. The "only if" direction is trivial. Denoting by "$\wedge$" passage to the formal completion at ∞, we have

$$V_L \text{ special} \Leftrightarrow V_L \xrightarrow{\sim} ((V_L)^\wedge)^{\mathrm{can}},$$

while for any V on $\mathbf{G}_m \otimes k/k$ we have

$$((V^\wedge)^{\mathrm{can}})_L \simeq ((V_L)^\wedge)^{\mathrm{can}}.$$

Therefore if V_L is special, we have an L-isomorphism

$$V_L \simeq ((V^\wedge)^{\mathrm{can}})_L.$$

In view of (4.1.2) (proven later, but with no circularity!), the existence of the above L-isomorphism implies the existence of a k-isomorphism $V \xrightarrow{\sim} (V^\wedge)^{\mathrm{can}}$, whence V is special if V_L is. Q.E.D.

2.5. The local differential galois group; upper numbering

(2.5.1) Fix a k-valued fibre functor ω on D.E. (K/k). We denote by

(2.5.1.1) $$I \stackrel{\mathrm{dfn}}{=\!=} \mathscr{A}ut^{\otimes}(\omega),$$

read – the local differential galois group, – the pro-algebraic affine algebraic group over k whose finite-dimensional k-representations "are" the objects of D.E. (K/k). We think of I as being the differential analogue of the local galois group of a local field of finite residue characteristic. [Though our notation, "I" for "inertia" is only reasonable when k is algebraically closed.]

(2.5.2) Pursuing this analogy, we next define the "upper numbering filtration" on I. For every real number $x \geqq 0$, we denote by

(2.5.2.1) $$\mathrm{D.E.}^{(\leqq x)}(K/k)$$

the full subcategory of D.E. (K/k) of objects all of whose slopes are $\leqq x$. Similarly, for real $x > 0$, we denote by $\mathrm{D.E.}^{(<x)}(K/k)$ the full subcategory of D.E. (K/k) of objects all of whose slopes are $< x$. As is obvious from (2.2.11.3–4) and (2.2.6.2) both $\mathrm{D.E.}^{(\leqq x)}$ and $\mathrm{D.E.}^{(<x)}$ are stable by internal hom, tensor product, and subquotients.

(2.5.3) Dual to the inclusions

(2.5.3.1) $$\begin{gathered} \mathrm{D.E.}^{(<x)}(K/k) \subset \mathrm{D.E.}(K/k) \\ \mathrm{D.E.}^{(\leqq x)}(K/k) \subset \mathrm{D.E.}(K/k) \end{gathered}$$

we have homomorphisms of the corresponding groups

(2.5.3.2) $$\begin{gathered} I \to \mathscr{A}ut^{\otimes}(\omega \mid \mathrm{D.E.}^{(<x)}(K/k)) \\ I \to \mathscr{A}ut^{\otimes}(\omega \mid \mathrm{D.E.}^{(\leqq x)}(K/k)), \end{gathered}$$

both of which are faithfully flat (by [De-Mi], 2.21). Their kernels are closed normal subgroups of I, denoted $I^{(x)}$ and $I^{(x+)}$ respectively (defined for $x > 0$ and for $x \geqq 0$ respectively). Concretely, if for each object V in D.E. (K/k) we denote by ρ_V: $I \to \mathscr{GL}(\omega(V))$ the corresponding representation of I, we have

(2.5.3.3) $$I^{(x)} = \bigcap_{V \in \mathrm{D.E.}^{(<x)}(K/k)} \mathrm{Ker}(\rho_V)$$

(2.5.3.4) $$I^{(x+)} = \bigcap_{V \in \mathrm{D.E.}^{(\leqq x)}(K/k)} \mathrm{Ker}(\rho_V).$$

By Tannakian duality, we have, for a given object V in D.E. (K/k), the following equivalences:

(2.5.3.5) For $x > 0$, V has all slopes $< x \Leftrightarrow I^{(x)} \subset \mathrm{Ker}(\rho_V)$.

(2.5.3.6) For $x \geqq 0$, V has all slopes $\leqq x \Leftrightarrow I^{(x+)} \subset \mathrm{Ker}(\rho_V)$.

(2.5.4) For $0 < x < y$, we have

(2.5.4.1) $$I \supset I^{(0+)} \supset I^{(x)} \supset I^{(x+)} \supset I^{(y)},$$

which we view as the differential analogue of the "upper numbering filtration" (cf. [Se-1], pp. 80–82).

(2.5.5) Given an object V in D.E. (K/k), its slope decomposition

(2.5.5.1) $$V = \bigoplus_{y \geqq 0} V(y)$$

is characterized in terms of the representation ρ_V of I on $\omega(V)$ as follows: its image under ω is the unique I-stable decomposition of $\omega(V)$ in which

(2.5.5.2) $\quad I^{(0+)}$ operates trivially on $\omega(V(0))$, for $y>0$, $I^{(y+)}$ acts trivially on $\omega(V(y))$, but $I^{(y)}$ has no non-zero invariants in $\omega(V(y))$.

(2.5.6) Given a non-zero object V of D.E. (K/k), we denote by $\rho_V\colon I\to \mathscr{GL}(\omega(V))$ the corresponding representation of I, and by

(2.5.6.1) $$G_{\mathrm{loc}}(V,\omega)\overset{\mathrm{dfn}}{=}\rho_V(I)\subset \mathscr{GL}(\omega(V))$$

the image of this representation. Equivalently (just as in (1.1.5)), we have (cf. (2.4.13))

(2.5.6.2) $$G_{\mathrm{loc}}(V,\omega)=\mathscr{A}ut^{\otimes}(\omega\,|\langle V\rangle).$$

(2.5.7) The largest slope of V may be described in terms of the representation ρ_V as

(2.5.7.1) $$=\text{g.l.b. }\{\text{real } x\geqq 0 \text{ such that } I^{(x+)}\subset \mathrm{Ker}(\rho_V)\}.$$

It is thus an intrinsic invariant of the subgroup $\mathrm{Ker}(\rho_V)$.

(2.5.8) In view of (2.5.2), the largest slope of V is equal to the sup, over all objects W in $\langle V\rangle$, of the largest slope of W.

(2.5.9) **Proposition.** *Let V be a non-zero object of* D.E. (K/k), *with largest slope written a/N in lowest terms (i.e., $a\geqq 0$ in $\mathbf{Z}$, $N\geqq 1$ in $\mathbf{Z}$, and $(a,N)=1$).*

(2.5.9.1) *The $\bar{k}$-algebraic group $G_{\mathrm{loc}}(V,\omega)\otimes_k \bar{k}$ has no faithful $\bar{k}$-representation of dimension $<N$.*

(2.5.9.2) *If $N=\dim(V)$, the given inclusion of $G_{\mathrm{loc}}(V,\omega)$ in $\mathscr{GL}(\omega(V))$ is an absolutely irreducible representation of $G_{\mathrm{loc}}(V,\omega)$.*

(2.5.9.3) *If $N=\dim(V)$, the given inclusion of $G_{\mathrm{loc}}(V,\omega)$ in $\mathscr{GL}(\omega(V))$, viewed as a representation of $G_{\mathrm{loc}}(V,\omega)$, is not isomorphic to a tensor product of two strictly lower-dimensional representations of $G_{\mathrm{loc}}(V,\omega)$.*

Proof. Since any two k-valued fibre functors are $\bar{k}$-isomorphic, we may suppose ω is the fibre functor constructed by picking a uniformizing parameter t and taking the fibre at a point of $\mathbf{G}_m(k)$ of the corresponding canonical extension. If we extend scalars from $k((t))=K$ to $\bar{k}((t))$, the slopes of V do not change, while $G_{\mathrm{loc}}(V,\omega)$ is replaced by $G_{\mathrm{loc}}(V,\omega)\otimes_k \bar{k}$ (cf. 2.4.15). So it certainly suffices to prove the proposition in the case $k=\bar{k}$.

(1) If Λ is a faithful k-representation of $G_{\mathrm{loc}}(V,\omega)$, corresponding to an object W of $\langle V\rangle$, then W has the same largest slope, a/N, as V (because $\rho_W=\Lambda$, ρ_V has $\mathrm{Ker}(\rho_W)=\mathrm{Ker}(\rho_V)$; alternately, because $\langle W\rangle=\langle V\rangle$, cf. (2.5.8)). By the fundamental integrality property of slopes (2.2.7.3), the multiplicity of a/N as slope of W is multiple of N, so $\geqq N$.

(2) If $N=\dim V$, then a/N is the unique slope of V – there is room for no others by (2.2.7.3) – and so V is irreducible by (2.2.8).

(3) If $V = V_1 \otimes V_2$ with V_1 and V_2 in $\langle V \rangle$, and with $N_i \overset{\text{dfn}}{=} \dim(V_i) < N = \dim(V)$ for $i = 1,2$, we obtain a contradiction as follows. Let λ_i denote the largest slope of V_i. Then $\lambda_i \leqq a/N$ (because $V_i \in \langle V \rangle$, cf. (2.5.8)). But λ_i has exact denominator $\leqq N_i$, so its exact denominator is certainly $< N$, while by hypotheses a/N has exact denominator N. Therefore $\lambda_i \neq a/N$, so we must have $\lambda_i < a/N$. But the largest slope of $V = V_1 \otimes V_2$ is $\leqq \sup(\lambda_1, \lambda_2)$ (cf. (2.2.11.4)), contradiction. Q.E.D.

2.6. *Upper numbering and change of field; structure theorems*

(2.6.1) Let $N \geqq 1$, and suppose k contains the N'th roots of unity. Denote by K_N the extension $k((t^{1/N}))$ of K, and by $\mathrm{D.E.}(K_N/K/k)$ the full subcategory of $\mathrm{D.E.}(K/k)$ consisting of the objects which become trivial over K_N. The diagram of functors

(2.6.1.1) $$\mathrm{D.E.}(K_N/k) \xleftarrow[K]{\otimes K_N} \mathrm{D.E.}(K/k) \xleftarrow{\text{incl.}} \mathrm{D.E.}(K_N/K/k)$$

gives rise to an exact sequence of local differential galois groups (with respect to compatible fibre functors)

(2.6.1.2) $$1 \to I(K_N/k) \to I(K/k) \to \mu_N \to 1,$$

cf. (1.4.4).

(2.6.2) **Proposition.** *For any real numbers $x \geqq 0$, $y > 0$, the above inclusion of $I(K_N/k)$ in $I(K/k)$ induces isomorphisms of "upper numbering" subgroups*

(2.6.2.1) $$I(K_N/k)^{(Nx+)} \xrightarrow{\sim} I(K/k)^{(x+)}$$

(2.6.2.2) $$I(K_N/k)^{(Ny)} \xrightarrow{\sim} I(K/k)^{(y)}.$$

Proof. The diagram of functors

(2.6.2.3) $$\mathrm{D.E.}(K_N/k)^{(\leqq Nx)} \xleftarrow[K]{\otimes K_N} \mathrm{D.E.}(K/k)^{(\leqq x)} \xleftarrow{\text{incl.}} \mathrm{D.E.}(K_N/K/k)$$

gives, by (2.2.11.5) and the same argument as in (1.4.3–4), an exact sequence

(2.6.2.4) $$1 \to I(K_N/k)/I(K_N/k)^{(Nx+)} \to I(K/k)/I(K/k)^{(x+)} \to \mu_N \to 1$$

which is the quotient of the exact sequence (2.6.1.2) above, so the snake lemma gives the assertion for $(x+)$. Similarly for (y). Q.E.D.

(2.6.3) **Proposition.** *Suppose k is algebraically closed. Then for every integer $N \geqq 1$, $I(K_N/k)$ is the unique closed subgroup of index N in I.*

Proof. Let $\Gamma \subset I$ be a closed subgroup of index N, and consider the permutation representation of I in I/Γ. Take the associated N-dimensional k-linear representation of I; the corresponding object W in $\mathrm{D.E.}(K/k)$ has $G_{\mathrm{loc}}(W, \omega)$ a finite group. Pick a parameter t of K, i.e., a k-isomorphism $K \simeq k((t))$, and consider the associated canonical extension W^{can} on $\mathbf{G}_m \otimes k$. By construction of the canonical extension, we have

$$G_{\mathrm{loc}}(W,\omega) \xrightarrow{\sim} G_{\mathrm{gal}}(W^{\mathrm{can}},\tilde{\omega}).$$

Therefore W^{can} on $\mathbf{G}_m \otimes k$ has its G_{gal} finite, so it becomes trivial on a finite etale covering of $\mathbf{G}_m \otimes k$, necessarily of the form $[M]$: $\mathbf{G}_m \otimes k \to \mathbf{G}_m \otimes k$, $x \mapsto x^M$, for some $M \geqq 1$. Returning to W itself, we see that W becomes trivial on some extension field K_M, i.e., $I(K_M/k)$ acts trivially on I/Γ. Therefore we have inclusions

$$I(K_M/k) \subset \Gamma \subset I.$$

For any $M \geqq 1$, $I/I(K_M/k)$ is cyclic of order M, canonically isomorphic to $\mathrm{Gal}(K_M/k)$. Therefore $N|M$, and $\Gamma/I(K_M/k)$ is the unique subgroup of index N in $I/I(K_M/k)$; whence $\Gamma = I(K_N/k)$ by unicity. Q.E.D.

(2.6.4) **Proposition.** *Suppose that k is algebraically closed. If*

$$\rho\colon I \to \mathscr{A}ut(M)$$

is any finite-dimensional k-representation of I, then (2.6.4.1) *the restriction of ρ to $I^{(0+)}$ is diagonalizable.*

(2.6.4.2) *For any real numbers $x \geqq 0$ and $y > 0$, the images $\rho(I^{(x+)})$ and $\rho(I^{(y)})$ in $\mathscr{A}ut(M)$ are connected tori.*

Proof. In virtue of (2.6.2), we may replace ρ by its restriction to any subgroup $I(K_N/k) \subset I$, for any integer $N \geqq 1$. In virtue of (2.2.2), there exists an integer $N \geqq 1$ such that for the object V in $\mathrm{D.E.}(K/k)$ corresponding to ρ, $V \underset{k}{\otimes} K_N$ has a direct sum decomposition

$$V \underset{k}{\otimes} K_N \simeq \bigoplus L_i \otimes U_i$$

where the L_i are one-dimensional objects of $\mathrm{D.E.}(K_N/k)$, and where the U_i, successive extensions of the trivial object, have all their slopes equal to zero. Therefore as representation of $I(K_N/k)$, we have

$$M \simeq \bigoplus (\text{a char. of } I(K_n/k)) \otimes \begin{pmatrix} \text{a rep. of } I(K_n/k) \\ \text{trivial on } I(K_N/k)^{(0+)} \end{pmatrix}.$$

This proves (1), and shows that the restriction of ρ to $I^{(0+)}$ is of the form

$$\begin{pmatrix} \chi_1 & & 0 \\ & \ddots & \\ 0 & & \chi_n \end{pmatrix}$$

where the χ_i are characters of $I^{(0+)}$ which extend to characters of $I(K_N/k)$ for some $N \geqq 1$.

To prove that for any real numbers $x \geqq 0$ and $y > 0$, the images $\rho(I^{(x+)})$ and $\rho(I^{(y)})$ are connected tori, we must show that for any integers $a_1, \ldots, a_n$, the character

$$\chi = (\chi_1)^{a_1} \cdots (\chi_n)^{a_n},$$

when restricted to either $I^{(x+)}$ or to $I^{(y)}$, is *either* trivial or is *not* of finite order.

Interpreting such a character χ as arising from a one-dimensional object L of $\mathrm{D.E.}(K_N/k)$, recall (2.5.3.5–6) that χ is trivial on $I^{(x+)}$ (resp. on $I^{(y)}$) if and only the slope of L, as object of D.E.(K_N/k), is $\leqq Nx$ (resp. $< Ny$). But for any integer $k \neq 0$, $L^{\otimes k}$ has the *same* slope as L does $\left[\text{if } L \text{ is } \left(K_N, t\frac{d}{dt}+f\right) \text{ with } f \in K_N, \text{ then } L^{\otimes k} \text{ is } \left(K_N, t\frac{d}{dt}+k\cdot f\right)\right]$. This means exactly that if χ is of finite order on either $I^{(x+)}$ or on $I^{(y)}$, then χ is already trivial on that group. Q.E.D.

(2.6.5) **Corollary.** *Suppose that k is algebraically closed, and denote by $K_\infty = \bigcup_{N \geq 1} K_N$ the algebraic closure of K, and by $\mathcal{O}_\infty = \bigcup \mathcal{O}_{K_N}$ the ring of integers in K_∞. Then, in terms of a uniformizing parameter t of K, we have*

(2.6.5.1) *$I^{(0+)}$ is the pro-torus over k whose character group is*

$$K_\infty/\mathcal{O}_\infty = \bigcup_{N \geq 1} K_n/\mathcal{O}_{K_N}$$

(2.6.5.2) *$I^{(x+)}$, for any real $x \geqq 0$, is the pro-torus over k whose character group is*

$$K_\infty/\{f \in k_\infty \text{ with } \operatorname{ord}_t(f) \geqq -x\}.$$

(2.6.5.3) *$I^{(y)}$, for any real $y > 0$, is the pro-torus over k whose character group is*

$$K_\infty/\{f \in K_\infty \text{ with } \operatorname{ord}_t(f) > -y\}.$$

In this identification, an object L in some D.E.(K_N/k) of the form $\left(K_N, t\frac{d}{dt}+f\right)$ with $f \in K_N$ gives rise to the character named by the image of f in the named quotient of K_∞.

Proof. Because the groups $I^{(0+)}$, $I^{(x+)}$, $I^{(y)}$ are all closed subgroups of I, they are the inverse limits of their images in "all" the finite dimensional representations of I. Therefore each is a pro-torus. The preceding proof shows that any character χ of one of these groups which occurs in a finite-dimensional representation of I extends to a character of $I(K_N/k)$ for some $N \geqq 1$. This, together with (2.3.8.3) and (2.5.3.3–4), gives the asserted formulas for the character groups. Q.E.D.

(2.6.6) **Theorem.** *Suppose k is algebraically closed, and*

$$\rho\colon I \to \mathcal{A}ut(M)$$

is an irreducible finite-dimensional k-representation of I, of dimension $n \geqq 1$. Then

(2.6.6.1) *the restriction of ρ to $I(K_n/k)$ is the direct sum of n distinct characters $\chi_1, \ldots, \chi_n$ of $I(K_n/k)$.*

(2.6.6.2) *The conjugation-induced action of $\boldsymbol{\mu}_n = I/I(K_n/k)$ on $I(K_n/k)$ by outer automorphism is transitive on the characters $\chi_1, \ldots, \chi_n$.*

(2.6.6.3) *If $n \geqq 2$, the restrictions to $I^{(0+)}$ of the characters $\chi_1, \ldots, \chi_n$ are all non-trivial and are all distinct.*

(2.6.6.4) *If $n \geqq 2$ and if the unique (by (2.3.4)) slope of ρ is r/n with $(r, n) = 1$, then the restrictions to $I^{(r/n)}$ of the characters $\chi_1, \ldots, \chi_n$ are all non-trivial and all distinct, and $\rho(I^{(r/n)})$ is a connected torus of dimension $\varphi(n) = \deg(\mathbf{Q}(\zeta_n)/\mathbf{Q})$.*

Proof. We first prove (1) and (2).

By Levelt, any representation of I is, on some $I(K_N/k)$, a successive extension of characters. Because ρ is irreducible on I, it remains semi-simple on any of the normal subgroups $I(K_N/k)$. Therefore, for some $N \geqq 1$, $\rho \mid I(K_N/k)$ is the direct sum of characters. Pick such an N, and consider the $I(K_N/k)$-isotypical decomposition of M:

(2.6.6.5) $$M = \bigoplus_{\alpha \in R} M_\alpha.$$

Because M is I-irreducible, I acts transitively on the set of these R isotypical components. Consequently, $d = \dim(M_\alpha)$ is independent of α, and

(2.6.6.6) $$n = dr, \qquad r = \# R.$$

Let $S_\alpha \subset I$ be the stabilizer of M_α. Then $I(K_N/k) \subset S_\alpha$, and $I/S_\alpha \xrightarrow{\sim} R$, so by uniqueness we have $S_\alpha = I(K_r/k)$, independent of α. Therefore as representation of I we have

(2.6.6.7) $$M = \mathrm{Ind}^I_{I(K_r/k)}(M_\alpha) \qquad \text{for any } \alpha \in R.$$

Because M is irreducible on I, M_α is certainly irreducible on $I(K_r/k)$. We must show $\dim(M_\alpha) = 1$. [For then $n = r = \# R$ by (2.6.6.6), the characters χ_α of $I(K_n/k)$ on M_α are distinct because they are distinct on the subgroup $I(K_N/k)$ of $I(K_n/k)$, and $\boldsymbol{\mu}_n = I/I(K_n/k)$ acts transitively on the χ_α's, because M is I-irreducible.]

Renaming M_α, $I(K_r/k)$ as M, I, we are reduced to the situation: M is an irreducible representation of I of dimension $n \geqq 1$, but for some $N \geqq 1$, $I(K_N/k)$ acts on M by scalar matrices, i.e., there exists a character χ of $I(K_N/k)$ such that
$$\rho(\gamma)(m) = \chi(\gamma)\, m \qquad \text{for} \quad \gamma \in I(K_N/k).$$

The character χ is invariant by I-conjugation (because it is equal to $(1/n)$ trace (ρ)). Because $I/I(K_N/k)$ is cyclic, and k is algebraically closed, χ extends to a character $\tilde{\chi}$ of I.

Twisting M by the inverse of $\tilde{\chi}$, we reduce to the case where M is an irreducible representation of I which is trivial on $I(K_N/k)$. As the quotient $I/I(K_N/k)$ is cyclic, and K is algebraically closed, we find $\dim(M) = 1$, as required. This concludes the proof of (1) and (2).

We next prove (3). Fix an n'th root $t^{1/n}$ of t, and write
$$s = 1/t^{1/n}.$$
Then K_n is $k((1/s))$, $\boldsymbol{\mu}_n$ acts by $s \mapsto \zeta s$, and the characters χ_i of $I(K_n/k)$ which occur in ρ correspond to one-dimensional objects L_i in D.E.(K_n/k) which are transivitely permuted among themselves by the action $s \mapsto \zeta s$ of $\boldsymbol{\mu}_n$. Fixing one such L_i, say
$$L_1 \sim \left(K_n, t\frac{d}{dt} + P(s)\right)$$
with
$$P(s) \in k[s],$$

all the L_i which occur are precisely the n objects

$$L_\zeta \stackrel{\mathrm{dfn}}{=} \left(K_n, \frac{td}{dt} + P(\zeta s)\right), \qquad \zeta \in \boldsymbol{\mu}_n .$$

To prove (3), we must show that if $\zeta_1 \neq \zeta_2$ are two distinct n'th roots of unity, then the corresponding ratio of characters is non-trivial on $I^{(0+)}$, i.e., that $L_{\zeta_1} \otimes (L_{\zeta_2})^\vee$ has slope >0, i.e., that

$$\deg(P(\zeta_1 s) - P(\zeta_2 s)) > 0 .$$

We have already proven that the corresponding ratio of characters is non-trivial on $I(K_n/k)$, so certainly we have

$$P(\zeta_1 s) - P(\zeta_2 s) \neq 0 .$$

But this can only happen if $P(\zeta_1 s) - P(\zeta_2 s)$ actually has degree >0, because its constant term is zero.

We now prove (4). The hypothesis of (4) is

$$\deg(P(s)) = r, \qquad (r, n) = 1 .$$

Therefore for $\zeta \in \boldsymbol{\mu}_n$,

$$P(\zeta s) \equiv \zeta^r P(s) \bmod (\deg < r \text{ in } s) .$$

To show that the χ_i remain distinct on $I^{(r/n)}$, we must show that for $\zeta_1 \neq \zeta_2$ distinct elements of $\boldsymbol{\mu}_n$, we have

$$\deg(P(\zeta_1 s) - P(\zeta_2 s)) = r ,$$

which is the case because

$$P(\zeta_1 s) - P(\zeta_2 s) = (\zeta_1^r - \zeta_2^r)\, P(s) + \text{lower terms},$$

and $(r, n) = 1$.

Similarly, a monomial in the L_ζ's,

$$\bigotimes_{\zeta \in \boldsymbol{\mu}_n} (L_\zeta)^{a_\zeta}, \qquad a_\zeta \in \mathbf{Z}$$

is trivial as character of $I^{(r/n)}$ if and only if

$$\text{degree}\left(\sum_\zeta a_\zeta P(\zeta s)\right) < r .$$

But this holds if and only if

$$\sum_\zeta a_\zeta \zeta^r = 0 \quad \text{in} \quad k .$$

Therefore the character group of $\rho(I^{(r/n)})$ is, via L_1, the $\mathbf{Z}$-submodule of k spanned by the ζ^ras ζ runs over $\boldsymbol{\mu}_n$. Because $(r, n) = 1$, this is precisely the ring $\mathbf{Z}[\zeta_n]$ of cyclotomic integers in $\mathbf{Q}(\zeta_n)$. Q.E.D.

(2.6.7) **Corollary.** *With the hypotheses and notations of part* (4) *of the Theorem* (2.6.6), *suppose further that n is odd. Then*

(2.6.7.1) *there exists no non-zero $I^{(r/n)}$-invariant k-bilinear form $M \times M \to k$.*

(2.6.7.2) *Every slope of $M \bigotimes_k M$ as I-representation is r/n.*

Proof. Clearly (2) $\Rightarrow$ (1). To prove (2), we must show that, in the notations of the proof of (2.6.6.3) above, $L_i \otimes L_j$ has slope r (with respect to K_r), for every $0 \leqq i, j \leqq n-1$. But $L_j \otimes L_j$ is, for ζ a suitable primitive n'th root of unity,

$$\left(K_n,\ t\frac{d}{dt} + P(\zeta^i s) + P(\zeta^j s)\right),$$

and

$$P(\zeta^i s) + P(\zeta^j s) \equiv (\zeta^{ir} + \zeta^{jr}) P(s) \bmod (\deg < r).$$

Because ζ^{ir} and ζ^{jr} are each n'th roots of unity with n odd, we have $\zeta^{ir} + \zeta^{jr} \neq 0$. Q.E.D.

(2.6.8) *Remark.* Under the hypotheses of part (4) of the Theorem (2.6.6), if n is *even* then the above argument shows that of the n^2 slopes of $M \bigotimes_k M$ as I-representation,

(2.6.8.1)
$$\begin{array}{l}\text{there are } n^2 - n \text{ slopes} = r/n \\ \text{there are } n \text{ slopes} < r/n.\end{array}$$

2.7. Local groups as subgroups of global ones

(2.7.1) Let k be an algebraically closed field of characteristic zero, C/k a proper smooth connected curve, $D \subset C$ a finite set of closed points, and $U = C - D$. For each "point at ∞" $x \in D$, we denote by K_x the completion of the function field $k(C)$ at the discrete valuation which "is" x.

(2.7.2) Given an object V in D.E.(U/k), we denote by V_x its inverse image in D.E.(K_x/k). If we chose a k-valued fibre functor ω on D.E.(K_x/k), then $V \mapsto \omega(V_x)$ is a k-valued fibre functor, say $\tilde{\omega}$, on D.E.(U/k). Given an object V in D.E.(U/k), the inverse image functor maps

(2.7.2.1)
$$\langle V \rangle \to \langle V_x \rangle.$$

By ([De-Mi], 2.21), the dual homomorphism of k-algebraic groups

(2.7.2.2)
$$G_{\mathrm{loc}}(V_x, \omega) \to G_{\mathrm{gal}}(V, \tilde{\omega})$$

is a closed immersion.

(2.7.3) Because k is algebraically closed, any two k-valued fibre functors on D.E.(U/k) (resp. on D.E.(K_x/k)) are isomorphic. Therefore for *any* k-valued fibre functors ω' and ω on D.E.(U/k) and on D.E.(K_x/k) respectively, and any isomorphism α from $\omega \circ$ (inverse image) to ω', we have a closed subgroup

(2.7.3.1)
$$G_{\mathrm{loc}}(V_x, \omega) \hookrightarrow G_{\mathrm{gal}}(V, \omega'),$$

the "inertia group at x" to speak figuratively, whose conjugacy class in $G_{\mathrm{gal}}(V, \omega')$ is independent of the chice of (ω, α), and compatible with change of ω'.

III. Interlude: cyclically minuscule representations

3.1. The Lie algebra setting

(3.1.1) Throughout this chapter, we fix an algebraically closed field k of characteristic zero. A "Lie algebra" will mean a finite-dimensional Lie algebra over k.

(3.1.2) Let $\mathfrak{G}$ be a semi-simple Lie algebra over k, $\mathfrak{H} \subset \mathfrak{G}$ a Cartan subalgebra, and W the Weyl group of the root system of $(\mathfrak{G}, \mathfrak{H})$. If V is any finite-dimensional k-representation of $\mathfrak{G}$, the restriction of V to $\mathfrak{H}$ is a direct sum of characters (i.e., one-dimensional representations) of $\mathfrak{H}$. The characters λ of $\mathfrak{H}$ which occur in V are called the weights of V; for each weight λ of V, the dimension of V^{λ} is called the multiplicity of λ in V. The Weyl group W acts on the set of weights of V; for λ a weight of V, and $w \in W$, λ and $w(\lambda)$ have the same multiplicity in V. If V is a faithful representation of $\mathfrak{G}$, then the set of weights of V is a faithful permutation representation of W (simply because for V faithful, the $\mathbf{Q}$-span of its weights in $\mathfrak{H}^*$ is equal to the $\mathbf{Q}$-span of the roots). If V is an irreducible representation of $\mathfrak{G}$, then at least one of its weights has multiplicity one (e.g., its "highest weight" in any ordering of the $\mathbf{Q}$-span of the roots).

(3.1.3) A finite-dimensional k-representation V of a semi-simple $\mathfrak{G}$ is called cyclically minuscule (CM) if it satisfies the following three conditions:

(3.1.3.1) V is irreducible.

(3.1.3.2) V is faithful.

(3.1.3.3) There exists an element $w \in W$ which cyclically permutes the weights of V, i.e., the cyclic subgroup $\Gamma \subset W$ generated by w acts transitively on the set of weights of V.

(3.1.4) Under these conditions, we say that V is CM of type w.

(3.1.5) **Lemma.** *Suppose V is* CM *of type w, and $\Gamma \subset W$ is the cyclic subgroup generated by w. Then*:

(3.1.5.1) *every weight of V has multiplicity one*

(3.1.5.2) *the order of w is* $\dim(V)$, i.e., $\#(\Gamma) = \dim(V)$

(3.1.5.3) *Γ acts simply transitively on the set of weights of V.*

Proof. Because V is irreducible, it has some weight of multiplicity one; because V is CM, every weight of V is a W-transform of this one, so it also has multiplicity one. Because V is faithful, W is a faithfully represented as a permutation group on the set of the $\dim(V)$ weights of V, so w generates a cyclic subgroup which acts transitively if and only if w is itself a cyclic permutation of the $\dim(V)$ weights. Q.E.D.

(3.1.6) **Lemma.** *Suppose $\mathfrak{G} = \mathfrak{G}_1 \times \mathfrak{G}_2$ is the product of two semi-simple Lie algebras. Let V be a faithful irreducible k-representation of $\mathfrak{G}$, $V = V_1 \otimes V_2$ its unique expression as the tensor product of faithful irreducible k-representations V_i of $\mathfrak{G}_i$, for $i = 1,2$.*

The following conditions are equivalent:

(3.1.6.1) *V is a* CM-*representation of* $\mathfrak{G}$.

(3.1.6.2) *V_1 and V_2 are* CM-*representations of $\mathfrak{G}_1$ and $\mathfrak{G}_2$ respectively, and their dimensions are relatively prime*: $(\dim V_1, \dim V_2) = 1$.

Proof. Choose Cartan subalgebras $\mathfrak{H}_i \subset \mathfrak{G}_i$ for $i = 1,2$, and take $\mathfrak{H} = \mathfrak{H}_1 \times \mathfrak{H}_2$. Any character of $\mathfrak{H}$ is uniquely of the form (λ_1, λ_2): $(h_1, h_2) \to \lambda_1(h_1) + \lambda_2(h_2)$, where λ_i is a character of $\mathfrak{H}_i$. The Weyl group W of $\mathfrak{G}$ is canonically the product $W_1 \times W_2$ of the Weyl groups of $\mathfrak{G}_1$ and $\mathfrak{G}_2$ respectively. The set S of weights of V is the product set $S_1 \times S_2$, $S_i = \{\text{weights of } V_i\}$, on which $W = W_1 \times W_2$ operates by the product action

$$(w_1, w_2): (\lambda_1, \lambda_2) \mapsto (w_1 \lambda_1, w_2 \lambda_2).$$

Counting weights, we see that V has all its weights of multiplicity one if and only if both V_1 and V_2 have their weights of multiplicity one. In view of (3.1.5.1), to prove (1) $\Leftrightarrow$ (2) we need only consider the case when V, V_1, V_2 each have all their weights of multiplicity one. We henceforth suppose this to be the case.

Thus $\#(S_i) = \dim(V_i)$ for $i = 1,2$, and $W_i \subset \mathrm{Aut}(S_i)$ is a subgroup. But an element $(w_1, w_2) \in W_1 \times W_2$ cyclically permutes $S_1 \times S_2$, if and only if all the following conditions hold: $(\#S_1, \#S_2) = 1$, and for both $i = 1,2$, w_i cyclically permutes S_i. [The $\Leftarrow$ direction is obvious. For $\Rightarrow$, suppose (w_1, w_2) cyclically permutes all of $S_1 \times S_2$. Then each component w_i must cyclically permute its S_i. But then w_i has order $= \#S_i$, so (w_1, w_2) has order l.c.m. $(\#S_1, \#S_2)$ in $W_1 \times W_2$. But (w_1, w_2) cyclically permutes $S_1 \times S_2$, so has order $(\#S_1)(\#S_2)$. Comparing, we see $\#S_1$ and $\#S_2$ must ve relatively prime.] Q.E.D.

3.2. The group setting

(3.2.1) Now let G be a connected semi-simple algebraic group over k, $\mathfrak{G}$ its Lie algebra, T a maximal torus in G, $\mathfrak{H} = \mathrm{Lie}(T)$, $N(T)$ the normalizer of T in G. Then $\mathfrak{H}$ is a Cartan subalgebra of $\mathfrak{G}$, and $N(T)/T$ is the Weyl group W.

(3.2.2) Let

$$\rho\colon\ G \to \mathscr{GL}(V)$$

be a finite dimensional k-representation of G. We say that ρ is a CM-representation of G if $\mathrm{Lie}(\rho)$ is a CM-representation of $\mathfrak{G}$. Concretely, then, ρ is a CM-representation of G if and only if all of the following conditions are satisfied.

(3.2.2.1) ρ is irreducible

(3.2.2.2) $\mathrm{Ker}(\rho)$ is finite

(3.2.2.3) the restriction of ρ to T is the direct sum of $\dim(V)$ distinct characters of T

(3.2.2.4) there exists an element $w \in N(T)$ which, acting by conjugation on T, cyclically permutes the above characters.

(3.2.3) Let $\mathfrak{G} = \mathfrak{G}_1 \times \mathfrak{G}_2$ be a product of two semi-simple Lie algebras over k. Let $\tilde{G}, \tilde{G}_1, \tilde{G}_2$ be the connected, simply connected semi-simple groups over k with Lie algebras $\mathfrak{G}, \mathfrak{G}_1, \mathfrak{G}_2$ respectively. Thus $\tilde{G} = \tilde{G}_1 \times \tilde{G}_2$. Suppose we are given a CM-representation V of $\mathfrak{G}$. Then we can write it as $V_1 \otimes V_2$, where V_1 and V_2 are CM-representations of $\mathfrak{G}_1$ and $\mathfrak{G}_2$ respectively, and where $(\dim V_1, \dim V_2) = 1$. The representations V, V_1, V_2 of $\mathfrak{G}, \mathfrak{G}_1, \mathfrak{G}_2$ "integrate" to representations ρ, ρ_1, ρ_2 of $\tilde{G}$, $\tilde{G}_1$, $\tilde{G}_2$ on V, V_1, V_2, and we have $\rho = \rho_1 \otimes \rho_2$ on $\tilde{G} = \tilde{G}_1 \otimes \tilde{G}_2$.

(3.2.4) **Lemma.** *Hypotheses as above,* $\operatorname{Ker}(\rho) = \operatorname{Ker}(\rho_1) \times \operatorname{Ker}(\rho_2)$ *(in* $\tilde{G} = \tilde{G}_1 \otimes \tilde{G}_2$).

Proof. Let Z_i denote the center of $\tilde{G}_i$, for $i = 1,2$. Then $Z = Z_1 \times Z_2$ is the center of $\tilde{G}$. Because $\operatorname{Lie}(\rho)$, $\operatorname{Lie}(\rho_1)$, $\operatorname{Lie}(\rho_2)$ are faithful, the kernels of ρ, ρ_1, ρ_2, lie in Z, Z_1, Z_2 respectively. Now ρ_i is an irreducible representation of $\tilde{G}_i$ on V_i, and (as $\tilde{G}_i$ is connected and semi-simple) $\rho(\tilde{G}_i) \subset \mathscr{SL}(V_i)$, so by irreducibility $\rho_i(Z_i) \subset Z(\mathscr{SL}(V_i)) = \boldsymbol{\mu}_{\dim(V_i)}$. In other words, the restriction of ρ_i to Z_i is a character χ_i of order dividing $\dim(V_i)$. Thus $\rho = \rho_1 \otimes \rho_2$ on $Z = Z_1 \otimes Z_2$ is the character $(z_1, z_2) \mapsto \chi_1(z_1)\chi_2(z_2)$. But $\dim V_1$ and $\dim V_2$ are relatively prime, so $\chi_1(z_1)\chi_2(z_2) = 1$ if and only if $\chi_1(z_1) = \chi_2(z_2) = 1$. Q.E.D.

(3.2.5) **Corollary.** *Let G be a connected semi-simple group over k, $\mathfrak{G} = \operatorname{Lie}(G)$, ρ: $G \to \mathscr{GL}(V)$ a faithful* CM-*representation of G. Let $\mathfrak{G} = \mathfrak{G}_1 \times \mathfrak{G}_2$ be a product of semi-simple Lie algebras. Then we have a canonical product decomposition $G = G_1 \times G_2$, $\rho = \rho_1 \otimes \rho_2$ on $V = V_1 \otimes V_2$, where ρ_i is a faithful* CM-*representation of G_i,* $\operatorname{Lie}(G_i) = \mathfrak{G}_i$, *and* $(\dim V_1, \dim V_2) = 1$.

Proof. In the notations of the preceding proof, we have $G = \tilde{G}/\operatorname{Ker}(\rho)$, $G_i = \tilde{G}_i/\operatorname{Ker}(\rho_i)$ for $i = 1,2$. Q.E.D.

(3.2.6) **Corollary.** *Let G be a non-trivial connected semi-simple group over k, ρ: $G \to \mathscr{GL}(V)$ a faithful* CM-*representation of G.*

(3.2.6.1) *If ρ is not isomorphic to a tensor product $\rho_1 \otimes \rho_2$ of two strictly lower-dimensional representations of G whose dimensions are relatively prime, then G is simple* (*i.e.*, $\operatorname{Lie}(G)$ *is simple*).

(3.2.6.2) *If* $\mathfrak{G} = \operatorname{Lie}(G)$ *is not simple, then $G = \Pi G_i$ and $\rho \simeq \otimes \rho_i$, where G_i is simple* (*i.e.*, $\operatorname{Lie}(G_i)$ *is simple*) *and where ρ_i is a faithful* CM-*representation of G_i. The dimensions* $\dim(\rho_i)$ *are pairwise relatively prime.*

(3.2.7) *Classification theorem. Let V be an n-dimensional* CM-*representation of a simple* Lie *algebra $\mathfrak{G}$ over k. Then $(\mathfrak{G}, V)$ is one of*

$$\mathfrak{G} = \mathfrak{SL}(n), \quad \textit{std. rep. or its contragredient}$$
$$\mathfrak{G} = \mathfrak{Sp}(n), \quad \textit{std. rep. if } n \textit{ is even.}$$

Proof. It is clear that the named candidates are CM-representations. To show that there are no others, we must use classification. Because V is CM, it is certainly minuscule, (i.e., the entire Weyl group acts transitively on the weights of V). By Bourbaki, Lie VIII, §7.4, Prop. 8, we have the following list of all pairs $(\mathfrak{G}, V)$ with

$\mathfrak{G}$ simple and V minuscule:

$$\begin{array}{ll} A_\ell,\ \ell \geqq 1; & \omega_1, \ldots, \omega_\ell \\ B_\ell,\ \ell \geqq 3; & \omega_\ell \\ C_\ell,\ \ell \geqq 2; & \omega_1 \\ D_\ell,\ \ell \geqq 4; & \omega_1,\ \omega_{\ell-1},\ \omega_\ell \\ E_6; & \omega_1,\ \omega_6 \\ E_7; & \omega_7 \end{array}$$

So what we must check is that *none* of

$$\begin{array}{ll} A_\ell,\ \ell \geqq 3; & \omega_2,\ \omega_3, \ldots, \omega_{\ell-1} \\ B_\ell,\ \ell \geqq 3; & \omega_\ell \\ D_\ell,\ \ell \geqq 4; & \omega_1,\ \omega_{\ell-1},\ \omega_\ell \\ E_6; & \omega_1,\ \omega_6 \\ E_7; & \omega_7 \end{array}$$

is CM.

We first eliminate the cases

$$\begin{array}{ll} B_\ell,\ \ell \geqq 3 & \omega_\ell (\dim = 2^\ell) \\ D_\ell,\ \ell \geqq 5 & \omega_{\ell-1} \text{ and } \omega_\ell (\dim = 2^{\ell-1}) \\ E_6 & \omega_1 \text{ and } \omega_6 \ (\dim = 27) \\ E_7 & \omega_7 \ (\dim = 56) \end{array}$$

by showing that in these cases, the Weyl group contains no element whose order is equal to the dimension of the representation in question. For $\mathfrak{G}$ of rank ℓ, the action of its Weyl group W on the free $\mathbf{Z}$-module of rank ℓ spanned by the roots gives an injective homomorphism

$$W \hookrightarrow \mathscr{GL}(\ell, \mathbf{Z}).$$

Any element $w \in \mathscr{GL}(\ell, \mathbf{Z})$ of finite order N has a monic characteristic polynomial $P(T)$ in $\mathbf{Z}[T]$ of degree ℓ, all of whose roots are N'th roots of unity. Therefore $P(T)$ is some product of cyclotomic polynomials, say

$$P(T) = \prod_{i=1}^{r} \Phi_{d_i}(T), \quad \text{all} \quad d_i \mid N.$$

If we define

$$d = \text{l.c.m. (the } d_i)$$

then w^d has all eigenvalues 1, so (being of finite order), $w^d = 1$, i.e.,

$$\text{the order of } w = \text{l.c.m. } (d_1, \ldots, d_r).$$

If w has order a power of a prime, say p^e, then all the d_i divide p^e, and at least one $d_i = p^e$, since p_e is their l.c.m. Therefore

$$\deg(P) = \ell = \sum_{i=1}^{r} \varphi(d_i) \geqq \varphi(p^e).$$

In particular, if the Weyl group W contains an element of order p^e, then

$$\ell \geqq \varphi(p^e).$$

Taking $p^e = 2^\ell$ rules out the spin representation ω_ℓ of B_ℓ, $\ell \geqq 3$.

Taking $p^e = 2^{\ell-1}$ rules out the spin representations $\omega_{\ell-1}$, ω_ℓ of D_ℓ, $\ell \geqq 5$.

Taking $p^e = 3^3 = 27$ rules out the representations ω_1, ω_6 of E_6.

Similarly, if w has order $p^a q^b$ with p, q two distinct primes, either $p^a q^b$ divides some d_i, or p^a divides some d_i and q^b divides another. So if W contains an element of order $p^a q^b$, then

$$\text{either } \ell \geqq \varphi(p^a q^b) \text{ or } \ell \geqq \varphi(p^a) + \varphi(q^b).$$

Taking $p^a q^b = 2^3\, 7 = 56$ rules out the representation ω_7 of E_7.

It remains to eliminate the cases

$$\begin{array}{ll} A_\ell,\ \ell \geqq 3 & \omega_i \text{ for } 2 \leqq i \leqq \ell - 1 \\ D_4 & \omega_3 \text{ and } \omega_4 \\ D_\ell,\ \ell \geqq 4 & \omega_1 \end{array}$$

For (D_ℓ, ω_1), i.e., the standard representation of $\mathfrak{S}0(2\ell)$, the 2ℓ weights are $\pm\varepsilon_1, \ldots, \pm\varepsilon_\ell$. An element $w \in W$ permutes the $\varepsilon_1, \ldots, \varepsilon_\ell$ and then changes an even number of their signs. If $w \in W$ is to cyclically permute all the $\pm\varepsilon_i$'s, its "underlying permutation" of $\{1, \ldots, \ell\}$ must itself be cyclic, so renumbering $\varepsilon_1, \ldots, \varepsilon_\ell$, the effect of w is

$$\begin{array}{ll} \varepsilon_1 \to a_1 \varepsilon_2 & a_1 = \pm 1 \\ \varepsilon_2 \to a_2 \varepsilon_3 & a_2 = \pm 1 \\ \vdots & \\ \varepsilon_\ell \to a_\ell \varepsilon_1 & a_\ell = \pm 1, \end{array}$$

with the auxiliary condition $a_1 a_2 \cdots a_\ell = 1$. But $a_1 \cdots a_\ell = 1$ insures that w^ℓ fixes each ε_i, so $w^\ell = \mathrm{id}$, and so w cannot cyclically permute the 2ℓ weights $\{\pm\varepsilon_i\}_{i=1,\ldots,\ell}$.

For D_4, the Weyl group contains no element of order 8 [indeed the preceding permutation argument shows that if ℓ is a power of two, then the Weyl group of D_ℓ contains no element of order 2ℓ].

For the A_ℓ case, we put $n = \ell + 1$. We must eliminate the Λ^i (std. rep.) of $\mathfrak{S}1(n)$ for $n \geqq 4$, $2 \leqq i \leqq n-2$. By duality of Λ^i with Λ^{n-i}, the weights of Λ^i are just the negatives of those in Λ^{n-i}, so it suffices to treat the case $i \geqq 2$, $2i \leqq n$. An element $w \in W$ is just a permutation of $\varepsilon_1, \ldots, \varepsilon_n$. The weights of Λ^i are the $\binom{n}{i}$ sums $\varepsilon_{\pi(1)} + \cdots + \varepsilon_{\pi(i)}$ where $1 \leqq \pi(1) < \pi(2) < \cdots \leqq n$. Write w as a product of disjoint cycles (including cycles of length one!) arranged in decreasing length; after re-numbering, w becomes

$$(1,2,\ldots,d_1)\,(d_1+1,\ldots,d_1+d_2)\cdots(d_1+\cdots+d_{r-1}+1,\ldots,d_1+\cdots+d_r),$$

with $d_1 \geqq d_2 \geqq \cdots \geqq d_r$ and $d_1 + \cdots + d_r = n$. If w is not a single cycle, i.e. if $r \geqq 2$, then $2(d_1 + \cdots + d_{r-1}) \geqq n$, so $d_1 + \cdots + d_{r-1} \geqq i$. Therefore in Λ^i, the orbit of $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i$ under w contains only weights which involve *none* of the last d_r variables. Finally, if w is a single cycle, i.e. if $r = 1$, then $w^n = 1$, but $\dim(\Lambda^i) = \binom{n}{i} > n$ for $2 \leqq i \leqq n-2$ ("binomial coefficients increase toward the middle"). Q.E.D.

(3.2.8) **Corollary.** *Let G be a non-trivial connected semi-simple group over k, ρ: $G \to \mathcal{GL}(V)$ a* CM-*representation of G. Then $G = \Pi G_i$, $V = \otimes V_i$, $\rho = \otimes \rho_i$ where each (G_i, ρ_i) is one of the following*:

$$\mathcal{SL}(n_i), \quad \rho_i = \text{std. or its contragredient}$$
$$\mathcal{Sp}(n_i), \quad \rho_i = \text{std. if } n_i \text{ even}$$

and where the n_i are pairwise relatively prime and $\geqq 2$. In particular, if ρ is not isomorphic to the tensor product of two strictly lower-dimensional representations of G, then (G, ρ) is one of

$$\mathcal{SL}(n), \quad \rho = \text{std. or its contragredient}$$
$$\mathcal{Sp}(n), \quad \rho = \text{std. if } n \text{ even}.$$

(3.2.9) **CM Criterion.** *Let k be an algebraically closed field of characteristic zero, V a k-vector space of finite dimension $n \geqq 2$, $G \subset \mathcal{SL}(V)$ a Zariski closed subgroup of the special linear group, $H \subset G$ a Zariski closed subgroup of G, and $H_+ \subset H$ a Zariski closed normal subgroup of H.*

A. *Suppose that*

(1) *G is connected*

(2) *H_+ is a connected torus*

(3) *as representation of H_+, V is the direct sum of n distinct characters of H_+.*

(4) *there exists an element $h \in H$ whose action on H_+ by conjugation cyclically permutes the n characters of H_+ acting on V.*

Then G is semi-simple, G acts irreducibly on V, and V is a faithful CM-*representation of G.*

B. *If in addition we suppose that*

(5) *as representation of H, V is not isomorphic to the tensor product of two strictly lower-dimensional representations of H whose dimensions are relatively prime,*

then G is simple (i.e., Lie(G) *is simple).*

Proof. Conditions (3) and (4) guarantee that H acts irreducibly on V. Therefore G acts irreducibly on V, so by (1) G is a connected irreducible subgroup of $\mathcal{SL}(V)$, whence G is semi-simple (cf. [Ka-3], 11.5.3.2). Now consider the decomposition of V as the direct sum of the n one-dimensional eigenspaces V_i for the n distinct characters $\chi_1, \ldots, \chi_n$ of H_+ which occur in V:

$$V = \oplus V_i; \ H_+ \text{ acts on } V_i \text{ by } \chi_i.$$

Because $\chi_1, \ldots, \chi_n$ are *distinct* characters of H_+, the centralizer $Z_G(H_+)$ of H_+ in G maps each V_i to itself, and any element in the normalizer $N_G(H_+)$ of H_+ in G permutes the V_i, i.e., in terms of any basis $v_1, \ldots, v_n$ of V with $v_i \in V_i$, we have

$$Z_G(H_+) \subset G \cap (\text{diagonal matrices})$$
$$N_G(H_+) \subset G \cap (\text{permutation-shape matrices}).$$

Now let $T \subset G$ be a maximal torus of G which contains the torus H_+. Because T centralizes H_+, we have $T \subset Z_G(H_+)$, whence $T \subset G \cap (\text{diagonal matrices})$. By maximality, we must have

$$T = (G \cap (\text{diagonal matrices}))^0.$$

This explicit description of T makes it clear that

$$G \cap (\text{permutation-shape matrices}) \subset N_G(T).$$

The torus T acts on the V_i by n distinct characters (because $H_+ \subset T$, and these characters are already distinct on T). Therefore replacing H_+ by T, we find

$$N_G(T) \subset G \cap (\text{permutation-shape matrices}),$$

whence we have

$$H \subset N_G(H_+) \subset G \cap \text{ (permutation-shape matrices)} = N_G(T).$$

Therefore if $h \in H$ cyclically permutes the n characters χ_i of H_+ in V, the matrix of h is (after suitably ordering the χ_i) of the shape

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & * \\ * & 0 & & 0 & 0 \\ 0 & * & & \cdot & \cdot \\ \cdot & 0 & & \cdot & \cdot \\ \cdot & \cdot & & 0 & \cdot \\ 0 & 0 & & * & 0 \end{bmatrix}.$$

Therefore this same h cyclically permutes the n distinct characters of T which occur in V. This proves A. Part B follows from A and part (1) of (3.2.6). Q.E.D.

(3.2.10) **Corollary.** *If hypotheses* (1), (2), (3), (4), (5) *of* 3.2.9 *all hold, then either*

$$G = \mathscr{SL}(V)$$

or

V is even-dimensional, and there exists a non-degenerate alternating form $\langle\, ,\, \rangle : V \times V \to k$ on V with respect to which $G = \mathscr{Sp}(V, \langle\, ,\, \rangle)$.

3.3. Remarks and questions on CM-representations

(3.3.1) Let $\mathfrak{G}$ be a simple Lie algebra, and V a faithful irreducible representation of $\mathfrak{G}$. In view of our explicit determination of which V's are CM, a glance at the tables shows that

(3.3.1.1) V is CM $\Leftrightarrow \dim(V) = h$, the Coxeter number of $\mathfrak{G}$. (And when V is CM, it is CM of type c where $c \in W$ is any Coxeter element). Can this equivalence be proven a priori?

(3.3.2) If we make use of the classification (cf. [Ka-3], 11.6) of V's which are irreducible when restricted to the "principal $\mathfrak{S}\mathfrak{L}(2)$" of $\mathfrak{G}$, we obtain the equivalence

(3.3.2.1) V is CM $\Leftrightarrow \begin{cases} V \text{ is minuscule } \textit{and} \\ V \text{ is irreducible for the principal } \mathfrak{S}\mathfrak{l}(2) \text{ in } \mathfrak{G}. \end{cases}$

Can this equivalence be proven a priori?

IV. Application to global differential galois groups

4.1. The main theorem

(4.1.1) Let k be a field of characteristic zero. X/k a proper smooth geometrically connected curve over k, $S \subset X$ a finite set of closed points of X, $U = X - S$, $K = k(X)$ the function field of X. We suppose that $U(k)$ is non-empty.

(4.1.2) **Lemma.** *Let V and W be objects in* D.E.(U/k), and let L/k *be an over-field of* k.

(1) *There exists an isomorphism* $V \simeq W$ *in* D.E.(U/k) *if and only if there exists an isomorphism* $V \bigotimes_k L \simeq W \bigotimes_k L$ *in* D.E.$\left(U \bigotimes_k L/L\right)$.

(2) $\mathrm{Hom}_{\mathrm{D.E.}(U/k)}(V, W) \bigotimes_k L \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D.E.}}\left(V \bigotimes_k L, W \bigotimes_k L\right)$.

Proof. Interpreting $\mathrm{Hom}_{\mathrm{D.E.}(U/k)}(V, W)$ as $H^0_{DR}(U/k, V^\vee \otimes W)$, we see that formation of the Hom commutes with arbitrary extension of scalars $k \to L$. We may suppose V and W have the same non-zero rank n, otherwise there is nothing to prove. For any point $x \in U(k)$, we have an injective k-linear map

$$\begin{aligned} \mathrm{Hom}_{\mathrm{D.E.}(U/k)}(V, W) &\hookrightarrow \mathrm{Hom}_k(V(x), W(x)), \\ \varphi &\mapsto \varphi(x), \end{aligned}$$

and φ is an isomorphism if and only if $\varphi(x)$ is. Pick bases $e_1, \ldots, e_n$ of $V(x)$, $f_1, \ldots, f_n$ of $W(x)$, so we can speak of the determinant of an element of $\mathrm{Hom}_{(k)}(V(x), W(x))$. Then $\varphi \mapsto \det(\varphi(x))$ is a polynomial function on the finite-dimensional k-vector space $Z = \mathrm{Hom}_{\mathrm{D.E.}(U/k)}(V, W)$. Because k is infinite, this polynomial function vanishes identically (and so at every point of $Z \bigotimes_k L$) if and only if vanishes at every point of Z itself. Q.E.D.

(4.1.3) We say that V is self-dual if it admits an isomorphism with its dual. By the above Lemma 4.1.2, this notion is invariant under field extension.

(4.1.4) **Main theorem.** *Let V be an object of* D.E.(U/k) *of rank $n \geqq 2$. Suppose that*

(1) $\det(V)$ *is trivial*

(2) *there exists a rational point $s \in S(k)$ such that the slopes of $V \otimes K_s$ are all equal to r/n for some integer $r \geqq 1$ prime to n*

(3) *there exists an embedding $k \hookrightarrow \mathbf{C}$ for which the geometric monodromy group G_{mono} of $(V_{\mathbf{C}})^{\mathrm{an}}$ on $(U_{\mathbf{C}})^{\mathrm{an}}$ is connected.*

Then

(1) *If n is odd, V is not self-dual.*

(2) *If n is even, and if V is self-dual, the space* $\mathrm{Hom}_{\mathrm{D.E.}(U/k)}(V, V^\vee)$ *is one-dimensional over k, and every non-zero element, viewed as a bilinear form on V, is non-degenerate and alternating.*

(3) *For any k-valued fibre-functor ω on* D.E.(U/k), *the differential galois group*

$$G_{\mathrm{gal}} = G_{\mathrm{gal}}(V, \omega) \subset \mathscr{GL}(\omega(V)) = \mathscr{GL}(n) \text{ is given by}$$

$$G_{\mathrm{gal}} = \begin{cases} \mathscr{Sp}(n) & n \text{ even and } V \text{ self-dual} \\ \mathscr{SL}(n) & \text{if not.} \end{cases}$$

Proof. We first reduce to the case when $k=\mathbf{C}$ and the embedding $k\hookrightarrow\mathbf{C}$ is the identity. Suppose the theorem true over $\mathbf{C}$. Then by Lemma (4.1.2), parts (1) and (2) hold over the subfield $k\hookrightarrow\mathbf{C}$. Therefore, over k we have à priori inclusions $G_{\mathrm{gal}}\subset\mathscr{SL}(n)$ if V is not self-dual, $G_{\mathrm{gal}}\hookrightarrow\mathscr{Sp}(n)$ if V is self-dual. That these inclusions of k-groups be equalities may be checked after extending scalars from k to $\bar{k}$, so the question is independent of the choice of k-valued fibre-functor ω. Threfore we may take for ω "fibre at $x\in U(k)$". By (1.3), G_{gal} doesn't change when we extend scalars, i.e., we have $G_{\mathrm{gal}}(V_{\mathbf{C}},x_{\mathbf{C}})=G_{\mathrm{gal}}(V,x)\bigotimes_k\mathbf{C}$, so we may reduce to $k=\mathbf{C}$.

Suppose now $k=\mathbf{C}$. Because $\det(V)$ is trivial, G_{gal} lies in $\mathscr{SL}(n)$. Because G_{mono} is connected, G_{gal} is connected (1.2.5.2). We apply (3.2.9) to $G=G_{\mathrm{gal}}$, $H=G_{\mathrm{loc}}$ – the image in G_{gal} of the inertia group I_s at s, $H_+=$ the image of $I_s^{(0+)}$. Because $V\otimes k_s$ has slopes of exact denominator n, it follows from (2.6.6) that conditions (2), (3), (4), and (5) of (3.2.9) are satisfied [as is condition (1), the connectedness of G, c.f. above]. Therefore by (3.2.10), G_{gal} is $\mathscr{SL}(n)$ for n odd, and is $\mathscr{SL}(n)$ or $\mathscr{Sp}(n)$ for n even. Conclusions (1) and (2) are immediate consequences of this list of possible G_{gal}'s. Q.E.D.

(4.1.5) We now give some "concrete" examples to which the Main Theorem applies. All our examples occur on $U=X-S$ with $X=\mathbf{P}^1$, because it is only on $\mathbf{P}^1-S$ that we know any "checkable" conditions which guarantee the connectness of G_{mono}, the Zariski closure of the monodromy group.

(4.1.6) **Theorem.** *Let k be a field of characteristic zero, $T\subset\mathbf{A}^1(k)=k$ a finite set of rational points of $\mathbf{A}^1_k$, $U=\mathbf{A}^1_k-T=\mathbf{P}^1_k-\{\infty,T\}$. Let V in* D.E.(U/k) *have rank $n\geqq 2$, and suppose*

(1) $\det(V)$ *is trivial*

(2) *at ∞, all the slopes of $V\otimes K_\infty$ are r/n, with $(r,n)=1$*

(3) *at every point $t\in T$, V is regular singular (i.e., all its slopes $=0$) and its exponents (cf.* [Ka-6], 12.0) *all lie in* $\mathbf{Z}$.

Then

$$G_{\mathrm{gal}}=\begin{cases}\mathscr{Sp}(n) & \text{if } n \text{ is even and } V \text{ is self-dual}\\ \mathscr{SL}(n) & \text{if not.}\end{cases}$$

Proof. By the invariance property (1.3.2) of G_{gal} under field extension, we may first replace k by an absolutely finitely generated subfield k_0 over which everything is defined, then embed $k_0\hookrightarrow\mathbf{C}$ to reduce to the case $k=\mathbf{C}$. The point then is that condition (3) means precisely that the *local* monodromy around each point $t\in T$ is unipotent. Because $\pi_1(\mathbf{C}-T,x)$ is generated by these local monodromies and all their conjugates, its image in $\mathscr{GL}(n)$ is generated by their unipotent images, and therefore the Zariski closure of this image, G_{mono}, is connected. Now apply the Main Theorem (4.1.4). Q.E.D.

Here are two minor but useful variations on the preceding theorem.

(4.1.7) **Theorem** *Let k be a field of characteristic zero, $T\subset\mathbf{A}^1(k)=k$ a finite set of rational points of $\mathbf{A}^1_k$, $U=\mathbf{A}^1_k-T=\mathbf{P}^1_k-\{\infty,T\}$. Let V in* D.E.(U/k) *have* rank $n\geqq 2$, *and suppose*

(1) det V *is trivial*

(2) *at ∞, all the slopes of $V\otimes k_\infty$ are r/n with $(r,n)=1$.*

(3) *at every point* $t \in T$, V *is regular singular*

(4) *at all but at most one point of* T, *all the exponents* (cf. [Ka-6], 12.0) *of* V *lie in* $\mathbf{Z}$, *and there exists an integer* $N \geqq 1$ *with* $(n, N) = 1$ *such that at every point of* T *the exponents of* V *lie in* $(1/N)\mathbf{Z}$.

Then

$$G_{\mathrm{gal}} = \begin{cases} \mathscr{S}\!\mathscr{p}(n) & \text{if } n \text{ is even and } V \text{ is self-dual} \\ \mathscr{S}\mathscr{L}(n) & \text{if not.} \end{cases}$$

Proof. As in the proof of (4.1.6) above, we reduce easily to the case $k = \mathbf{C}$. We need only treat the case when there is an exceptional point $t_0 \in T$. Consider the finite etale μ_N-covering $U_N \xrightarrow{\pi} U$ defined by $(x - t_0)^{1/N}$; via $(x - t_0)^{1/N}$ as coordinate, we have $U_N = \mathbf{A}^1 - \pi^{-1}(T)$. The inverse image $\pi^* V$ of V on U_N is easily seen to satisfy all the hypotheses of (4.1.6) $\langle$it is in checking (2) that we need $(n, N) = 1\rangle$, and hence

$$G_{\mathrm{gal}}(\pi^* V) = \begin{cases} \mathscr{S}\!\mathscr{p}(n) & \text{if } n \text{ is even and } \pi^* V \text{ is self-dual} \\ \mathscr{S}\mathscr{L}(n) & \text{if not.} \end{cases}$$

In view of the obvious inclusions

$$G_{\mathrm{gal}}(\pi^* V) \subset G_{\mathrm{gal}}(V) \subset \mathscr{S}\mathscr{L}(n),$$

the asserted theorem is obvious except in the case when n is even and $\pi^* V$ is self-dual; in that case we have

$$G_{\mathrm{gal}}(\pi^* V) = \mathscr{S}\!\mathscr{p}(n) \subset G_{\mathrm{gal}}(V) \subset \mathscr{S}\mathscr{L}(n).$$

In this case we use the fact that $G_{\mathrm{gal}}(\pi^* V)$ is a normal closed subgroup of finite index dividing N (1.4.5) in $G_{\mathrm{gal}}(V)$. Then $G_{\mathrm{gal}}(V) = G$ has $G^0 = \mathscr{S}\!\mathscr{p}(n)$, and G acts on $G^0 = \mathscr{S}\!\mathscr{p}(n)$ by conjugation. Because every automorphism of $\mathscr{S}\!\mathscr{p}(n)$ is inner, and the standard representation of $\mathscr{S}\!\mathscr{p}(n)$ is irreducible, we have $G \subset \mathscr{S}\!\mathscr{p}(n) \cdot$ (scalars). Because $G \subset \mathscr{S}\mathscr{L}(n)$, we have $G \subset \mathscr{S}\!\mathscr{p}(n) \cdot \mu_n$. Because the index of $\mathscr{S}\!\mathscr{p}(n)$ in G divides N and $(n, N) = 1$, we have $G \subset \mathscr{S}\!\mathscr{p}(n)$, whence $G = \mathscr{S}\!\mathscr{p}(n)$. Q.E.D.

(4.1.8) **Theorem.** *Hypotheses and notations as in the previous theorem* (4.1.7), *let* L *be a rank-one object of* D.E.(U/k), *and denote by*

$$\Gamma \subset \mathbf{G}_m = \mathscr{G}\mathscr{L}(1)$$

the G_{gal} *for* L *itself (concretely,* $\Gamma = \mathbf{G}_m$ *unless* $L^{\otimes N}$ *is trivial for some integer* $N \geqq 1$, *in which case* $\Gamma = \mu_N$ *for the least such* N). *Then* $G_{\mathrm{gal}}(V \otimes L)$ *is given by*

$$\begin{aligned} G_{\mathrm{gal}}(V \otimes L) &= G_{\mathrm{gal}}(V) \cdot \Gamma \subset \mathscr{G}\mathscr{L}(n) \\ &= \begin{cases} \mathscr{S}\!\mathscr{p}(n) \cdot \Gamma & \text{if } n \text{ is even and } V \text{ is self-dual} \\ \mathscr{S}\mathscr{L}(n) \cdot \Gamma & \text{if not.} \end{cases} \end{aligned}$$

Proof. The differential galois group of $V \otimes L$ is a subgroup K of $G_{\mathrm{gal}}(V) \times G_{\mathrm{gal}}(L)$ which projects onto each factor. We claim that K is equal to this product. Because $G_{\mathrm{gal}}(V)$ is either $\mathscr{S}\mathscr{L}(n)$ or $\mathscr{S}\!\mathscr{p}(n)$, $G_{\mathrm{gal}}(V)$ is equal to its own commutator subgroup, while $G_{\mathrm{gal}}(L)$ is (trivially) commutative. Therefore the commutator subgroup $[K, K]$ of K is equal to $G_{\mathrm{gal}}(V) \times \{1\}$; because K maps onto $G_{\mathrm{gal}}(L)$, we must have $K = G_{\mathrm{gal}}(V) \times G_{\mathrm{gal}}(L)$.

If we now view $V \otimes L$ as an object of $\langle V \oplus L \rangle$, its $G_{\mathrm{gal}}(V \otimes L)$ is the image of $K = G_{\mathrm{gal}}(V \oplus L)$ in the corresponding representation ρ of K, which is the composite homomorphism

$$\begin{tikzcd} K = G_{\mathrm{gal}}(V \oplus L) \arrow[r, hook] \arrow[ddr, dashed, "\rho"'] & G_{\mathrm{gal}}(V) \times G_{\mathrm{gal}}(L) \arrow[d] \\ & G_{\mathrm{gal}}(V) \cdot G_{\mathrm{gal}}(L) \\ & \mathscr{GL}(n). \arrow[u, phantom, "\bigcap"] \end{tikzcd}$$

Because $K = G_{\mathrm{gal}}(V) \times G_{\mathrm{gal}}(L)$, we find $\rho(K) = G_{\mathrm{gal}}(V) \cdot G_{\mathrm{gal}}(L)$, as asserted. Q.E.D.

4.2. Equations of Airy type on $\mathbf{A}^1$

(4.2.1) Let k be a field of characteristic zero, $\mathbf{A}_k^1 = \mathrm{Spec}(k[x])$ the affine line over k,

(4.2.1.1) $$\partial = \frac{d}{dx},$$

and

(4.2.1.2) $$\mathscr{D} = k[x, \partial] \text{ the Weyl algebra.}$$

Given a polynomial

(4.2.1.3) $$P(x) = \sum a_i x^i \in k[x],$$

we denote by

(4.2.1.4) $$P(\partial) = \sum a_i \partial^i$$

the corresponding constant-coefficient differential operator on $\mathbf{A}_k^1$.

(4.2.2) Let $n \geqq 1$ and $m \geqq 1$ be strictly positive integers, $P_n(x)$ and $Q_m(x)$ polynomials in $k[x]$ of degrees n and m respectively. The differential operator on $\mathbf{A}_k^1$ defined by

(4.2.2.1) $$P_n(\partial) + Q_m(x) \in \mathscr{D}$$

will be called an Airy operator of bidegree (n, m). The corresponding cyclic object $\mathscr{D}/\mathscr{D}(P_n(\partial) + Q_m(x))$ of D.E.$(\mathbf{A}_k^1/k)$ will be an Airy D.E. of bidegree (n, m); its rank as D.E. is n, and all of its slopes at ∞ are equal to $(n+m)/n$.

(4.2.3) If $n \geqq 2$, the determinant of the Airy equation corresponding to the Airy operator

(4.2.3.1) $$P_n(\partial) + Q_m(x) = a_n \partial^n + a_{n-1} \partial^{n-1} + \cdots + Q_m(x)$$

is the cyclic D.E. of rank-one corresponding to the constant-coefficient operator

(4.2.3.2) $$\partial + (a_{n-1}/a_n).$$

(4.2.4) For any $\alpha \in k$, and any Airy operator $P_n(\partial) + Q_m(x)$, the tensor product of the rank-one D.E.

(4.2.4.1) $$L(\alpha) = \mathscr{D}/\mathscr{D}(\partial + \alpha)$$

with the Airy D.E.

(4.2.4.2) $$\mathscr{D}/\mathscr{D}(P_n(\partial) + Q_m(x))$$

is the Airy D.E.

(4.2.4.3) $$\mathscr{D}/\mathscr{D}(P_n(\partial + \alpha) + Q_m(x)).$$

(4.2.5) Finally, recall that on $\mathbf{A}^1_k$, the group (under $\otimes$) of isomorphism classes of rank-one D.E.'s is isomorphic to the additive group underlying $k[x]$, with $f \in k[x]$ corresponding to $\mathscr{D}/\mathscr{D}(\partial + f)$. Because this group is torsion-free, the differential galois group of an object $\mathscr{D}/\mathscr{D}(\partial + f)$ is equal to $\mathbf{G}_m$ if $f \neq 0$, and to $\{1\}$ if $f = 0$. Because the group of rank-one D.E.'s is uniquely divisible, being a k-vector space, we may speak of fractional powers of its elements. Putting all this together, we find

(4.2.6) **Lemma.** *Let V be an Airy* D.E. *on $\mathbf{A}^1_k$ of bidegree (n, m).*

(1) *If $n = 1$, then $G_{\mathrm{gal}}(V) = \mathbf{G}_m$.*

(2) *If $n \geqq 2$, then V may be written uniquely in the form $V \simeq V_0 \otimes L$, where V_0 is an Airy* D.E. *of bidegree (n, m) whose determinant is trivial, and where $L = (\det(V))^{1/n}$. In terms of the Airy operator*

$$P_n(\partial) + Q_m(x) = a_n \partial^n + a_{n-1} \partial^{n-1} + \cdots + Q_m(x)$$

defining V, V_0 is defined by the Airy operator

$$P_n(\partial - (1/n)(a_{n-1}/a_n)) + Q_m(x),$$

and L is defined by the first order operator

$$\partial + (1/n)(a_{n-1}/a_n).$$

(4.2.7) **Theorem.** *Let*

$$P_n(\partial) + Q_m(x) = a_n \partial^n + a_{n-1} \partial^{n-1} + \cdots + Q_m(x)$$

be an Airy operator of bidegree (n, m) on $\mathbf{A}^1$, V the corresponding Airy D.E., *and V_0 the Airy* D.E. *corresponding to the Airy operator*

$$P_n(\partial - (1/n)(a_n/a_{n-1})) + Q_m(x).$$

Suppose that n and m are relatively prime. Then the G_{gal} of V is given by

(1) *if $n = 1$, $G_{\mathrm{gal}} = \mathbf{G}_m$*

(2) *if $n \geqq 3$ is odd,* $G_{\mathrm{gal}} = \begin{cases} \mathscr{SL}(n) & \text{if } a_{n-1} = 0 \\ \mathscr{GL}(n) & \text{if } a_{n-1} \neq 0 \end{cases}$

(3) *if n is even, then*

$$G_{\mathrm{gal}} = \begin{cases} \mathscr{Sp}(n) & \textit{if } a_{n-1} = 0 \textit{ and } V_0 \textit{ is self-dual} \\ \mathscr{GSp}(n) & \textit{if } a_{n-1} \neq 0 \textit{ and } V_0 \textit{ is self-dual} \\ \mathscr{SL}(n) & \textit{if } a_{n-1} = 0 \textit{ and } V_0 \textit{ is not self-dual} \\ \mathscr{GL}(n) & \textit{if } a_{n-1} \neq 0 \textit{ and } V_0 \textit{ is not self-dual,} \end{cases}$$

where $\mathscr{GSp}(n) = \mathbf{G}_m \cdot \mathscr{Sp}(n)$ denotes the group of symplectic similitudes.

Proof. If $n=1$, the fact that $m \geqq 1$ insures that V is a non-trivial rank-one D.E. on $\mathbf{A}_k^1$, so its G_{gal} is necessarily $\mathbf{G}_m$.

If $n \geqq 2$, we write $V = V_0 \otimes L$ as in (4.2.6). Because n and m are relatively prime, the slopes of V_0 at ∞ are all $(n+m)/n$, of exact denominator n. So we simply apply (4.1.6) to V_0, and then combine this with (4.1.8). Q.E.D.

In general, we do *not* know how to decide whether or not V_0 is self-dual when n is even. The next corollary gives a sufficient condition.

(4.2.8) **Corollary.** *Hypotheses and notations as in the Theorem* (4.2.7) *above, suppose in addition that n is even and that the polynomial in* ∂, $P_n(\partial - (1/n)(a_{n-1}/a_n))$, *is a polynomial in* ∂^2 (*i.e., it is invariant under* $\partial \mapsto -\partial$, *or equivalently, it is formally self-adjoint*). *Then* V_0 *is self-dual, and* (*consequently*) *the* G_{gal} *of V is given by*

$$G_{\mathrm{gal}} = \begin{cases} \mathscr{S}\!\mathscr{p}(n) & \text{if } a_{n-1} = 0 \\ \mathscr{G}\mathscr{S}\!\mathscr{p}(n) & \text{if } a_{n-1} \neq 0. \end{cases}$$

Proof. The auto-duality of V_0 is immediate from the compatibility (1.5.3) of formal adjoint and dual. The determination of G_{gal} for V then results from the theorem (4.2.7). Q.E.D.

(4.2.9) *Example.* For $n \geqq 2$, $m \geqq 1$, $\lambda \in k^\times$, and n and m relatively prime, the Airy D.E. given by $\partial^n + \lambda x^m$ has G_{gal} equal to $\mathscr{S}\mathscr{L}(n)$ for n odd, and to $\mathscr{S}\!\mathscr{p}(n)$ for n even.

In the case $m=1$, we have a converse to the above Corollary (4.2.8).

(4.2.10) **Corollary.** *Hypotheses and notations as in the Theorem* (4.2.7) *above, suppose in addition that n is even,* $m=1$, *and the polynomial in* ∂, $P_n(\partial - (1/n)(a_{n-1}/a_n))$ *is not invariant under* $\partial \to -\partial$. *Then* V_0 *is not self-dual, and* (*consequently*) *the* G_{gal} *of V is given by*

$$G_{\mathrm{gal}} = \begin{cases} \mathscr{S}\mathscr{L}(n) & \textit{if } a_{n-1} = 0 \\ \mathscr{G}\mathscr{L}(n) & \textit{if } a_{n-1} \neq 0. \end{cases}$$

Proof. We know by (1.5.3) that the *dual* of V_0 is the Airy D.E. corresponding to the Airy operator

$$P_n(-\partial - (1/n)(a_{n-1}/a_n)) + Q_m(x);$$

by hypothesis this operator is *not* a $k^\times$-multiple of the Airy operator

$$P_n(\partial - (1/n)(a_{n-1}/a_n)) + Q_m(x)$$

which gives rise to V_0 itself. The required result is then the special case $m=1$ of part (2) of the following proposition.

(4.2.11) **Proposition.** *Let* A_1 *and* A_2 *be two Airy operators on* $\mathbf{A}_k^1$, *and let* V_1 *and* V_2 *be the corresponding Airy* D.E.'s on $\mathbf{A}_k^1$. *Suppose that there exists an isomorphism* $V_1 \simeq V_2$ *of* D.E.'s *on* $\mathbf{A}_k^1$. *Then*

(1) *the operations* A_1 *and* A_2 *have the same bidegree, say* (n, m).

(2) *If* $n=1$ *or if* $m=1$, *then* $A_1 = \lambda A_2$ *for some* $\lambda \in k^\times$ (*i.e.,* $A_1 = A_2$ *if both* A_1 *and* A_2 *are monic in* ∂).

Proof. We recover the bidegree (n_i, m_i) of A_i) of A_i from the isomorphism class of V_i by the rules $n_i = \operatorname{rank}(V_i)$, $(n_i + m_i)/n_i =$ the unique slope at ∞ of V_i. This proves (1). Assertion (2) is obvious for $n=1$ (c.f., the above discussion of rank-one D.E.'s on $\mathbf{A}_k^1$), and follows from this case for $m=1$ by formal Fourier transform (the automorphism ($x \mapsto \partial$, $\partial \mapsto -x$) of the Weyl algebra), which interchanges Airy operators of bidegree (n, m) with those of bidegree (m, n). Q.E.D.

4.3. An open problem

If we suppose only that n and m are relatively prime, does it remain true that $V_1 \simeq V_2$ implies $A_1 = \lambda A_2$ for some $\lambda \in k^\times$? This can be proven (by brute force) if $n=2$ (and so for $m=2$ by Fourier transform), but the general case remains unclear. On the other hand, if we drop the requirement that n and m be relatively prime, $V_1 \simeq V_2$ need *not* imply the $k^\times$-proportionality of A_1 and A_2. Here is a simple counter-example: pick $a \in k$, $a \neq -1$, and consider

$$A_1 = \partial^2 - x^2 - a$$
$$A_2 = \partial^2 - x^2 - 2 - a.$$

Direct calculation shows that if e is a cyclic vector for V_1 with $e'' = (x^2 + a)e$ then $xe + e'$ is another cyclic vector for V_1 which satisfies $(xe + e')'' = (x^2 + 2 + a)$ $(xe + e')$. Therefore if f is a cyclic vector for V_2 with $f'' = (x^2 + 2 + a) f$, then $f \mapsto xe + e'$ defines an isomorphism $V_2 \xrightarrow{\sim} V_1$.

4.4. Equations of Kloosterman type on $\mathbf{G}_m$

(4.4.1) Let k be a field of characteristic zero, $\mathbf{G}_{m,k} = \operatorname{Spec}(k[x, x^{-1}])$ the multiplicative group over k,

(4.4.1.1) $$D = x\frac{d}{dx},$$

$\mathscr{D}\!\mathit{iff} = k[x, x^{-1}, D]$ the ring of differential operators on $\mathbf{G}_{m,k}$.

Given a polynomial

(4.4.1.2) $$P(x) = \sum a_i x^i \in k[x],$$

we denote by

(4.4.1.3) $$P(D) = \sum a_i D^i$$

the corresponding constant-coefficient (w.r.t. D) operator on $\mathbf{G}_{m,k}$.

(4.4.2) Let $n \geqq 1$ and $m \geqq 1$ be strictly positive integers, $P_n(x) \in k[x]$ a polynimial of degree n, and $Q_m(x) \in k[x]$ a polynomial in $k[x]$ of degree m with $Q_m(0) = 0$. The differential operator on $\mathbf{G}_{m,k}$ defined by

(4.4.2.1) $$P_n(D) + Q_m(x)$$

will be called a Kloosterman operator of bidegree (n, m). The corresponding cyclic object $\mathscr{D}\!\mathit{iff}/\mathscr{D}\!\mathit{iff}(P_n(D) + Q_m(x))$ of D.E.$(\mathbf{G}_{m,k}/k)$ will be called a Kloosterman D.E. of bidegree (n, m); its rank as D.E. is n, all of its slopes at ∞ are equal to m/n, it has a regular singular point at $x=0$ and (because $Q_m(0)=0$) its exponents (cf. [Ka-6], 12.0) at $x=0$ are the zeroes of the polynomial $P_n(x)$.

(4.4.3) On $\mathbf{G}_{m,k}$, the group (under $\otimes$) of rank-one D.E.'s is the quotient $k[x,x^{-1}]/\mathbf{Z}$, with $f\in k[x,x^{-1}] \bmod \mathbf{Z}$ corresponding to $\mathscr{D}iff/\mathscr{D}iff(D+f)$. (The isomorphism between $\mathscr{D}iff/\mathscr{D}iff(D+f)$ and $\mathscr{D}iff/\mathscr{D}iff(D+f+n)$ is given by "multiplication by x^n".)

(4.4.4) If $n\geqq 2$, the determinant of the Kloosterman D.E. corresponding to the operator

(4.4.4.1) $$P_n(D)+Q_m(x)=a_nD^n+a_{n-1}D^{n-1}+\cdots+Q_m(x)$$

is the cyclic D.E. of rank-one corresponding to the constant-coefficient operator

(4.4.4.2) $$D+(a_{n-1}/a_n).$$

(4.4.5) For any $\alpha\in k$, and any Kloosterman operator $P_n(D)+Q_m(x)$, the tensor product of the rank-one D.E.

(4.4.5.1) $$K(\alpha)=\mathscr{D}iff/\mathscr{D}iff(D+\alpha)$$

with the Kloosterman D.E.

(4.4.5.2) $$\mathscr{D}iff/\mathscr{D}iff(P_n(D)+Q_m(x))$$

is the Kloosterman D.E.

(4.4.5.3) $$\mathscr{D}iff/\mathscr{D}iff(P_n(D+\alpha)+Q_m(x)).$$

(4.4.6) **Theorem.** *Let*

$$P_n(D)+Q_m(x)$$

be a Kloosterman operator on $\mathbf{G}_{m,k}$, *V the corresponding Kloosterman* D.E. *Suppose that*

(a) *n and m are relatively prime*

(b) *if $n\geqq 2$, all the zeroes of $P_n(x)$ lie in* $\mathbf{Z}$.

Then the differential galois group G_{gal} of V is given by

(1) *if $n=1$,* $G_{\text{gal}}=\mathbf{G}_m$

(2) *if $n\geqq 3$ is odd,* $G_{\text{gal}}=\mathscr{SL}(n)$.

(3) *if n is even,* $G_{\text{gal}}=\begin{cases}\mathscr{S}\!p(n) & \text{if } V \text{ is self-dual}\\ \mathscr{SL}(n) & \text{if not.}\end{cases}$

Proof. If $n=1$, the fact that $m\geqq 1$ insures that no strictly positive tensor power of V is trivial as D.E. on $\mathbf{G}_{m,k}$ (i.e., a polynomial in x of degree $m\geqq 1$ is a non-torsion element in $k[x,x^{-1}]/\mathbf{Z}$), so the G_{gal} of V is necessarily $\mathbf{G}_m$.

If $n\geqq 2$, the hypothesis that $P_n(x)$ has all roots in $\mathbf{Z}$ insures that a_{n-1}/a_n, being $\pm$ (the sum of the roots), lie in $\mathbf{Z}$, and hence that $\det(V)$ is trivial. It also insures that at $x=0$, which is a regular singular point of V, all the exponents of V lie in $\mathbf{Z}$. Because n and m are relatively prime, the slopes of V at ∞ are all m/n, of exact denominator n. So we simply apply (4.2.6), with $T=\{0\}$. Q.E.D.

(4.4.7) **Corollary.** *Hypotheses and notations as in the Theorem* (4.4.6) *above, suppose that n is even, and that for some integer $a\in\mathbf{Z}$, $P_n(D+a)$ is a polynomial in D^2. Then V is self-dual, and (consequently) $G_{\text{gal}}=\mathscr{S}\!p(n)$.*

Proof. Replacing D by $D+a$ amounts to tensoring with the rank-one object $K(a)$ corresponding to $D+a$, which for $a\in\mathbf{Z}$ is trivial. Thus we may reduce to the case when $P_n(D)$ is a polynomial in D^2, in which case the operator $P_n(D)+Q_m(x)$ is formally self-adjoint in the sense of (1.5.3). The autoduality of V then follows from (1.5.3). Q.E.D.

(4.4.8) **Corollary.** *Let n and m be integers $\geqq 1$, $\lambda\in k^\times$, and consider the Kloosterman operator on $\mathbf{G}_{m,k}$*

$$\mathbf{D}^n+\lambda x^m,$$

V the corresponding D.E. *(We do not assume n and m to be relatively prime.) Then the G_{gal} of V is given by*

$$G_{\text{gal}}=\begin{cases}\mathbf{G}_m & \text{if } n=1\\ \mathscr{SL}(n) & \text{if } n\geqq 3 \textit{ is odd}\\ \mathscr{Sp}(n) & \text{if } n \textit{ is even.}\end{cases}$$

Proof. We easily reduce to the case $k=\mathbf{C}$ (compare (4.1.6)).

If $m=1$ (indeed if m is prime to n), this a special case of (4.4.6–7). Consider the Kloosterman operator

$$m^nD^n+\lambda x$$

on $\mathbf{G}_{m,k}$, and the corresponding D.E. W. Under the finite etale $\boldsymbol{\mu}_m$ covering π: $\mathbf{G}_m\to\mathbf{G}_m$ given by $x\mapsto x^m$, we have

$$\pi^*W=V,$$

so, by (1.4.5), $G_{\text{gal}}(V)$ is an open subgroup of finite index in $G_{\text{gal}}(W)$. By the result with $m=1$, applied to W, we see that $G_{\text{gal}}(W)$ is connected, so $G_{\text{gal}}(V)=G_{\text{gal}}(W)$ is as asserted. Q.E.D.

4.5. A special class of Kloosterman equations on $\mathbf{G}_m$

(4.5.1) We continue to work over a field k of characteristic zero. Fix an integer $n\geqq 1$. Given an element $\lambda\in k^\times$, and elements $a_1,\dots a_n$ in k, consider the Kloosterman operator

$$\prod_{i=1}^{n}(D-a_i)+\lambda x, \tag{4.5.1.1}$$

and the associated Kloosterman D.E. on $\mathbf{G}_{m,k}$, which we denote

$$\mathrm{Kl}_\lambda(a_1,\dots,a_n). \tag{4.5.1.2}$$

(4.5.2) **Proposition.** *There exists an isomorphism*

$$\mathrm{Kl}_\lambda(a_1,\dots,a_n)\simeq\mathrm{Kl}_\mu(b_1,\dots,b_n)$$

in D.E.$(\mathbf{G}_{m,k}/k)$ *if and only if both of the following conditions are satisfied*:

(1) $\lambda=\mu$ in $k^\times$

(2) *after possibly renumbering the b's, we have $a_i-b_i\in\mathbf{Z}$ (inside k) for $i=1,\dots,n$.*

Proof. To prove the "if" direction, it suffices to exhibit an isomorphism

$$\mathrm{Kl}_\lambda(a_1,\dots,a_n)\xleftarrow{\sim}\mathrm{Kl}_\lambda(a_1+1,a_2,\dots,a_n).$$

In terms of the standard cyclic vector e for $\mathrm{Kl}_\lambda(a_1,\ldots,a_n)$, which is annihilated by

$$\prod_1^n (D-a_i)+\lambda x,$$

one verifies easily that $(D-a_1)e$ is another cyclic vector for $\mathrm{Kl}_\lambda(a_1,\ldots,a_n)$, and that $(D-a_1)e$ is annihilated by

$$(D-a_1-1)\prod_{i=2}^n (D-a_i)+\lambda x.$$

Therefore if we view each Kl_λ as $\mathscr{D}\!\mathit{iff}/\mathscr{D}\!\mathit{iff}$ (the Kl_λ operator), then right multiplication by $D-a_1$ defines the required isomorphism. [The inverse isomorphism is right multiplication by $(-1/\lambda x)\prod_{i=2}^n (D-a_i)$.]

Suppose now that we are given an isomorphism

$$\mathrm{Kl}_\lambda(a_1,\ldots,a_n)\simeq \mathrm{Kl}_\mu(b_1,\ldots,b_n).$$

Dropping first to an absolutely finitely generated subfield k_0 of k over which this isomorphism is defined, and then embedding k_0 in $\mathbf{C}$, we may reduce to the case $k=\mathbf{C}$. Then the n eigenvalues of "local monodromy around $x=0$" of $\mathrm{Kl}_\lambda(a_1,\ldots,a_n)$ are the n numbers $\exp(-2\pi i a_j)$, $j=1,\ldots,n$ (cf. [Ka-6], 12.0). Therefore the a_i mod $\mathbf{Z}$ are determined by the isomorphism class of $\mathrm{Kl}_\lambda(a_1,\ldots,a_n)$. Consequently, if $\mathrm{Kl}_\lambda(a_1,\ldots,a_n)\simeq \mathrm{Kl}_\mu(b_1,\ldots,b_n)$, then after renumbering the b's we have $a_i\equiv b_i \bmod \mathbf{Z}$. Using the "if" part, we may construct an isomorphism $\mathrm{Kl}_\mu(b_1,\ldots,b_n)\simeq \mathrm{Kl}_\mu(a_1,\ldots,a_n)$. Thus we obtain an isomorphism

$$\mathrm{Kl}_\lambda(a_1,\ldots,a_n)\simeq \mathrm{Kl}_\mu(a_1,\ldots,a_n).$$

We must prove that $\lambda=\mu$. But in terms of the multiplicative translation automorphisms $T_\alpha\colon x\mapsto \alpha x$ of $\mathbf{G}_{m,k}$ given by each $\alpha\in k^\times$, we clearly have

$$(T_{\mu/\lambda})^*\,\mathrm{Kl}_\lambda(a_1,\ldots,a_n)=\mathrm{Kl}_\mu(a_1,\ldots,a_n),$$

simply because $D=x\dfrac{d}{dx}$ is translation-invariant on $\mathbf{G}_m$. Thus we obtain an isomorphism

$$(T_{\mu/\lambda})^*\,\mathrm{Kl}_\lambda(a_1,\ldots,a_n)\simeq \mathrm{Kl}_\lambda(a_1,\ldots,a_n).$$

Because $V=Kl_\lambda(a_1,\ldots,a_n)$ has $\mathrm{Irr}_\infty(V)=1$, and is irreducible at ∞ (because it has rank n, and all slopes $1/n$ at ∞), the existence of the above isomorphism contradicts (2.3.8.2) unless $\mu/\lambda=1$. Q.E.D.

(4.5.3) **Theorem.** *Fix an integer $n\geqq 2$, an element $\lambda\in k^\times$, and n elements $a_1,\ldots,a_n$ in k.*

Suppose that

(1) *there exists an integer $N\geqq 1$ prime to n such that*

$$Na_i\in\mathbf{Z}\quad \textit{for}\quad i=1,\ldots,n.$$

(2) $\sum_{i=1}^n a_i$ *lies in* $\mathbf{Z}$.

Then the G_{gal} *of* $Kl_\lambda(a_1, \dots, a_n)$ *is given by*

$$G_{\mathrm{gal}} = \begin{cases} \mathscr{S}\!\mathscr{p}(n) & \text{if } n \text{ is even and } \{a_i \bmod \mathbf{Z}\} = \{-a_i \bmod \mathbf{Z}\} \\ \mathscr{S}\mathscr{L}(n) & \text{otherwise.} \end{cases}$$

Proof. In view of (4.5.2) above, this is just an application of (4.1.7). Q.E.D.

(4.5.4) *Variant.* Notations as in the above theorem, suppose only that there exists an integer $N \geqq 1$ prime to n with all $Na_i \in \mathbf{Z}$ for $i = 1, \dots, n$. The $\sum a_i$ lies in $(1/n)\mathbf{Z}$. Because N is prime to n, there exists an element $b \in (1/N)\mathbf{Z}$, unique modulo $\mathbf{Z}$, which satisfies

$$\sum a_i - nb \in \mathbf{Z}.$$

In terms of the rank-one D.E.

$$K(b) = \mathscr{D}\!\mathit{iff}/\mathscr{D}\!\mathit{iff}(D - b),$$

we have

$$\mathrm{Kl}_\lambda(a_1, \dots, a_n) \simeq \mathrm{Kl}_\lambda(a_1 - b, \dots, a_n - b) \otimes K(b).$$

For $\mathrm{Kl}_\lambda(a_1 - b, \dots, a_n - b)$, all the hypotheses of (4.5.3) are verified. For $K(b)$, G_{gal} is the unique subgroup "$\langle e^{2\pi ib}\rangle$" of $\boldsymbol{\mu}_N$ of order the exact denominator of $b \bmod \mathbf{Z}$. So by (4.1.8) we find that $\mathrm{Kl}_\lambda(a_1, \dots, a_n)$ has G_{gal} equal to

$$\begin{cases} \langle e^{2\pi ib}\rangle \mathscr{S}\!\mathscr{p}(n) & \text{if } n \text{ is even and} \\ & \{a_i - b \bmod \mathbf{Z}\} = \{b - a_i \bmod \mathbf{Z}\} \\ \langle e^{2\pi ib}\rangle \mathscr{S}\mathscr{L}(n) & \text{if not.} \end{cases}$$

Appendix: A table of analogies

Connected smooth curve C over $\mathbf{C}$, with marked point "∞"	connected smooth curve C over an algebraically closed field of characteristic $p > 0$, with marked point "∞".
D.E. on $C - \infty$	lisse ℓ-adic sheaf on $C - \infty$, for $\ell \neq p$, i.e., an ℓ-adic rep'n. of $\pi_1(C - \infty)$.
G_{gal}	$G_{\mathrm{geom}} \overset{\mathrm{dfn}}{=}$ Zariski closure of $\mathrm{Im}(\rho)$.
"Regular singular" at ∞	tamely ramified at ∞.
Local diff. galois group I_∞, with its upper numbering filtration	inertia group I_∞, with its upper numbering filtration.
The subgroup $I_\infty^{(0+)}$ of I_∞	the wild inertia subgroup P_∞ of I_∞.
$I_\infty^{(0+)}$ is a connected torus; every rep'n is a $\oplus$ of characters, every char. of finite order is trivial	P_∞ is a p-group; every rep'n of dim $< p$ is $\oplus$ of char.'s, every character of finite order $< p$ is trivial.
Slopes at ∞	breaks at ∞.
Irregularity at ∞	Swan conductor at ∞.
Deligne's Euler-Poincaré formula ([De-1], II, 6.21)	Grothendieck's Euler-Poincaré formula.
"Canonical extension" to $\mathbf{G}_m$ of a D.E. on a punctured formal nbd. of ∞	"canonical extension" to $\mathbf{G}_m$ of a lisse sheaf on a punctured formal neighborhood of ∞.

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Oblatum 6-XI-1985