Inventiones math. 18, 1–118 (1972)

Algebraic Solutions of Differential Equations (p-Curvature and the Hodge Filtration)

Nicholas M. Katz (Bures-sur-Yvette)

Introduction

This paper grew out of an attempt to answer the following question, first raised by Grothendieck. Consider a linear homogeneous $n \times n$ system of first-order differential equations

$$\frac{d}{dz}(Y) = A(z)\, Y$$

in which $A(z)$ is an $n \times n$ matrix of rational functions of z. To fix ideas, suppose that the coefficients of the entries of $A(z)$ all lie in an algebraic number field K. Then for almost all primes $\mathfrak{p}$ of K, it makes sense to reduce this equation modulo $\mathfrak{p}$, obtaining a differential equation over $\mathbf{F}_q(z)$.

(I) *Suppose that for almost all primes* $\mathfrak{p}$, *the reduced equation has a full set of solutions (i.e., has n solutions in $(\mathbf{F}_q(z))^n$ which are linearly independent over $\mathbf{F}_q(z)$). Does the original equation admit a full set of solutions in algebraic functions of z?*

For example, the equation

$$\frac{d}{dz}(y) = \frac{1}{az}\, y$$

with $a \in \mathbf{Z}$ may be reduced modulo p for all those primes p not dividing a, and the reduced equation admits the solution z^b for any integer b such that $ab \equiv 1$ modulo p. The original equation has for its solution the function $z^{1/a}$. Of course, (I) may be reformulated in greater apparent generality. Let R be a subring of $\mathbf{C}$ which, as a ring, is finitely generated over $\mathbf{Z}$. Let S be a smooth R-scheme with geometrically connected fibres, and consider a differential equation (M, ∇) on S/R, by which we understand a locally free sheaf M on S of finite rank together with an R-linear integrable connection $\nabla: M \to \Omega^1_{S/R} \otimes M$. (We considered above the case $R = \mathcal{O}[1/n]$, $\mathcal{O}$ the ring of integers in an algebraic number field, S an open set in $\mathbf{P}^1_R$, $M = \mathcal{O}^n_S$, and $\nabla: M \to \Omega^1_{S/R} \otimes M$ given by

$$\nabla m = dm - dz \otimes A(z) \cdot m.)$$

Any maximal ideal $\mathfrak{p}$ of R has a finite residue field $\mathbf{F}_q$ of characteristic $p>0$. Reducing modulo $\mathfrak{p}$, we obtain from (M, ∇) a differential equation on a scheme which is smooth over a (finite) field of characteristic $p>0$. In order for the differential equation $(M/\mathfrak{p}M, \nabla)$ in characteristic p to have a full set of solutions, in the sense that $M/\mathfrak{p}M$ is spanned by its subsheaf of horizontal sections; it is necessary and sufficient that a certain p-linear homomorphism, the p-curvature of $(M/\mathfrak{p}M, \nabla)$, *vanish.*

(I bis) *Suppose that for every maximal ideal* $\mathfrak{p}$ *of* R, *the* p*-curvature of* $(M/\mathfrak{p}M, \nabla)$ *vanishes. Does the complex differential equation* $(M, \nabla)_{\mathbf{C}}$ *on the smooth* $\mathbf{C}$*-scheme* $S_{\mathbf{C}}$ *have a full set of algebraic solutions, in the sense that it becomes trivial on a finite etale covering of* $S_{\mathbf{C}}$*?*

It is proved in ([24], Theorem 13.0) that the vanishing of the p-curvature for all maximal ideals $\mathfrak{p}$ of R implies that the complex differential equation $(M, \nabla)_{\mathbf{C}}$ on $S_{\mathbf{C}}$ has only regular singular points, and that its local monodromy groups around the "branches at ∞", in any smooth compactification $\bar{S}_{\mathbf{C}}$ of $S_{\mathbf{C}}$ such that $D=\bar{S}_{\mathbf{C}}-S_{\mathbf{C}}$ is a divisor with normal crossings in $\bar{S}_{\mathbf{C}}$, are all *finite* groups. In view of the fact that differential equations with regular singular points are determined by their global monodromy groups, (I bis) may be reformulated

(I ter) *Suppose that for every maximal ideal* $\mathfrak{p}$ *of* R, *the* p*-curvature of* $(M/\mathfrak{p}M, \nabla)$ *vanishes. Does the differential equation* $(M, \nabla)_{\mathbf{C}}$ *on* $S_{\mathbf{C}}$ *have a finite global monodromy group?*

An easy argument, via restricting to curves, projecting and specializing parameters in R shows that if the original question (I) always has an affirmative answer, then (I bis) and (I ter) always have affirmative answers. Unfortunately, (I) is far from being resolved.

Consider the special case $(M, \nabla)=(\mathcal{O}_S, \nabla)$ of a rank one equation. The connection ∇ is of the form $\nabla(f)=df+f\nabla(1)$; its integrability implies that the one-form $\omega=\nabla(1)$ is necessarily closed. This equation admits a solution mod $\mathfrak{p}$ if and only if ω is logarithmic mod $\mathfrak{p}$. It admits a solution on a finite étale covering of $S_{\mathbf{C}}$ if and only if an integral multiple $n\omega$, $n\geqq 1$ of ω is logarithmic on $S_{\mathbf{C}}$ [for if $\omega=-df/f$ with f an algebraic function of degree n, then putting $g=\operatorname{Norm}(1/f)$, we have $n\omega=dg/g$]. Thus a special case of (I bis), which is interesting even (or perhaps *especially*) when S/R is an open subset of a *curve* of genus $g\geqq 1$ (c.f. (7.4.4) for $g=0$) is

(I log) *If* $\omega\in\Gamma(S, \Omega^1_{S/R})$ *is closed, and is logarithmic modulo* $\mathfrak{p}$ *for every maximal ideal* $\mathfrak{p}$ *of* R, *is an integral multiple* $n\omega$, $n\geqq 1$ *of* ω *logarithmic on* $S_{\mathbf{C}}$*?*

In applications, it is sometimes more natural to reverse the point of view. Given a smooth connected $\mathbf{C}$-scheme $S_{\mathbf{C}}$, and a differential equation $(M, \nabla)_{\mathbf{C}}$ on $S_{\mathbf{C}}$, we can always find an affine open set $\mathcal{U}_{\mathbf{C}}\subset S_{\mathbf{C}}$, a finitely generated subring $R\subset\mathbf{C}$, a smooth R-scheme $\mathcal{U}$ with geometrically connected fibres and complex fibre $\mathscr{U}_{\mathbf{C}}$, and a differential equation (M, ∇) on $\mathscr{U}/R$ which induces $(M, \nabla)_{\mathbf{C}}|\mathscr{U}_{\mathbf{C}}$ on $\mathscr{U}_{\mathbf{C}}$. Suppose there exists an affine open set $\mathscr{V} \subset \mathscr{U}$ such that, for any maximal ideal $\mathfrak{p}$ of R, the p-curvature of $(M/\mathfrak{p}M|V, \nabla)$ vanishes. If such a $\mathscr{V}$ exists for one set of choices $(\mathscr{U}, R, (M, \nabla))$, then such a $\mathscr{V}$ exists for any set of choices. It's existence is thus an intrinsic property of the germ of $(M, \nabla)_{\mathbf{C}}$ at the generic point of $S_{\mathbf{C}}$, which we call "having p-curvature zero for almost all p".

Because "p-curvature zero" is a property which is local for the étale topology, it follows that if $(M, \nabla)_{\mathbf{C}}$ becomes trivial on a finite étale covering of $S_{\mathbf{C}}$, then it has p-curvature zero for almost all p in the above sense. Grothendieck's question is whether the converse is true:

(I quat) *If $(M, \nabla)_{\mathbf{C}}$ on $S_{\mathbf{C}}$ has p-curvature zero for almost all p, does it become trivial on a finite étale covering of $S_{\mathbf{C}}$?*

Our main result is that (I bis) admits an affirmative answer when the differential equation involved is a Picard-Fuchs equation, or a suitable direct factor of one. Recall that if $K/\mathbf{C}$ is any function field, and U/K any smooth K-variety, the finite-dimension K-spaces of algebraic de Rham cohomology $H^n_{DR}(U/K)$ are each endowed with a canonical integrable connection ∇, that of Gauss-Manin ("differentiation of cohomology classes with respect to parameters"). The resulting differential equations $(H^n_{DR}(U/K), \nabla)$ are called the Picard-Fuchs equations.

The suitable direct factors are the following. Suppose a finite group G acts as K-automorphisms of U. Then it acts on the de Rham cohomology $H^n_{DR}(U/K)$ in a horizontal way (i.e., it respects ∇). For any irreducible $\mathbf{C}$-representation χ of G and any automorphism σ of $\mathbf{C}$, let χ^σ denote the representation deduced from χ by applying σ to its matrix coefficients. We say that χ and χ^σ are $\mathbf{Q}$-conjugate. Let $\chi_1, \ldots, \chi_r$ be the non-isomorphic irreducible representation of G which are $\mathbf{Q}$-conjugate to χ. Let $P(\chi_i)(H^n_{DR}(U/K), \nabla)$ denote the χ_i-isotypical component of $(H^n_{DR}(U/K), \nabla)$, i.e. the part of $H^n_{DR}(U/K)$ which transforms by χ_i, with its induced Gauss-Manin connection. Then $\bigoplus_{i=1}^{r} P(\chi_i)(H^n_{DR}(U/K), \nabla)$ is what we mean by a suitable direct factor of $(H^n_{DR}(U/K), \nabla)$.

The proof is based upon the somewhat striking fact that in characteristic p, a suitable associated graded form of the p-curvature of the Gauss-Manin connection is a "twisted" form of the mapping "cup-product with the Kodaira-Spencer class". The proof of this fact is unfortunately computational, and rather long. It depends essentially upon the identity of Hochschild, which asserts that for any derivation D of a commutative $\mathbf{F}_p$-algebra, and for any element X of that algebra, we have $D^{p-1}(X^{p-1}DX)+(D(X))^p = X^{p-1}D^p(X)$. Indeed, in the case of an H^1, this identity is the proof.

The idea of looking at the associated graded mapping defined by the p-curvature was suggested by Clemens' attempt to generalize the Picard-Lefschetz formula.

The proof of the theorem is concluded by a transcendental argument, based essentially on the Hodge Index Theorem. The argument in the non-proper case is due to Deligne, and depends upon his theory of mixed Hodge structures. Our motivation for extending the theorem to the non-proper case, and to the suitable direct factors defined by the action of a finite group, was Dwork's suggestion that our results should imply an affirmative answer to (I bis) for the classical hypergeometric differential equation. Indeed, by studying the one-parameter families of curves $Y^n = X^A(X-1)^B(X-\lambda)^C$, $Y \neq 0$ cover $\mathbf{C}(\lambda)$, we show that the hypergeometric equation with parameters $a, b, c \in \mathbf{C}$

$$\lambda(1-\lambda)\frac{d^2 f}{d\lambda^2} + \big(c-(a+b+1)\,\lambda\big)\frac{df}{d\lambda} - abf = 0$$

has two algebraic solutions if and only if $a, b, c \in \mathbf{Q}$ and, for almost all primes p, the reduced equation has two solutions in $\mathbf{F}_p(\lambda)$ which are linearly independent over $\mathbf{F}_p(\lambda)$.

Consider the "general case", which in the theory of the hypergeometric equation means that none of the exponent differences $1-c$, $c-a-b$, $a-b$ is an integer. As we learned from Dwork, the existence of two mod p solutions for almost all p is equivalent to the condition that 1, $\zeta_a = \exp(2\pi i a)$, $\zeta^b = \exp(2\pi i b)$, $\zeta_c = \exp(2\pi i c)$ be distinct roots of unity, and that for any automorphism σ of $\mathbf{C}$, the counterclockwise arc on the unit circle leading from 1 to ζ_c^σ contains either ζ_a^σ or ζ_b^σ, but not both. The problem of when the hypergeometric equation has two algebraic solutions is of course a classical one, and was solved completely by Schwartz, whose solution (Schwartz's list) is stated with uncommon clarity by Goursat ([10], p. 40–41). It is by no means evident *a priori* that our solution is equivalent to Schwartz's.

The classical theory of Reimann's p-scheme shows that any second order differential equation on an open set of $\mathbf{P}^1_{\mathbf{C}}$ which has at most three singular points, all regular, is isomorphic to (the inverse image, by an automorphism of $\mathbf{P}^1_{\mathbf{C}}$, of) a hypergeometric equation. Thus (I bis) has an affirmative answer for all such equations.

We would like to point out that, by projection, the problem (I log) on a hyperelliptic curve of genus $g \geqq 1$ is equivalent to the problem (I bis) for certain second-order equations on $\mathbf{P}^1_{\mathbf{C}}$ having $3g+2$ singular points. In the final section, we explain how (I log) on an elliptic curve over $\mathbf{Q}$ which has a nontrivial rational point of order two, is equivalent to an intriguing diophantine problem on the elliptic curve, which we view as being (if true) an arithmetic analogue of Manin's "theorem of the Kernel" for abelian varieties over function fields ([29]). For this reason, problem (I bis) for second order differential equations on $\mathbf{P}^1_{\mathbf{C}}$ with five singular points is already of great interest, despite its mundane appearance.

Let us briefly indicate the contents of the various sections.

In the first section, we explain in intrinsic terms the well-known relation between the Gauss-Manin connection and the cup-product with the Kodaira-Spencer class ((1.3.2), (1.4.1.6), (1.4.1.7)).

In the second section, we recall the Cartier operation, which completely analyses the local de Rham cohomology of a smooth morphism in characteristic $p>0$ (cf. (2.1.1)). After an interlude of general nonsense on degeneration of spectral sequences after base change (2.2.1.11), we return to reality and study the "conjugate" spectral sequence of de Rham cohomology in characteristic $p>0$ (cf. (2.3.2)). We then explain the relation between this spectral sequence and the classical Hasse-Witt matrix (cf. (2.3.4.1)), introduce the sometimes-defined "higher" Hasse-Witt matrices (2.3.4.22) and extend to them the theorem of Igusa-Manin (2.3.6.3). This divertimento is concluded by a "numerical example" (2.3.7–8), where we calculate the "Hasse invariant" of a hypersurface of geometric genus one.

In the third section, we prove the main technical result, relating the p-curvature and the Kodaira-Spencer class (3.2), (3.3). Before giving the proof, we recall some of the basic facts about the modular representation theory of finite groups of order prime to p (3.2.1–3).

The fourth section reviews Deligne's theory of mixed Hodge structures. It is independent of the three preceeding sections, and takes place entirely over $\mathbf{C}$. The principal results are (4.1.2), (4.3.3), (4.3.4), (4.3.5), (4.4.2).

In the fifth section, we put together the results of the two preceeding sections, to prove our main theorems ((5.1), (5.3), (5.5), (5.7)) on Picard-Fuchs equations.

The sixth section is devoted to the proof of (I bis) for the hypergeometric equation (6.2). The first two parts (6.0–1) give a useful elementary dictionary between "modules with integrable connections" and "n-th order differential equations". The rest of the section is devoted to the proof of (6.2). The essential proposition (6.8.6) is essentially contained in Messing [32]. The entire section can be read independently of the preceeding ones – the result (5.7) is only applied in a formal way.

The final section is devoted to the problem (I log), especially on elliptic curves. The focal point is Conjecture (7.5.11), which emerged in conversations with Tate and Mazur.

The appendix gives a simple (not elementary) proof of a useful case of Riemann's Existence Theorem.

1. Generalities on the Kodaira-Spencer Class and the Gauss-Manin Connection

1.0. The Geometric Setting

Throughout this Section 1, we will consider the situation

$$
\begin{tikzcd}
D \arrow[r, hook, "i"] & X \arrow[d, "f"] & U = X - D \arrow[l, hook', "j"'] \\
& S \arrow[d, "g"] & \\
& T &
\end{tikzcd}
\tag{1.0.1}
$$

in which T is an arbitrary base scheme, S is a smooth T-scheme (via g), which will play the role of a parameter space, X is a smooth S-scheme (via f), whose fibres over S are "parameterized" by S, and D is (via i) a union of divisors D_i in X, each of which is smooth over S (hence also over T), and which have normal crossings relative to S (hence also relative to T). This situation persists after arbitrary change of base $T' \to T$.

In *practice*, X is usually proper over S, and should be thought of as a particularly nice compactification of the smooth "open" S-scheme $U = X - D$, which is psychologically prior to X. We allow D to be the empty divisor, corresponding to U being proper over S. We do *not assume* X proper over S except when we explicitly so state.

(1.0.2) Let $Der_D(X/T)$ (resp. $Der_D(X/S)$) denote the locally free sheaf on X of germs of T-linear (resp. S-linear) derivations of $\mathcal{O}_X$ to $\mathcal{O}_X$ which preserve the ideal sheaf of each branch D_i of D. We may now define the sheaf of germs of relative (to T, resp. to S) Kahler differentials on X with logarithmic singularities along D, by

$$
\begin{aligned}
\Omega^1_{X/T}(\log D) &= Hom_{\mathcal{O}_X}(Der_D(X/T), \mathcal{O}_X) \\
\Omega^1_{X/S}(\log D) &= Hom_{\mathcal{O}_X}(Der_D(X/S), \mathcal{O}_X).
\end{aligned}
\tag{1.0.2.1}
$$

For every integer $p \geqq 0$, we define

$$
\begin{aligned}
\Omega^p_{X/T}(\log D) &= \Lambda^p_{\mathcal{O}_X} \Omega^1_{X/T}(\log D) \\
\Omega^p_{X/S}(\log D) &= \Lambda^p_{\mathcal{O}_X} \Omega^1_{X/S}(\log D).
\end{aligned}
\tag{1.0.2.2}
$$

(1.0.3) In fact, $\Omega^\bullet_{X/T}(\log D)$ (resp. $\Omega^\bullet_{X/S}(\log D)$) is a *subcomplex* of $j_* \Omega^\bullet_{U/T}$ (resp. $j_* \Omega^\bullet_{U/S}$), where $j \colon U \hookrightarrow X$ denotes the inclusion. To fix ideas, let us explicate these sheaves in terms of local coordinates. We may cover S by affine open sets $\mathscr{U}_i$, and cover X by affine open sets V_i such that

each $\mathscr{U}=\mathscr{U}_i$ is étale over $\mathbf{A}_T^r$ (r depending on i) via local coordinates $s_1, \ldots, s_r$

(1.0.3.1) each $V=V_i$ is étale over $\mathbf{A}^n_{\mathscr{U}_i}$ (n depending on i) via local coordinates $x_1, \ldots, x_n$

the branches of D which meet V_i are defined by the equation $x_\nu=0$, $\nu=1, \ldots, \alpha$ (α depending on i).

Then, over, V the sheaf $Der_D(X/T)$ is a free $\mathscr{O}_V$-module with basis

$$(1.0.3.2)\quad x_\nu \frac{\partial}{\partial x_\nu}\ (\nu=1, \ldots, \alpha),\quad \frac{\partial}{\partial x_j}\ (j=\alpha+1, \ldots, n),\quad \frac{\partial}{\partial s_\mu}\ (\mu=1, \ldots, r)$$

and $Der_D(X/S)$ is a free $\mathscr{O}_V$-module with basis

$$(1.0.3.3)\quad x_\nu \frac{\partial}{\partial x_\nu}\ (\nu=1, \ldots, \alpha),\quad \frac{\partial}{\partial x_j}\ (j=\alpha+1, \ldots, n).$$

Thus, over V, $\Omega^1_{X/T}(\log D)$ is free on $\mathscr{O}_V$ with basis

$$(1.0.3.4)\quad \frac{dx_\nu}{x_\nu}\ (\nu=1, \ldots, \alpha),\quad dx_j\ (j=\alpha+1, \ldots, r),\quad ds_\mu\ (\mu=1, \ldots, r)$$

while $\Omega^1_{X/S}(\log D)$ is free on $\mathscr{O}_V$ with basis

$$(1.0.3.5)\quad \frac{dx_\nu}{x_\nu}\ (\nu=1, \ldots, \alpha),\quad dx_j\ (j=\alpha+1, \ldots, n).$$

(1.0.3.6) Clearly the formation of $\Omega^\bullet_{X/S}$ $(\log D)$ commutes with arbitrary change of base $S' \to S$.

(1.0.3.7) *Remark.* (When S is of characteristic zero, the complex $\Omega^\bullet_{X/S}(\log D)$ is quasi-isomorphic to $j_* \Omega^\bullet_{U/S}$. This is not the case in general, because $j_* \Omega^\bullet_{U/S}$ has "too much" cohomology, while $\Omega^\bullet_{X/S}(\log D)$ has only "geometrically meaningful" cohomology, whence our "preference" for $\Omega^\bullet_{X/S}(\log D)$.)

1.1. The Kodaira-Spencer Class

From the definitions, it follows that we have an exact sequence of locally free sheaves on X

$$(1.1.1)\quad 0 \to f^*(\Omega^1_{S/T}) \to \Omega^1_{X/T}(\log D) \to \Omega^1_{X/S}(\log D) \to 0$$

which gives rise to an element

$$(1.1.2)\qquad \rho \in \operatorname{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}(\log D), f^*(\Omega^1_{S/T})) \simeq H^1(X, \mathit{Der}_D(X/S) \otimes_{\mathcal{O}_X} f^*(\Omega^1_{S/T})).$$

This element is called the *Kodaira-Spencer class* of the situation 1.0. It's image in

$$(1.1.3)\qquad \begin{aligned} &H^0(S, R^1 f_*(\mathit{Der}_D(X/S) \otimes_{\mathcal{O}_X} f^*(\Omega^1_{S/T}))) \\ &\simeq H^0(S, R^1 f_*(\mathit{Der}_D(X/S)) \otimes_{\mathcal{O}_S} \Omega^1_{S/T}) \\ &\simeq \operatorname{Hom}_{\mathcal{O}_S}(\mathit{Der}(S/T)), R^1 f_*(\mathit{Der}_D(X/S)) \end{aligned}$$

is the *Kodaira-Spencer mapping* (but still denoted ρ!), which may be explicated as follows. Let D be a section of $\mathit{Der}(S/T)$ over an affine open set $\mathscr{U} \subset S$. Then $\rho(D)$ is an element of $H^1(f^{-1}(\mathscr{U}), \mathit{Der}_D(f^{-1}(\mathscr{U})/\mathscr{U}))$ which may be given explicitly as follows. Let $\{V_i\}$ be an affine open cover of $f^{-1}(\mathscr{U})$. By the exactness of the dual of (1.1.1) over each V_i (or, more "directly", by the explicit description (1.0.3) via local coordinates), we may choose, for each i, a derivation $D_i \in H^0(V_i, \mathit{Der}_D(X/T))$ which *extends* the given derivation $D \in H^0(\mathscr{U}, \mathit{Der}(S/T))$. Because D_i and D_j extend the *same* derivation D of $\mathscr{U}$, the difference $D_i - D_j$ lies in $H^0(V_i \cap V_j, \mathit{Der}_D(X/S))$. Thus $\{D_i - D_j\}$ defines a 1-cocycle on the covering $\{V_i\}$ with coefficients in $\mathit{Der}_D(f^{-1}(\mathscr{U})/\mathscr{U})$, whose cohomology class is $\rho(D)$. In particular, $\rho(D) = 0$ if and only if there exists a derivation in $H^0(f^{-1}(\mathscr{U}), \mathit{Der}_D(X/T))$ which extends D.

1.2. The Koszul Filtration, Interior Product, and Cup Product

(1.2.0) This section is devoted to recording some compatibilities among wellknown constructions from exterior algebra as they occur in homological algebra. For ease of later reference, we record a few more compatibilities than necessary for our application in Section 3.

(1.2.1) Let X be an arbitrary scheme (for even a ringed topos, if that be more to the taste of the reader), and let

$$(1.2.1.1)\qquad 0 \to \mathscr{G} \xrightarrow{\alpha} \mathscr{H} \xrightarrow{\beta} \mathscr{F} \to 0$$

be an exact sequence of locally free $\mathcal{O}_X$-modules of finite rank. (In the application, it will be the exact sequence (1.1.1).) The *Koszul filtration* of the exterior algebra $\Lambda^\bullet_{\mathcal{O}_X} \mathscr{H}$, which we will henceforth denote simply $\Lambda^\bullet \mathscr{H}$ is defined by the ideals

$$(1.2.1.2)\qquad K^i(\Lambda^\bullet \mathscr{H}) = \text{image of } \Lambda^i \mathscr{G} \otimes \Lambda^{\bullet - i} \mathscr{H} \to \Lambda^\bullet \mathscr{H}.$$

The associated graded module is given by

(1.2.1.3) $$\mathrm{gr}^i_K(\Lambda^\bullet \mathscr{H}) \simeq \Lambda^i \mathscr{G} \otimes \Lambda^{\bullet - i} \mathscr{F}.$$

Consider the exact sequence

(1.2.1.4) $$0 \to \mathrm{gr}^1_K \to K^0/K^2 \to \mathrm{gr}^0_K \to 0$$

whose term of degree ν is

(1.2.1.5) $$0 \to \mathscr{G} \otimes \Lambda^{\nu-1} \mathscr{F} \to K^0/K^2(\Lambda^\nu \mathscr{H}) \to \Lambda^\nu \mathscr{F} \to 0.$$

(1.2.1.6) For each integer $\nu \geqq 1$, we define in this way a functor Λ^ν from the category $\mathrm{EXT}(\mathscr{F}, \mathscr{G})$ of extensions of $\mathscr{F}$ by $\mathscr{G}$ to the category

$$\mathrm{EXT}(\Lambda^\nu \mathscr{F}, \mathscr{G} \otimes \Lambda^{\nu-1} \mathscr{F})$$

of extensions of $\Lambda^\nu \mathscr{F}$ by $\mathscr{G} \otimes \Lambda^{\nu-1} \mathscr{F}$. Passing to the (groups of) isomorphism classes of objects of these categories, we obtain a morphism, still denoted Λ^ν,

(1.2.1.7) $$\Lambda^\nu \colon \mathrm{Ext}^1_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\Lambda^\nu \mathscr{F}, \mathscr{G} \otimes \Lambda^{\nu-1} \mathscr{F}).$$

Because $\mathscr{F}$ is locally free, the sheaf $\mathit{Ext}^1_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ vanishes, and the local $\Rightarrow$ global spectral sequence of Ext furnishes us with an isomorphism

(1.2.1.8) $$\mathrm{Ext}^1_{\mathcal{O}_S}(\mathscr{F}, \mathscr{G}) \simeq H^1(X, \mathit{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})).$$

Similarly, the local freeness of $\Lambda^\nu \mathscr{F}$ furnishes an isomorphism

(1.2.1.8bis) $$\mathrm{Ext}^1_{\mathcal{O}_X}(\Lambda^\nu \mathscr{F}, \mathscr{G} \otimes \Lambda^{\nu-1} \mathscr{F}) \simeq H^1(H, \mathit{Hom}_{\mathcal{O}_X}(\Lambda^\nu \mathscr{F}, \mathscr{G} \otimes \Lambda^{\nu-1} \mathscr{F})).$$

Let us make explicit the two identifications (1.2.1.8) and (1.2.1.8 bis), then use them to make explicit the morphism Λ^ν (1.2.1.7).

(1.2.1.9) Since $\mathscr{F}$ is locally free of finite rank, the exactness of (1.2.1.1) assures that there is an open covering V_i of X and, over each V_i, a morphism $\varphi_i \colon \mathscr{F}|V_i \to \mathscr{H}|V_i$ which is a section of $\beta \colon \mathscr{H} \to \mathscr{F}$ (i.e. such that $\beta \circ \varphi_i = \mathrm{id}_{\mathscr{F}|V_i}$). The difference $\varphi_i - \varphi_j$ defines a morphism from $\mathscr{F}|V_i \cap V_j$ to $\mathscr{H}|V_i \cap V_j$ whose image is in fact contained in $\mathrm{Ker}(\beta)|V_i \cap V_j$ (because on $V_i \cap V_j$, $\beta \circ (\varphi_i - \varphi_j) = \beta \circ \varphi_i - \beta \circ \varphi_j = \mathrm{id}_{\mathscr{F}} - \mathrm{id}_{\mathscr{F}} = 0$). Thus we may interpret $\varphi_i - \varphi_j$ as defining a morphism from $\mathscr{F}|V_i \cap V_j$ to $\mathscr{G}|V_i \cap V_j$ ($\underset{\alpha}{\rightleftarrows} \mathrm{Ker}(\beta)|V_i \cap V_j$), whence $\{\varphi_i - \varphi_j\}$ is a one-cocycle for the covering $\{V_i\}$ with coefficients in $\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$. The cohomology class in $H^1(X, \mathit{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}))$ of this cocycle is the element corresponding to the extension class of (1.2.11) via the isomorphism 1.2

(1.2.1.10) Let us recall that there is another "standard" isomorphism

(1.2.1.10.1) $$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) \simeq H^1(X, \mathit{Hom}(\mathscr{F}, \mathscr{G}))$$

which is the *negative* of the isomorphism (1.2.1.8) explicated in (1.2.1.9), and which we will *not* use.

In the setting of (1.2.1.9), we may choose over each V_i a morphism ψ_i: $\mathscr{H}|V_i \to \operatorname{Ker}(\beta)|V_i \underset{\alpha}{\tilde{\leftarrow}} \mathscr{G}|V_i$ which is a section of α: $\mathscr{G} \hookrightarrow \mathscr{H}$ (i.e., such that $\psi_i \cdot \alpha = \mathrm{id}_{\mathscr{G}|V_i}$).

(1.2.1.10.2) In fact, we may simply choose ψ_i so that $\alpha \circ \psi_i = \mathrm{id}_{\mathscr{H}|V_i} - \varphi_i \circ \beta$, this being possible because, over V_i, $\mathrm{id}_{\mathscr{H}} - \varphi_i \circ \beta$ is a projection onto $\ker(\beta)$. Then the difference $\psi_i - \psi_j$ defines a morphism from $\mathscr{H}|V_i \cap V_j$ to $\mathscr{G}|V_i \cap V_j$ which vanishes on the image of α, thus defining by passage to quotients a morphism from $\mathscr{F}|V_i \cap V_j$ to $\mathscr{G}|V_i \cap V_j$, still denoted $\psi_i - \psi_j$. The cohomology class of $\{\psi_i - \psi_j\}$ in $H^1(X, Hom(\mathscr{F}, \mathscr{G}))$ is the element corresponding to the extension class of (1.2.1.1) via the isomorphism (1.2.1.10.1). To see that *this* isomorphism is the negative of (1.2.1.9), it suffices to recall that

(1.2.1.10.3) $$\alpha \circ \psi_i = \mathrm{id}_{\mathscr{H}|V_i} - \varphi_i \circ \beta$$

whence

(1.2.1.10.4) $$\alpha \circ (\psi_i - \psi_j) = -(\varphi_i - \varphi_j) \circ \beta$$

which gives the equality of the two cocycles $\{\psi_i - \psi_j\}$ and $\{-(\varphi_i - \varphi_j)\}$.

(1.2.1.11) The "second" isomorphism (1.2.1.10) is the *dual* of the first (1.2.1.8), in the sense that the diagram

$$\begin{array}{ccc} \mathrm{Ext}^1(\mathscr{F}, \mathscr{G}) & \xrightarrow[\sim]{(1.2.1.8)} & H^1(X, Hom(\mathscr{F}, G)) \\ \downarrow \vee & & \downarrow \vee \\ \mathrm{Ext}^1(\check{\mathscr{G}}, \check{\mathscr{F}}) & \xrightarrow[\sim]{(1.2.1.10)} & H^1(X, Hom(\check{\mathscr{G}}, \check{\mathscr{F}})) \end{array}$$

in which the left vertical map associates to the class of $0 \to \mathscr{G} \to \mathscr{H} \to \mathscr{F} \to 0$ the class of the dual extension $0 \to \check{\mathscr{F}} \to \check{\mathscr{H}} \to \check{\mathscr{G}} \to 0$ and in which the right vertical arrow is deduced by passage to cohomology from the canonical isomorphism of duality $Hom(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} Hom(\check{\mathscr{G}}, \check{\mathscr{F}})$, is commutative. To see this let us remark simply that if $\{\varphi_i - \varphi_j\}$ is a cocycle representing the class of $0 \to \mathscr{G} \to \mathscr{H} \to \mathscr{F} \to 0$ via (1.2.1.8), which is to say, via splitting $\mathscr{H} \to \mathscr{F}$, then $\{\check{\varphi}_i - \check{\varphi}_j\}$ is a cocycle arising from the class of $0 \to \check{\mathscr{F}} \to \check{\mathscr{H}} \to \check{\mathscr{G}} \to 0$ by splitting $\check{\mathscr{F}} \to \check{\mathscr{H}}$, which is to say, via (1.2.1.10). Keaping in mind that the isomorphisms (1.2.1.8) and (1.2.1.10) are the negative of each other, we find:

(1.2.1.12) **Proposition.** *Let $\xi \in H^1(X, Hom(\mathscr{F}, \mathscr{G}))$ be the class of the extension $0 \to \mathscr{G} \to \mathscr{H} \to \mathscr{F} \to 0$ via* (1.2.1.8), *and let $\check{\xi} \in H^1(X, Hom(\check{\mathscr{G}}, \check{\mathscr{F}}))$ be the class of the dual extension $0 \to \check{\mathscr{F}} \to \check{\mathscr{H}} \to \check{\mathscr{G}} \to 0$, via* (1.2.1.8). *Then via the canonical isomorphism $Hom(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} Hom(\check{\mathscr{G}}, \check{\mathscr{F}})$, $\check{\xi} = -\xi$.*

(1.2.2) Interior Product and Cup Product.

Let us now recall the morphism I of *interior product* for each integer $\nu \geqq 1$.

(1.2.2.1) $$I\colon Hom_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \to Hom_{\mathcal{O}_X}(\Lambda^\nu \mathcal{F}, \mathcal{G}\otimes\Lambda^{\nu-1}\mathcal{F}).$$

In terms of local sections $f_1,\dots,f_\nu$ of $\mathcal{F}$ and φ of $Hom(\mathcal{F},\mathcal{G})$,

(1.2.2.2) $$I(\varphi)(f_1\wedge\cdots\wedge f_\nu)=\sum_{j=1}^{\nu}(-1)^{j-1}\varphi(f_j)\otimes f_1\wedge\cdots\wedge\widehat{f_j}\wedge\cdots\wedge f_\nu.$$

(1.2.2.3) **Proposition.** *The diagram*

$$\begin{array}{ccc}\mathrm{Ext}^1(\mathcal{F},\mathcal{G}) & \xrightarrow{\Lambda^\nu} & \mathrm{Ext}^1(\Lambda^\nu\mathcal{F},\mathcal{G}\otimes\Lambda^{\nu-1}\mathcal{F}) \\ \wr\| & & \wr\| \\ H^1(X, Hom(\mathcal{F},\mathcal{G})) & \xrightarrow{I} & H^1(X, Hom(\Lambda^\nu\mathcal{F},\mathcal{G}\otimes\Lambda^{\nu-1}\mathcal{F}))\end{array}$$

in which the vertical isomorphisms are (1.2.1.8) *and* (1.2.1.8 bis), *is commutative.*

Proof. Just as in (1.2.1.9), let us choose a covering V_i of X and sections $\varphi_i\colon \mathcal{F}|V_i\to\mathcal{H}|V_i$ of $\beta\colon\mathcal{H}\to\mathcal{F}$. Abusing notation, we denote by $\Lambda^\nu\varphi_i$ the *composition*

(1.2.2.3.1) $$\Lambda^\nu\mathcal{F}|V_i \xrightarrow{\Lambda^\nu(\varphi_i)} \Lambda^\nu\mathcal{H}|V_i \xrightarrow{\text{proj.}} K^0/K^2(\Lambda^\nu\mathcal{H})|V_i$$

(i.e., $\Lambda^\nu\varphi_i \overset{\text{def'n}}{=} \Lambda^\nu(\varphi_i)$ modulo K^2). Clearly the $\Lambda^\nu\varphi_i$ give local sections of $\Lambda^\nu\beta\colon K^0/K^2(\Lambda^\nu\mathcal{H})\to\Lambda^\nu\mathcal{F}$, and hence 1-cocycle $\{\Lambda^\nu\varphi_i-\Lambda^\nu\varphi_j\}$ with coefficients in $Hom_{\mathcal{O}_X}(\Lambda^\nu\mathcal{F},\mathcal{G}\otimes\Lambda^{\nu-1}\mathcal{F})$ has as cohomology class in $H^1(X, Hom_{\mathcal{O}_X}(\Lambda^\nu\mathcal{F},\mathcal{G}\otimes\Lambda^{\nu-1}\mathcal{F}))\simeq\mathrm{Ext}^1(\Lambda^\nu\mathcal{F},\mathcal{G}\otimes\Lambda^{\nu-1}\mathcal{F})$ the class of $\Lambda^\nu(0\to\mathcal{G}\to\mathcal{H}\to\mathcal{F}\to 0)$. To conclude the proof, it remains only to notice that

(1.2.2.3.2) $$\Lambda^\nu\varphi_i-\Lambda^\nu\varphi_j=I(\varphi_i-\varphi_j).$$

To see this, we calculate, for local sections $f_1,\dots,f_\nu$ of $\mathcal{F}$

$$\begin{aligned}(\Lambda^\nu\varphi_i-\Lambda^\nu\varphi_j)(f_1\wedge\cdots\wedge f_\nu) &= \varphi_i(f_1)\wedge\cdots\wedge\varphi_i(f_\nu)-\varphi_j(f_1)\wedge\cdots\wedge\varphi_j(f_\nu) \\ &= (\varphi_j(f_1)+(\varphi_i-\varphi_j)(f_1))\wedge\cdots\wedge(\varphi_j(f_\nu)+(\varphi_i-\varphi_j)(f_\nu)) \\ &\quad -\varphi_j(f_1)\wedge\cdots\wedge\varphi_j(f_\nu) \\ &= \sum_{a=1}^{\nu}(-1)^{a-1}(\varphi_i-\varphi_j)(f_a)\otimes f_1\wedge\cdots\wedge\widehat{f_a}\wedge\cdots\wedge f_\nu \\ &\quad +\text{terms in } K^2(\Lambda^\nu\mathcal{H}) \\ &= I(\varphi_i-\varphi_j)(f_1\wedge\cdots\wedge f_\nu).\quad \text{Q.E.D.}\end{aligned}$$

(1.2.2.4) Let us reinterpret the interior product mapping I as a pairing

(1.2.2.4.1) $$I\colon \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})\otimes \Lambda^{\nu}\mathcal{F}\to\mathcal{G}\otimes\Lambda^{\nu-1}\mathcal{F}.$$

From this pairing we deduce a cup-product pairing for all $p, q \geqq 0$

(1.2.2.4.2) $$H^p(X, \mathcal{H}om(\mathcal{F},\mathcal{G}))\otimes H^q(X,\Lambda^{\nu}\mathcal{F})\to H^{p+q}(X,\mathcal{G})\otimes\Lambda^{\nu-1}\mathcal{F}).$$

(1.2.2.4.3) Taking $p=1$, consider the element $\xi\in H^1(X,\mathcal{H}om(\mathcal{F},\mathcal{G}))$ which corresponds via (1.2.1.8) to the class of the extension (1.2.1.2),

(1.2.2.4.4) $$0\to\mathcal{G}\to\mathcal{H}\to\mathcal{F}\to 0.$$

(1.2.2.4.5) **Proposition.** *The mapping "cup-product with ξ"*

$$\xi\colon H^q(X,\Lambda^{\nu}\mathcal{F})\to H^{q+1}(X,\mathcal{G}\otimes\Lambda^{\nu-1}\mathcal{F})$$

deduced from (1.2.2.4.2) *is equal to the coboundary in the long exact cohomology sequence arising from the short exact sequence* (1.2.1.5)

(1.2.2.4.6) $$0\to\mathcal{G}\otimes\Lambda^{\nu-1}\mathcal{F}\to K^0/K^2(\Lambda^{\nu}\mathcal{H})\to\Lambda^{\nu}\mathcal{F}\to 0.$$

Proof. Indeed by definition this map is none other than the cup-product with the image of $\xi\in H^1(X,\mathcal{H}om(\mathcal{F};\mathcal{G}))$ under

$$I\colon H^1(X,\mathcal{H}om(\mathcal{F},\mathcal{G}))\to H^1(X,\mathcal{H}om(\Lambda^{\nu}\mathcal{F},\mathcal{G}\otimes\Lambda^{\nu-1}\mathcal{F})).$$

According to (1.2.2.3), the image $I(\xi)$ is none other than the class of the extension (1.2.2.4.6) (via the isomorphism (1.2.1.8 bis)).

(1.2.2.4.7) But it is a general fact that the coboundary mapping $H^q(X,C)\to H^{q+1}(X,A)$ associated to *any* short exact sequence $0\to A\to B\to C\to 0$ of $\mathcal{O}_X$-modules is none other than the cup-product with the element of $\operatorname{Ext}^1_{\mathcal{O}_X}(C,A)$ (this last operation has a sense, thanks to the isomorphism of functors $H^q(X,-)\xrightarrow{\sim}\operatorname{Ext}^q_{\mathcal{O}_X}(\mathcal{O}_X,-)$). Q.E.D.

1.3. Application to Hodge Cohomology

(1.3.1) **Proposition.** *In the geometric situation of* 1.0, *consider the short exact sequence* (1.1.1)

(1.3.1.1) $$0\to f^*(\Omega^1_{S/T})\to\Omega^1_{X/T}(\log D)\to\Omega^1_{X/S}(\log D)\to 0$$

which gives rise, via the Λ^p construction (1.2.1.6), *to a short exact sequence*

(1.3.1.2) $$0\to f^*(\Omega^1_{S/T})\otimes\Omega^{p-1}_{X/S}(\log D)\to K^0/K^2(\Omega^p_{X/T}(\log D))\to\Omega^p_{X/S}(\log D)\to 0.$$

The coboundary mapping associated to (1.3.1.2)

(1.3.1.3) $\partial\colon H^q(X, \Omega^p_{X/S}(\log D)) \to H^{q+1}(X, f^*(\Omega^1_{S/T}) \otimes \Omega^{p-1}_{X/S}(\log D))$

is the cup-product with the Kodaira-Spencer class (1.1.2)

$$\begin{aligned}\rho \in & \operatorname{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}(\log D), f^*(\Omega^1_{S/T})) \\ & \simeq H^1(X, \mathit{Hom}(\Omega^1_{X/S}(\log D), f^*(\Omega^1_{S/T})) \simeq H^1(X, f^*(\Omega^1_{S/T}) \otimes \operatorname{Der}_D(X/S)).\end{aligned}$$

Proof. This is just (1.2.2.4.5) applied to the exact sequence (1.3.1.1), in which case the element ξ of (1.2.2.4.5) *is* the Kodaira-Spencer class.

Localizing on S, we have

(1.3.2) **Corollary.** *Hypotheses as in* (1.3.1), *the coboundary mapping*

(1.3.2.1)
$$\begin{array}{c}\partial\colon R^q f_*(\Omega^p_{X/S}(\log D)) \to R^{q+1} f_*(f^*(\Omega^1_{S/T}) \otimes \Omega^{p-1}_{X/S}(\log D)) \\ \wr\| \\ \Omega^1_{S/T} \otimes R^{q+1} f_*(\Omega^{p-1}_{X/S}(\log D))\end{array}$$

associated to (1.3.1.1) *is given by cup-product with the Kodaira-Spencer mapping* (1.1.3), *viewed as an element*

(1.3.2.2)
$$\begin{array}{c}\rho \in \operatorname{Hom}_{\mathcal{O}_S}(\mathit{Der}(S/T)), R^1 f_*(\mathit{Der}_D(X/S)) \\ \wr\| \\ H^0(S, \Omega^1_{S/T}) \otimes R^1 f_*(\mathit{Der}_D(X/S)).\end{array}$$

1.4. Application to the Gauss-Manin Connection

(1.4.0) The construction of the Gauss-Manin connection on

$$H_{DR}(X/S(\log D)) = \mathbf{R} f_*(\Omega_{X/S}(\log D))$$

is based on the fact that the Koszul filtration of $\Omega^\bullet_{X/T}(\log D)$ arising from the exact sequence (1.1.1)

(1.4.0.1) $0 \to f^*(\Omega^1_{S/T}) \to \Omega^1_{X/T}(\log D) \to \Omega^1_{X/S}(\log D) \to 0$

is a filtration by subcomplexes, and that the associated graded complexes are given by

(1.4.0.2) $\operatorname{gr}^i_K(\Omega^\bullet_{X/T}(\log D)) \simeq f^*\, \Omega^i_{S/T} \otimes \Omega^{\bullet - i}_{X/S}(\log D).$

In fact, it is defined ([35]) as the coboundary map ∇ in the long exact sequence of the $\mathbf{R}^i f_*$ and the short exact sequence

(1.4.0.3)
$$\begin{aligned} 0 \to f^*(\Omega^1_{S/T}) \otimes \Omega^{\bullet-1}_{X/S}(\log D) \to K^0/K^2(\Omega^\bullet_{X/T}(\log D)) \\ \to \Omega^\bullet_{X/S}(\log D) \to 0. \end{aligned}$$

This "makes sense", because the coboundary is a mapping

(1.4.0.4)
$$\begin{tikzcd} \nabla\colon \mathbf{R}^q f_*(\Omega^\bullet_{X/S}(\log D)) \arrow[r] & \mathbf{R}^{q+1} f_*(f^*(\Omega^1_{S/T}) \otimes \Omega^{\bullet-1}_{X/S}(\log D)) \\ & \Omega^1_{S/T} \otimes \mathbf{R}^q f_*(\Omega^\bullet_{X/S}(\log D)). \arrow[u, "\wr"'] \end{tikzcd}$$

(The final isomorphism thanks to the local freeness of $\Omega^1_{S/T}$.)

(1.4.1) *The Hodge Filtration.* Let us recall that for any complex $L^\bullet$, the Hodge filtration of $L^\bullet$ is the filtration by the subcomplexes $F^i(L^\bullet)$, where by definition

(1.4.1.1)
$$F^i(L^j) = \begin{cases} 0 & \text{if } j \equiv i \\ L^j & \text{if } j \geqq i. \end{cases}$$

(1.4.1.2) Let us denote by $L^\bullet[n]$ the complex $L^{\bullet+n}$ whose term of degree i is L^{i+n}, whence

(1.4.1.3)
$$F^i(L^\bullet[n]) = (F^{i+n}(L^\bullet))[n].$$

Applying F^i to the exact sequence (1.4.0.3), we obtain the exact sequence

(1.4.1.4)
$$\begin{aligned} 0 \to (f^*(\Omega^1_{S/T}) \otimes F^{i-1}(\Omega^\bullet_{X/S}(\log D)))[-1] \\ \to F^i(K^0/K^2(\Omega^\bullet_{X/T}(\log D)))\, F^i(\Omega^\bullet_{X/S}(\log D)) \to 0. \end{aligned}$$

Thus the coboundary maps for the $\mathbf{R}^i f_*$ and the exact sequence (1.4.0.3) and (1.4.1.4) "fit together" to form a commutative diagram

(1.4.1.5)
$$\begin{tikzcd} \mathbf{R}^q f_*(\Omega^\bullet_{X/S}(\log D)) \arrow[r, "\nabla"] & \Omega^1_{S/T} \otimes \mathbf{R}^q f_*(\Omega^\bullet_{X/S}(\log D)) \\ \mathbf{R}^q f_*(F^i(\Omega^\bullet_{X/S}(\log D))) \arrow[r, "\partial"] \arrow[u] & \Omega^1_{S/T} \otimes \mathbf{R}^q f_*(F^{i-1}(\Omega^\bullet_{X/S}(\log D))). \arrow[u] \end{tikzcd}$$

Thus we find:

(1.4.1.6) **Proposition** *(Griffith's Transversality Theorem). The Gauss-Manin connection respects the Hodge filtration up to a shift of one, i.e.*

(1.4.1.6.1)
$$\nabla(F^i\, \mathbf{R}^q f_*(\Omega^\bullet_{X/S}(\log D))) \subset \Omega^1_{S/T} \otimes (\mathbf{R}^q f_*(\Omega^\bullet_{X/S}(\log D))).$$

As a corollary of (1.3.2) and the *definition* of the Gauss-Manin connection, we have

(1.4.1.7) **Proposition.** *Suppose that the Hodge $\Rightarrow$ De Rham spectral sequence*

(1.4.1.7.1) $$E_1^{p,q} = R^q f_*(\Omega^p_{X/S}(\log D)) \Rightarrow \mathbf{R}^{p+q} f_*(\Omega^\bullet_{X/S}(\log D))$$

is degenerate at E_1, i.e., that $\mathrm{gr}^p_F \mathbf{R}^{p+q} f_*(\Omega^\bullet_{X/S}(\log D)) = R^q f_*(\Omega^p_{X/S})$. *Then the associated graded mapping induced by the Gauss-Manin connection is the cup-product with the Kodaira-Spencer mapping* (1.3.2.2)

$$\rho \in H^0(S, \Omega^1_{S/T} \otimes R^1 f_*(Der_D(X/S)));$$

i.e., the diagram

(1.4.1.7.2) $$\begin{array}{ccc} \mathrm{gr}^p_F \mathbf{R}^{p+q} f_*(\Omega^\bullet_{X/S}(\log D)) & \xrightarrow{\nabla} & \Omega^1_{S/T} \otimes \mathrm{gr}^{p-1}_F \mathbf{R}^{p+q} f_*(\Omega^\bullet_{X/S}(\log D)) \\ \wr\| & & \wr\| \\ R^q f_*(\Omega^p_{X/S}(\log D)) & \xrightarrow{\rho} & \Omega^1_{S/T} \otimes R^{q+1} f_*(\Omega^{p-1}_X(\log D)) \end{array}$$

commutes.

(1.4.1.8) *Remark.* In case X/S is proper, it follows from Deligne's mixed Hodge theory [8] and a slight modification of his argument ([6], Theorem 5.5) that

(1.4.1.8.1) If S is any scheme of characteristic zero, the spectral sequence (1.4.1.7.1) is degenerate at E_1, all of its terms $E_1^{p,q}$, $E_\infty^{p,q}$ are locally free, and its formation commutes with arbitrary change of base $S' \to S$.

(1.4.1.8.2) If S is any reduced and irreducible scheme whose generic point is of characteristic zero, there exists a non-void Zariski open set $\mathscr{U}$ in S over which the assertions of (1.4.1.8.1) are valid. We conclude this section by stating explicitly a very useful corollary of (1.4.1.7).

(1.4.1.9) **Corollary.** *Hypotheses as in* (1.4.1.7), *fix an integer $n \geqq 0$, and suppose that $M \subseteqq \mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))$ is an $\mathcal{O}_S$-submodule stable under the Gauss-Manin connection, i.e., that*

(1.4.1.9.1) $$\nabla(M) \subset \Omega^1_{S/T} \otimes M.$$

(We then say that M is horizontal.)

Let us define the induced Hodge filtration of M, $F^i(M)$, by

(1.4.1.9.2) $$F^i(M) = M \cap F^i\, \mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)).$$

Then

$$\begin{tikzcd} \mathrm{gr}_F^p(M) = F^p(M)/F^{p+1}(M) \arrow[r, hook] & \mathrm{gr}_F^p\, \mathbf{R}^{p+q} f_*(\Omega^\bullet_{X/S}(\log D)) \arrow[d, "\wr"] \\ & R^q f_*(\Omega^p_{X/S}(\log D)). \end{tikzcd} \tag{1.4.1.9.3}$$

Because M is horizontal, it follows from (1.4.1.7) *that we have a commutative diagram*

$$\begin{tikzcd} \mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)) \arrow[r, "\nabla"] & \Omega^1_{S/T} \otimes \mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)) \\ F^p(M) \arrow[u, "\cup"'] \arrow[r, "\nabla"] \arrow[d, two heads] & \Omega^1_{S/T} \otimes F^{p-1}(M) \arrow[u, "\cup"'] \arrow[d, two heads] \\ \mathrm{gr}_F^p(M) \arrow[r, "\rho"] \arrow[d, "\cap"] & \Omega^1_{S/T} \otimes \mathrm{gr}_F^{p-1}(M) \arrow[d, "\cap"] \\ R^q f_*(\Omega^p_{X/S}(\log D)) \arrow[r, "\rho"] & \Omega^1_{S/T} \otimes R^{q+1} f_*(\Omega^{p-1}_{X/S}(\log D)) \end{tikzcd} \tag{1.4.1.9.4}$$

We deduce from it that the Hodge filtration $F^i(M)$ of M is horizontal (i.e., each $F^i(M)$ is horizontal) if and only if the restriction to $\oplus_p \mathrm{gr}_F^p(M)$ of the mapping "cup-product with the Kodaira-Spencer class"

$$\rho\colon \bigoplus_p \mathrm{gr}_F^p(M) \to \bigoplus_p \Omega^1_{S/T} \otimes \mathrm{gr}_F^{p-1}(M) \tag{1.4.1.9.5}$$

vanishes.

(1.4.1.10) *Remark.* In practice, the M in (1.4.1.9) will be either all of $\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))$, or the *primitive part* of $\mathbf{R}^n f_*(\Omega^\bullet_{X/S})$ in case D is void and X/S is projective and smooth, or the part of $\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))$ which transforms according to a prechosen irreducible representation of a finite group (of order prime to all residue characteristics of S) which acts as a group of S-automorphisms of X and preserves D. This last case will arise when we discuss Schwartz's list (cf. 6.0).

2. The Cartier Operation and the Conjugate Spectral Sequence

2.0. Throughout this section, we will consider the situation of 1.0, with the additional assumption that S is a scheme of characteristic p (a prime number), i.e. that $p \cdot 1_S = 0$ in $\mathcal{O}_S$.

(2.0.1) Recall that for any S-scheme $\pi\colon Y \to S$, the S-scheme $Y^{(p)}$ is by definition the fibre product of $\pi\colon Y \to S$ and the *absolute* Frobenius morphism $F_{\text{abs}}: S \to S$ (so on the ring level, F_{abs} is just "raising to the p-th power"). Thus by construction $Y^{(p)}$ sits in a cartesian diagram

(2.0.1.1)
$$\begin{tikzcd} Y^{(p)} \arrow[r, "\sigma = \pi^*(F_{\text{abs}})"] \arrow[d, "\pi^{(p)} = F_{\text{abs}}^*(\pi)"] & Y \arrow[d, "\pi"] \\ S \arrow[r, "F_{\text{abs}}"'] & S \end{tikzcd}$$

The pair of morphisms

$$F_{\text{abs}}: Y \to Y$$
$$\pi: Y \to S$$

defines a morphism $F: Y \to Y^{(p)}$, the *relative Frobenius*, which fits into a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "F"] \arrow[dr, "\pi"'] & Y^{(p)} \arrow[r, "\sigma"] \arrow[d, "\pi^{(p)}"] & Y \arrow[r, "F"] \arrow[d, "\pi"] & Y^{(p)} \arrow[dl, "\pi^{(p)}"] \\ & S \arrow[r, "F_{\text{abs}}"] & S & \end{tikzcd}$$

in which $F \cdot \sigma$ is $F_{\text{abs}}: Y^{(p)} \to Y^{(p)}$ and $\sigma \cdot F$ is $F_{\text{abs}}: Y \to Y$. Intuitively, F raises the "vertical coordinates" to the p-th power, and σ raises the "S coordinates" to the p-th power.

(2.0.2) Consider now the special case $Y = X$ is a smooth S-scheme, say of relative dimension n. Then:

(2.0.2.1) $F_*(\mathcal{O}_X)$ is a locally free $\mathcal{O}_{X^{(p)}}$ module of rank p^n; indeed if $x_1, \ldots, x_n$ are local coordinates on an open set $V \subset X$ (i.e., an étale morphism $V \to \mathbf{A}_S^n$), then a base of $F_*(\mathcal{O}_V)$ as $\mathcal{O}_{V^{(p)}}$ module is given by the p^n monomials $x_1^{w_1} \ldots x_n^{w_n}$ having $0 \leqq w_i \leqq p-1$.

(2.0.2.2) $F^{-1}(\mathcal{O}_{X^{(p)}})$ is precisely the subsheaf of $\mathcal{O}_X$ which is killed by all of $Der(X/S)$.

(2.0.2.3) $F_*(\Omega_{X/S}^\bullet(\log D))$ is an $\mathcal{O}_{X^{(p)}}$-linear complex of locally free coherent sheaves on $X^{(p)}$.

2.1. The Cartier Operation

The fundamental fact about De Rham cohomology in characteristic p is the following theorem of Cartier whose statement we recall (cf. [3], chpt. 2, and [24], 7.2).

(2.1.1) **Theorem.** *There is a unique isomorphism of $\mathcal{O}_{X^{(p)}}$ modules for each integer $i \geqq 0$*

(2.1.1.1) $\mathscr{C}^{-1}: \Omega^i_{X^{(p)}/S}(\log D^{(p)}) \xrightarrow{\sim} \mathscr{H}^i(F_*(\Omega^\bullet_{X/S}(\log D)))$

which satisfies

(2.1.1.2) $\mathscr{C}^{-1}(1)=1,$

(2.1.1.3) $\mathscr{C}^{-1}(\omega \wedge \tau)=\mathscr{C}^{-1}(\omega) \wedge \mathscr{C}^{-1}(\tau),$

(2.1.1.4) $\mathscr{C}^{-1}(d(\sigma^{-1}(x)))=$ the class of $x^{p-1}\,dx$.

(2.1.2) Putting together all the conditions, we see that, in terms of the local coordinates $x_1, \dots, x_n$ chosen in (1.0.3.1), we have

$\mathscr{C}^{-1}(h)=$ the class of $F^{-1}(h)$ for h a local sections of $\mathcal{O}_{X^{(p)}}$

(2.1.2.1) $\mathscr{C}^{-1}\left(\dfrac{d\sigma^{-1}(x_\nu)}{\sigma^{-1}(x_\nu)}\right)=$ the class of $\dfrac{dx_\nu}{x_\nu}$, for $\nu=1, \dots, \alpha$

$\mathscr{C}^{-1}(d\sigma^{-1}(x_j))=$ the class of $x_j^{p-1}\,dx_j$ for $j=\alpha+1, \dots, n$.

2.2. General Nonsense

We must now give a tautology on functorial spectral sequences, whose precise formulation is somewhat lengthy.

(2.2.1) Suppose given $f\colon Y \to Z$ an arbitrary morphism of schemes. For every Z-scheme Z_1, we form the fibre product $Y_{Z_1}=Y x_Z Z_1$, which sits in the cartesian diagram

$$\begin{tikzcd} Y_{Z_1} \arrow[r] \arrow[d, "f_{Z_1}"] & Y \arrow[d, "f"] \\ Z_1 \arrow[r] & Z. \end{tikzcd}$$

Suppose we are give, for every Z-scheme Z_1, a finitely filtered complex (non-zero only in positive degrees). (K_{Z_1}, F) of $f_{Z_1}^{-1}(\mathcal{O}_{Z_1})$ modules on Y_{Z_1}, which is functorial in the variable Z-scheme Z_1 in the following sense:

(2.2.1.2) For any morphism $\varphi\colon Z_2 \to Z_1$ of Z-schemes, denote by $\varphi_Y\colon Y_{Z_2} \to Y_{Z_1}$ the induced morphism, which sits in the commutative diagram

(2.2.1.3)
$$\begin{tikzcd} & Y_{Z_2} \arrow[rr, "\varphi_Y"] \arrow[dl] \arrow[d, "f_{Z_2}"] & & Y_{Z_1} \arrow[d, "f_{Z_1}"] \arrow[dr] & \\ Y \arrow[drr, "f"'] & Z_2 \arrow[rr, "\varphi"] \arrow[dr] & & Z_1 \arrow[dl] & Y \arrow[dll, "f"] \\ & & Z & & \end{tikzcd}$$

We are to be given a morphism $K(\varphi)$ of filtered complexes of $f_{Z_2}^{-1}(\mathcal{O}_{Z_2})$-modules on Y_{Z_2}

$$\varphi_Y^{-1}(K_{Z_1}, F) \underset{\varphi_Y^{-1} f_{Z_1}^{-1}(\mathcal{O}_{Z_1})}{\otimes} f_{Z_2}^{-1}(\mathcal{O}_{Z_2}) \xrightarrow{K(\varphi)} (K_{Z_2}, F) \tag{2.2.1.4}$$

which satisfies the natural transitivity condition for a composition of morphisms of Z-schemes.

Consider now the spectral sequence of $\mathcal{O}_{Z_1}$-modules

$$E_1^{p,q}(Z_1) = \mathbf{R}^{p+q} f_{Z_1 *}(\mathrm{gr}_F^p(K_{Z_1})) \Rightarrow \mathbf{R}^{p+q} f_{Z_1 *}(K_{Z_1}), \tag{2.2.1.5}$$

on the Z-scheme Z_1, whose $E_r^{p,q}$ term we denote $E_r^{p,q}(Z_1)$. From the given functoriality of (K_{Z_1}, F) in Z_1, we deduce, for every morphism $\varphi\colon Z_2 \to Z_1$ of Z-schemes, morphisms of $\mathcal{O}_{Z_2}$-modules called "change of base morphisms",

$$\varphi^*(E_r^{p,q}(Z_1)) \xrightarrow{K(\varphi)} E_r^{p,q}(Z_2) \tag{2.2.1.6}$$

which render commutative all diagrams

$$\begin{tikzcd} \varphi^*(E_r^{p,q}(Z_1)) \arrow[r, "K(\varphi)"] \arrow[d, "\varphi^*(d_r^{p,q})"'] & E_r^{p,q}(Z_2) \arrow[d, "d_r^{p,q}"] \\ \varphi^*(E^{p+r,q+1-r}(Z_1)) \arrow[r, "K(\varphi)"] & E_r^{p+r,q+1-r}(Z_2). \end{tikzcd} \tag{2.2.1.7}$$

These morphisms are compatible with the usual isomorphism of E_{r+1} with the cohomology of (E_r, d_r), in the sense that the diagram

$$\begin{tikzcd} \varphi^*(\mathrm{Ker}\, d_r^{p,q} \text{ in } E_r^{p,q}(Z_1)) \arrow[r, "K(\varphi)"] \arrow[d, "\varphi^*(\text{canonical protection})"'] & \mathrm{Ker}\, d_r^{p,q} \text{ in } E_r^{p,q}(Z_2) \arrow[d, "\text{canonical projection}"] \\ \varphi^*(E_{r+1}^{p,q}(Z_1)) \arrow[r, "K(\varphi)"] & E_{r+1}^{p,q}(Z_2) \end{tikzcd} \tag{2.2.1.8}$$

commutes.

Further, the induced mapping on E_∞ is the associated graded of the change of base morphism deduced from (2.2.1.4):

$$\varphi^* \mathbf{R}^p f_{Z_1 *}(K_{Z_1}) \xrightarrow{K(\varphi)} \mathbf{R}^p f_{Z_2 *}(K_{Z_2}). \tag{2.2.1.9}$$

For each integer $r_0 \geqq 1$, we say that the formation of E_{r_0} commutes with base change if for every Z-scheme $\varphi\colon Z_1 \to Z$, and all pairs (p, q) of integers, the morphism (2.2.1.6)

$$\varphi^*(E_{r_0}^{p,q}(Z_1)) \xrightarrow{K(\varphi)} E_{r_0}^{p,q}(Z_2) \tag{2.2.1.10}$$

is an *isomorphism*. We say that the formation of the spectral sequence from E_{r_0} on commutes with base change if for all $r \geqq r_0$, the formation of E_r commutes with base change.

(2.2.1.11) **Tautology.** *Hypotheses as above, suppose that for an integer* $r_0 \geqq 1$, *the formation of* E_{r_0} *commutes with base change, and that the spectral sequence over* Z

(2.2.1.12) $$E_1^{p,q}(Z) = \mathbf{R}^{p+q} f_*(\operatorname{gr}_F^p(K_Z)) \Rightarrow \mathbf{R}^{p+q} f_*(K_Z)$$

is degenerate at E_{r_0}. *Then after any arbitrary change of base* $\varphi: Z_1 \to Z$, *the spectral sequence*

(2.2.1.13) $$E_1^{p,q}(Z_1) = \mathbf{R}^{p+q} f_{Z_1 *}(\operatorname{gr}_F^p(K_{Z_1})) \Rightarrow \mathbf{R}^{p+q} f_{Z_1 *}(K_{Z_1})$$

is degenerate at E_{r_0}. *Furthermore its formation from* E_{r_0} *on commutes with arbitrary change of base.*

(2.2.2) *Examples.* Let's return to the geometric situation 1.0, so that our morphism $f: Y \to Z$ in (2.2.1) becomes the morphism $f: X \to S$ of 1.0. We take for K the complex $\Omega^\bullet_{X/S}(\log D)$), and for an arbitrary S-scheme S', we take $K_{S'} = \Omega^\bullet_{X'/S'}(\log D')$, the symbol $'$ denoting fibre product with S' over S. There are two filtrations in which we shall be interested. The first is the Hodge filtration (1.4.1), (the "bestial" one in the terminology of [8]). The second is the one, noted $\tau_{\leqq \bullet}$ and called "canonical" in [8], which is defined by

(2.2.2.1) $$\tau_{\leqq p}(K) = \begin{cases} K^n & \text{if } n < p \\ \operatorname{Ker}(d) & \text{if } n = p \\ 0 & \text{if } n > p. \end{cases}$$

The spectral sequence defined by this filtration

(2.2.2.2) $$_{\tau \leqq \bullet}E_1^{p,q} = \mathbf{R}^{p+q} f_*(\operatorname{gr}^p_{\tau \leqq \bullet} K) \Rightarrow \mathbf{R}^{p+q} f_*(K)$$

is the *décalage* (cf. [8], 1.3.3) of the "second spectral sequence of hypercohomology"

(2.2.2.3) $$_{\mathrm{II}}E_2^{p,q} = R^p f_*(\mathscr{H}^q(K)) \Rightarrow \mathbf{R}^{p+q} f_*(K),$$

which by definition means that we have isomorphisms, compatible with the d_r and with the standard isomorphism $E_{r+1} \simeq H(E_r, d_r)$,

(2.2.2.4) $$_{\tau \leqq \bullet}E_r^{p,q} \simeq {}_{\mathrm{II}}E_{r+1}^{q+2p,-p}$$

for each integer $r \geqq 1$ (cf. [8]).

2.3. The Conjugate Spectral Sequence

(2.3.0) Recall that the entire "second spectral sequence" of hypercohomology

(2.3.0.1) $$E_2^{p,q} = R^p f_*(\mathscr{H}^q(\Omega^\bullet_{X/S}(\log D))) \Rightarrow \mathbf{R}^{p+q} f_*(\Omega^\bullet_{X/S}(\log D))$$

is endowed with the Gauss-Manin connection (cf. [24], 3.5). For reasons which will soon become apparent (cf. (2.3.3) below), we call this the conjugate spectral sequence, and the corresponding filtration of $\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))$ the conjugate filtration, noted F^i_{con}. This filtration is *horizontal*, and is every bit as interesting as the Hodge filtration.

(2.3.1) Let us explicate the E_2 term of the conjugate spectral sequence with the aid of Cartier's isomorphism (2.1.1).

(2.3.1.1) **Lemma.** *The Cartier isomorphism $\mathscr{C}$ defines an isomorphism of $\mathcal{O}_S$-modules for each pair a, b of non-negative integers*

(2.3.1.1.1) $${}_{\mathrm{con}}E_2^{a,b} = R^a f_*(\mathscr{H}^b(\Omega^\bullet_{X/S}(\log D))) \xrightarrow[\sim]{\mathscr{C}} R^a f^{(p)}_*(\sigma^*(\Omega^b_{X/S}(\log D))).$$

Proof. From the commutativity of the diagram

(2.3.2.1.1.1) $$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "f"'] & X^{(p)} \arrow[dl, "f^{(p)}"] \\ S & \end{tikzcd}$$

and the fact that F is a homeomorphism, we have isomorphisms

(2.3.1.1.2) $${}_{\mathrm{con}}E_2^{a,b} = R^a f_*(\mathscr{H}^b(\Omega^\bullet_{X/S}(\log D))) \simeq R^a f^{(p)}_*(F_* \mathscr{H}^b(\Omega^\bullet_{X/S}(\log D)))$$

and

(2.3.1.1.3) $$F_* \mathscr{H}^b(\Omega^\bullet_{X/S}(\log D)) \simeq \mathscr{H}^b(F_*(\Omega^\bullet_{X/S}(\log D)))$$

which combine to give an isomorphism

(2.3.1.1.4) $${}_{\mathrm{con}}E_2^{a,b} \simeq R^a f^{(p)}_*(\mathscr{H}^b(F_*(\Omega^\bullet_{X/S}(\log D)))).$$

Composing (2.3.1.1.4) with Cartier's isomorphism

(2.3.1.1.5) $$\mathscr{H}^b(F_*(\Omega^\bullet_{X/S}(\log D))) \xrightarrow[\sim]{\mathscr{C}} \Omega^b_{X^{(p)}/S}(\log D^{(p)})$$

and the inverse of the canonical isomorphism

(2.3.1.1.6) $$\sigma^*(\Omega^b_{X/S}(\log D)) \xrightarrow{\sim} \Omega^b_{X^{(p)}/S}(\log D^{(p)})$$

gives the asserted isomorphism 2.3.1.1.1. Q.E.D.

(2.3.1.2) **Corollary.** *Suppose that either the absolute Frobenius $F_{\mathrm{abs}}: S \to S$ is flat (which is the case if S is a regular scheme) or that for all pairs a, b of non-negative integers, the Hodge cohomology sheaves $R^a f_*(\Omega^b_{X/S}(\log D))$ are flat $\mathcal{O}_S$-modules (for instance, if they are locally free $\mathcal{O}_S$-modules). Then via* (2.2.1.1) *and the inverse of the base-changing isomorphism, we have an isomorphism.*

(2.3.1.2.1) $${}_{\mathrm{con}}E_2^{a,b} \xrightarrow[\sim]{\mathscr{C}} R^a f^{(p)}_*(\sigma^*(\Omega^b_{X/S}(\log D))) \xrightarrow{\sim} F^*_{\mathrm{abs}} R^a f_*(\Omega^b_{X/S}(\log D)).$$

Proof. The only point is that under either of the hypotheses, the canonical morphism of base change

(2.3.1.2.2) $\quad F_{\mathrm{abs}}^* R^a f_*(\Omega_{X/S}^b(\log D)) \to R^a f_*^{(p)}(\sigma^*(\Omega_{X/S}^b(\log D)))$

which comes from the cartesian diagram (2.0.1.1)

(2.3.1.2.3)
$$\begin{tikzcd} X^{(p)} \arrow[r, "\sigma"] \arrow[d, "f^{(p)}"] & X \arrow[d, "f"] \\ S \arrow[r, "F_{\mathrm{abs}}"] & S \end{tikzcd}$$

is an isomorphism.

(2.3.1.3) *Remark.* Under the isomorphism (2.3.1.2.1), the Gauss-Manin connection deduced on $F_{\mathrm{abs}}^* R^a f_*(\Omega_{X/S}^b(\log D))$ annihilates the *image* under F_{abs}^* of $R^a f_*(\Omega_{X/S}^b(\log D))$ (compare [24], 5.1.1).

(2.3.2) **Proposition.** *In the geometric situation* 1.0, *suppose that X is proper over S* (*and that S is a scheme of characteristic p, as it has been throughout Section* 2). *Suppose further that*

(2.3.2.1) *Each of the Hodge cohomology sheaves $R^a f_*(\Omega_{X/S}^b(\log D))$ is a locally free sheaf of finite rank on S and* (hence) *that its formation commutes with arbitrary change of base $S' \to S$.*

(2.3.2.2) *The Hodge $\Rightarrow$ De Rham spectral sequence*

$$E_1^{a,b} = R^b f_*(\Omega_{X/S}^a(\log D)) \Rightarrow \mathbf{R}^{a+b} f_*(\Omega_{X/S}^\bullet(\log D))$$

is degenerate at E_1.

Then the conjugate spectral sequence, which, thanks to (2.3.1.2.1) *and the hypothesis* (2.3.2.1), *may be written*

(2.3.2.3) $\quad {}_{\mathrm{con}}E_2^{a,b} = F_{\mathrm{abs}}^* R^a f_*(\Omega_{X/S}^b(\log D)) \Rightarrow \mathbf{R}^{a+b} f_*(\Omega_{X/S}^\bullet(\log D)),$

is degenerate at E_2.

Proof. By (2.3.1.4.1), it follows that the conjunction of the hypotheses (2.3.2.1) and (2.3.2.2) remains true after an arbitrary change of base $S' \to S$, and implies that the formation of the Hodge $\Rightarrow$ De Rham spectral sequence commutes with arbitrary change of base $S' \to S$. From (2.2.1.2) it follows also that the formation of the ${}_{\mathrm{con}}E_2^{a,b}$ commutes with arbitrary change of base, while by general principles the formation of the entire conjugate spectral sequence commutes with any *flat* base change $S' \to S$.

(2.3.2.4) We may assume that S is affine, because the question is local on S. We wish to reduce to the case in which S is noetherian. So suppose $S = \mathrm{Spec}(A)$. Clearly there exists a subring $A_0 \subset A$ which is finitely generated over $\mathbf{Z}$, a proper and smooth A_0-scheme X_0, and smooth (over A_0) divisors $D_{i,0}$ in X_0 which cross normally relative to $S_0 = \mathrm{Spec}(A_0)$, such that the geometric situation (1.0) over S (for the purposes of 2.3.2, the base scheme T figuring in (1.0) may be taken to be S itself) comes from the analogous situation over S_0 by the change of base $S \to S_0$ (cf. EGA IV, 8.9.1, 8.10.5, and 7.7.9). We must show that after replacing A_0 by a larger subring A_1, $A \supset A_1 \supset A_0$, which is still finitely generated over $\mathbf{Z}$, the hypotheses (2.3.2.1.2), that the Hodge $\Rightarrow$ De Rham spectral sequence degenerate at E_1 and have E_1 locally free of finite rank, are valid over A_1, which is a noetherian ring. (For then, once the theorem is proved over A_1, it remains true over A by (2.2.1.11).) In fact, it suffices to find such an A_1 over which (2.3.2.1) holds; then (2.3.2.2) follows. For if we suppose that over A_1 the E_1 terms noted $E_1^{a,b}(A_1)$, are *locally free of finite rank*, the differential

(2.3.2.4.1) $$d_1\colon E_1^{a,b}(A_1) \to E_1^{a+1,b}(A_1)$$

must vanish, because after extension of scalars to $A \supset A_1$, this differential becomes zero, because of the commutative diagram (cf. (2.2.1.7))

(2.3.2.4.2)
$$\begin{array}{ccc} E_1^{a,b}(A_1)\otimes_{A_1} A & \xrightarrow{d_1} & E_1^{a+1,b}(A_1)\otimes_{A_1} A \\ \wr\downarrow & & \wr\downarrow \\ E_1^{a,b}(A) & \xrightarrow{d_1} & E_1^{a+1,b}(A) \end{array}$$

But clearly if a homomorphism between locally free modules becomes zero after extending scalars by an *inclusion* $A_1 \subset A$, the homomorphism is zero. Thus $d_1 =$ over A_1, and hence $E_2 \simeq E_1$, whence E_2 is locally free of finite rank, and it formation commutes with all change of base. Repeating the above argument, we get $d_2 = 0$ over A_1, and inductively $d_r = 0$ for all $r \geqq 1$.

So now let's prove that we can find an A_1 over which all the E_1 terms are locally free of finite rank. We'll prove this by descending induction on the integer b of $E_1^{a,b}$. Because $E_1^{a,b}(A_0) = H^b(X_0, \Omega^a_{X_0/S_0}(\log D))$ *vanishes* unless a and b are in the interval $[0, N]$, N being the relative dimension of X_0/A_0 it suffices to find, for each integer $0 \leqq a \leqq N$, an A_1 which "works" for all the $E_1^{a,b}$ with fixed a. By the standard base-changing theorems (cf. [34], pp. 46–55) we know that, for fixed a,

(2.3.2.4.3) if, for all $b > b_0$, the module $E_1^{a,b}(A_0)$ is locally free of finite rank on A_0, then the formation of E_1^{a,b_0} commutes with arbitrary change of base.

Since $E_1^{a,b}(A_0)$ vanishes for $b > N$, our descending induction will work if we can show that, for (a, b) fixed:

(2.3.2.4.4) if the formation of $E_1^{a,b}$ commutes with arbitrary change of base, then the fact that $E_1^{a,b}(A) = E_1^{a,b}(A_0)\otimes_{A_0} A$ is a locally free A-module of finite rank implies that for a suitable subring A_1 of A, $A_0 \subset A_1 \subset A$, which is finitely generated over $\mathbf{Z}$, the A_1-module $E_1^{a,b}(A_1)$ is locally free of finite rank, or what is the same, flat, since in any case it of finite type over the noetherian ring A_1.

The truth of this last assertion is a particularly simple case of (EGA IV, 11.2.6.1), in the notation of which $B_0 = A_0$, $M_0 = E_1^{a,b}(A_0)$, and the A_λ are all the absolutely finitely generated subrings of A with $A_0 \subset A_\lambda \subset A$.

(2.3.2.5) Having reduced to the case in which S is the spectrum of a noetherian ring, we may further assume that S is the spectrum of a noetherian local ring (again because the question is local on S). We may next suppose S to be the spectrum of a complete noetherian local ring (by faithful flatness of the completion), and finally that S is the spectrum of an artinian local ring. Let us first explain this last reduction step.

Suppose $S = \operatorname{Spec}(A)$, A a complete noetherian local ring with maximal ideal $\mathfrak{m}$, and suppose that for each integer $n \geqq 0$, the conjugate spectral sequence over $S_n = \operatorname{Spec}(A/\mathfrak{m}^{n+1})$ is degenerate at E_2. Let's denote by $({}_{\mathrm{con}}E_r^{a,b}(n), d_r(n))$ the conjugate spectral over S_n (including $n = \infty$, putting $S_\infty = S$). Then as remarked above, we have

(2.3.2.6) $$ {}_{\mathrm{con}}E_2^{a,b}(\infty) \simeq \varprojlim_n {}_{\mathrm{con}}E_2^{a,b}(n), $$

whence

(2.3.2.7) $$ d_2(\infty) = \varprojlim_n d_2(n) = 0 $$

suppose $d_2(\infty) = \cdots = d_r(\infty) = 0$. Then

(2.3.2.8) $$ {}_{\mathrm{con}}E_{r+1}^{a,b}(\infty) = {}_{\mathrm{con}}E_2^{a,b}(\infty) = \varprojlim_n {}_{\mathrm{con}}E_2^{a,b}(n) = \varprojlim_n {}_{\mathrm{con}}E_{r+1}^{a,b}(n) $$

whence

(2.3.2.9) $$ d_{r+1}(\infty) = \varprojlim d_{r+1}(n) = 0 . $$

This shows inductively that $({}_{\mathrm{con}}E_r^{a,b}(\infty), d_r(\infty))$ is degenerate at E_2, and completes the proof of validity of the reduction to the case in which S is the spectrum of an artinian local ring.

(2.3.2.10) Suppose now that $S = \operatorname{Spec}(A)$ with A artin local, and denote by $\operatorname{lng}_A(M)$ the *length* of an A-module M. Then a necessary and sufficient condition that the conjugate spectral sequence degenerate at E_2 is that

(2.3.2.11) $$ \sum_{a,b} \operatorname{lng}_A({}_{\mathrm{con}}E_2^{a,b}) = \sum_{a,b} \operatorname{lng}_A({}_{\mathrm{con}}E_\infty^{a,b}) . $$

The second term in (2.3.2.11) is

(2.3.2.12) $$ \sum_{a,b} \operatorname{lng}_A({}_{\mathrm{con}}E_\infty^{a,b}) = \sum_{n} \operatorname{lng}_A(\mathbf{R}^n f_*(\Omega_{X/S}^\bullet(\log D))) $$

while the first term is, by (2.3.1.2.1),

$$\sum_{a,b} \lg_A({}_{\mathrm{con}}E_2^{a,b}) = \sum_{a,b} \lg_A(F_{\mathrm{abs}}^* R^a f_*(\Omega^b_{X/S}(\log D))). \tag{2.3.2.13}$$

Now by hypothesis (2.3.2.1), each of the A-modules $R^a f_*(\Omega^b_{X/S}(\log S))$ is free of finite rank, and hence each of the A-modules $F_{\mathrm{abs}}^* R^a f_*(\Omega^b_{X/S}(\log D))$ is free of the *same* finite rank. In particular, we have, for each a, b

$$\lg_A(F_{\mathrm{abs}}^* R^a f_*(\Omega^b_{X/S}(\log D))) = \lg_A(R^a f_*(\Omega^b_{X/S}(\log D))). \tag{2.3.2.14}$$

Putting together (2.3.2.12–14), the criterion (2.3.2.11) for degeneration at E_2 may be written

$$\sum_{a,b} \lg_A(R^a f_*(\Omega^b_{X/S}(\log D))) = \sum_n \lg_A(\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))). \tag{2.3.2.15}$$

This last equality holds, in virtue of the hypothesis (2.3.2.2) that the Hodge $\Rightarrow$ De Rham spectral sequence is degenerate at E_1. This concludes the proof of degeneration. Q.E.D.

(2.3.2.16) **Corollary.** *Under the hypotheses of proposition* (2.3.2), *the formation of the conjugate spectral sequence* (2.3.0.1) *commutes with arbitrary change of base* $S' \to S$.

Proof. By (2.3.2), the conjugate spectral sequence is degenerate at E_2, and by (2.3.1.2) the formation of its E_2 term commutes with arbitrary change of $S' \to S$. The result follows by (2.2.1.11).

(2.3.3) We are now in a position to explain the terminology "conjugate" spectral sequence. With the assumptions of proposition (2.3.2), suppose further that S is the spectrum of a field K of characteristic p, and that the divisor D is void. The Hodge $\Rightarrow$ De Rham spectral sequence

$$E_1^{a,b} = H^b(X, \Omega^a_{X/K}) \Rightarrow \mathbf{H}^{a+b}(X, \Omega^\bullet_{X/K}) \tag{2.3.3.1}$$

being degenerate at E_1, we have

$$\mathrm{gr}_F^a(\mathbf{H}^n(X, \Omega^\bullet_{X/K})) \simeq H^{n-a}(X, \Omega^a_{X/K}). \tag{2.3.3.2}$$

The degeneracy at E_2 of the conjugate spectral sequence

$${}_{\mathrm{con}}E_2^{a,b} = F_{\mathrm{abs}}^* H^a(X, \Omega^b_{X/K}) \Rightarrow \mathbf{H}^{a+b}(X, \Omega^\bullet_{X/K}) \tag{2.3.3.3}$$

which we prefer to rewrite as

$${}_{\mathrm{con}}E_2^{a,b} = H^a(X^{(p)}, \Omega^b_{X^{(p)}/K}) \Rightarrow \mathbf{H}^{a+b}(X, \Omega^\bullet_{X/K}), \tag{2.3.3.4}$$

gives an isomorphism

$$\mathrm{gr}^a_{F_{\mathrm{con}}} \mathbf{H}^n(X, \Omega^\bullet_{X/K}) \simeq H^a(X^{(p)}, \Omega^{n-a}_{X^{(p)}/K}). \tag{2.3.3.5}$$

Putting together (2.3.3.5) and (2.3.3.3) (for $X^{(p)}$) we find an isomorphism

$$\mathrm{gr}^a_{F_{\mathrm{con}}} \mathbf{H}^n(X, \Omega^\bullet_{X/K}) \simeq \mathrm{gr}_F^{n-a} \mathbf{H}^n(X^{(p)}, \Omega^\bullet_{X^{(p)}/K}). \tag{2.3.3.6}$$

In order to explain the transcendental analogue of (2.3.3.6), let Y be a a proper and smooth $\mathbf{C}$-scheme, and denote by Y^{an} the "underlying" complex manifold. By GAGA ([39, 36]) and Poincaré's lemma ([14]), we have isomorphisms

(2.3.3.7) $$\mathbf{H}^n(Y, \Omega^\bullet_{Y/\mathbf{C}}) \xrightarrow{\sim} \mathbf{H}^n(Y^{an}, \Omega^{\bullet an}_{Y/\mathbf{C}}) \xleftarrow{\sim} H^n(Y^{an}, \mathbf{C}) \xleftarrow{\sim} H^n(Y^{an}, \mathbf{Z}) \otimes \mathbf{C}$$

by means of which any automorphism of the field $\mathbf{C}$ operates on $\mathbf{H}^n(Y, \Omega^\bullet_{Y/\mathbf{C}})$ (by transporting by (2.2.3.6) its action on $H^n(Y^{an}, \mathbf{Z}) \otimes \mathbf{C}$ through the second factor). In particular, the automorphism "complex conjugation", denoted

(2.3.3.8) $$F_{\mathrm{abs}}: \mathbf{C} \to \mathbf{C},$$

operates on $\mathbf{H}^n(Y, \Omega^\bullet_{Y/\mathbf{C}})$, furnishing a canonical (albeit transcendental) isomorphism

(2.3.3.9) $$F^*_{\mathrm{abs}} \mathbf{H}^n(Y, \Omega^\bullet_{Y/\mathbf{C}}) \simeq \mathbf{H}^n(Y, \Omega^\bullet_{Y/\mathbf{C}}).$$

The complex conjugate of the Hodge filtration, noted ${}_{\mathrm{con}}F^i$, is by definition the image under (2.3.3.8) of the filtration $F^*_{\mathrm{abs}}(F^i)$ of $F^*_{\mathrm{abs}} \mathbf{H}^n(Y, \Omega^\bullet_{Y/\mathbf{C}})$. According to Hodge theory ([42, 5]), we have, for $i = 1, \ldots, n$, a direct sum decomposition

(2.3.3.10) $$\mathbf{H}^n(Y, \Omega^\bullet_{Y/\mathbf{C}}) = F^i \mathbf{H}^n(Y, \Omega^\bullet_{Y/\mathbf{C}}) \oplus F^{n+1-i}_{\mathrm{con}} \mathbf{H}^n(Y, \Omega^\bullet_{Y/\mathbf{C}})$$

or, what is the same, a bigraduation (Hodge decomposition)

(2.3.3.10 bis) $$\mathbf{H}^n(Y, \Omega^\bullet_{Y/\mathbf{C}}) = \bigoplus_{i=0}^{n} (F^i \cap F^{n-i}_{\mathrm{con}})(\mathbf{H}^n(Y, \Omega^\bullet_{Y/\mathbf{C}})).$$

This bigraduation gives (transcendental) isomorphisms

(2.3.3.11) $$\mathrm{gr}^a_{F_{\mathrm{con}}}(\mathbf{H}^n(Y, \Omega^\bullet_{Y/\mathbf{C}})) \xleftarrow{\sim} F^{n-a} \cap F^a_{\mathrm{con}} \xrightarrow{\sim} \mathrm{gr}^{n-a}_F(\mathbf{H}^n(Y, \Omega^\bullet_{Y/\mathbf{C}}))$$

which we regard as the transcendental analogue of (2.3.3.6).

(2.3.4) We must however hasten to point out that the analogue over a field K of characteristic p of the Hodge decomposition (2.3.3.10) provided by F and F_{con} over $\mathbf{C}$ is generally *false*. Indeed, the extent to which the filtrations F and F_{con} fail to be transversal is an interesting arithmetic invariant. In order to explain this point more fully, it is convenient to first recall the Hass-Witt "operators" and their relation to the conjugate spectral sequence.

(2.3.4.1) We return to the sheltering hypotheses of proposition (2.3.2). Fixing an integer $n \geqq 0$, the degeneration of the conjugate spectral sequence at E_2 gives us an inclusion

(2.3.4.1.1) $$F^*_{\mathrm{abs}} R^n f_*(\mathcal{O}_X) \xrightarrow[\sim]{\mathscr{C}^{-1}} {}_{\mathrm{con}}E_2^{n,0} \hookrightarrow \mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))$$

and the degeneration of the Hodge $\Rightarrow$ De Rham spectral sequence gives us a surjection

(2.3.4.1.2) $$\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)) \twoheadrightarrow R^n f_*(\mathcal{O}_X) = E_1^{0,n}.$$

Putting these together, we obtain the diagram which defines the Hasse-Witt operations:

(2.3.4.1.3)
$$\begin{tikzcd} {}_{\mathrm{con}}E_2^{n,0} \arrow[r, hook] & \mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)) \arrow[r, two heads] & E_1^{0,n} \arrow[d, equal] \\ F^*_{\mathrm{abs}} R^n f_*(\mathcal{O}_X) \arrow[u, "\wr"] \arrow[rr, dotted, "\text{H-W}"] & & R^n f_*(\mathcal{O}_X) \end{tikzcd}$$

The matrix of this operation in a local base of $R^n f_*(\mathcal{O}_X)$ is called the Hasse-Witt matrix of X/S in dimension n.

The composite p-linear mapping

(2.3.4.1.4) $$R^n f_*(\mathcal{O}_X) \xrightarrow{F^*_{\mathrm{abs}}} F^*_{\mathrm{abs}} R^n f_*(\mathcal{O}_X) \xrightarrow{\text{H-W}} R^n f_*(\mathcal{O}_X)$$

is the one induced by the p-th power endomorphism of $\mathcal{O}_X$.

From the definition (2.2.4.1.3), it follows that we have:

(2.3.4.1.5) **Proposition.** *Hypotheses as in* (2.3.2), *the kernel of the Hasse-Witt operation*

(2.3.4.1.6) $$\text{H-W}: F^*_{\mathrm{abs}} R^n f_*(\mathcal{O}_X) \to R^n f_*(\mathcal{O}_X)$$

is none other than the intersection $(F^n_{\mathrm{con}} \cap F^1)(\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)))$.

(2.3.4.1.7) **Corollary** (*Assumptions as in* (2.3.2)). *In order to have a direct sum decomposition*

(2.3.4.1.8) $$\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)) \xleftarrow{\sim} F^1 \oplus F^n_{\mathrm{con}}$$

it is necessary and sufficient that the Hasse-Witt operation

(2.3.4.1.9) $$\text{H-W}: F^*_{\mathrm{abs}} R^n f_*(\mathcal{O}_X) \to R^n f_*(\mathcal{O}_X)$$

be an isomorphism.

Proof. Since the statements whose equivalence is asserted are both of the form "a certain homomorphism of locally free S-modules of finite rank is an isomorphism", and because the formation of these modules commutes with all changes of base $S' \to S$, we are immediately reduced to the case in which S is the spectrum of a field K.

The isomorphisms

(2.3.4.1.10) $${}_{\mathrm{con}}E_2^{a,b} \xrightarrow[\sim]{\mathscr{C}} F^*_{\mathrm{abs}} E_1^{b,a}$$

together with the respective degenerations (2.3.2.2–3) imply, for every $0 \leqq i \leqq n$, the formula

(2.3.4.1.11) $$\dim_K \mathbf{H}^n(X, \Omega^\bullet_{X/K}(\log D)) = \dim_K F^{i+1} + \dim_K F^{n-i}_{\mathrm{con}}.$$

For the homomorphisms (2.3.4.1.8), source and target have the same dimension, just as for (2.3.4.1.9), and both homomorphisms have the same kernel, namely $F^1 \cap F^n_{\mathrm{con}}$. Q.E.D.

(2.3.4.2) We now discuss the sometimes defined "higher" Hasse-Witt operations (which include the usual one as a special case), still supposing the hypotheses of (2.3.2). As before, we fix an integer $n \geqq 0$. For each integer i, we denote by $h(i)$ the composite mapping

(2.3.4.2.1)
$$\begin{tikzcd}
F^{n-i}_{\mathrm{con}}(\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))) \arrow[r, hook] \arrow[dr, "h(i)"] & \mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)) \arrow[d] \\
& \mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))/F^{i+1}
\end{tikzcd}$$

(2.3.4.2.2) **Proposition.** *Hypotheses as in* (2.3.2), *and* $n \geqq 0$ *fixed as above, suppose that for an integer* i, *the mapping* $h(i)$ (2.2.4.2.1) *is an isomorphism. Then there is a unique mapping of locally free* $\mathcal{O}_S$*-modules, the* $i+1$*-st Hasse-Witt operation*

(2.3.4.2.3) $$\text{H-W}(i+1)\colon F^*_{\mathrm{abs}} R^{n-i-1} f_*(\Omega^{i+1}_{X/S}(\log D)) \to R^{n-i-1} f_*(\Omega^{i+1}_{X/S}(\log D))$$

which renders commutative the following diagram:

(2.3.4.2.4)
$$\begin{tikzcd}
& 0 & 0 \arrow[d] \\
F^*_{\mathrm{abs}}(R^{n-i-1} f_*(\Omega^{i+1}_{X/S}(\log D))) \arrow[r, "\mathscr{C}^{-1}", "\sim"'] \arrow[rr, bend left=40, "\text{H-W}(i+1)"] & {}_{\mathrm{con}}E_2^{n-i-1,i+1} \arrow[u] & E^{i+1,n-i-1} \simeq R^{n-i-1} f_*(\Omega^{i+1}_{X/S}(\log D)) \arrow[d] \\
& F^{n-i-1}_{\mathrm{con}}(\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))) \arrow[r, "h(i+1)"] \arrow[u] & \mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))/F^{i+2} \arrow[d] \\
& F^{n-i}_{\mathrm{con}}(\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))) \arrow[r, "h(i)", "\sim"'] \arrow[u] & \mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))/F^{i+1} \arrow[d] \\
& 0 \arrow[u] & 0
\end{tikzcd}$$

(2.3.4.2.5) *Furthermore, $h(i+1)$ is an isomorphism if and only if* H-W$(i+1)$ *is an isomorphism.*

Proof. Follows formally from the diagram.

(2.3.4.3) **Corollary.** *Assumptions as in* (2.3.2), *the canonical map*

(2.3.4.3.1) $$F_{\mathrm{con}}^{n-i} \oplus F^{i+1} \to \mathbf{R}^n f_*(\Omega^{\bullet}_{X/S}(\log D))$$

is an isomorphism if and only if the map $h(i)$ (2.3.4.2.1)

(2.3.4.3.2) $$h(i)\colon F_{\mathrm{con}}^{n-i} \to \mathbf{R}^n f_*(\Omega^{\bullet}_{X/S}(\log D))/F^{i+1}$$

is an isomorphism. If this is the case, then the canonical map

(2.3.4.3.3) $$F_{\mathrm{con}}^{n-i-1} \oplus F^{i+2} \to \mathbf{R}^n f_*(\Omega^{\bullet}_{X/S}(\log D))$$

is an isomorphism if and only if the $i+1$'st Hasse-Witt operation (which is defined, because $h(i)$ is supposed an isomorphism)

(2.3.4.3.4) $$\text{H-W}(i+1)\colon F_{\mathrm{abs}}^* R^{n-i-1} f_*(\Omega^{i+1}_{X/S}(\log D)) \to R^{n-i-1} f_*(\Omega^{i+1}_{X/S}(\log D))$$

is an isomorphism.

Proof. The second equivalence follows from the first (applied to $i+1$), in virtue of (2.3.4.2.5). Precisely as in the proof of (2.3.4.1.7), the proof of the first equivalence is immediately reduced to the case in which S is the spectrum of a field K. Thanks to (2.3.4.1.11), the homomorphism (2.3.4.3.1) has source and target of the same dimension, as does (2.3.4.3.2), and both homomorphisms have the same kernel, namely $F^{i+1} \cap F_{\mathrm{con}}^{n-i}$. Q.E.D.

(2.3.4.4) **Corollary.** *Assumptions as in* (2.3.2), *and $n \geqq 0$ fixed, suppose that the map $h(i)$ figuring in* (2.3.4.2.4) *is an isomorphism. Then:*

(2.3.4.4.1) *The $\mathcal{O}_S$-module $F^{i+1} \cap F_{\mathrm{con}}^{n-i-1}(\mathbf{R}^n f_*(\Omega^{\bullet}_{X/S}(\log D)))$ is locally free, and its formation commutes with arbitrary change of base $S' \to S$.*

(2.3.4.4.2) *The canonical mapping*

$$F_{\mathrm{con}}^{n-i-1} \mathbf{R}^n f_*(\Omega^{\bullet}_{X/S}(\log D)) \leftarrow F_{\mathrm{con}}^{n-i} \otimes F^{i+1} \cap F_{\mathrm{con}}^{n-i-1}$$

is an isomorphism.

Proof. It suffices to prove (2.3.4.4.2), since by hypothesis $F_{\mathrm{con}}^{n-i-1}/F_{\mathrm{con}}^{n-i}$ is locally free, and its formation commutes with arbitrary change of base $S' \to S$. The composite mapping, formed from the bottom half of the diagram (2.3.4.2.4),

$$\begin{tikzcd}
F_{\mathrm{con}}^{n-i-1}(\mathbf{R}^n f_*(\Omega^{\bullet}_{X/S}(\log D))) \arrow[r, "h(i+1)"] & \mathbf{R}^n f_*(\Omega^{\bullet}_{X/S}(\log D))/F^{i+2} \arrow[d] \\
F_{\mathrm{con}}^{n-i}(\mathbf{R}^n f_*(\Omega^{\bullet}_{X/S}(\log D))) & \mathbf{R}^n f_*(\Omega^{\bullet}_{X/S}(\log D))/F^{i+1} \arrow[l, "h(i)^{-1}", "\sim"']
\end{tikzcd}$$

is a *section* of the inclusion $F_{\text{con}}^{n-i} \hookrightarrow F_{\text{con}}^{n-i-1}$, whose kernel is none other then $F^{i+1} \cap F_{\text{con}}^{n-i-1}$. This shows that the canonical mapping $F^{i+1} \cap F_{\text{con}}^{n-i-1} \to F_{\text{con}}^{n-i-1}/F_{\text{con}}^{n-i}$ is an isomorphism, which proves (2.3.4.4.2).

(2.3.5) In this section we wish to explain the differential equations satisfied by certain of the higher Hasse-Witt matrices introduced in (2.3.4). These differential equations were first noticed by Igusa [22] in the case of elliptic curves, then later explained quite generally by Manin [28].

(2.3.5.1) We place ourselves under the hypotheses of (2.3.2), and assume further that the divisor D is void, and that the geometric fibres of X/S are connected, of dimension N. Under these hypotheses, we have

(2.3.5.1.1) $\mathbf{R}^{2N} f_*(\Omega_{X/S}^\bullet) \xrightarrow{\sim} \mathbf{R}^N f_*(\Omega_{X/S}^N)$ is locally free of rank 1 and even canonically isomorphic to the structural sheaf $\mathcal{O}_S$, via the *trace morphism* (cf. [15] and [42])

(2.3.5.1.2) $$\operatorname{tr}\colon \mathbf{R}^{2N} f_*(\Omega_{X/S}^\bullet) \to \mathcal{O}_S$$

which carries the Gauss-Manin connection on $\mathbf{R}^{2N} f_*(\Omega_{X/S}^\bullet)$ into the standard connection on $\mathcal{O}_S$ (the one given by exterior differentiation $d\colon \mathcal{O}_S \to \Omega_{S/T}^1$).

The cup-product pairings

(2.3.5.1.3) $$\mathbf{R}^n f_*(\Omega_{X/S}^\bullet) \otimes \mathbf{R}^{2N-n} f_*(\Omega_{X/S}^\bullet) \longrightarrow \mathbf{R}^{2N} f(\Omega_{X/S}^\bullet) \xrightarrow[\sim]{\operatorname{tr}} \mathcal{O}_S$$

are perfect dualities of coherent locally free S modules, for which the filtrations F and F_{con} are both self-dual in the sense that

(2.3.5.1.4) $$(F^i \mathbf{R}^n f_*(\Omega_{X/S}^\bullet))^\perp = F^{N+1-n} \mathbf{R}^{2N-n} f_*(\Omega_{X/S}^\bullet)$$

(2.3.5.1.5) $$(F_{\text{con}}^i \mathbf{R}^n f_*(\Omega_{X/S}^\bullet))^\perp = F_{\text{con}}^{N+1-n} \mathbf{R}^{2N-n} f_*(\Omega_{X/S}^\bullet).$$

The associated graded pairings

(2.3.5.1.6) $$\begin{tikzcd} \operatorname{gr}_F^i(\mathbf{R}^n f_*(\Omega_{X/S}^\bullet)) \otimes \operatorname{gr}_F^{N-i}(\mathbf{R}^{2N-n} f_*(\Omega_{X/S}^\bullet) \arrow[r] \arrow[d, dash, "\wr"'] & \operatorname{gr}_F^N(\mathbf{R}^{2N} f_*(\Omega_{X/S}^\bullet))) \arrow[d, dash, "\wr"'] & \\ R^{n-i} f_*(\Omega_{X/S}^i) \otimes R^{N+i-n} f_*(\Omega_{X/S}^{N-i}) \arrow[r] & R^N f_*(\Omega_{X/S}^N) \arrow[r, "\operatorname{tr}", "\sim"'] & \mathcal{O}_S \end{tikzcd}$$

and

(2.3.5.1.7) $$\begin{tikzcd} \operatorname{gr}_{F_{\text{con}}}^{n-i}(\mathbf{R}^n f_*(\Omega_{X/S}^\bullet)) \otimes \operatorname{gr}_{F_{\text{con}}}^{N+i-n}(\mathbf{R}^{2N} f_*(\Omega_{X/S}^\bullet)) \arrow[r] \arrow[d, "\wr"'] & \operatorname{gr}_{F_{\text{con}}}^N(\mathbf{R}^{2N} f_*(\Omega_{X/S}^\bullet)) \arrow[d, "\wr"'] & \\ F_{\text{abs}}^*(R^{n-i} f_*(\Omega_{X/S}^i)) \otimes F_{\text{abs}}^*(R^{N+i-n} f_*(\Omega_{X/S}^{N-i})) \arrow[r] & F_{\text{abs}}^* R^N f_*(\Omega_{X/S}^N) \arrow[r, "F_{\text{abs}}^*(\operatorname{tr})", "\sim"'] & \mathcal{O}_S \end{tikzcd}$$

are the usual perfect pairings of Serre duality, and the lower row of (2.3.5.1.7) is simply the inverse image by F_{abs} of the lower row of (2.3.5.1.6).

(2.3.5.2) **Proposition.** *Hypotheses as in* (2.3.5.1), *fix integers* $n \geqq 0$ *and* i. *The morphism*

(2.3.5.2.1) $$h(i)\colon F_{\mathrm{con}}^{n-i}(\mathbf{R}^n f_*(\Omega^\bullet_{X/S})) \to \mathbf{R}^n f_*(\Omega^\bullet_{X/S})/F^{i+1}$$

is an isomorphism if and only if the morphism

(2.3.5.2.2) $$h(N-i-1)\colon F_{\mathrm{con}}^{N+1+i-n}(\mathbf{R}^{2N-n} f_*(\Omega^\bullet_{X/S})) \to \mathbf{R}^{2N-n} f_*(\Omega^\bullet_{X/S})/F^{N-i}$$

is an isomorphism.

Proof. Just as in the proof of (2.3.4.1.7), we are immediately reduced to the case in which S is the spectrum of a field K. By the autoduality of the Hodge and conjugate filtrations (2.3.5.1.4–5), the *dual* of (2.3.5.2.1) is the natural mapping

(2.3.5.2.3) $$F^{N-i}(\mathbf{R}^{2N} f_*(\Omega^\bullet_{X/S})) \to \mathbf{R}^{2N-n} f_*(\Omega^\bullet_{X/S})/F_{\mathrm{con}}^{N+1+i-n}.$$

Thus (2.3.5.2.1) and (2.3.5.2.3) are isomorphisms or not together. But (2.3.5.2.2) and (2.3.5.2.3) have the same kernel, $(F^{N-i} \cap F_{\mathrm{con}}^{N+1-i-n}) \cdot (\mathbf{R}^{2N-n} f_*(\Omega^\bullet_{X/S}))$. Since the source and target of (2.3.5.2.2) have the same dimension over K, and the same is true of (2.3.5.2.2), it follows that (2.3.5.2.2) and (2.3.5.2.3) are isomorphisms or not together. Q.E.D.

(2.3.5.2.4) **Corollary.** *Hypotheses as in* (2.3.5.1), *suppose in addition that the relative dimension of* X/S *is* 2. *If the Hasse-Witt operation*

(2.3.5.2.5) $$\text{H-W}\colon F_{\mathrm{abs}}^* R^2 f_*(\mathcal{O}_X) \to R^2 f_*(\mathcal{O}_X)$$

is an isomorphism, then the higher Hasse-Witt operation

(2.3.5.2.6) $$\text{H-W}(1)\colon F_{\mathrm{abs}}^*(R^1 f_*(\Omega^1_{X/S})) \to R^1 f_*(\Omega^1_{X/S})$$

is an isomorphism.

Proof. (2.3.5.2) and (2.3.4.2.5).

(2.3.6) We are now ready to discuss the differential equations promised in (2.3.5). We recall our situation: T, a scheme of characteristic p, S a smooth T-scheme, $f\colon X \to S$ a proper and smooth morphism of relative dimension N with geometrically connected fibres, whose Hodge cohomology is locally free, and whose Hodge $\Rightarrow$ De Rham spectral sequence is degenerate at E_1. Fix an integer $n \geqq 0$, and denote by a the smallest integer such that $R^{n-a} f_*(\Omega^a_{X/S})$ is *not zero.* (We also assume n so chosen that $\mathbf{R}^n f_*(\Omega^\bullet_{X/S})$ is not zero, so that such an a exists.) The a'th Hasse-Witt operation

(2.3.6.1) $$\text{H-W}(a)\colon F_{\mathrm{abs}}^*(R^{n-a} f_*(\Omega^a_{X/S})) \to R^{n-a} f_*(\Omega^a_{X/S})$$

is defined (because, in dimension n, all the "lower" ones H-W(i), $i<a$, have source and target the zero module, hence are isomorphisms). In fact, it is none other than the composite (analogous to (2.3.4.1.3))

$$\begin{tikzcd} {}_{\mathrm{con}}E_2^{n-a,a} \arrow[r, hook] & \mathbf{R}^n f_*(\Omega^\bullet_{X/S}) \arrow[r, two heads] & E_1^{a,n-a} \arrow[d, equal] \\ F^*_{\mathrm{abs}}(R^{n-a}f_*(\Omega^a_{X/S})) \arrow[u, "\wr"] \arrow[rr, dotted, "\text{H-W}(a)"] & & R^{n-a}f_*(\Omega^a_{X/S}). \end{tikzcd} \tag{2.3.6.2}$$

(2.3.6.3) **Proposition** (Igusa, Manin). *Assumptions as in* (2.3.6), *suppose S affine and so small that all the Hodge cohomology sheaves are free $\mathcal{O}_S$-modules. Let $\omega_1, \dots, \omega_\ell$ be a base of $R^{n-a}f_*(\Omega^a_{X/S})$, and denote by (a_{ij}) the matrix in $M_\ell(\Gamma(S, \mathcal{O}_S))$ of* H-W(a) *with respect to this base:*

$$\text{H-W}(a)(F^*_{\mathrm{abs}}(\omega_i)) = \sum_i a_{ji}\,\omega_j. \tag{2.3.6.4}$$

(2.3.6.5) *Consider the dual basis $\omega_1^*, \dots, \omega_\ell^*$ of $R^{N+a-n}f_*(\Omega^{N-a}_{X/S})$. By duality and the definition of the integer a, it is also the least integer with $R^{N+a-n}f_*(\Omega^{N-a}_{X/S})$ non-zero, so that the degeneration of the Hodge$\Rightarrow$De Rham spectral sequence gives an inclusion*

$$R^{N+a-n}f_*(\Omega^{N-a}_{X/S}) \hookrightarrow \mathbf{R}^{2N-n}f_*(\Omega^\bullet_{X/S}). \tag{2.3.6.6}$$

(2.3.6.7) *Let $\mathcal{D}_1, \dots, \mathcal{D}_\ell$ be T-linear differential operators on S which are in the algebra generated by $Der(S/T)$, which operate on the De Rham cohomology sheaves $\mathbf{R}^i f_*(\Omega_{X/S})$ via the Gauss-Manin connection ∇. Suppose that $\omega_1^*, \dots, \omega_\ell^*$, considered as sections of $\mathbf{R}^{2N-n}f_*(\Omega^\bullet_{X/S})$, satisfy the differential equation*

$$\sum_j \nabla(\mathcal{D}_j)(\omega_j^*) = 0 \quad \text{in } \mathbf{R}^{2N-n}f_*(\Omega^\bullet_{X/S}). \tag{2.3.6.8}$$

Then each column of the matrix (a_{ij}) satisfies the differential equation

$$\sum_j \mathcal{D}_j(a_{ji}) = 0, \quad \text{for } i = 1, \dots, \ell. \tag{2.3.6.9}$$

Proof. The proof is based on the fact that the composite p-linear mapping

$$R^{n-a}f_*(\Omega^a_{X/S}) \xrightarrow{F^*_{\mathrm{abs}}} F^*_{\mathrm{abs}}(R^{n-a}f_*(\Omega^a_{X/S})) \xrightarrow{\text{H-W}(a)} R^{n-a}f_*(\Omega^a_{X/S}) \tag{2.3.6.9bis}$$

may be factored through the subsheaf $\mathbf{R}^n f_*(\Omega^\bullet_{X/S})^\nabla$ of *horizontal* sections (cf. (2.3.1.3)), as expressed in the commutative diagram

$$\begin{tikzcd}
{}_{\mathrm{con}}E_2^{n-a,a} \arrow[r] & \mathbf{R}^n f_*(\Omega^\bullet_{X/S}) \arrow[r, two heads] & R^{n-a}f_*(\Omega^a_{X/S}) \\
F^*_{\mathrm{abs}}(R^{n-a}f_*(\Omega^a_{X/S})) \arrow[u] \arrow[urr, bend left=60, "\text{H-W}(a)"] & \mathbf{R}^n f_*(\Omega_{X/S})^{\nabla} \arrow[u] \arrow[ur, dashed, "\mathrm{pr}"'] & \\
R^{n-a}f_*(\Omega^a_{X/S}) \arrow[u, "F^*_{\mathrm{abs}}"'] \arrow[ur, dashed, "\alpha"'] & &
\end{tikzcd}$$

Let $(\ ,\ )$ denote the cup-product pairing (2.3.5.1.3) of De Rham cohomology. We have, for each i,

$$0=\Big(\alpha(\omega_i), \sum_j \nabla(\mathscr{D}_j)(\omega_j^*)\Big) \tag{2.3.6.10}$$

and, because $\alpha(\omega_i)$ is *horizontal*, we have

$$0=\sum_j \mathscr{D}_j\big((\alpha(\omega_i), \omega_j^*)\big). \tag{2.3.6.11}$$

Because each ω_j^* has Hodge filtration $\geqq N-a$, the cup-product $(\alpha(\omega_i), \omega_j^*)$ depends only on the class of $\alpha(\omega_i)$ modulo $F^{a+1}\mathbf{R}^n f_*(\Omega^\bullet_{X/S})$, by (2.3.5.1.4), which is to say on $\mathrm{pr}.\,\alpha(\omega_i)=\text{H-W}(a)(F^*_{\mathrm{abs}}(\omega_i))$. Furthermore denoting by $\langle\,,\rangle$ the cup-product pairing (2.3.6.1.6) of Hodge cohomology, we have

$$\begin{aligned}(\alpha(\omega_i), \omega_j^*)&=\langle \mathrm{pr}.\,\alpha(\omega_i), \omega_j^*\rangle=\langle \text{H-W}(a)(F^*_{\mathrm{abs}}(\omega_i)), \omega_j^*\rangle\\ &=\Big\langle\sum_k a_{ki}\,\omega_k, \omega_j^*\Big\rangle=a_{ji}. \quad \text{Q.E.D.}\end{aligned} \tag{2.3.6.12}$$

(2.3.7) A Numerical Example-Certain Hypersurfaces of Geometric Genus One

(2.3.7.0) Recall that for any base scheme S, and any hypersurface X in $\mathbf{P}_S^{n+1}$ which is smooth over S, the Hodge cohomology sheaves. $R^a f_*(\Omega^b_{X/S})$ ($f\colon X\to S$ the structural morphism) are locally free $\mathcal{O}_S$-modules of finite rank (whose formation consequently commutes with all change of base) and the Hodge $\Rightarrow$ De Rham spectral sequence degenerates at E_1 (cf. [5a]). In terms of a system of homogeneous coordinates $X_1, \dots, X_{n+2}$ on $\mathbf{P}_S^{n+1}$, we can write an equation for X (i.e., an isomorphism between $\mathcal{O}_{\mathbf{P}}(-d)$, d being the *degree* of X, and the ideal $I(X)$ defining X in $\mathbf{P}_S^{n+1}$), at least locally on S. Localizing on S, we may and will assume that X is defined by a homogeneous form of degree d,

$$H=H(X_1,\ldots,X_{n+2})\in\Gamma(S,\mathcal{O}_S)[X_1,\ldots,X_{n+2}].$$

The corresponding short exact sequence on $\mathbf{P}\overset{\text{def'n}}{=}\mathbf{P}_S^{n+1}$

$$0\to\mathcal{O}_{\mathbf{P}}(-d)\xrightarrow{H}\mathcal{O}_{\mathbf{P}}\to\mathcal{O}_X\to 0 \tag{2.3.7.1}$$

gives an isomorphism of cohomology sheaves on S (π: $\mathbf{P}_S^{n+1}\to S$ denoting the projection) via coboundary:

$$R^n f_*(\mathcal{O}_X)\xrightarrow{\sim}R^{n+1}\pi_*(\mathcal{O}_{\mathbf{P}}(-d)). \tag{2.3.7.2}$$

Using the standard covering of projective space, the $\mathcal{O}_S$-module $R^{n+1}\pi_*(\mathcal{O}_{\mathbf{P}}(-d))$ is easily computed: it's the free $\mathcal{O}_S$-module

(2.3.7.3) "forms" of degree $-d$ in $\mathcal{O}_S[X_1,\ldots,X_{n+2}X_1^{-1},\ldots,X_{n+2}^{-1}]$ / the $\mathcal{O}_S$-span of those monomials $X^W=X_1^{W_1}\ldots X_{n+2}^{W_{n+2}}$ for which $\sum W_i=-d$ but $W_i\geqq 0$ for some i

which admits as base the monomials

$$m(W)=X^W=X_1^{W_1}\ldots X_{n+2}^{W_{n+2}},\quad \sum W_i=-d,\quad W_i<0\quad\text{for all } i. \tag{2.3.7.4}$$

We denote by the same symbols $m(W)$ the corresponding basis of $R^n f_*(\mathcal{O}_X)$ via the inverse of the isomorphism (2.3.7.2).

The "residue" exact sequence in highest degree

$$\begin{array}{c} 0\to\Omega_{\mathbf{P}/S}^{n+1}\to\Omega_{\mathbf{P}/S}^{n+1}(\log X)\to\Omega_{X/S}^{n}\to 0\\ \Vert\\ \Omega_{\mathbf{P}/S}^{n+1}\otimes I(X)^{-1}\end{array} \tag{2.3.7.5}$$

gives us an isomorphism, the "Poincaré residue"

$$\pi_*(\Omega_{\mathbf{P}/S}^{n+1}\otimes I(X)^{-1})\xrightarrow{\sim}f_*(\Omega_{X/S}^{n}). \tag{2.3.7.6}$$

In terms of the local coordinates $x_i=X_i/X_{n+2}$, the global section of $\Omega_{\mathbf{P}/S}^{n+1}(n+2)$ given by

$$X_1\ldots X_{n+2}\frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_{n+1}}{x_{n+1}} \tag{2.3.7.7}$$

defines an isomorphism

$$\mathcal{O}_{\mathbf{P}}\xrightarrow{\sim}\Omega_{\mathbf{P}/S}^{n+1}(n+2) \tag{2.3.7.8}$$

from which it follows that $\pi_*(\Omega^{n+1}_{\mathbf{P}/S}\otimes I(X)^{-1})$ is a free $\mathcal{O}_S$-module, with basis the differentials

$$\frac{X^W}{H}\frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_{n+1}}{x_{n+1}},\quad \sum W_i=d,\quad W_i>0 \text{ for all } i. \tag{2.3.7.9}$$

The image of these differentials under the residue isomorphism (2.3.7.6) are the differentials on X:

$$\begin{aligned}\omega(W)&=\frac{X^W}{X_{n+1}\dfrac{\partial H}{\partial X_{n+1}}}\frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_n}{x_n}, \quad \sum W_i=d,\ W_i>0 \text{ for all } i\\ &=\frac{x_1^{W_1-1}\dots x_{n+1}^{W_{n+1}-1}dx_1\wedge\cdots\wedge dx_n}{\dfrac{\partial h}{\partial x_{n+1}}}.\end{aligned} \tag{2.3.7.10}$$

The bases $m(W)$ of $R^n f_*(\mathcal{O}_X)$ (2.3.7.4) and $\omega(W)$ of $f_*(\Omega^n_{X/S})$ are dual to each other under Serre Duality:

$$\langle m(-W),\omega(V)\rangle=\begin{cases}1 & \text{if } V=W\\ 0 & \text{if } V\neq W.\end{cases} \tag{2.3.7.11}$$

(2.3.7.12) Suppose once again that S is of characteristic p. Then from the commutative diagram of sheaves on $\mathbf{P}^{n+1}_S$

$$\begin{array}{ccccccccc} 0\to & \mathcal{O}_{\mathbf{P}}(-d) & \xrightarrow{H} & \mathcal{O}_{\mathbf{P}} & \longrightarrow & \mathcal{O}_X & \to 0\\ & \downarrow{\scriptstyle H^{p-1}\cdot F_{\mathrm{abs}}} & & \downarrow{\scriptstyle F_{\mathrm{abs}}} & & \downarrow{\scriptstyle F_{\mathrm{abs}}} & \\ 0\to & \mathcal{O}_{\mathbf{P}}(-d) & \xrightarrow{H} & \mathcal{O}_{\mathbf{P}} & \longrightarrow & \mathcal{O}_X & \to 0\end{array} \tag{2.3.7.13}$$

it follows that the p-linear endomorphism of $R^n f_*(\mathcal{O}_X)$ induced by the p-th power mapping $\mathcal{O}_X\to\mathcal{O}_X$ corresponds via the coboundary isomorphism (2.3.7.2) to the p-linear endomorphism of $R^{n+1}\pi_*(\mathcal{O}_{\mathbf{P}}(-d))$ induced by the composite $\mathcal{O}_{\mathbf{P}}(-d)\xrightarrow{p\text{-th power}}\mathcal{O}_{\mathbf{P}}(-pd)\xrightarrow{H^{p-1}}\mathcal{O}_{\mathbf{P}}(-d)$. This permits the calculation of the Hasse-Witt matrix of a hypersurface:

(2.3.7.14) **Algorithm.** *Assumptions as in* (2.3.7.0) *and* (2.3.7.12), *the Hasse-Witt matrix in dimension n of a smooth hypersurface* $X\subset\mathbf{P}^{n+1}_S$ *defined by an equation* $H=H(X_1,\dots,X_{n+2})\in\Gamma(S,\mathcal{O}_S)[X_1,\dots,X_{n+2}]$ *may be computed with respect to the basis* $\{m(-W)|\sum W_i=d,\ W_i>0$ for all $i\}$ *of* $R^n f_*(\mathcal{O}_X)$ *as follows:*

Raise H to $p-1$-st *power, and write it explicitly as a sum of monomials*

$$H^{p-1}=\sum A_U X^U. \tag{2.3.7.15}$$

Then the Hass-Witt matrix is given by

$$\text{(2.3.7.16)} \qquad \text{H-W}(F^*_{\mathrm{abs}}(m(-W))) = \sum A_{pW-V}\, m(-V).$$

(2.3.7.17) **Special Case.** Hypotheses as in (2.3.7.14), suppose that X has degree $d=n+2$. Then $R^n f_*(\mathcal{O}_X)$ is free of rank one on S, with base $m(-1,-1,\ldots,-1)$ and the Hasse-Witt matrix of X/S in dimension n (or the Hasse invariant, as we shall call it in this case) with respect to the basis $m(-1,-1,\ldots,-1)$ is given by the coefficient of $(X_1\ldots X_{n+2})^{p-1}$ in H^{p-1}.

We now apply this to a particularly beautiful family of hypersurfaces.

(2.3.7.18) **Corollary.** *Let $d\geqq 2$ be relatively prime to p, and put $S=$* Spec$(\mathbf{F}_p[\lambda][1/1-\lambda^d])$. *Consider the smooth (over S) hypersurface X in $\mathbf{P}_S^{d-1}$ of equation*

$$\text{(2.3.7.19)} \qquad \sum_{i=1}^{d} X_i^d - d\lambda X_1\ldots X_d.$$

It's Hasse invariant is given by the truncated hypergeometric series

$$\text{(2.3.7.20)} \qquad (d\lambda)^{p-1}\sum_{a=0}^{\left[\frac{p-1}{d}\right]} \frac{(1/d)_a(2/d)_a\ldots(d-1/d)_a}{a!\,a!\ldots a!}\lambda^{-ad}$$

where for $\alpha\neq 0$, we put $(\alpha)_0=1$, $(\alpha)_n=\alpha(\alpha+1)\ldots(\alpha+n-1)$ if $n\geqq 1$. This may be expressed in terms of the "full" hypergeometric series in $\mathbf{Z}_p[[\lambda^{-1}]][\lambda]$ by means of Dwork's congruence (cf. [8a], pp. 36–37)

$$\text{(2.3.7.21)} \quad (d\lambda)^{p-1}\sum_{a=0}^{\left[\frac{d-1}{d}\right]} \frac{(1/d)_a(2/d)_a\ldots(d-1/d)_a}{a!\,a!\ldots a!}\lambda^{-ad} \equiv G(\lambda)/G(\lambda^p)\ \text{modulo}\ (p)$$

where $G(\lambda)=(d\lambda)^{-1}F\begin{pmatrix}1/d,2/d,\ldots,d-1/d\\ 1,\ldots,1\end{pmatrix};\lambda^{-d}$ is the element of $\mathbf{Z}_p[[\lambda^{-1}]]$ given by

$$\text{(2.3.7.22)} \quad G(\lambda)=\lambda^{-1}\sum_{a\geqq 0}(-1)^{a(d-1)}\binom{-1/d}{a}\binom{-2/d}{a}\cdots\binom{-(d-1)/d}{a}\lambda^{-ad}.$$

Proof. By direct calculation, one finds that (2.3.7.20) *is* the coefficient of $(\prod_i X_i)^{p-1}$ in $\sum X_i^d - d\lambda\prod_i X_i$. To apply the congruences of Dwork, we need only observe that in the sum (2.3.7.20), we could have let a run from 0 all the way to $p-1$ (which is the *usual* first truncation point for hypergeometric-type series), because, for $a\leqq p-1$ but $ad\geqq p$, we have

$$\text{(2.3.7.23)} \quad (1/d)_a(2/d)_a\ldots(d-1/d)_a=\frac{(ad)!}{d^{ad}\cdot a!}\equiv 0 \quad \text{modulo}\ p. \quad \text{Q.E.D.}$$

(2.3.8) *The example, continued.* According to (2.3.6), the Hasse invariant of the hypersurface (2.3.7.19) is to satisfy every differential equation satisfied by the differential $\omega = \omega(1, \dots, 1)$ (in the notation (2.3.7.10)) considered as a section of $\mathbf{R}^{d-2} f_*(\Omega^\bullet_{X/S})$. By Dwork's congruence (2.3.7.21), this is equivalent to the formal series $G(\lambda)$ (2.3.7.21) satisfying every such differential equation. Because the formal series $G(\lambda)$ is universal, i.e., independent of p, and the formation of the Hodge and De Rham cohomologies of a smooth hypersurface commutes with arbitrary change of base, it follows that we have:

(2.3.8.1) **Corollary.** *Consider the hypersurface smooth over the spectrum* S *of* $\mathbf{Z}[\lambda][1/d(1-\lambda^d)]$ *given by the equation* (2.3.7.19). *For every differential equation*

$$\sum a_i(\lambda)\, \nabla \left(\frac{d}{d\lambda}\right)^i (\omega) = 0 \quad \text{in } \mathbf{R}^{d-2} f_*(\Omega^\bullet_{X/S}) \tag{2.3.8.2}$$

satisfied by $\omega = \omega(1, \dots, 1)$ *in* $f_*(\Omega^{d-2}_{X/S})$, $a_i(\lambda) \in \mathbf{Z}[\lambda][1/d(1-\lambda^d)]$, *we have*

$$\sum a_i(\lambda) \left(\frac{d}{d\lambda}\right)^i (G(\lambda)) = 0 \quad \text{in } \mathbf{Z}\left[\frac{1}{d}\right][[\lambda^{-1}]], \tag{2.3.8.3}$$

where $G(\lambda) \in \mathbf{Z}\left[\frac{1}{d}\right][[\lambda^{-1}]]$ *is the series*

$$G(\lambda) = (d\lambda)^{-1} \sum \frac{(1/d)_a \dots (d-1/d)_a}{a! \dots a!} \lambda^{-ad}. \tag{2.3.8.4}$$

(2.3.8.5) **Remark.** In fact, "the" differential equation satisfied by ω is

$$\nabla \left(\frac{d}{d\lambda}\right)^{d-1} (\omega) = \nabla \left(\lambda \frac{d}{d\lambda}\right)^{d-1} (\lambda \omega) \tag{2.3.8.6}$$

as may be deduced from [23a] and an immediate calculation shows that indeed

$$\left(\frac{d}{d\lambda}\right)^{d-1} (G(\lambda)) = \left(\lambda \frac{d}{d\lambda}\right)^{d-1} (\lambda G(\lambda)). \tag{2.3.8.7}$$

(2.3.9) We refer to forthcoming works of B. Mazur for the congruence relations between the higher Hasse-Witt matrices and the zeta function, which generalize the "ordinary" congruence formula [25].

2.4. The Question of Quasicoherence of the Conjugate Spectral Sequence

(2.4.0) We return now to the geometric situation of 1.0, and, to fix ideas, we suppose that S is affine. The conjugate spectral sequence

$$E_2^{a,b} = R^a f_*(\mathcal{H}^b(\Omega^\bullet_{X/S}(\log D))) \Rightarrow \mathbf{R}^{a+b} f_*(\Omega^\bullet_{X/S}(\log D)) \tag{2.4.0.1}$$

is by definition the "second spectral sequence of hypercohomology" (cf. (2.2.2.2)). In general, we do not know whether or not either the E_2 terms or the E_∞ terms of this spectral sequence are quasi-coherent $\mathcal{O}_S$-modules (though of course the $\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))$ *are* quasicoherent).

(2.4.0.2) Consider a covering $\{V_i\}$ of X by affine open sets. From it we construct a Cech bicomplex of quasicoherent $\mathcal{O}_S$-modules

(2.4.0.3)
$$C^{\bullet\bullet} = C^{\bullet\bullet}(\{V_i\}, \Omega^\bullet_{X/S}(\log D));$$
$$C^{a,b} = C^a(\{V_i\}, \Omega^b_{X/S}(\log D)) = \coprod_{\sigma_a} (f|\sigma_a)_*(\Omega^b_{X/S}(\log D)|\sigma_a)$$

where σ_a runs over the a-simplices $V_{i_0} \cap \cdots \cap V_{i_a}$ of the nerve of the covering. The homology sheaves of the associated simple complex are the $\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))$, just because f is quasi-compact and separated, the various $\Omega^b_{X/S}(\log D)$ are quasi-coherent $\mathcal{O}_X$-modules, and the d in $\Omega^\bullet_{X/S}(\log D)$ is $f^{-1}(\mathcal{O}_S)$-linear (cf. [12], III).

(2.4.0.4) The "first" spectral sequence of this bicomplex, the one associated to the "Hodge" filtration

(2.4.0.5)
$$F^j C^{\bullet\bullet} = \sum_{b \geqq j} C^\bullet(\{V_i\}, \Omega^b_{X/S}(\log D)),$$

gives the usual Hodge $\Rightarrow$ De Rham spectral sequence:

(2.4.0.6)
$$E_1^{a,b} = R^b f_*(\Omega^a_{X/S}(\log D)).$$

The second spectral sequence of this bicomplex, the one associated to the "conjugate" filtration

(2.4.0.7)
$$F^k C^{\bullet\bullet} = \sum_{a \geqq k} C^a(\{V_i\}, \Omega^\bullet_{X/S}(\log D))$$

gives rise to a spectral sequence

(2.4.0.8)
$$E_2^{a,b} = \widetilde{\check{H}^a(\{V_i\}, \mathscr{H}^b_{\text{presheaf}}(\Omega^\bullet_{X/S}(\log D)))} \Rightarrow \mathbf{R}^{a+b} f_*(\Omega^\bullet_{X/S}(\log D))$$

of quasi-coherent $\mathcal{O}_S$-modules ($\sim$ denoting the quasi-coherent sheaf associated to a $\Gamma(S, \mathcal{O}_S)$-module), which maps canonically to the conjugate spectral sequence (2.4.0.1),

(2.4.0.9)
$$E_2^{a,b} = R^a f_*(\mathscr{H}^b(\Omega^\bullet_{X/S}(\log D))) \Rightarrow \mathbf{R}^{a+b} f_*(\Omega^\bullet_{X/S}(\log D)).$$

Of course, for each affine open covering of X, the spectral sequence (2.4.0.8) *is* a spectral sequence of quasi-coherent $\mathcal{O}_S$-modules, but the canonical mapping from (2.4.0.9) need not be an isomorphism, even if we replace (2.4.0.8) by its direct limit over all affine open coverings of X.

(2.4.1) In the special case when S is a schema of characteristic p, the conjugate spectral sequence (2.4.0.1) *is* the second spectral sequence of hypercohomology for the $\mathcal{O}_{X^{(p)}}$-linear complex $F_*(\Omega^\bullet_{X/S}(\log D))$ on $X^{(p)}$, and the functors $\mathbf{R}^n f^{(p)}_*$. Because $f^{(p)}: X^{(p)} \to S$ is quasi-compact and separated, the various $F_*(\Omega^b_{X/S}(\log D))$ are quasi-coherent $\mathcal{O}_X$-modules, and $F_*(d)$ is $\mathcal{O}_{X^{(p)}}$-linear, the second spectral sequence (2.4.0.8) of the Cech bicomplex *does* map isomorphically to the conjugate spectral sequence (2.4.0.9), which *is* a spectral sequence of quasi-coherent $\mathcal{O}_S$-modules.

The simple interpretation of the conjugate spectral sequence in characteristic p as the second spectral sequence of the Cech bicomplex (2.4.0.3) makes possible effective calculations, as we shall see. On the contrary, the conjugate spectral sequence over $\mathbf{C}$ remains in shadow.

3. The Main Technical Result on the p-Curvature of the Gauss-Manin Connection

3.0. We return to the geometric situation 1.0, and assume as before that S, and hence T, is a scheme of characteristic p. Recall that for any $\mathcal{O}_S$-module with integrable T-connection, (M, ∇), its p-curvature ψ is the p-linear homomorphism of $\mathcal{O}_S$-modules (cf. [24], 5.2)

$$\psi: Der(S/T) \to End_{\mathcal{O}_S}(M) \tag{3.0.1}$$

defined by

$$\psi(D) = (\nabla(D))^p - \nabla(D^p). \tag{3.0.2}$$

Equivalently, we may view ψ as defining by transposition a homomorphism of $\mathcal{O}_S$-modules, also noted ψ,

$$\psi: M \to F^*_{\text{abs}}(\Omega^1_{S/T}) \otimes M \tag{3.0.3}$$

(where F_{abs} denotes the absolute Frobenius endomorphism of S).

The significance of p-curvature (due to Cartier; cf. [24], 5.1) is the following: (M, ∇) has p-curvature zero if and only if M is spanned over $\mathcal{O}_S$ by the subsheaf M^∇ of horizontal sections; more precisely, denoting by $S^{(p)}$ the fibre product of $g: S \to T$ and the absolute Frobenius F_{abs}: $T \to T$, if and only if the canonical mapping

$$(M^\nabla) \otimes_{\mathcal{O}_{S^{(p)}}} \mathcal{O}_S \to M \tag{3.0.4}$$

is an isomorphism.

3.1. Taking for M the De Rham cohomology $\mathbf{R}^n f_*(\Omega_{X/S}(\log D))$, and for ∇ the Gauss-Manin connection, we recall once again (cf. [24], 3.5) that the entire conjugate spectral sequence

$$E_2^{a,b} = R^a f_*(\mathcal{H}^b(\Omega^\bullet_{X/S}(\log D))) \Rightarrow \mathbf{R}^{a+b} f_*(\Omega^\bullet_{X/S}(\log D)) \tag{3.1.1}$$

is endowed with the action of the Gauss-Manin connection, and that on the E_2 terms, the p-curvature is *zero*. This implies that the p-curvature also vanishes on the E_∞ terms, and hence that

$$\psi\left(F^i_{\mathrm{con}}\,\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))\right) \subset F^*_{\mathrm{abs}}(\Omega^1_{S/T}) \otimes F^{i+1}_{\mathrm{con}}\,\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)). \tag{3.1.2}$$

Passing to the associated graded, there is an induced mapping, again denoted ψ

$$\psi\colon \mathrm{gr}^i_{F_{\mathrm{con}}}(\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))) \to F^*_{\mathrm{abs}}(\Omega^1_{S/T}) \otimes \mathrm{gr}^i_{F_{\mathrm{con}}}(\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))). \tag{3.1.3}$$

Our main technical result 3.2 *identifies* this mapping, under suitable hypotheses with a twisted form of the Kodaira-Spencer mapping (1.3.2.1).

3.2. Theorem. *Under the hypotheses of* (2.3.2), *the diagram below is commutative*

$$\begin{tikzcd}
\mathrm{gr}^a_{F_{\mathrm{con}}}(\mathbf{R}^{a+b} f_*(\Omega^\bullet_{X/S}(\log D))) \arrow[r, "\psi"] \arrow[d, equal] & F^*_{\mathrm{abs}}(\Omega^1_{S/T}) \otimes \mathrm{gr}^{a+1}_{F_{\mathrm{con}}}(\mathbf{R}^{a+b} f_*(\Omega^\bullet_{X/S}(\log D))) \arrow[d, equal] \\
{}_{\mathrm{con}}E^{a,b}_\infty \arrow[d, equal] & F^*_{\mathrm{abs}}(\Omega^1_{S/T}) \otimes {}_{\mathrm{con}}E^{a+1,b-1}_\infty \arrow[d, equal] \\
{}_{\mathrm{con}}E^{a,b}_2 & F^*_{\mathrm{abs}}(\Omega^1_{S/T}) \otimes {}_{\mathrm{con}}E^{a+1,b-1}_2 \\
F^*_{\mathrm{abs}}(R^a f_*(\Omega^b_{X/S}(\log D))) \arrow[u, "\wr", "\mathscr{C}^{-1}"'] \arrow[r, "(-1)^{b+1}F^*_{\mathrm{abs}}(\rho)"] & F^*_{\mathrm{abs}}(\Omega^1_{S/T}) \otimes F^*_{\mathrm{abs}}(R^{a+1} f_*(\Omega^{b-1}_{X/S}(\log D))) \arrow[u, "\wr", "i\otimes\mathscr{C}^{-1}"'] \\
R^a f_*(\Omega^b_{X/S}(\log D)) \arrow[u, "F^*_{\mathrm{abs}}"'] \arrow[r, "(-1)^{b+1}\rho"] & \Omega^1_{S/T} \otimes R^{a+1} f_*(\Omega^{b-1}_{X/S}(\log D)) \arrow[u, "F^*_{\mathrm{abs}}\otimes F^*_{\mathrm{abs}}"']
\end{tikzcd}$$

in which ρ is the cup-product with the Kodaira-Spencer class (1.1.3), *viewed as a global section over S of $\Omega^1_{S/T} \otimes R^1 f_*(Der_D(X/S))$.*

Before proceeding to the proof, we will recall some basic facts about the modular representation theory of finite groups of order prime to p, and then restate 3.2 "with a group of operators".

(3.2.1) Let G be a finite group of order prime to p, and k a field of characteristic p. Let V be a finite-dimensional k-space on which G acts as a group of k-automorphisms, through a homomorphism $\chi\colon G \to GL(V)$. Denoting by $F_{\mathrm{abs}}\colon k \to k$ the absolute Frobenius endomorphism of k, the representation $\chi^{(p)}$ of G on $F^*_{\mathrm{abs}}(V) = V \underset{k}{\otimes} k$ (where k is a module over itself by $F_{\mathrm{abs}}\colon k \to k$) is given by

(3.2.1.1) $$\chi^{(p)}(g)(v \otimes a) \overset{\text{def'n}}{=} \chi(g)(v) \otimes a .$$

In matricial terms, the matrix coefficients of $\chi^{(p)}$ are the p-th powers of the matrix coefficients of χ. The character of $\chi^{(p)}$ is the p-th power of the character of χ.

(3.2.2) **Proposition.** *Hypotheses as in* (3.2.1), *if χ is an absolutely irreducible representation of G in a finite-dimensional k-space, then its degree divides the order of G, and in particular is prime to p.*

Proof. This will be a simple consequence of the analogous fact for representation in characteristic zero. In fact, extending scalars if necessary, we may suppose that k is the residue field of a discrete valuation ring $(\mathcal{O}, \mathfrak{p})$ whose fraction field K has characteristic zero, and is such that every irreducible representation of G in a finite-dimensional K-space is absolutely irreducible. In fact, we will prove

(3.2.2 bis) **Proposition.** *Let $(\mathcal{O}, \mathfrak{p})$ be a discrete valuation ring whose fraction field K has characteristic zero, and whose residue field k has characteristic $p > 0$. Let G be a finite group, of order prime to p, such that every irreducible representation of G in a finite-dimensional K-space is absolutely irreducible. Then every irreducible representation of G in a finite-dimensional k-space is the "reduction* mod $\mathfrak{p}$" *of a representation of G in a free $\mathcal{O}$-module of finite rank, which is irreducible over K, and which is determined over K up to isomorphism.*

Proof. Because G has order prime to p, the group-ring $k[G]$ is semi-simple, as is $K[G]$, and the isomorphism classes of irreducible representations of G in finite dimensional k-spaces (resp. K-spaces) are in bijective correspondence with the indecomposable central idempotents $f_1, \dots, f_r$ (resp. $e_1, \dots, e_s$) in $k[G]$ (resp. $K[G]$), which give the projections onto the "isotypique" components of any finite dimensional $k[G]$ (resp. $K[G]$) module. We have

(3.2.2.1) $$1 = \sum e_i, \qquad e_i e_j = \delta_{ij} e_i$$

and for any central function f in $K[G]$,

(3.2.2.2) $$f = \sum a_i e_i, \qquad a_i \in K .$$

We will first show that the e_i give an $\mathcal{O}$-base for the central functions with values in $\mathcal{O}$. The e_i are given by the following explicit formulaes, in which χ_i denotes the corresponding irreducible representation:

(3.2.2.3) $$e_i = \frac{\deg(\chi_i)}{\# G} \sum_g \operatorname{trace}(\chi_i(g^{-1})) \cdot g .$$

Because G is finite, any representation of G on a finite dimensional K-space comes by extension of scalars from a representation of G in a free $\mathcal{O}$-module of finite rank (take the lattice generated by the G-translates of any lattice), and hence its *character* takes values in $\mathcal{O}$. It follows that the functions e_i take values in $\mathcal{O}$; i.e., lie in $\mathcal{O}[G]$. Because the χ_i are absolutely irreducible, we have

$$\deg(\chi_i) \mid \# G, \qquad \text{hence } p \nmid \deg(\chi_i). \tag{3.2.2.4}$$

Hence the values of the e_i at the identity element $1 \in G$ are units in $\mathcal{O}$, because

$$e_i(1) = \frac{\deg(\chi_i)}{\# G} \cdot \operatorname{trace}(\chi_i(1)) = \frac{(\deg(\chi_i))^2}{\# G}. \tag{3.2.2.5}$$

Thus if $f \in \mathcal{O}[G]$ is central, $f = \sum a_i e_i$ and

$$a_i = \frac{(a_i e_i)(1)}{e_i(1)} = \frac{(f e_i)(1)}{e_i(1)} \in \mathcal{O}. \tag{3.2.2.6}$$

Thus the e_i give an $\mathcal{O}$-base of the center of $\mathcal{O}[G]$.

Now let $f_1 \in k[G]$ be the indecomposable central idempotent corresponding to an irreducible representation χ. We can certainly lift f_1 to a *central function* f with values in $\mathcal{O}$, which we may write

$$f = \sum a_i e_i, \qquad a_i \in \mathcal{O}. \tag{3.2.2.7}$$

Because $f_1 \cdot f_1 = f_1$, we have

$$\sum a_i^2 e_i = f^2 \equiv f = \sum a_i e_i \mod (\mathfrak{p}[G]) \tag{3.2.2.8}$$

and multiplying both sides by e_i, we get

$$a_i^2 e_i \equiv a_i e_i \mod \mathfrak{p}[G]. \tag{3.2.2.9}$$

Evaluating at the identity element $1 \in G$, we have (by (3.2.2.4))

$$a_i^2 \equiv a_i \mod \mathfrak{p}, \tag{3.2.2.10}$$

so that each coefficient a_i is congruent to either 0 or 1 mod $\mathfrak{p}$. Thus we may lift f_1 to a central idempotent

$$\sum \varepsilon_i e_i, \qquad \varepsilon_i = 0 \quad \text{or } 1 \tag{3.2.2.11}$$

in $\mathcal{O}[G]$. Because the e_i are a *basis* of the center of $\mathcal{O}[G]$, the indecomposability of f_1 as central idempotent in $k[G]$ implies that ε_i differs from zero for only *one* value of i, say $i=1$. This shows that e_i is the *unique* central idempotent in $\mathcal{O}[G]$ lifting f_1. This shows that χ is the "reduction mod $\mathfrak{p}$" of a unique irreducible representation χ_1 of G in a finite-dimensional K-space. Q.E.D.

(3.2.3) **Corollary.** *Hypotheses and notations as in* (3.2.2), *the indecomposable central idempotents in* $K[G]$, *which all lie in* $\mathcal{O}[G]$, *have reductions modulo* $\mathfrak{p}$ *in* $k[G]$ *which remain indecomposable.*

Proof. The number of e_i is equal to the number of f_i, both being the number of conjugacy classes in G. Hence *every* e_i lifts an f_i. Q.E.D.

In terms of representations, this gives:

(3.2.3 bis) **Corollary.** *Hypotheses as in* (3.2.2 bis), *the reduction* mod $\mathfrak{p}$ *of any* K*-irreducible representation of* G *in a free* $\mathcal{O}$*-module of finite rank is irreducible (and every irreducible representation of* G *in a finite-dimensional* k*-space arises this way).*

(3.2.3.1) **Corollary.** *Hypotheses as in* (3.2.2 bis), *every irreducible representation of* G *in a finite-dimensional* k*-space is absolutely irreducible.*

Proof. This follows from (3.3.0 bis), applied to arbitrary finite extensions K' of K, and arbitrary extensions of the valuation of K to K', by which we can realize arbitrary finite extensions k' of k as residue field.

(3.2.3.2) **Corollary.** *Hypotheses in* (3.2.2 bis), *the indecomposable central idempotent (noted* $P(\chi)$*) associated to an irreducible representation* χ *in a finite-dimensional* k*-space is given by the formula*

$$P(\chi)=\frac{\deg(\chi)}{\# G}\sum_{g}\operatorname{trace}(\chi(g^{-1}))\cdot g. \tag{3.2.3.3}$$

(3.2.4) We return to the hypotheses of (2.3.2), and suppose given in addition:

(3.2.4.0) a finite group G of order prime to p, which acts as a group of S-automorphisms of X and preserves the divisor D (though not necessarily the individual D_i);

(3.2.4.1) a subfield k of $\Gamma(T, \mathcal{O}_T)$, and an absolutely irreducible representation χ of G in a finite-dimensional k-space.

(3.2.4.2) By functoriality, the group G acts on the De Rham cohomology sheaves $\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))$ as a group of $\mathcal{O}_S$-linear horizontal (for the Gauss-Manin connection), and respects both the Hodge and the conjugate filtrations. (In fact, G acts on both the Hodge $\Rightarrow$ De Rham and the conjugate spectral sequences.)

(3.2.4.3) The functoriality of the diagram (3.2.0) shows immediately that it is a diagram of G-morphisms (though the lower vertical arrows F^*_{abs}, being p-linear, are not $k[G]$-morphisms).

This said, we may now "restate" 3.2.

(3.3)(=3.2 bis) Theorem. *Under the hypotheses of* (3.2.4), *the following subdiagram of* (3.2.0) *is commutative.*

$$\begin{tikzcd}
P(\chi^{(p)})({}_{\mathrm{con}}E_\infty^{a,b}) \arrow[r, "\psi"] & (1\otimes P(\chi^{(p)}))(F_{\mathrm{abs}}^*(\Omega_{S/T}^1)\otimes({}_{\mathrm{con}}E_\infty^{a,b})) \\
P(\chi^{(p)})F_{\mathrm{abs}}^*(R^a f_*(\Omega_{X/S}^b)) \arrow[u, "\wr" ', "\mathscr{C}^{-1}"] \arrow[r, "(-1)^{b+1}F_{\mathrm{abs}}^*(\rho)"] & (1\otimes P(\chi^{(p)}))(F_{\mathrm{abs}}^*(\Omega_{S/T}^1))\otimes F_{\mathrm{abs}}^*(R^{a+1}f_*(\Omega_{X/S}^{b-1}(\log D))) \arrow[u, "\wr" ', "1\otimes\mathscr{C}^{-1}"] \\
P(\chi)R^a f_*(\Omega_{X/S}^b(\log D)) \arrow[u, "F_{\mathrm{abs}}^*"] \arrow[r, "(-1)^{b+1}\rho"] & (1\otimes P(\chi))(\Omega_{S/S}^1\otimes R^{a+1}f_*(\Omega_{X/S}^{b-1}(\log D))). \arrow[u, "F_{\mathrm{abs}}^*\otimes F_{\mathrm{abs}}^*"]
\end{tikzcd}$$

Proof (Assuming 3.2). Thu upper square is deduced from the upper square of (3.2.0), which is a diagram of $k[G]$-modules, by applying the projector $P(\chi^{(p)})$. The lower square is deduced from the lower square of (3.2.0) by applying $P(\chi)$ to the lower horizontal line, and noting that

$$\begin{aligned}
F_{\mathrm{abs}}^*\circ(P(\chi)) &= F_{\mathrm{abs}}^*\left(\frac{\deg(\chi)}{\#G}\sum \mathrm{trace}(\chi(g^{-1}))\cdot g\right)\circ F_{\mathrm{abs}}^* \\
&= \left(\frac{\deg(\chi)}{\#G}\right)^p \sum(\mathrm{trace}\,\chi(g^{-1}))^p g\circ F_{\mathrm{abs}}^* = P(\chi^{(p)})F_{\mathrm{abs}}^*. \quad \text{Q.E.D.}
\end{aligned} \tag{3.3.1}$$

(3.3.2) **Corollary.** *Hypotheses as in* (3.2.4), *suppose that the absolute Frobenius endomorphism of $\mathcal{O}_S$ is injective (which is the case if T is reduced, for example). If, for a fixed integer n, the p-curvature of the Gauss-Manin connection on the "part of $\mathbf{R}^n f_*(\Omega_{X/S}^\bullet(\log D))$ which transforms by $\chi^{(p)}$", i.e., on the submodule $P(\chi^{(p)})(\mathbf{R}^n f_*(\Omega_{X/S}^\bullet(\log D)))$ is zero, then the induced Hodge filtration on $P(\chi)(\mathbf{R}^n f_*(\Omega_{X/S}^\bullet(\log D)))$ is horizontal, i.e., stable by the Gauss-Manin connection.*

3.4. The Proof of 3.2: Reduction Steps

(3.4.0) The question being local on S, we may and will assume S affine. As explained in 2.4, the De Rham cohomology sheaves on S are the total homology of the Cech bicomplex associated to any affine open covering of X, and the Hodge and conjugate spectral sequences are the "first" and "second" spectral sequences of the Cech bicomplex. The proof we will give is not at all intrinsic, but rather depends on explicit calculations on the level of the Cech bicomplex itself. It would be of considerable interest to give an intrinsic proof.

(3.4.1) We choose a finite covering of X by affine open sets V_i (as in (1.0.3.1)), each étale over $\mathbf{A}_S^n$ via "local coordinates" $x_i(i), \ldots, x_n(i)$, such that those branches of D meeting V_i are defined by the equations $x_1(i)=0, \ldots, x_\alpha(i)=0$, α depending on V_i.

We next define a (not necessarily integrable) T-connection ∇ on the simple complex deduced by "totalization" from the Cech bicomplex, which yields the Gauss-Manin connection upon passage to homology (cf. [24, 35]).

For any $D \in \mathrm{Der}(S/T)$, we denote

(3.4.1.1) $D(i)$ = the unique element of $\mathrm{Der}_D(V_i/T)$ which extends D and which annihilates the chosen local coordinates $x_1(i), \ldots, x_n(i)$, and by

$$\text{(3.4.1.2)} \qquad \mathrm{Lie}(D(i)):\ \Omega^\bullet_{V_i/S}(\log D) \to \Omega^\bullet_{V_i/S}(\log D)$$

the "Lie derivative with respect to D_i".

For each pair of integers $i<j$, we denote by

$$\text{(3.4.1.3)} \qquad I(D(i)-D(j)):\ \Omega^\bullet_{V_i \cap V_j/S}(\log D) \to \Omega^{\bullet-1}_{V_i \cap V_j/S}(\log D)$$

the operator "interior product with the S-derivation $D(i)-D(j)$" (cf. (1.2.2)). The connection ∇ on $C^\bullet(\{V_i\}, \Omega^\bullet_{X/S}(\log D))$ is given as follows. For a fixed integer b, a section

$$\text{(3.4.1.4)} \qquad \omega \in \sum_a C^a(\{V_i\}, \Omega^b_{X/S}(\log D))$$

and a simplex $i_0 < \cdots < i_q$,

$$\text{(3.4.1.5)} \qquad \begin{aligned}(\nabla(D)(\omega))(i_0, \ldots, i_q) &= \mathrm{Lie}(D(i_0))(\omega(i_0, \ldots, i_q)) \\ &\quad + (-1)^b I(D(i_0)-D(i_1))(\omega(i_1, \ldots, i_q)).\end{aligned}$$

Thus $\nabla(D)$ is the sum of two terms. The first, of bidegree $(0,0)$, is the cup-product with the 0-chain $\{\mathrm{Lie}(D(i))\}$. The second, of bidegree $(1,-1)$, is the cup-product with the 1-cocycle $I(D(i)-D(j))$. This explicit construction makes clear computationally the truth of (1.4.1.6), (1.4.1.7), and (2.3.0.1).

In order to prove 3.2, it suffices by linearity to establish the commutativity of the outermost square of (3.2.0), and after "contraction" with any $D \in \mathrm{Der}(S/T)$. Thus we must prove commutative the diagram

$$\text{(3.4.1.6)} \qquad \begin{array}{ccc} R^a f_*(\mathscr{H}^b(\Omega^\bullet_{X/S}(\log D))) & \xrightarrow{\psi(D)} & R^{a+1} f_*(\mathscr{H}^{b-1}(\Omega^\bullet_{X/S}(\log D))) \\ \uparrow {\scriptstyle \mathscr{C}^{-1} \circ F^*_{\mathrm{abs}}} & & \uparrow {\scriptstyle \mathscr{C}^{-1} \circ F^*_{\mathrm{abs}}} \\ R^a f_*(\Omega^b_{X/S}(\log D)) & \xrightarrow{(-1)^{b+1}\rho(D)} & R^{a+1} f_*(\Omega^{b-1}_{X/S}(\log D)). \end{array}$$

Let us explicate the arrows in this diagram. The vertical ones are deduced from the morphisms of sheaves

$$\text{(3.4.1.7)} \qquad \Omega^b_{X/S}(\log D) \xrightarrow{\sigma^*} \Omega^b_{X^{(p)}/S}(\log D^{(p)}) \xrightarrow[\sim]{\mathscr{C}^{-1}} \mathscr{H}^b(\Omega^\bullet_{X/S}(\log D)).$$

In order to render the vertical arrows still more concrete, we introduce, for each open set V_i, a p-linear endomorphism $\mathscr{F}_i$ of $\Omega^b_{V_i/S}(\log D)$, whose image lies in the subsheaf of closed forms, and which "lifts" the mapping 3.4.1.7. Indeed, the formulas (2.1.2.1) give such a lifting: we require that

$$(3.4.1.8)\qquad \begin{aligned} &\mathscr{F}_i(1)=1\\ &\mathscr{F}_i\left(\frac{dx_\nu(i)}{x_\nu(i)}\right)=\frac{dx_\nu(i)}{x_\nu(i)} \quad \text{for } i=1,\dots,\alpha\\ &\mathscr{F}_i(dx_\nu(i))=(x_\nu(i))^{p-1}\,dx_\nu(i) \quad \text{for } i=\alpha+1,\dots,n\\ &\mathscr{F}_i(\omega\wedge\tau)=\mathscr{F}_i(\omega)\wedge\mathscr{F}_i(\tau)\\ &\mathscr{F}_i(\omega+\tau)=\mathscr{F}_i(\omega)+\mathscr{F}_i(\tau)\\ &\mathscr{F}_i(h\,\omega)=h^p\,\mathscr{F}_i(\omega) \quad \text{for } h\in\mathcal{O}_{V_i}. \end{aligned}$$

We then have our desired lifting

$$(3.4.1.9)\qquad \begin{array}{ccc} \Omega^b_{V_i/S}(\log D) & \xrightarrow{\mathscr{F}_i} & \text{closed forms} \subset \Omega^b_{V_i/S}(\log D)\\ & \searrow & \Big\downarrow \text{canonical projection}\\ & & \mathscr{H}^b(\Omega^\bullet_{V_i/S}(\log D)) \end{array}$$

(3.4.1.10) The lower horizontal arrow in (3.4.1.6) is, as previously (cf. (1.1.3)) noted, $(-1)^{b+1}$ times the cup-product with 1-cocycle

$$\{I(D(i)-D(j))\}.$$

The upper horizontal arrow in (3.4.1.6) is slightly less straightforward to explicate. Let α be a section of $R^a f_*(\mathscr{H}^b(\Omega^\bullet_{X/S}(\log D)))$. We may represent α by an *a-cochain* τ of *closed* forms

$$(3.4.1.11)\qquad \tau\in C^a(\{V_i\},\Omega^b_{X/S}(\log D)),\quad d\tau=0$$

whose *image* in

$$C^a(\{V_i\},\mathscr{H}^b(\Omega^\bullet_{X/S}(\log D)))$$

is a cocycle representing α. By the degeneration of the conjugate spectral sequence at E_2, we may choose τ to be the component of bidegree (a,b) in a *total* $a+b$ cocycle σ which lies in

$$(3.4.1.12)\qquad F^a_{\text{con}}=\sum_{i\geq 0} C^{a+i}(\{V_i\},\Omega^{b-i}_{X/S}(\log D)).$$

The section $\psi(D)(\alpha)$ of $R^{a+1}f_*(\mathscr{H}^{b-1}(\Omega^\bullet_{X/S}(\log D)))$ is then represented by the component of bidegree $(a+1,b-1)$ in the total cocycle

$$(3.4.1.13)\qquad (\nabla(D)^p-\nabla(D^p))(\sigma).$$

Recall that

$$(3.4.1.14) \qquad (\nabla(D)^p - \nabla(D^p))(F^a_{\mathrm{con}}(C^{\bullet\bullet})) \subset F^{a+1}_{\mathrm{con}}(C^{\bullet\bullet})$$

(because $\nabla(D)(F^a_{\mathrm{con}}(C^{\bullet\bullet})) \subset F^a_{\mathrm{con}}(C^{\bullet\bullet})$, and $\nabla(D)$ on $\mathrm{gr}^a_{F_{\mathrm{con}}}(C^{\bullet\bullet})$ is an integrable connection of p-curvature zero, given on $V_{i_0} \cap \cdots \cap V_{i_q}$ by $D \to \mathrm{Lie}(D(i_0))$).

Since σ is congruent to τ modulo $F^{a+1}_{\mathrm{con}}(C^{\bullet\bullet})$, it follows from (3.4.1.13) that

$$(3.4.1.15) \quad (\nabla(D)^p - \nabla(D^p))(\sigma) \equiv (\nabla(D)^p - \nabla(D^p))(\tau) \mod F^{a+2}_{\mathrm{con}}.$$

Thus the section $\psi(D)(\alpha)$ is represented by the component of bidegree $(a+1, b-1)$ in $(\nabla(D)^p - \nabla(D^p))(\tau)$.

(3.4.2) **Lemma.** *Let* $\tau \in C^a(\{V_i\}, \Omega^b_{X/S}(\log D))$, *and* $i_0 < i_1 < \cdots < i_{a+1}$. *For each integer* $n \geqq 1$, *the element*

$$(3.4.2.0) \qquad \nabla(D)^n(\tau) \in \sum_{i \geqq 0} C^{a+i}(\{V_i\}, \Omega^{b-i}_{X/S}(\log D))$$

has its components of bidegree (a, b) and $(a+1, b-1)$ given by

$$(3.4.2.1) \qquad \nabla(D)^n(\tau)(i_0, \ldots, i_a) = \mathrm{Lie}(D(i_0))^n(\tau(i_0, \ldots, i_a)),$$

$$(3.4.2.2) \qquad \begin{aligned} &\nabla(D)^n(\tau)(i_0, \ldots, i_a) \\ &= (-1)^b \sum_{k+\ell=n-1} \mathrm{Lie}(D_{i_0})^k I(D(i_0) - D(i_1)) \, \mathrm{Lie}(D_{i_1})^\ell (\tau(i_1, \ldots, i_a)). \end{aligned}$$

Proof. The proof, by induction, is immediate; the case $n=1$ is the definition. Q.E.D.

Putting $n = p$, we find

$$(3.4.2.3) \qquad \begin{aligned} &(\nabla(D)^p - \nabla(D^p))(\tau)(i_0, \ldots, i_a) \\ &= [(\mathrm{Lie}(D(i_0)))^p - \mathrm{Lie}(D(i_0)^p)](\tau(i_0, \ldots, i_a)) = 0, \\ &\text{since } (D(i_0))^p = D^p(i_0) \end{aligned}$$

$$(3.4.2.4) \qquad \begin{aligned} &(\nabla(D)^p - \nabla(D^p))(\tau)(i_0, \ldots, i_{a+1}) \\ &= (-1)^b \sum_{k+\ell=p+1} \mathrm{Lie}(D_{i_0})^k I(D(i_0) - D(i_1)) \mathrm{Lie}(D_{i_1})^\ell (\tau(i_1, \ldots, i_{a+1})) \\ &\quad - (-1)^b I(D(i_0)^p - D(i_1)^p)(\tau(i_1, \ldots, i_{a+1})). \end{aligned}$$

Thus (3.4.2.4) gives a formula for a cocycle representing $\psi(D)(\alpha)$ in $R^{a+1} f_*(\mathscr{H}^{b-1}(\Omega^\bullet_{X/S}(\log D)))$, when we take for τ a particular representing cochain for α.

(3.4.3) Combining all our explications (3.4.1.9), (3.4.1.10), and (3.4.2.4), we see that the commutativity of (3.4.1.6), and hence the truth of 3.0, is implied by the following assertion:

(3.4.3.0) **Assertion.** *For any cochain* $\tau \in C^a(\{V_i\}, \Omega^b_{X/S}(\log D))$, *the cochain in* $C^{a+1}(\{V_i\}, \Omega^{b-1}_{X/S}(\log D))$ *which assigns to* $(i_0, \ldots, i_{a+1})$ *the* $b-1$ *form*

$$(-1)^b \sum_{k+\ell=p-1} \operatorname{Lie}(D(i_0))^k I(D(i_0)-D(i_1)) \operatorname{Lie}(D(i_1))^\ell (\mathscr{F}_{i_1}(\tau(i_1, \ldots, i_{a+1})))$$

$$-(-1)^b I(D(i_0)^p - D(i_1)^p)(\mathscr{F}_{i_1}(\tau(i_1, \ldots, i_{a+1}))) \tag{3.4.3.1}$$

$$-(-1)^{b+1} \mathscr{F}_{i_0}(I(D(i_0)-D(i_1))(\tau(i_1, \ldots, i_{a+1})))$$

is a cocycle of closed forms, and is cohomologous to zero in

$$C^{a+1}(\{V_i\}, \mathscr{H}^{b-1}(\Omega^\bullet_{X/S}(\log D))).$$

(3.4.3.2) In fact, we will prove that the cochain (3.4.3.1) is in fact a cochain of *exact* forms, and so *vanishes* in $C^{a+1}(\{V_i\}, \mathscr{H}^{b-1}(\Omega^\bullet_{X/S}(\log D)))$. Notice that the occurence of $\mathscr{F}_{i_0}$ in the last line of (3.4.3.1) may be replaced by $\mathscr{F}_{i_1}$ without modifying the class modulo

$$d(\Gamma(V_{i_0} \cap \cdots \cap V_{i_{a+1}}, \Omega^{b-2}_{X/S}(\log D)))$$

of the cochain (3.4.3.1), because both $\mathscr{F}_{i_0}$ and $\mathscr{F}_{i_1}$ restricted to $V_{i_0} \cap V_{i_1}$ are *liftings* of $\mathscr{C}^{-1} \circ \sigma^*$ (3.4.1.7). This change made, the truth of (3.4.3.0), and hence of 3.2, results from the following proposition (3.5.0), which may or may not prove to be of any independent interest. In it, $V_{i_0} \cap \cdots \cap V_{i_a}$ is renamed V, $D(i_0)$ and $D(i_1)$ are renamed P and Q, $\tau(i_1, \ldots, i_{a+1})$ is renamed τ, $\mathscr{F}_{i_1}$ is renamed $\mathscr{F}$, and the coordinates $x_1(i_1), \ldots, x_n(i_1)$ are renamed $x_1, \ldots, x_n$.

3.5. Conclusion of the Proof of 3.2

(3.5.0) **Proposition.** *Let* S *be an affine scheme of characteristic* p, *and* V *a scheme étale over* $\mathbf{A}^n_S$, *by means of "coordinates"* $x_1, \ldots, x_n$ *on* V. *Fix a (possibly empty) subset* α *of* $\{1, \ldots, n\}$, *and let* $D_\alpha \subset V$ *be the divisor with normal crossings relative to* S *defined by* $\prod_{i \in \alpha} x_i = 0$ *(we put* $D_\emptyset = \emptyset$*).*

Let $D \in \operatorname{Der}(S/\mathbf{F}_p)$ *be any derivation of* S, $P \in \operatorname{Der}_{D_\alpha}(V/\mathbf{F}_p)$ *an arbitrary extension of* D *to a derivation of* V *which preserves the ideals* $x_i \mathcal{O}_V$, *for* $i \in \alpha$, *and* $Q \in \operatorname{Der}_{D_\alpha}(V/\mathbf{F}_p)$ *the unique extension which annihilates* $x_1, \ldots, x_n$.

We denote by $\mathscr{F}$: $\Omega^\bullet_{V/S}(\log D_\alpha) \to \Omega^\bullet_{V/S}(\log D_\alpha)$ *the unique* p*-linear endomorphism of* $\Omega^\bullet_{V/S}(\log D_\alpha)$ *as* $\mathcal{O}_V$*-module which satisfies*

$$\begin{aligned}
&\mathscr{F}(1) = 1 \\
&\mathscr{F}\left(\frac{dx_i}{x_i}\right) = \frac{dx_i}{x_i}, \qquad i \in \alpha \\
&\mathscr{F}(dx_i) = x_i^{p-1}\, dx_i, \qquad i \notin \alpha \\
&\mathscr{F}(\omega \wedge \tau) = \mathscr{F}(\omega) \wedge \mathscr{F}(\tau).
\end{aligned} \tag{3.5.0.1}$$

Then for any differential form

(3.5.0.2) $$\tau \in \Gamma(V, \Omega^b_{V/S}(\log D_\alpha)),$$

we have

(3.5.0.3) $$\sum_{k+\ell=p-1} \mathrm{Lie}(P)^k\, I(P-Q)\, \mathrm{Lie}(Q)^\ell(\mathscr{F}(\tau)) - I(P^p-Q^p)(\mathscr{F}(\tau))$$
$$\equiv -\mathscr{F}\big(I(P-Q)(\tau)\big) \text{ modulo } d\Gamma(V, \Omega^{b-1}_{V/S}(\log D_\alpha)).$$

Proof. Both sides of the asserted congruence (3.5.0.3) are p-linear in τ, and exterior differentiation is linear over p-th powers, so it suffices to check the case in which τ is a *product* of the one-forms

(3.5.0.4) $$\begin{matrix} \dfrac{dx_i}{x_i}, & i \in \alpha \\ dx_i, & i \notin \alpha. \end{matrix}$$

We first introduce some auxilary notation:

(3.5.0.5) $$\tau(i) = \begin{cases} \dfrac{dx_i}{x_i}, & \text{if } i \in \alpha \\ dx_i, & \text{if } i \notin \alpha, \end{cases}$$

(3.5.0.6) $$\sigma(i) = \begin{cases} \dfrac{dx_i}{x_i}, & \text{if } i \in \alpha \\ x_i^{p-1}\, dx, & \text{if } i \notin \alpha, \end{cases}$$

(3.5.0.7) $$f(i) = \begin{cases} \dfrac{P(x_i)}{x_i}, & \text{if } i \in \alpha \\ x_i^{p-1}\, P(x_i) & \text{if } i \notin \alpha, \end{cases}$$

(3.5.0.8) $$g(i) = \begin{cases} \dfrac{P(x_i)}{x_i}, & \text{if } i \in \alpha \\ P(x_i), & \text{if } i \notin \alpha, \end{cases}$$

(3.5.0.9) $$h(i) = \begin{cases} \dfrac{P^p(x_i)}{x_i}, & \text{if } i \in \alpha \\ x_i^{p-1}\, P^p(x_i), & \text{if } i \notin \alpha. \end{cases}$$

(3.5.1) **Lemma.** *The following relations hold among the "quantities"* $\tau(i), \sigma(i), f(i), g(i), h(i)$

(3.5.1.1) $$\mathscr{F}(\tau(i)) = \sigma(i),$$

(3.5.1.2) $$\mathrm{Lie}(P)(\sigma(i)) = d f(i),$$

(3.5.1.3) $\qquad \operatorname{Lie}(Q)(\sigma(i))=0,$

(3.5.1.4) $\qquad I(P-Q)(\tau(i))=g(i),$

(3.5.1.5) $\qquad I(P-Q)(\sigma(i))=f(i),$

(3.5.1.6) $\qquad I(P^p-Q^p)(\sigma(i))=h(i).$

Proof. (3.5.1.1) is by definition (3.5.0.1) of $\mathscr{F}$. As for (3.5.0.2), we simply calculate

$$\text{if } i\in\alpha, \quad \operatorname{Lie}(P)(\sigma(i))=\operatorname{Lie}(P)\left(\frac{dx_i}{x_i}\right)=\frac{\operatorname{Lie}(P)(dx_i)}{x_i}-\frac{P(x_i)}{x_i}\frac{dx_i}{x_i}$$

(3.5.1.7)
$$=\frac{dP(x_i)}{x_i}-\frac{P(x_i)}{x_i}\frac{dx_i}{x_i}$$
$$=d\left(\frac{P(x_i)}{x_i}\right)=df(i).$$

(3.5.1.8) if $i\notin\alpha$, we multiply the above calculation (3.5.1.7) by x_i^p.

(3.5.1.3) holds because $Q(x_i)=0$ by definition (3.5.0) of Q. As for (3.5.1.4), we calculate

(3.5.1.9) if $i\in\alpha$, $I(P-Q)(\tau(i))=I(P-Q)\left(\frac{dx_i}{x_i}\right)=\frac{(P-Q)(x_i)}{x_i}=\frac{P(x_i)}{x_i}=g(i),$

(3.5.1.10) if $i\notin\alpha$, $I(P-Q)(\tau(i))=I(P-Q)(dx_i)=(P-Q)(x_i)=P(x_i)=g(i)$

and (3.5.1.5) follows similarly:

(3.5.1.11) $\qquad$ if $i\in\alpha$, $\quad I(P-Q)(\sigma(i))=I(P-Q)\left(\frac{dx_i}{x_i}\right)=\frac{P(x_i)}{x_i}=f(i),$

(3.5.1.12) $\qquad$ if $i\notin\alpha$, $\quad I(P-Q)(\sigma(i))=I(P-Q)(x_i^{p-1}dx_i)=x_i^{p-1}P(x_i)=f(i).$

The proof of (3.5.1.6) is identical to that of (3.5.1.5), depending only on the fact that $Q^p(x_i)=0$. Q.E.D.

(3.5.2) We now return to the verification of the congruence (3.5.0.3) for a product of the $\tau(i)$. To begin, we'll check the case $\tau=\tau(1)$. In this case $(b=1)$, the congruence (3.5.0.3) becomes an assertion of equality:

(3.5.2.0)
$$\sum_{k+\ell=p-1}\operatorname{Lie}(P)^k I(P-Q)\operatorname{Lie}(Q)^\ell(\mathscr{F}(\tau(1)))-I(P^p-Q^p)\mathscr{F}(\tau(1))$$
$$=-\mathscr{F}(I(P-Q)(\tau(1))).$$

Substituting via (3.5.1), (3.5.2.0) becomes

(3.5.2.1)
$$\sum_{k+\ell=p-1}\operatorname{Lie}(P)^k I(P-Q)\operatorname{Lie}(Q)^\ell(\sigma(1))-I(P^p-Q^p)(\sigma(1))=-\mathscr{F}(g(1))$$

and substituting via (3.5.1) and especially via (3.5.1.3), (3.5.2.1) becomes

$$\text{Lie}(P)^{p-1} I(P-Q)(\sigma(1)) - h(1) = -g(1)^p. \tag{3.5.2.2}$$

A final substitution via (3.5.1) gives the assertion

$$P^{p-1}(f(1)) - h(1) = -g(1)^p. \tag{3.5.2.3}$$

We now "decode" (3.5.2.3) by returning to the definitions (3.5.0.7–9) of f, h, g; (3.5.2.3) becomes the assertion

$$\text{if } 1 \in \alpha, \quad P^{p-1}\left(\frac{P(x_1)}{x_1}\right) - \frac{P^p(x_1)}{x_1} = -\left(\frac{P(x_1)}{x_1}\right)^p, \tag{3.5.2.4}$$

$$\text{if } 1 \notin \alpha, \quad P^{p-1}(x_1^{p-1} P(x_1)) - x_1^{p-1} P^p(x_1) = -(P(x_1))^p. \tag{3.5.2.5}$$

Both (3.5.2.4) and (3.5.2.5) are in fact true, and follow ((3.5.2.5) directly, (3.5.2.4) after dividing by x_1^p) from Hochschild's identity (cf. [20]), according to which, if P is *any* derivation of *any* commutative ring A of characteristic $p>0$, then for *any* element $x \in A$,

$$P^{p-1}(x^{p-1} P(x)) - x^{p-1} P^p(x) = -(P(x))^p. \tag{3.5.2.6}$$

This concludes the proof of (3.5.0.3) in case $b=1$. In the following, we will make use of it, through the identity (3.5.2.3).

(3.5.3) Because our proof of (3.5.0.3) in the general case is so unenlightening we first present the proof in the case $b=2$, which is somewhat more intelligible.

The assertion to be verified is

$$\begin{aligned} &\sum_{k+\ell=p-1} \text{Lie}(P)^k I(P-Q)\, \text{Lie}(Q)^\ell(\mathscr{F}(\tau(1) \wedge \tau(2))) \\ &\quad - I(P^p - Q^p)(\mathscr{F}(\tau(1) \wedge \tau(2))) \\ &\quad \equiv -\mathscr{F}(I(P-Q)(\tau(1) \wedge \tau(2))) \text{ modulo } d\Gamma(V, \mathcal{O}_v). \end{aligned} \tag{3.5.3.0}$$

Substituting via (3.5.1.1) and noting that by (3.5.1.3) the terms under the summation sign vanish for $\ell \neq 0$, (3.5.3.0) becomes

$$\begin{aligned} &\text{Lie}(P)^{p-1}(I(P-Q))(\sigma(1) \wedge \sigma(2)) - I(P^p - Q^p)(\sigma(1) \wedge \sigma(2)) \\ &\quad \equiv -\mathscr{F}(I(P-Q)(\tau(1) \wedge \tau(2))) \text{ modulo } d\Gamma(V, \mathcal{O}_v). \end{aligned} \tag{3.5.3.1}$$

Expanding the interior products and substituting via (3.5.1), (3.5.3.1) becomes

$$\begin{aligned} &\text{Lie}(P)^{p-1}(f(1)\,\sigma(2) - f(2)\,\sigma(1)) - (h(1)\,\sigma(2) - h(2)\,\sigma(1)) \\ &\quad \equiv -\mathscr{F}(g(1)\,\tau(2) - g(2)\,\tau(1)) \text{ modulo } d\Gamma(V, \mathcal{O}_v). \end{aligned} \tag{3.5.3.2}$$

A final substitution gives

$$\text{(3.5.3.3)} \quad \begin{aligned} &\operatorname{Lie}(P)^{p-1}(f(1)\,\sigma(2)-f(2)\,\sigma(1))-(h(1)\,\sigma(2)-h(2)\,\sigma(1)) \\ &\equiv -g(1)^p\,\sigma(2)+g(2)^p\,\sigma(1) \text{ modulo } d\Gamma(V,\mathcal{O}_v). \end{aligned}$$

Availing ourselves of (3.5.2.3), (3.5.3.3) becomes

$$\text{(3.5.3.4)} \quad \begin{aligned} \operatorname{Lie}(P)^{p-1}(f(1)\,\sigma(2)-f(2)\,\sigma(1)) &\equiv P^{p-1}(f(1))\,\sigma(2)-P^{p-1}(f(2))\,\sigma(1) \\ &\text{modulo } d\Gamma(V,\mathcal{O}_v). \end{aligned}$$

Now the right hand member of (3.5.3.4) is the "first term" in the expansion of the left hand member by Leibniz's rule, so that, expanding, (3.5.3.4) becomes

$$\text{(3.5.3.5)} \quad \begin{aligned} \sum_{k+\ell=p-1,\,\ell\neq 0} \binom{p-1}{k} &[P^k(f(1))\operatorname{Lie}(P)^\ell(\sigma(2)) \\ &-P^k(f(2))\operatorname{Lie}(P)^\ell(\sigma(1))]\in d\Gamma(V,\mathcal{O}_v). \end{aligned}$$

By (3.5.1.2), we may substitute

$$\text{(3.5.3.6)} \quad \begin{aligned} \operatorname{Lie}(P)^\ell(\sigma(2)) &= \operatorname{Lie}(P)^{\ell-1}\,df(2)=dP^{\ell-1}(f(2)) \\ \operatorname{Lie}(P)^\ell(\sigma(1)) &= \operatorname{Lie}(P)^{\ell-1}\,df(1)=dP^{\ell-1}(f(1)). \end{aligned}$$

Doing so, and remembering that

$$\text{(3.5.3.7)} \quad \binom{p-1}{k}\equiv(-1)^k \bmod p \quad \text{for } 0\leqq k\leqq p-1$$

we must show

$$\text{(3.5.3.8)} \quad \begin{aligned} \sum_{k+\ell=p-1,\,\ell\neq 0} (-1)^k &[P^k f(1)\,dP^{\ell-1} f(2) \\ &-P^k(f(2))\,dP^{\ell-1} f(1)]\in d\Gamma(V,\mathcal{O}_v) \end{aligned}$$

Re-indexing the summation by k and $m=\ell-1$, (3.5.3.8) becomes (remembering that $(-1)^{k+1}\equiv(-1)^m \bmod p$)

$$\text{(3.5.3.9)} \quad \begin{aligned} &\sum_{k+m=p-2} (-1)^k P^k f(1)\,dP^m f(2) \\ &+\sum_{k+m=p-2} (-1)^m P^k(f(2))\,dP^m f(1)\in d\Gamma(V,\mathcal{O}_v). \end{aligned}$$

This is the case; in fact, the left member of (3.5.3.9) is

$$\text{(3.5.3.10)} \quad d\Big(\sum_{k+m=p-2} (-1)^k P^k(f(1))\,P^m(f(2))\Big).$$

This proves (3.5.0.3) in case $b=2$, and gives a hint of the combinatorial rearrangements necessary in the general case.

(3.5.4) We now turn to the general case. We adopt the convention that a product indexed by a subset of $\mathbf{Z}$ is to be taken in increasing order, and that, unless otherwise specified, all indexing variables run over the set $\{1, \dots, b\}$.

We must verify the congruence (3.5.0.3) in the case

(3.5.4.0) $$\tau = \tau(1) \wedge \tau(2) \wedge \cdots \wedge \tau(b) = \prod_{\nu} \tau(\nu).$$

Using the substitutions (3.5.1) and the identity (3.5.2.3) just as in the case $b=2$, we see that the congruence in question is equivalent to the congruence

(3.5.4.1) $$\begin{aligned} &\operatorname{Lie}(P)^{p-1}\Big(\sum_i (-1)^{i+1} f(i) \prod_{\nu \neq i} \sigma(\nu)\Big) \\ &\equiv \sum_i (-1)^{i+1} P^{p-1}\big(f(i)\big) \prod_{\nu \neq i} \sigma(\nu) \text{ modulo } d\Gamma\big(V, \Omega^{\bullet}_{V/S}(\log D_\alpha)\big). \end{aligned}$$

Expanding the left member of (3.5.4.1) bei Leibniz's rule, and remembering (3.5.3.7), (3.5.4.1) becomes

(3.5.4.2) $$\begin{aligned} &\sum_i (-1)^{i+1} \sum_{k=1}^{p-1} (-1)^k P^{p-1-k}\big(f(i)\big) \\ &\quad \cdot \operatorname{Lie}(P)^k\Big(\prod_{\nu \neq i} \sigma(\nu)\Big) \in d\Gamma\big(V, \Omega^{\bullet}_{V/S}(\log D_\alpha)\big). \end{aligned}$$

Our next task is to expand each of the terms

(3.5.4.3) $$\operatorname{Lie}(P)^k\Big(\prod_{\nu \neq i} \sigma(\nu)\Big)$$

using Leibniz's rule. To facilitate this, we introduce the notations

(3.5.4.4) $\ell = (\ell_1, \dots, \ell_b)$, a b-tuple of non-negative integers

(3.5.4.5) $$|\ell| = \sum_i \ell_i$$

(3.5.4.6) $$\binom{|\ell|}{\ell} = \frac{(|\ell|)!}{\prod_i (\ell_i)!} = \frac{(\sum \ell_i)!}{\prod (\ell_i)!}$$

(3.5.4.7) $$S(\ell) = \{i \mid \ell_i \neq 0\}$$

(3.5.4.8) for any nonempty subset $\Delta \subset \{1, \dots, b\} - \{i\}$ we denote by $\operatorname{sgn}(\Delta, \mathrm{i})$ the *sign* of the permutation

$$\begin{pmatrix} \Delta, \{1, \dots, b\} - \{i\} - \Delta \\ \{1, \dots, b\} - \{i\} \end{pmatrix}.$$

(We make the convention that subsets of $\mathbf{Z}$ are to be enumerated in increasing order.)

Returning to the expansion of (3.5.4.3), we find

$$\begin{aligned}
\operatorname{Lie}(P)^k\Big(\prod_{v\neq i}\sigma(v)\Big) &= \sum_{\ell;\,|\ell|=k,\,i\notin S(\ell)} \binom{|\ell|}{\ell} \prod_{v\neq i} \operatorname{Lie}(P)^{\ell_v}(\sigma(v)) \\
(3.5.4.9)\qquad &= \sum_{\emptyset\neq\Delta\subset\{1,\dots,b\}-\{i\}} \operatorname{sgn}(\Delta,i) \sum_{\ell;\,|\ell|=k,\,S(\ell)=\Delta} \binom{|\ell|}{\ell} \\
&\quad\cdot \prod_{v\in\Delta} \operatorname{Lie}(P)^{\ell_v}(\sigma(v)) \prod_{v\notin\Delta\cup\{i\}} \sigma(v).
\end{aligned}$$

Substituting (3.5.4.9) into (3.5.4.2), we are faced with proving that $d\Gamma(V, \Omega^{\bullet}_{V/S}(\log D_\alpha))$ contains

$$\begin{aligned}
(3.5.4.10)\qquad &\sum_i (-1)^{i+1} \sum_{k=1}^{p-1} (-1)^k P^{p-1-k}(f(i)) \sum_{\emptyset\neq\Delta\not\ni i} \operatorname{sgn}(\Delta,i) \sum_{\ell;\,|\ell|=k,\,S(\ell)=\Delta} \binom{|\ell|}{\ell} \\
&\quad\cdot \prod_{v\in\Delta} \operatorname{Lie}(P)^{\ell_v}(\sigma(v)) \prod_{v\notin\Delta\cup\{i\}} \sigma(v).
\end{aligned}$$

The key point now is to reindex the expression (3.5.4.10) by the subsets $\Delta'=\Delta\cup\{i\}$ *having at least two elements*; then (3.5.4.10) becomes

$$\begin{aligned}
&\sum_{\Delta'} \sum_{i\in\Delta'} (-1)^{i+1} \operatorname{sgn}(\Delta'-i,i) \sum_{k=1}^{p-1} (-1)^k P^{p-1-k}(f(i)) \\
(3.5.4.11)\qquad &\cdot \sum_{\ell;\,|\ell|=k,\,S(\ell)=\Delta'-\{i\}} \binom{|\ell|}{\ell} \prod_{v\in\Delta'-\{i\}} \operatorname{Lie}(P)^{\ell_v}(\sigma(v)) \prod_{v\notin\Delta'} \sigma(v).
\end{aligned}$$

We now calculate the $\operatorname{sign}(-1)^{1+1}\operatorname{sgn}(\Delta'-i,i)$ as a function of the position of i in Δ' and of the position of Δ' in $\{1,\dots,b\}$.

$$\begin{aligned}
(-1)^{i+1}\operatorname{sgn}(\Delta'-i,i) &= \operatorname{sgn}\begin{pmatrix} i,\{1,\dots,b\}-\{i\} \\ 1,\dots,b \end{pmatrix} \operatorname{sgn}\begin{pmatrix} \Delta'-i,\{1,\dots,b\}-\Delta' \\ \{1,\dots,b\}-\{i\} \end{pmatrix} \\
&= \operatorname{sgn}\begin{pmatrix} i,\{1,\dots,b\}-\{i\} \\ 1,\dots,b \end{pmatrix} \operatorname{sgn}\begin{pmatrix} i,\Delta'-i,\{1,\dots,b\}-\Delta' \\ i,\{1,\dots,b\}-i \end{pmatrix} \\
(3.5.4.12)\qquad &= \operatorname{sgn}\begin{pmatrix} i,\Delta'-\{i\},\{1,\dots,b\}-\Delta' \\ 1,\dots,b \end{pmatrix} \\
&= \operatorname{sgn}\begin{pmatrix} i,\Delta'-\{i\},\{1,\dots,b\}-\Delta' \\ \Delta',\{1,\dots,b\}-\Delta' \end{pmatrix} \operatorname{sgn}\begin{pmatrix} \Delta',\{1,\dots,b\}-\Delta' \\ 1,\dots,b \end{pmatrix} \\
&= \operatorname{sgn}\begin{pmatrix} i,\Delta'-\{i\} \\ \Delta' \end{pmatrix} \operatorname{sgn}\begin{pmatrix} \Delta',\{1,\dots,\mathrm{b}\}-\Delta' \\ 1,\dots,b \end{pmatrix}.
\end{aligned}$$

Because each $\sigma(v)$ is a closed form, in order to show that (3.5.4.11) lies in $d\Gamma(V, \Omega^{\bullet}_{V/S}(\log D_\alpha))$, it suffices to show that, for each subset $\Delta'\subset\{1,\dots,b\}$

of two or more elements, $d\Gamma(V, \Omega^{\bullet}_{V/S}(\log D_\alpha))$ contains

$$\sum_{i\in\Delta'} \operatorname{sgn}\begin{pmatrix} i, \Delta'-\{i\} \\ \Delta' \end{pmatrix} \sum_{k=1}^{p-1} (-1)^k P^{p-1-k}(f(i)) \sum_{\ell;\, |\ell|=k,\, S(\ell)=\Delta'-\{i\}} \binom{|\ell|}{\ell} \cdot \prod_{\nu\in\Delta'-\{i\}} \operatorname{Lie}(P)^{\ell_\nu}(\sigma(\nu)). \tag{3.5.4.13}$$

With no loss in generality, we may suppose $\Delta'=\{1,2,\ldots,b\}$, where $b=\#(\Delta')$; then

$$\operatorname{sgn}\begin{pmatrix} i, \Delta'-\{i\} \\ \Delta' \end{pmatrix} = (-1)^{i+1}. \tag{3.5.4.14}$$

Let us introduce a final notation:

(3.5.4.15) $m=(m_1,\ldots,m_b)$, a b-triple of integers satisfying $m_i \geqq 1$, and $\sum m_i = p$.

A multi-index ℓ occurring in (3.5.4.13) with $|\ell|=k$ and $S(\ell)=\Delta'-\{i\}$ gives rise to such an m by putting

$$m_\nu = \begin{cases} \ell_\nu & \text{if } \nu \neq i \\ p-k & \text{if } \nu = i. \end{cases} \tag{3.5.4.16}$$

In terms of m, the multinomial coefficient $(-1)^k \binom{|\ell|}{\ell}$ is easily calculated:

$$\begin{aligned} (-1)^k \binom{|\ell|}{\ell} &\equiv \frac{(p-1)!}{k!\,(p-1-k)!} \cdot \frac{k!}{\ell_1!\ldots\ell_b!} = \frac{(p-1)!\,(p-k)}{m_1!\ldots m_b!} \\ &\equiv \frac{-1}{\prod_\nu (m_\nu)!}\, m_i \text{ modulo } p. \end{aligned} \tag{3.5.4.17}$$

Using (3.5.4.14) and (3.5.4.17), the congruence (3.5.4.13) may be rewritten

$$\sum_m \frac{-1}{\prod_\nu (m_\nu)!} \sum_i (-1)^{i+1} m_i P^{m_i-1}(f(i)) \cdot \prod_{\nu\neq i} \operatorname{Lie}(P)^{m_\nu}(\sigma(\nu)) \in d\Gamma(V, \Omega^{\bullet}_{V/S}(\log D_\alpha)). \tag{3.5.4.18}$$

The proof of (3.5.4.18), and hence of (3.5.0), will be concluded the following calculation.

(3.5.5) *Calculation.* For any m as in (3.5.4.13),

$$\begin{aligned} &\sum_i (-1)^{i+1} m_i P^{m_i-1}(f(i)) \prod_{\nu\neq i} \operatorname{Lie}(P)^{m_\nu}(\sigma(\nu)) \\ &= d\Big(P^{m_1-1}(f(1)) \sum_{i\neq 1} (-1)^{i+1} m_i P^{m_i-1}(f(i)) \prod_{\nu\neq 1, i} \operatorname{Lie}(P)^{m_\nu}(\sigma(\nu))\Big). \end{aligned} \tag{3.5.5.0}$$

Proof. According to (3.5.1.2), we have

(3.5.5.1) $$\mathrm{Lie}(P)(\sigma(v)) = df(v)$$

whence for any integer $m_v \geqq 1$, we have

(3.5.5.2) $$\mathrm{Lie}(P)^{m_v}(\sigma(v)) = \mathrm{Lie}(P)^{m_v-1}(df(v)) = d\left(P^{m_v-1}(f(v))\right).$$

Let us expand the second member of (3.5.5.0), remembering that the $\sigma(v)$, and hence the $\mathrm{Lie}(P)^{m_v}(\sigma(v))$, are *closed* forms.

(3.5.5.3) $$\begin{aligned} &d\Big(P^{m_1-1}(f(1))\sum_{i\neq 1}(-1)^{i+1} m_i P^{m_i-1}(f(i)) \prod_{v\neq 1,i} \mathrm{Lie}(P)^{m_v}(\sigma(v))\Big)\\ &= \mathrm{Lie}(P)^{m_1}(\sigma(1))\sum_{i\neq 1}(-1)^{i+1} m_i P^{m_i-1}(f(i)) \prod_{v\neq 1,i} \mathrm{Lie}(P)^{m_v}(\sigma(v))\\ &\quad + P^{m_1-1}(f(1))\sum_{i\neq 1}(-1)^{i+1} m_i \mathrm{Lie}(P)^{m_i}(\sigma(i)) \prod_{v\neq 1,i} \mathrm{Lie}(P)^{m_v}(\sigma(v)). \end{aligned}$$

The first term in (3.5.5.4) is the part of the first member of (3.5.5.0) corresponding to $i \neq 1$. Thus it remains to see that

(3.5.5.4) $$\begin{aligned} &m_1 P^{m_1-1}(f(1)) \prod_{v\neq 1} \mathrm{Lie}(P)^{m_v}(\sigma(v))\\ &= P^{m_1-1}(f(1))\sum_{i\neq 1}(-1)^{i+1} m_i \mathrm{Lie}(P)^{m_i}(\sigma(i)) \prod_{v\neq 1,i} \mathrm{Lie}(P)^{m_v}(\sigma(v)). \end{aligned}$$

Rearrainging the final product in the second member, (3.5.5.5) may be rewritten

(3.5.5.5) $$\begin{aligned} &m_1 P^{m_1-1}(f(1)) \prod_{v\neq 1} \mathrm{Lie}(P)^{m_v}(\sigma(v))\\ &= -P^{m_1-1}(f(1))\sum_{i\neq 1} m_i \prod_{v\neq 1} \mathrm{Lie}(P)^{m_v}(\sigma(v)). \end{aligned}$$

Because $\sum m_i = p$, $\sum_i m_i \equiv 0$ (p), and (3.5.5.6) is true. This concludes the proof of (3.5.5), hence of (3.5.0), and hence of Theorem 3.2 and its Corollaries 3.3 and (3.3.1). Q.E.D.

4. Review of Deligne's Mixed Hodge Structures ([8])

4.0. The Weight Filtration

(4.0.1) We return to the geometric situation (1.0). The *weight filtration* of the complex $\Omega^\bullet_{X/S}(\log D)$ is by definition the finite increasing filtration defined by

(4.0.1.0) $$W_i(\Omega^\bullet_{X/S}(\log D)) = \text{image of } \Omega^i_{X/S}(\log D) \otimes \Omega^{\bullet - i}_{X/S} \to \Omega^\bullet_{X/S}(\log D).$$

Notice that

(4.0.1.1) $$W_i \wedge W_j \subset W_{i+j}.$$

The "Poincaré residue" (cf. [7, 8]) furnishes isomorphisms (which are „canonical" only after choosing an ordering of the finite set which indexes the smooth divisors D_i)

$$(4.0.1.2)\qquad \mathrm{gr}_n^W(\Omega^\bullet_{X/S}(\log D)) = \begin{cases} 0 & \text{if } n<0 \\ \bigoplus_{i_1<\cdots<i_n} \Omega^{\bullet-n}_{D_{i_1}\cap\cdots\cap D_{i_n}/S} & \text{if } n\geqq 1 \\ \Omega^\bullet_{X/S} & \text{if } n=0. \end{cases}$$

[We are guilty of an abuse of notation in (4.0.1.2) above, because, strictly speaking, gr_n^W is isomorphic to the direct sum of the *extensions by zero* from $D_{i_1}\cap\cdots\cap D_{i_n}$ to X of the complexes $\Omega^{\bullet-n}_{D_{i_1}\cap\cdots\cap D_{i_n}/S}$]. To avoid either further abuses or typographical disaster, we will henceforth adopt the following notation: eat $f(i_1, \ldots, i_n)$ denote the composite

$$(4.0.1.3)\qquad \begin{tikzcd} D_{i_1}\cap\cdots\cap D_{i_n} \arrow[r, hook] \arrow[drr, "f(i_1,\ldots,i_n)"'] & D \arrow[r, hook] & X \arrow[d, "f"] \\ & & S \end{tikzcd}$$

and let us abbreviate, for each integer $k \geqq 0$,

$$(4.0.1.4)\qquad H^k_{DR}(D_{i_1}\cap\cdots\cap D_{i_n}/S) \overset{\text{def'n}}{=} \mathbf{R}^k f(i_1, \ldots, i_n)_*(\Omega^\bullet_{D_{i_1}\cap\cdots\cap D_{i_n}/S}),$$

which is a quasicoherent sheaf of $\mathcal{O}_S$-modules.

The spectral sequence of the filtration W and the functor $\mathbf{R}f_*$ may be written, via (4.0.1.2), as

$$(4.0.1.5)\qquad {}_W E_1^{-n,k+n} = \bigoplus_{i_1<\cdots<i_n} H^{k-n}_{DR}(D_{i_1}\cap\cdots\cap D_{i_n}/S) \Rightarrow \mathbf{R}^k f_*(\Omega^\bullet_{X/S}(\log D)).$$

(4.0.2) **Proposition.** *The spectral sequence* (4.0.1.5) *is (naturally) a spectral sequence with cup-product in the category of quasi-coherent $\mathcal{O}_S$-modules with integrable connection relative to T. This connection is the Gauss-Manin connection on the E_1 terms and on the abutment. In particular, the W filtration on the $\mathbf{R}^k f_*(\Omega^\bullet_{X/S}(\log D))$ is horizontal for the Gauss-Manin connection.*

Proof. The product structure in the spectral sequence results from (4.0.1.1). The action of the Gauss-Manin connection on the spectral sequence may be seen directly as follows. The question being local on S, we may suppose S affine. Choose an affine open covering of X by sufficiently small coordinatized open sets V_ν as in (3.4.1).

The filtration W of $\Omega^\bullet_{X/S}(\log D)$ induces a filtration W on the Cech bicomplex of quasi-coherent $\mathcal{O}_S$-modules

$$(4.0.2.0)\qquad W_i C^\bullet(\{V_\nu\}, \Omega^\bullet_{X/S}(\log D)) \overset{\text{def'n}}{=} C^\bullet(\{V_\nu\}, W_i\Omega^\bullet_{X/S}(\log D))$$

whose assosciated graded is given by

$$\text{(4.0.2.1)}\quad \begin{aligned} &\mathrm{gr}_n^W\, C^\bullet(\{V_\nu\}, \Omega^\bullet_{X/S}(\log D)) \\ &\qquad = \bigoplus_{i_1<\cdots<i_n} C^\bullet(\{V_\nu \cap D_{i_1} \cap \cdots \cap D_{i_n}\}, \Omega^{\bullet-n}_{D_{i_1}\cap\cdots\cap D_{i_n}/S}). \end{aligned}$$

The spectral sequence (4.0.1.3) is the spectral sequence of the filtered complex of quasi-coherent $\mathcal{O}_S$-modules obtained from the filtered bicomplex (4.0.2.0) by totalization. The (not necessarily integrable) T-connection ∇ on the Cech bicomplex $C^\bullet(\{V_\nu\}, \Omega^\bullet_{X/S}(\log D))$ constructed in (3.4.1) preserves the filtration W. This provides a T-connection on the spectral sequence (4.0.1.5), which is that of Gauss-Manin on the abutment. To see that it is that of Gauss-Manin on the E_1 term, we simply note that on the assosciated graded for W,

$$\text{(4.0.2.2)}\qquad C^\bullet(\{V_\nu \cap D_{i_1} \cap \cdots \cap D_{i_n}\}, \Omega^\bullet_{D_{i_1}\cap\cdots\cap D_{i_n}/S})$$

it induces the same T-connection which the method of (3.4.1) would construct, viewing the affine open sets $V_\nu \cap D_{i_1} \cap \cdots \cap D_{i_n}$ as a coordinatized open covering of the smooth S-scheme $D_{i_1} \cap \cdots \cap D_{i_n}$ (the coordinates being those on V_ν whose restriction to $D_{i_1} \cap \cdots \cap D_{i_n}$ do not vanish).

Of course, the truth of (4.0.2) could also be perceived by "pure thought", by contemplating the definition of the Gauss-Manin connection in terms of the Koszul filtration (cf. (1.4)). We leave it to the reader. Q.E.D.

(4.0.3) **Corollary.** *Suppose also that T is (the spectrum of) a field of characteristic zero, and that $f: X \to S$ is proper. Then for every integer $r \geqq 1$, the E_r terms of* (4.0.1.6) *are locally free $\mathcal{O}_S$-modules of finite rank, and their formation commutes with arbitrary change of base $S' \to S$. Furthermore* (4.0.1.6) *is degenerate at E_2 (and hence by* (2.2.1.11) *remains so after an arbitrary base change).*

Proof. Because $f: X \to S$ is proper, (4.0.1.6) is a spectral sequence of coherent $\mathcal{O}_S$-modules with integrable connection relative to T. Because T is the spectrum of a field of characteristic zero, any coherent $\mathcal{O}_S$-module with integrable connection is a locally free $\mathcal{O}_S$-module of finite rank. Thus in particular the E_r terms of (4.0.1.6) are locally free for each $r \leqq 1$. To prove that their formation commutes with arbitrary base change, we first remark that it's true for $r=1$, by the usual base-changing theorems. Suppose inductively that it's true for all $r \leqq r_0$. We may universally identify $E^{p,q}_{r_0+1}$ with the first cohomology of the complex of locally free $\mathcal{O}_S$-modules of finite rank with integrable T-connection

$$\text{(4.0.3.0)}\qquad E_{r_0}^{p-r_0,q+r_0-1} \xrightarrow{d_{r_0}} E_{r_0}^{p,q} \xrightarrow{d_{r_0}} E_{r_0}^{p+r_0,q+1-r_0}.$$

Because the d_{r_0} are horizontal, all the cohomology groups of this complex are coherent $\mathcal{O}_S$-modules with integrable T-connection, hence are locally free, hence their formation commutes with arbitrary change of base.

To prove degeneration at E_2, it suffices, by the above, to prove it after an arbitrary change of base $S' \to S$ where S' is the spectrum of a field (because S is *reduced*). But when S is the spectrum of a field of characteristic zero, the degeneration is proved by Deligne in [8]. Q.E.D.

4.1. Topological Interpretation

Suppose, in addition to the hypotheses of (4.0.3), that $T = \mathrm{Spec}(\mathbf{C})$. By the regularity theorem (cf. [7, 11, 24]), the Gauss-Manin connection on the terms of the spectral sequence (4.0.1.6) has "regular singular points". By the fundamental comparison theorem of [7], the functor "germs of analytic local horizontal sections" from the category of coherent sheaves on S with integrable connections relative to $\mathbf{C}$, with regular singular points, to the category of local coefficient systems of finite-dimensional complex vector spaces on the "underlying" analytic space S^{an}

$$(4.1.0) \qquad (M, \nabla) \rightsquigarrow (M \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_{S^{\mathrm{an}}})^{\nabla^{\mathrm{an}}},$$

is an equivalence of categories. This means that we "know" the spectral sequence (4.0.1.6) once we know the assosciated spectral sequence of local coefficient systems on S^{an} to which it gives rise via the functor 4.1.0.

(4.1.1) **Proposition.** *The spectral sequence in local systems on S^{an} corresponding to* (4.0.1.6) *is the décalage* (*cf.* (2.2.2.4)) *of the Leray spectral sequence (relative to the base S^{an}) of the open immersion of S^{an}-schemes*

$$(4.1.1.0) \qquad \begin{array}{ccc} U^{\mathrm{an}} = X^{\mathrm{an}} - D^{\mathrm{an}} & \xrightarrow{j^{\mathrm{an}}} & X^{\mathrm{an}} \\ & \searrow^{\pi^{\mathrm{an}}} & \downarrow{\scriptstyle f^{\mathrm{an}}} \\ & & S^{\mathrm{an}} \end{array}$$

in complex cohomology:

$$(4.1.1.1) \qquad E_2^{p,q} = R^p f_*^{\mathrm{an}}(R^q j_*^{\mathrm{an}}(\mathbf{C})) \Rightarrow R^{p+q}\pi_*^{\mathrm{an}}(\mathbf{C})$$

(which consequently degenerates at E_3).

Proof. We will first reduce the problem to an analytic one. Notice that the entirity of Section 4.0 may be repeated word for word in the category of analytic spaces, by considering the filtration W on

$$(\Omega^\bullet_{X/S}(\log D))^{\mathrm{an}}.$$

The corresponding spectral sequence in locally free $\mathcal{O}_{S^{\mathrm{an}}}$-modules of finite rank

(4.1.1.2)
$$\begin{aligned} {}_W E_{1\,(\mathrm{an})}^{-n,k+n} &= \bigoplus_{i_1<\cdots<i_n} H_{DR}^{k-n}(D_{i_1}^{\mathrm{an}} \cap \cdots \cap D_{i_n}^{\mathrm{an}}/S^{\mathrm{an}}) \\ &\Rightarrow \mathbf{R}^k f_*^{\mathrm{an}}((\Omega_{X/S}^{\bullet}(\log D))^{\mathrm{an}}) \end{aligned}$$

is of formation compatible with arbitrary change of base $S' \to S^{\mathrm{an}}$ in the category of analytic spaces, thanks to (4.0.3).

The canonical morphism of spectral sequences

(4.1.1.3)
$$(4.0.1.6) \to (4.1.1.2)$$

induces horizontal (for the Gauss-Manin connections (4.0.2)) morphisms

(4.1.1.4)
$$E_r \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_{S^{\mathrm{an}}} \to E_r(an).$$

In fact, the morphisms (4.1.1.4) are all *isomorphisms*, because source and target are locally free $\mathcal{O}_{S^{\mathrm{an}}}$-modules of finite rank of formation compatible with arbitrary change of base, and because (by GAGA) the morphism (4.1.1.3) of spectral sequences is an isomorphism when $S = \mathrm{Spec}(\mathbf{C})$.

Thus the analytic spectral sequence (4.1.1.2) gives rise, via the functor "germs of local horizontal sections" (which is an equivalence of categories between coherent sheaves on $\mathcal{O}_{S^{\mathrm{an}}}$ with integrable connection and local systems of finite-dimensional $\mathbf{C}$-spaces on S^{an}) to the *same* spectral sequence in local systems on S^{an} as does (4.0.1). It will be convenient to think of this spectral sequence in local systems as a sub-spectral sequence of (4.1.1.2).

(4.1.1.5) Consider the "absolute" log complex $(\Omega_{X/\mathbf{C}}^{\bullet}(\log D))^{\mathrm{an}}$, together with its weight filtration W, defined as in (4.0.1.0). The spectral sequence of the filtration W and the functors $\mathbf{R} f_*^{\mathrm{an}}$,

(4.1.1.6)
$$\begin{aligned} {}_W E_1^{-n,k+n}(\mathrm{abs}) &= \bigotimes_{i_1<\cdots<i_n} R^{k-n} f(i_1, \ldots, i_n)_*^{\mathrm{an}}(\mathbf{C}) \\ &\Rightarrow \mathbf{R}^k f_*^{\mathrm{an}}((\Omega_{X/\mathbf{C}}(\log D))^{\mathrm{an}}) \end{aligned}$$

is a spectral sequence of sheaves of $\mathbf{C}$-vector spaces on S^{an}. [We have used Poincaré's lemma on the $D_{i_1} \cap \cdots \cap D_{i_n}$ to identify the E_1 term.]

The natural mapping of absolute to relative log complexes

(4.1.1.7)
$$(\Omega_{X/\mathbf{C}}^{\bullet}(\log D))^{\mathrm{an}} \to (\Omega_{X/S}^{\bullet}(\log D))^{\mathrm{an}}$$

is compatible with the W-filtration, and thus induces a morphism of spectral sequences

(4.1.1.8)
$$(4.1.1.6) \to (4.1.1.2).$$

(4.1.1.9) **Lemma.** *The morphism* (4.1.1.8) *of spectral sequences induces an isomorphism between* (4.1.1.6) *and the sub-spectral sequence in local systems of* (4.1.1.2) *obtained by taking germs of local horizontal sections.*

Proof. Let's denote by $E_r(\mathrm{an})$ the terms of (4.1.1.2) by $E_r(\mathrm{an})^{\nabla}$ their sheaves of germs of horizontal sections, and by $E_r(\mathrm{abs})$ the terms of (4.1.1.6). It follows from the *definition* of the Gauss-Manin connection in terms of the Koszul filtration (cf. (1.4)) that the canonical morphisms $E_r(\mathrm{abs}) \to E_r(\mathrm{an})$ factor through $E_r(\mathrm{an})^{\nabla}$. It remains only to prove inductively that the mappings

$$E_r(\mathrm{abs}) \to E_r(\mathrm{an})^{\nabla}$$

are isomorphisms.

For $r=1$, this is proven in [6]. Below we will indicate another conceptual proof (cf. (4.1.2)). Suppose the result for all $r \leqq r_0$. Because the functor "germs of horizontal sections" is *exact*, $E_{r_0+1}(\mathrm{an})^{\nabla}$ is the first cohomology sheaf of the complex

$$(4.1.1.10) \qquad E_{r_0}(\mathrm{an})^{\nabla} \xrightarrow{d_{r_0}(\mathrm{an})} E_{r_0}(\mathrm{an})^{\nabla} \xrightarrow{d_{r_0}(\mathrm{an})} E_{r_0}(\mathrm{an})^{\nabla}.$$

By induction, this complex receives isomorphically the complex

$$(4.1.1.11) \qquad E_{r_0}(\mathrm{abs}) \xrightarrow{d_{r_0}(\mathrm{abs})} E_{r_0}(\mathrm{abs}) \xrightarrow{d_{r_0}(\mathrm{abs})} E_{r_0}(\mathrm{abs}),$$

whose first cohomology sheaf is $E_{r_0+1}(\mathrm{abs})$. This implies that

$$E_{r_0+1}(\mathrm{abs}) \xrightarrow{\sim} E_{r_0+1}(\mathrm{an})^{\nabla},$$

and proves the lemma.

We may now conclude the proof of (4.1.1) by noting the filtered quasi-isomorphisms (cf. (2.2.2.1) and [8], (3.1.7.1) and Prop. (3.1.8))

$$(4.1.1.12) \qquad \begin{array}{ccc} ((\Omega^{\bullet}_{X/\mathbf{C}}(\log D))^{\mathrm{an}}, W) & \longleftarrow & ((\Omega^{\bullet}_{X/\mathbf{C}}(\log D))^{\mathrm{an}}, \tau_{\leqq} .) \\ & & \downarrow \\ & & (j^{\mathrm{an}}_{*}(\Omega^{\bullet \mathrm{an}}_{X/\mathbf{C}}), \tau_{\leqq} .). \end{array} \quad \text{Q.E.D.}$$

In the course of the above, we made use of the following fact, applied to X^{an} and the intersections $D^{\mathrm{an}}_{i_1} \cap \cdots \cap D^{\mathrm{an}}_{i_n}$ over S^{an}, in order to prove that (4.1.1.9) was an isomorphism for $r=1$.

(4.1.2) **Proposition.** *Let* $f: \mathscr{X} \to \mathscr{S}$ *be a proper and smooth morphism of complex manifolds. The canonical morphism of sheaves on* $\mathscr{S}$

$$(4.1.2.0) \qquad R^q f_*(\mathbf{C}) \xrightarrow{\sim} \mathbf{R}^q f_*(\Omega^{\bullet}_{\mathscr{X}/\mathbf{C}}) \longrightarrow \mathbf{R}^q f_*(\Omega_{\mathscr{X}/\mathscr{S}})$$

is an isomorphism between its source and the subsheaf $\mathbf{R}^q f_*(\Omega^\bullet_{\mathscr{X}/\mathscr{S}})^\nabla$ *of germs of horizontal sections (for the Gauss-Manin connection) of* $\mathbf{R}^q f_*(\Omega^\bullet_{\mathscr{X}/\mathscr{S}})$.

Proof. The proof is an exercise in the *definition* of the Gauss-Manin connection (cf. (1.4) and [35]). We consider the Koszul filtration K (cf. (1.4)) of the complex $\Omega^\bullet_{\mathscr{X}/\mathbf{C}}$, and the corresponding spectral sequence for the functors $\mathbf{R}f_*$

(4.1.2.1) $$E_1^{p,q} = \mathbf{R}^{p+q} f_*(\mathrm{gr}_K^p\, \Omega^\bullet_{\mathscr{X}/\mathbf{C}}) \Rightarrow \mathbf{R}^{p+q} f_*(\Omega^\bullet_{\mathscr{X}/\mathbf{C}}).$$

By (the analytic version of) (1.4.0.2), we have

(4.1.2.2) $$E_1^{p,q} = \Omega^p_{\mathscr{S}/\mathbf{C}} \underset{\mathcal{O}_{\mathscr{S}}}{\otimes} \mathbf{R}^q f_*(\Omega^\bullet_{\mathscr{X}/\mathscr{S}}).$$

By *definition* of the Gauss-Manin connection (cf. [35]), the differential

(4.1.2.3) $$d_1^{p,q}\colon\ E_1^{p,q} \to E_1^{p+1,q}$$

is the mapping

(4.1.2.4) $$d\otimes 1 + (-1)^p\, 1\otimes \nabla$$

deduced from the Gauss-Manin connection $\nabla = d_1^{0,q}$.

Because the $\mathbf{R}^q f_*(\Omega^\bullet_{\mathscr{X}/\mathscr{S}})$ are coherent sheaves on the complex manifold $\mathscr{S}$, they are locally free of finite rank, and the canonical mapping

(4.1.2.5) $$\mathcal{O}_{\mathscr{S}} \otimes_{\mathbf{C}} \mathbf{R}^q f_*(\Omega^\bullet_{\mathscr{X}/\mathscr{S}})^\nabla \to \mathbf{R}^q f_*(\Omega^\bullet_{\mathscr{X}/\mathscr{S}})$$

is an isomorphism. Thus we have an isomorphism

(4.1.2.6) $$E_1^{p,q} \simeq \Omega^p_{\mathscr{S}/\mathbf{C}} \otimes_{\mathbf{C}} \mathbf{R}^q f_*(\Omega^\bullet_{\mathscr{X}/\mathscr{S}})^\nabla$$

in terms of which

(4.1.2.7) $$d_1^{p,q} = d\otimes 1.$$

As $\mathbf{R}^q f_*(\Omega^\bullet_{\mathscr{X}/\mathscr{S}})^\nabla$ is a sheaf of $\mathbf{C}$-spaces, it is automatically *flat* over $\mathbf{C}$, and hence we have

(4.1.2.8) $$E_2^{p,q} \simeq \mathscr{H}^p(\Omega^\bullet_{\mathscr{S}/\mathbf{C}}) \oplus_{\mathbf{C}} \mathbf{R}^q f_*(\Omega^\bullet_{\mathscr{X}/\mathscr{S}})^\nabla.$$

By the Poincaré lemma,

(4.1.2.9) $$\mathscr{H}^p(\Omega^\bullet_{\mathscr{S}/\mathbf{C}}) = \begin{cases} \mathbf{C} & \text{if } p=0 \\ 0 & \text{if } p\neq 0. \end{cases}$$

Thus we have

(4.1.2.10) $$E_2^{p,q} = \begin{cases} \mathbf{R}^q f_*(\Omega^\bullet_{\mathscr{X}/\mathscr{S}})^\nabla & \text{if } p=0 \\ 0 & \text{if } p\neq 0. \end{cases}$$

Consequently, the spectral sequence (4.1.2.1) is degenerate at E_2, and the canonical mappings

$$\mathbf{R}^q f_*(\Omega^\bullet_{\mathscr{X}/\mathbf{C}}) \to E_\infty^{0,q} \to E_2^{0,q} \tag{4.1.2.11}$$

are isomorphisms. Q.E.D.

4.2. Families of Mixed Hodge Structures

(4.2.0) We recall that a pure Hodge structure over $\mathbf{Z}$ of weight n, H, is by definition, a $\mathbf{Z}$-module of finite type $H_\mathbf{Z}$, together with a (Hodge) filtration F of the complex vector space $H_\mathbf{C} \overset{\text{def'n}}{=} H_\mathbf{Z} \otimes_\mathbf{Z} \mathbf{C}$, which satisfies the condition

$$H_\mathbf{C} = \bigotimes_{\substack{p+q=n \\ p,q \in \mathbf{Z}}} F^p \cap \bar{F}^q(H_\mathbf{C}) \tag{4.2.0.0}$$

($\bar{F}^q$ denotes the complex conjugate of F^q. Complex conjugation, and indeed all of $\mathrm{Aut}(\mathbf{C})$, acts on $H_\mathbf{C} = H_\mathbf{Z} \otimes_\mathbf{Z} \mathbf{C}$ through the second factor!). For all pairs (p, q) of integers with $p+q=n$, we put

$$H^{p,q} = F^p \cap \bar{F}^q(H_\mathbf{C}), \tag{4.2.0.1}$$

the "subspace of $H_\mathbf{C}$ of type (p, q)". The bigraduation

$$H_\mathbf{C} = \bigoplus_{p+q=n} H^{p,q}; \quad H^{q,p} = \overline{H^{p,q}} \tag{4.2.0.2}$$

allows us to define an action s of $\mathbf{C}^*$ as real group on $H_\mathbf{C}$ by putting

$$s(z) = \text{multiplication by } z^p \bar{z}^q \quad \text{on} \quad H^{p,q}. \tag{4.2.0.3}$$

Following Weil, we put $C = s(i)$. For each $z \in \mathbf{C}^*$, the endomorphism $s(z)$ of $H_\mathbf{C}$ commutes with complex conjugation, and hence comes from an endomorphism still noted $s(z)$ of $H_\mathbf{R} = H_\mathbf{Z} \otimes_\mathbf{Z} \mathbf{R}$, $\mathbf{R}$ denoting the field of real numbers.

(4.2.0.4) A *morphism* φ between pure Hodge structures H and H' is a group homomorphism $\varphi_\mathbf{Z}$: $H_\mathbf{Z} \to H'_\mathbf{Z}$ whose $\mathbf{C}$-linear extension $\varphi_\mathbf{C}$: $H_\mathbf{C} \to H'_\mathbf{C}$ commutes with the action s of $\mathbf{C}^*$ (4.2.0.3). (Thus between pure Hodge structure of different weights there is only the zero morphism.)

(4.2.0.5) The category of pure Hodge structures has an internal Hom and a tensor product $\otimes$, defined in the expected way (cf. [8]).

(4.2.0.6) Tate's Hodge structure $\mathbf{Z}(n)$ is the rank one Hodge structure of weight $-2n$ which is purely of type $(-n, -n)$, and whose integral lattice $H_\mathbf{Z}$ is the subgroup

$$(2\pi i)^n \mathbf{Z} \subset \mathbf{C}. \tag{4.2.0.7}$$

A polarization of a pure Hodge structure H of weight n is a homomorphism of Hodge structures

(4.2.0.8) $$(\ ,\ )\, H\otimes H\to \mathbf{Z}(-n)$$

such that the real bilinear form on $H_{\mathbf{R}}$

(4.2.0.9) $$(2\pi i)^n(x, C y)$$

is symmetric and positive definite.

The positive-definiteness means that

(4.2.0.10) if $h\in H^{p,q}$, $h\neq 0$, then $o^{p-q}(2\pi i)^n(h,\bar{h})>0$.

(4.2.0.11) The category of pure Hodge structures up to isogeny is the category whose objects are pure Hodge structures, but whose morphisms are defined by

(4.2.0.12) $\operatorname{Hom}(\text{up to isogeny})(H, H')=\operatorname{Hom}(H, H')\otimes_{\mathbf{Z}}\mathbf{Q}$.

(4.2.0.13) The full subcategory of the category of pure Hodge structures up to isogeny consisting of the *polarizable* objects (i.e., those which admit at least one polarization) is a *semi-simple*. It is closed under the formation of internal hom, tensor products, finite direct sums of objects of the same weight, subobjects and quotient objects. Any objects isogenous to a polarizable one is polarizable.

(4.2.0.14) **Proposition.** *Let $(H, (\ ,\ ))$ be a polarized pure Hodge structure. Then its group of automorphisms is finite.*

Proof. Any automorphism φ determines an element $\bar{\varphi}$ in the group

(4.2.0.15) $$\operatorname{Aut}(H_{\mathbf{Z}}/\text{torsion})\cap \operatorname{Aut}(H_{\mathbf{R}}, (2\pi i)^n(x, C y))$$

which, being the intersection of a discrete and a compact subgroup of $\operatorname{Aut}(H_{\mathbf{R}})$, is finite. As $\varphi\to\bar{\varphi}$ is clearly a group homomorphism, it remains to prove its kernel is finite. But $\varphi\to\bar{\varphi}$ is *injective*, unless $H_{\mathbf{Z}}=$torsion. In the latter case, $\operatorname{Aut}(H_{\mathbf{Z}})$ is a finite group. Q.E.D.

(4.2.1) Let $\mathscr{S}$ be a topological space. A family of pure Hodge structures of weight n on $\mathscr{S}$ is by defintion a local system $H_{\mathbf{Z}}$ of $\mathbf{Z}$-modules of finite type, together with a continuously warying filtration F_s^i of $(H_{\mathbf{C}})_s$, the complexification of the stalk of $H_{\mathbf{Z}}$ at s, which point by point is a pure Hodge structure of weight n.

(4.2.1.1) A polarization of a family of pure Hodge structures of weight n is by definition a morphism of local systems on $\mathscr{S}$

(4.2.1.2) $$(\ ,\ )\colon\ H_{\mathbf{Z}}\otimes H_{\mathbf{Z}}\to \mathbf{Z}(-n)_{\mathscr{S}}$$

which point by point is a polarization (4.2.0.8). A family of pure Hodge structure is called *polarizable* if it admits at least one polarization. The considerations of (4.2.0.11–13) apply *mutatis mutandis*, in this context.

(4.2.1.3) **Proposition.** *Let $\mathscr{S}$ be a connected, locally arcwise connected, locally arcwise simply connected topological space (so that $\mathscr{S}$ "has" a universal covering). Let $(H_{\mathbf{Z}}, F)$ be a polarizable family of pure Hodge structures. Suppose that the filtration F is locally constant, in the sense that it comes from a filtration by sub-local systems of the complexified local system $H_{\mathbf{C}}$. Then there exists a finite étale covering $\pi\colon \mathscr{S}' \to \mathscr{S}$ such that the inverse image $\pi^*(H_{\mathbf{Z}}, F)$ of $(H_{\mathbf{Z}}, F)$ on $\mathscr{S}'$ is a constant family of pure Hodge structures.*

Proof. Fix a point $s_0 \in \mathscr{S}$. The (topological) fundamental group $\pi_1(\mathscr{S}, s_0)$ acts on the stalk $(H_{\mathbf{Z}})_{s_0}$, and by hypothesis it preserves the filtration $F^i_{s_0}$ of $(H_{\mathbf{C}})_{s_0}$. A polarization $(\ ,\ )$ on the family $(H_{\mathbf{Z}}, F)$ induces a polarization $(\ ,\ )_{s_0}$ on $(H_{\mathbf{Z}})_{s_0}$, and $\pi_1(\mathscr{S}, s_0)$ acting on $(H_{\mathbf{Z}})_{s_0}$ preserves this polarization. Thus $\pi_1(\mathscr{S}, s_0)$ acts on $(H_{\mathbf{Z}})_{s_0}$ through the automorphism group of the polarized Hodge structure $((H_{\mathbf{Z}})_{s_0}, F_{s_0}, (\ ,\ )_{s_0})$, which is a finite group (4.2.0.14). Thus there exists a finite étale covering $\pi\colon \mathscr{S}' \to \mathscr{S}$ such that $\pi^*(H_{\mathbf{Z}})$ is a constant local system. Hence $\pi^*(H_{\mathbf{C}})$ is constant, and so necessarily are its sub-local systems $\pi^* F^i(H_{\mathbf{C}})$. Q.E.D.

(4.2.2) Let $\mathscr{S}$ be a topological space. A family of *mixed Hodge structures* on $\mathscr{S}$ is by definition a finitely filtered object in the category of local systems of $\mathbf{Z}$-modules of finite type

(4.2.2.0) $$(H_{\mathbf{Z}}, W); \quad W_i H_{\mathbf{Z}} \subset W_{i+1} H_{\mathbf{Z}}$$

together with a continuously varying Hodge filtration F^i_s on the complexifications $(H_{\mathbf{C}})_s$, of the stalks of $H_{\mathbf{Z}}$, such that, for each integer n, the filtration induced by the F^i_s makes $\mathrm{gr}^W_n H_{\mathbf{Z}} = W_n H_{\mathbf{Z}}/W_{n-1} H_{\mathbf{Z}}$ into a family of pure Hodge structures of weight n.

(4.2.2.1) A morphism of families of mixed Hodge structures is a morphism of local systems which respects W and whose complexification respects the filtration F^i_s point by point.

(4.2.2.2) An essential fact (cf. [8]) about the category of families of mixed Hodge structures on a topological space $\mathscr{S}$ is that it is an *abelian category*, and in particular that any morphism φ is automatically *strictly compatible* with the filtration W (and also F, for that matter). This means that $W_i \cap \mathrm{image}(\varphi) = \varphi(W_i)$, for every integer i.

(4.2.2.3) **Proposition.** *Let $\mathscr{S}$ be a topological space as in* (4.2.1.3). *Let $(H_{\mathbf{Z}}, F, W)$ be a family of mixed Hodge structures on $\mathscr{S}$, such that each of the assosciated graded families $(\mathrm{gr}^W_n H_{\mathbf{Z}}, F)$ of pure Hodge structures is polarizable. Suppose that the Hodge filtration F is locally constant, in the* *sense that it comes from a filtration by sub-local systems of the complexified local system $H_{\mathbf{C}}$. Then there exists a finite étale covering $\pi\colon \mathscr{S}' \to \mathscr{S}$ such that the inverse image $\pi^*(H_{\mathbf{Z}}, W, F)$ of $(H_{\mathbf{Z}}, W, F)$ on $\mathscr{S}''$ is a constant family of mixed Hodge structures.*

Proof. Applying (4.2.1.3) to each of the finitely many non-zero gr_n^W, we find a finite étale covering $\pi\colon \mathscr{S}' \to \mathscr{S}$ on which the gr_n^W become constant families of Hodge structures. So replacing $\mathscr{S}$ by $\mathscr{S}'$, it suffices to show that if the gr_n^W are all constant, then $H_{\mathbf{Z}}$ is constant. The constancy of the gr_n^W signifies that under the action of $\pi_1(\mathscr{S}, s_0)$ on $(\mathbf{H}_{\mathbf{Z}})_{s_0}$, for any $\gamma \in \pi_1(\mathscr{S}, s_0)$;

(4.2.2.3.0) $$(1-\gamma)\big(W_n(H_{\mathbf{Z}})_{s_0}\big) \subset W_{n-1}(H_{\mathbf{Z}})_{s_0}.$$

By hypothesis γ preserves the filtrations W *and* F, hence γ and $1-\gamma$ are endomorphisms of the mixed Hodge structure $\big((H_{\mathbf{Z}})_{s_0}, W, F\big)$. Thus (4.2.2.2), $1-\gamma$ is strictly compatible with W. Combining (4.2.2.3.0) and strictness, we find

(4.2.2.3.1) $$(1-\gamma)(W_n) \subset W_{n-1} \cap \text{image}\,(1-\gamma) = (1-\gamma)(W_{n-1}).$$

Since $W_n = 0$ for $n \ll 0$, and $W_n = \text{all}$ for $n \gg 0$, this implies $1-\gamma = 0$, and hence $H_{\mathbf{Z}}$ is a constant local system. Q.E.D.

(4.2.2.4) *Remark.* Let us agree to call *polarizable* a family of mixed Hodge structures whose assosciated graded families gr_n^W are all polarizable. Because gr_n^W is *exact*, finite direct sums, sub-objects and quotient objects of polarizable families of mixed Hodge structures are polarizable. Clearly, any object *isogenous* to a polarizable one is polarizable.

4.3. Geometric Interpretation

(4.3.0) Let $T = \mathrm{Spec}(\mathbf{C})$, S a connected smooth $\mathbf{C}$-scheme, $f\colon X \to S$ a *projective* and smooth S-scheme, and $D = \bigcup D_i$ a union of divisors in X which are smooth over S and which cross normally relative to S

(4.3.0.0) $$\begin{tikzcd} D \arrow[r, hook, "i"] & X \arrow[d, "f"] & U = X - D \arrow[l, hook', "j"'] \arrow[dl, "\pi"] \\ & S \arrow[d] & \\ & T & \end{tikzcd}$$

(4.3.0.1) For each integer $n \geqq 0$, the sheaf $R^n \pi_*^{an}(\mathbf{Z})$ on S^{an} is a *local system* of $\mathbf{Z}$-modules of finite type. As we have seen (4.1.1), the corresponding rational local system

$$R^n \pi_*^{an}(Q) = R^n \pi_*^{an}(\mathbf{Z}) \otimes_{\mathbf{Z}} Q \tag{4.3.0.2}$$

is the abutment of the Leray spectral sequence in local systems on S^{an}.

$$E_2^{p,q} = R^p f_*^{an}(R^q j_*^{an}(\mathbf{Q})) \Rightarrow R^{p+q} \pi_*^{an}(\mathbf{Q}), \tag{4.3.0.3}$$

which is *degenerate* at E_3 (because by (4.1.1) its complexification is).

(4.3.0.4) As a temporary notational device, let us denote by N^i the decreasing filtration of $R^n \pi_*^{an}(\mathbf{Q})$ defined by the spectral sequence (4.3.0.3). We *define* the weight filtration W of the local system $R^n \pi_*^{an}(\mathbf{Z})$ by:

(4.3.0.5) $W_i R^n \pi_*^{an}(\mathbf{Z})$ = the inverse image of $N^{2n-i} R^n \pi_*^{an}(Q)$ under the canonical mapping $R^n \pi_*^{an}(\mathbf{Z}) \to R^n \pi_*^{an}(\mathbf{Q})$

(thus $W_i = 0$ if $i < n$, and W_i = all if $i \geqq 2n$).

The locally free coherent sheaf on S^{an}

$$\begin{aligned} R^n \pi_*^{an}(\mathbf{C}) \otimes_{\mathbf{C}} \mathcal{O}_{S^{an}} &= \mathbf{R}^n f_*^{an}((\Omega_{X/S}^{\bullet}(\log D))^{an}) \\ &\simeq \mathbf{R}^n f_*(\Omega_{X/S}^{\bullet}(\log D)) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}} \end{aligned} \tag{4.3.0.6}$$

is the abutment of the Hodge $\Rightarrow$ De Rham spectral sequence

$$\begin{array}{ccc} E_1^{p,q} = R^q f_*^{an}((\Omega_{X/S}^{p}(\log D))^{an}) & \Rightarrow & \mathbf{R}^{p+q} f_*^{an}((\Omega_{X/S}^{\bullet}(\log D))^{an}) \\ \| & & \| \\ R^q f_*(\Omega_{X/S}^{p}(\log D)) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}} & & \mathbf{R}^{p+q} f_*(\Omega_{X/S}^{\bullet}(\log D)) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}} \end{array} \tag{4.3.0.7}$$

which has E_1 locally free, and which degenerates at E_1 (cf. (1.4.1.8) and [5]). The corresponding filtration F of $R^n \pi_*^{an}(\mathbf{C}) \otimes_{\mathbf{C}} \mathcal{O}_{S^{an}}$ defines point by point a filtration F_s of the stalk

$$(R^n \pi_*^{an}(\mathbf{C}))_s = \mathbf{R}^n f_*^{an}((\Omega_{X/S}^{\bullet}(\log D))^{an}) \otimes_{\mathcal{O}_{S^{an}}} (\mathcal{O}_{S^{an}}/\mathfrak{m}_s),$$

where $\mathfrak{m}_s$ denotes the ideal defining the point $s \in S^{an}$.

(4.3.1) **Proposition** (Deligne-Hodge). *The triple* $(R^n \pi_*^{an}(\mathbf{Z}), W, F)$ *defined above is a polarizable family of mixed Hodge structures on* S^{an}.

Proof. That it is a family of mixed Hodge structures follows from the Deligne's theory, point by point. It remains to see that it is polarizable. Consider the Leray spectral sequence

$$E_2^{p,q} = R^p f_*^{an}(R^q j_*^{an} \mathbf{Z}) \Rightarrow R^{p+q} \pi_*^{an}(\mathbf{Z}). \tag{4.3.1.1}$$

When tensored with $\mathbf{Q}$, it degenerates at E_3 and defines (a renumbering of) the filtration W on $R^n\pi_*^{\mathrm{an}}(\mathbf{Q})$. Thus the assosciated graded families of Hodge structures gr_i^W of our family of mixed Hodge structures are *isogenous* to various of the E_3 terms of the Leray spectral sequence (4.3.1.0) over $\mathbf{Z}$.

As these latter are sub-quotients (i.e., quotients of sub-objects) of the E_2 terms of (4.3.1.0), it remains to polarize the E_2 terms.

In the notation of (4.0.1.4), we have ($(-q)$ denoting $\otimes_{\mathbf{Z}}\mathbf{Z}(-q)$, cf. (4.2.0)).

$$(4.3.1.2)\qquad E_2^{p,q}=\begin{cases}\bigoplus_{i_1<\cdots<i_q} R^p f^{\mathrm{an}}(i_1,\ldots,i_q)_*(\mathbf{Z})(-q) & \text{if } q\neq 0\\ R^p f_*^{\mathrm{an}}(\mathbf{Z}) \quad \text{if } q=0.\end{cases}$$

Because finite direct sums of polarizable families are polarizable, and Tate twists $H(-q)$ of polarizable families H are polarizable, we need only remark the following proposition, applied to X and to all intersections $D_{i_1}\cap\cdots\cap D_{i_q}$.

(4.3.1.3) **Proposition** (Hodge's Index Theorem). *Let $f\colon X\to S$ be a projective and smooth morphism of $\mathbf{C}$-schemes. Then for any integer $n\geqq 0$, $R^n f_*^{\mathrm{an}}\mathbf{Z}$ is polarizable.*

Proof. We may assume that X/S has geometrically connected fibres, and is of constant relative dimension N. Let $L\in H^2(X^{\mathrm{an}},\mathbf{Z})$ be the cohomology class of a hyperplane section, i.e. the inverse image under the composite $pr_2\circ i$

$$(4.3.1.4)\qquad \begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & S\times\mathbf{P}_{\mathbf{C}} \arrow[r, "pr_2"] \arrow[d, "pr_1"] & \mathbf{P}_{\mathbf{C}} \\ & S & \end{tikzcd}$$

of the class of a hyperplane in $H^2(\mathbf{P}_{\mathbf{C}}^{\mathrm{an}},\mathbf{Z})$.

By the "hard Lefschetz theorem" ([1]), the iterated cup-product with L

$$(4.3.1.5)\qquad L^i\colon R^{N-i}f_*^{\mathrm{an}}(\mathbf{Z})(-i)\to R^{N+i}f_*^{\mathrm{an}}(\mathbf{Z})$$

is an *isogeny* (i.e., becomes an isomorphism when tensored with $\mathbf{Q}$).

Thus it suffices to show that $R^n f_*^{\mathrm{an}}(\mathbf{Z})$ is polarizable for $n\leqq N$.

For $n\leqq N$, the primitive part of $R^n f_*^{\mathrm{an}}(\mathbf{Z})$ is defined by

$$(4.3.1.6)\qquad \begin{aligned}\mathrm{Prim}^n f_*^{\mathrm{an}}(\mathbf{Z}) = \text{Kernel of } L^{N-n+1}\colon & R^n f_*^{\mathrm{an}}(\mathbf{Z})\\ &\to R^{2N-n+2}f_*^{\mathrm{an}}(\mathbf{Z})(N-n+1).\end{aligned}$$

According to the "primitive decomposition" ([43], pp. 77–79), for $n \leqq N$ the mapping

(4.3.1.7) $$\bigoplus_{0 \leqq i \leqq [n/2]} \mathrm{Prim}^{n-2i} f_*^{\mathrm{an}}(\mathbf{Z})(-i) \xrightarrow{\Sigma L^i} R^n f_*^{\mathrm{an}}(\mathbf{Z})$$

is an *isogeny*. Thus it remains to polarize $\mathrm{Prim}^n f_*^{\mathrm{an}}(\mathbf{Z})$ for $n \leqq N$. According to the Hodge Index Theorem (cf. [43]), the pairing

(4.3.1.8) $$\begin{gathered} \mathrm{Prim}^n f_*^{\mathrm{an}}(\mathbf{Z}) \times \mathrm{Prim}^n f_*^{\mathrm{an}}(\mathbf{Z}) \to R^{2N} f_*^{\mathrm{an}}(\mathbf{Z}) \sim \mathbf{Z}(-N) \\ (x, y) \to x \cup L^{N-n} y, \qquad \cup = \text{cup-product} \end{gathered}$$

is a polarization. This concludes the proof of (4.3.1.6) and of (4.3.1).

(4.3.2) **Interpretation.** When $S = \mathrm{Spec}(\mathbf{C})$, the mixed Hodge structure (4.3.1) on $H^n(U^{\mathrm{an}}, \mathbf{Z})$ depends only on U, and *not* on the compactification $U \hookrightarrow X$ chosen to represent U as the complement in a proper smooth variety $X/\mathbf{C}$ of a divisor with normal crossings. This mixed Hodge structure is *functorial* in U, in the sense that if $h\colon U \to V$ is a morphism of smooth $\mathbf{C}$-schemes, the induced morphisms $h^*\colon H^n(V^{\mathrm{an}}, \mathbf{Z}) \to H^n(U^{\mathrm{an}}, \mathbf{Z})$ are morphisms of mixed Hodge structures. (4.3.2.1) By Hironaka [18], *any* quasi-projective smooth $\mathbf{C}$-scheme U may be compactified as above, and thus its integral cohomology $H^\bullet(U^{\mathrm{an}}, \mathbf{Z})$ carries a functorial mixed Hodge structures. The family (4.3.1) of mixed Hodge structures is just the "interpolation" of these mixed Hodge structures on the fibres of $\pi\colon U = X - D \to S$.

(4.3.3) **Proposition.** *Hypotheses as in* (4.3.0), *the following conditions are equivalent, for any integer* $n \geqq 0$.

(4.3.3.0) *The Hodge filtration F on the locally free sheaf* $\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))$ *on S is horizontal for the Gauss-Manin connection ∇.*

(4.3.3.1) *The Hodge filtration F of the family of mixed Hodge structures on* S^{an}, $(R^n \pi_*^{\mathrm{an}}(\mathbf{Z}), W, F)$, *is locally constant, i.e., it comes from a filtration of* $R^n \pi_*^{\mathrm{an}}(\mathbf{C})$ *by sub-local systems.*

(4.3.3.2) *There exists a finite étale covering* $\varphi\colon S' \to S$ *such that* $(\varphi^{\mathrm{an}})^*(R^n \pi_*^{\mathrm{an}}(\mathbf{Z}), W, F)$ *is a constant family of mixed-Hodge structures on* $(S')^{\mathrm{an}}$.

(4.3.3.3) *There exists a finite étale covering* $\varphi\colon S' \to S$ *such that*

$$\varphi^*(\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)), \nabla)$$

is isomorphic to $((\mathcal{O}_{S'})^{b_n}, d)$ *as a coherent $\mathcal{O}_{S'}$-module with connection* (b_n = rank of $\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))$).

Proof. By (4.3.0.6), we have (4.3.3.0) ⇒ (4.3.3.1).

By (4.2.2.3), (4.3.1), and "Riemann's existence theorem" (cf. [36] and the appendix) we have (4.3.3.1) $\Rightarrow$ (4.3.3.2).

To see that (4.3.3.2) $\Rightarrow$ (4.3.3.3), notice that (4.3.3.2) implies in particular that the local system $(\varphi^{\mathrm{an}})^*(R^n\pi_*^{\mathrm{an}}(\mathbf{C}))$ is *constant* on $(S')^{\mathrm{an}}$, i.e., isomorphic to $(\mathbf{C}^{b_n})_{(S')^{\mathrm{an}}}$.

Thus, both $\varphi^*(\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)), \nabla)$ and $((\mathcal{O}_{S'})^{b_n}, d)$ are coherent $\mathcal{O}_{S'}$-modules with integrable connections having regular singular points (cf. 4.1) and they both give rise via (4.1.0) to isomorphic local systems on $(S')^{\mathrm{an}}$. Hence they are isomorphic.

In order to prove that (4.3.3.3) $\Rightarrow$ (4.3.3.0), we first remark that it suffices to prove that (4.3.3.0) holds after any base change $\varphi: S' \to S$ by a finite étale morphism, because (4.3.3.0) is of a differential nature, hence local on S for the étale topology. The hypothesis (4.3.3.3) implies that the local system $R^n\pi_*^{\mathrm{an}}(\mathbf{C})$ becomes constant on a finite étale covering, hence also $R^n\pi_*^{\mathrm{an}}(\mathbf{Q})$ and $R^n\pi_*^{\mathrm{an}}(\mathbf{Z})$/torsion become constant. As the torsion in $R^n\pi_*^{\mathrm{an}}(\mathbf{Z})$ is a local system of finite groups, it follows that $R^n\pi_*^{\mathrm{an}}(\mathbf{Z})$ becomes constant on a finite étale covering $\varphi': S'' \to S$. After making the change of base by such a finite étale $\varphi': S'' \to S$, we are reduced to showing that if $R^n\pi_*^{\mathrm{an}}(\mathbf{Z})$ is a constant local system, then (4.3.3.0) (or equivalently (4.3.3.1)) holds. This is achieved by the following more general proposition.

(4.3.4) **Proposition.** *Hypotheses as in* (4.3.0), *the largest constant sub-local system of* $R^n\pi_*^{\mathrm{an}}(\mathbf{Z})$ *is a constant sub-family of mixed Hodge structures.*

Proof. In down-to-earth terms, this means the following:

(4.3.4.0) For any point $s \in S$, any element $h \in (R^n\pi_*^{\mathrm{an}}(\mathbf{C}))_s$ which is invariant under $\pi_1(S^{\mathrm{an}}, s)$, and any homotopy class of paths $\gamma_{s,t}$ from s to a second point $t \in S$, if $h \in F_s^i(R^n\pi_*^{\mathrm{an}}(\mathbf{C}))_s$, then $\gamma_{s,t}(h) \in F_t^i(R^n\pi_*^{\mathrm{an}}(\mathbf{C}))_t$, where $\gamma_{s,t}(h)$ denotes the element of $(R^n\pi_*^{\mathrm{an}}(\mathbf{C}))_t$ deduced by transporting h along the path $\gamma_{s,t}$.

In order to establish (4.3.4.0), it suffices to do so far all pairs (s, t) of nearby points ("nearby" in S^{an}), and for a single path $\gamma_{s,t}$ (because S^{an}, being a complex manifold, is locally simply connected). We can always find a connected nonsingular affine curve $C \subset S$ which contains a given pair (s, t) of nearby points. When we take the inverse image of our situation on C, our chosen element $h \in (R^n\pi_*^{\mathrm{an}}(\mathbf{C}))_s$ remains invariant under $\pi_1(C^{\mathrm{an}}, s)$ because the action of this group factors through $\pi_1(S^{\mathrm{an}}, s)$. Thus it suffices to prove (4.3.4) when S is a connected smooth affine *curve*, in which case it follows from the more precise

(4.3.5) **Proposition.** *Hypotheses as in* (4.3.0), suppose that S is a smooth connected affine curve. Then the canonical mapping of families of mixed Hodge structures on S^{an}

(4.3.5.0) $$(H^n(U^{\mathrm{an}}, \mathbf{Z}))_{S^{\mathrm{an}}} \to R^n \pi_*^{\mathrm{an}}(\mathbf{Z})$$

has as image a constant sub-family of mixed Hodge structures. Its image is the largest constant sub-local system of $R^n \pi_*^{\mathrm{an}}(\mathbf{Z})$.

Proof. The first assertion results from the fact that the category of families of mixed Hodge structures is an abelian subcategory of the category of local systems, and that the formation of images (as well as of kernels and cokernels) commutes with the inclusion of the category of families of mixed Hodge structures into that of local systems. The second assertion will result from the equality $E_\infty^{0,n} = E_2^{0,n}$ in the usual Leray spectral sequence in integral cohomology of π^{an}: $U^{\mathrm{an}} \to S^{\mathrm{an}}$, which itself follows from the following proposition.

(4.3.6) **Proposition.** *Hypotheses as in* (4.3.5), *the Leray spectral sequence*

(4.3.6.0) $$E_2^{p,q} = H^p(S^{\mathrm{an}}, R^q \pi_*^{\mathrm{an}}(\mathbf{Z})) \Rightarrow H^{p+q}(U^{\mathrm{an}}, \mathbf{Z})$$

is degenerate at E_2.

Proof. It suffices to show that $E_2^{p,q} = 0$ unless $p=0$ or $p=1$. This is true, because the $R^q \pi_*^{\mathrm{an}}(\mathbf{Z})$ are local systems on S^{an}, and because S^{an} is an Eilenberg-Maclane space $K(\pi, 1)$ with $\pi = \pi_1(S^{\mathrm{an}}, s)$ a free group. This concludes the proof of (4.3.5), hence of (4.3.4) and (4.3.3) as well. Q.E.D.

4.4. We now Wish to Restate 4.3.3 "with a Group of Operators"

(4.4.0) Let G be a finite group. The indecomposable central idempotents in the rational group-ring $\mathbf{Q}[G]$ are obtained in the following way. For every irreducible representation χ of G in a finite-dimensional $\mathbf{C}$-space, let

(4.4.0.0) $$P(\chi) = \frac{\deg(\chi)}{\# G} \sum_g \operatorname{trace}(\chi(g^{-1})) \cdot g$$

denote the corresponding indecomposable central idempotent in $\mathbf{C}[G]$.

(4.4.0.1) As σ runs over the group $\operatorname{Aut}(\mathbf{C})$, the irreducible representation $\chi^{(\sigma)}$ runs over a finite number of isomorphism classes of irreducible $\mathbf{C}$-representations of G, and the projectors $P(\chi^{(\sigma)})$ $(= \sigma \cdot P(\chi)$, viewed as a function on G with values in $\mathbf{C})$ run over a finite number of *distinct* indecomposable central idempotents in $\mathbf{C}[G]$. We shall call the $\chi^{(\sigma)}$ (resp. the $P(\chi^{(\sigma)})$) the $\mathbf{Q}$-conjugates of χ (resp. of $P(\chi)$).

For each $\mathbf{Q}$-conjugacy class Δ of irreducible $\mathbf{C}$-representations of G, the element

(4.4.0.2) $$P(\Delta) \overset{\text{def'n}}{=} \sum_{\chi\in\Delta} P(\chi)$$

of $\mathbf{C}[G]$ lies in fact in $\mathbf{Q}[G]$, and is an indecomposable central idempotent in $\mathbf{Q}[G]$. Every indecomposable central idempotent in $\mathbf{Q}[G]$ is of this form. We have

(4.4.0.3) $$P(\Delta)\,P(\Delta') = \begin{cases} P(\Delta) & \text{if } \Delta = \Delta' \\ 0 & \text{if not.} \end{cases}$$

(4.4.0.4) Let $H_{\mathbf{Z}}$ be a local system of $\mathbf{Z}$-modules of finite type on which G acts. We define the G-sub-local system

(4.4.0.5) $P(\Delta)\,H_{\mathbf{Z}} \overset{\text{def'n}}{=}$ the inverse image in $H_{\mathbf{Z}}$ of $P(\Delta)(H_{\mathbf{Q}}) \subset H_{\mathbf{Q}}$.

The canonical mapping of local systems

(4.4.0.6) $$\bigoplus_{\Delta} P(\Delta)\,H_{\mathbf{Z}} \to H_{\mathbf{Z}}$$

is a G-morphism and an *isogeny*, i.e., it becomes an isomorphism when tensored with $\mathbf{Q}$.

(4.4.1) If G acts on a family of mixed Hodge structures $(H_{\mathbf{Z}}, W, F)$ (meaning that its action on the local system $H_{\mathbf{Z}}$ respects W and F), then each $P(\Delta)\,H_{\mathbf{Z}}$ is a sub-family of mixed Hodge structures. If $(H_{\mathbf{Z}}, W, F)$ is a polarizable (4.2.2.4) family of mixed Hodge structures, so is each sub-family $P(\Delta)\,H_{\mathbf{Z}}$.

(4.4.2) **Proposition.** *Hypotheses as in* (4.3.0), *suppose G is a finite group of S-automorphisms of X which respects the divisor D. Let $n \geqq 0$ be an integer Δ a $\mathbf{Q}$-conjugacy class* (4.4.0.1) *irreducible representation of G in complex vector spaces, and $P(\Delta) \in \mathbf{Q}[G]$ the corresponding projector* (4.4.0.2).

The following conditions are equivalent.

(4.4.2.0) *The induced Hodge filtration F on the locally free sheaf $P(\chi)(\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)))$ on S is horizontal for the Gauss-Manin connection, for every irreducible representation χ in the given $\mathbf{Q}$-conjugacy class Δ.*

(4.4.2.1) *The Hodge filtration F of the family of mixed Hodge structures on S^{an}, $(P(\Delta)\,R^n \pi^{\mathrm{an}}_*(\mathbf{Z}), W, F)$, is locally constant, i.e., comes from a filtration of $P(\Delta)\,R^n \pi^{\mathrm{an}}_*(\mathbf{C})$ by sub-local systems.*

(4.4.2.2) *There exists a finite étale covering $\varphi\colon S' \to S$ such that $(\varphi^{\mathrm{an}})^*(P(\Delta)\,R^n \pi^{\mathrm{an}}_*(\mathbf{Z}), W, F)$ is a constant family of mixed Hodge structures on $(S')^{\mathrm{an}}$.*

(4.4.2.3) *There exists a finite étale covering $\varphi: S' \to S$ such that, for every irreducible χ on the* **Q**-*conjugary class Δ, $\varphi^*(P(\chi)\,\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)), \nabla)$ is isomorphic to $((\mathcal{O}_{S'})^{b_n(\chi)}, d)$ as a coherent $\mathcal{O}_{S'}$-module with connection $(b_n(\chi) = \text{rank of } P(\chi)(\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))))$.*

Proof. The proof of (4.3.3) applies almost verbatim. The implications (4.4.2.0)$\Leftrightarrow$(4.4.2.1)$\Rightarrow$(4.4.2.2) are proved as in (4.3.3), remembering that $(P(\Delta)\,R^n \pi^{\mathrm{an}}_*(\mathbf{Z}), W, F)$ is polarizable (by (4.4.1) and (4.3.1)). The implication (4.4.2.2)$\Rightarrow$(4.4.2.3) is proved as in (4.3.3), remembering that each $(P(\chi)\,\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)), \nabla)$ has regular singular points, being a sub-object of $(\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)), \nabla)$. This final implication (4.4.2.3)$\Rightarrow$(4.4.2.1) may be reduced, as in (4.3.3), to the case in which the local system $P(\Delta)\,R^n \pi^{\mathrm{an}}_*(\mathbf{Z})$ is *constant*. But in that case, we have ($P(\Delta)$ being idempotent).

(4.4.2.4) $P(\Delta)\,R^n \pi^{\mathrm{an}}_*(\mathbf{Z}) = P(\Delta) \cdot I$ where I = the largest constant sub-local system of $R^n \pi^{\mathrm{an}}_*(\mathbf{Z})$.

By (4.3.4), I is a constant sub-family of mixed Hodge structures, hence (4.4.1) so is $P(\Delta)\,I = P(\Delta)\,R^n \pi^{\mathrm{an}}_*(\mathbf{Z})$, and in particular (4.4.2.1) holds. Q.E.D.

5. Applications to the Question of Grothendieck

5.0. A Global Situation

(5.0.0) Let A be a subring of **C** which is finitely generated over **Z**, and put $T = \mathrm{Spec}(A)$. Let $g: S \to T$ be a smooth morphism with geometrically connected fibres, and $f: X \to S$ a projective and smooth morphism. Let $D = \bigcup D_i$ be a union of divisors in X, each smooth over S, which have normal crossings relative to S.

(5.0.0.0)
$$\begin{tikzcd} D \arrow[dr, hook] & \\ & X \arrow[d] \\ & S \arrow[d] \\ & T \end{tikzcd}$$

(5.0.1) Let $\mathcal{U} \subset S$ be an affine open neighborhood of the generic point of S over which each of the Hodge cohomology sheaves $R^p f_*(\Omega^q_{X/S})$ is a locally free module of finite rank. (Such a $\mathscr{U}$ always exists.) Then, *over* $\mathscr{U}$, the Hodge $\Rightarrow$ De Rham spectral sequence

$$(5.0.1.0) \qquad E_1^{p,q} = R^q f_*(\Omega^p_{X/S}(\log D)) \Rightarrow \mathbf{R}^{p+q} f_*(\Omega^\bullet_{X/S}(\log D))$$

is degenerate at E_1. Let us recall why this is so. By Deligne's mixed Hodge theory ([8], 3.2.13), this spectral sequence degenerates at E_1 over the generic point of $\mathscr{U}$. Because E_1 is locally free over $\mathscr{U}$, the vanishing of d_1 at the generic point implies its vanishing on all of $\mathscr{U}$. Then $E_2 = E_1$ is locally free on $\mathscr{U}$, so $d_2 = 0$ on $\mathscr{U}$ because $d_2 = 0$ at the generic point of $\mathscr{U}$... this proves inductively that all $d_r = 0$ over $\mathscr{U}$.

By (2.2.1.11), after any change of base $S' \to S$ which factors through $\mathscr{U}$, the spectral sequence (5.0.1.0) will continue to have E_1 locally free of finite rank, and to degenerate at E_1.

(5.1) Theorem. *Hypotheses as in* (5.0.0), *let* $n \geqq 0$ *be an integer. Suppose that there is some non-void affine open set* $\mathscr{U} \subset S$ *satisfying* (5.0.1), *and an infinite set* Σ *of prime numbers* p, *such that the following condition* (5.1.0) *holds*

(5.1.0) *For every point of* T *with values in (the spectrum of) a finite field* $\mathbf{F}_q$ *of characteristic* $p \in \Sigma$, *after the base change* $\operatorname{Spec}(\mathbf{F}_q) \to T$,

$$(5.1.0.0) \qquad \begin{tikzcd} D_{\mathscr{U}} \otimes \mathbf{F}_q \arrow[r] \arrow[d, hook] & D_{\mathscr{U}} \arrow[r] \arrow[d] & D \arrow[d, hook] \\ X_{\mathscr{U}} \otimes \mathbf{F}_q \arrow[r] \arrow[d] & X_{\mathscr{U}} \arrow[r] \arrow[d] & X \arrow[d] \\ \mathscr{U} \otimes \mathbf{F}_q \arrow[r] \arrow[d] & \mathscr{U} \arrow[r, hook] \arrow[d] & S \\ \operatorname{Spec}(\mathbf{F}_q) \arrow[r] & T & \end{tikzcd}$$

the p-*curvature of the free* $\mathcal{O}_{\mathscr{U}} \otimes_A \mathbf{F}_q$ *module with integrable connection relative to* $\mathbf{F}_q$

$$(5.1.0.1) \qquad (\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)) | \mathscr{U}, \nabla) \otimes \mathbf{F}_q$$

is zero, in other words, that the inverse image of $\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))$ *on* $\mathscr{U} \otimes \mathbf{F}_q$ *is spanned by its horizontal sections.*

(5.1.1) *Then after the change of base* $\mathrm{Spec}(\mathbf{C}) \to T$,

(5.1.1.0)
$$\begin{tikzcd}
D_{\mathbf{C}} \arrow[r] \arrow[d, hook] & D \arrow[d, hook] \\
X_{\mathbf{C}} \arrow[r] \arrow[d] & X \arrow[d] \\
S_{\mathbf{C}} \arrow[r] \arrow[d] & S \arrow[d] \\
\mathrm{Spec}(\mathbf{C}) \arrow[r] & T
\end{tikzcd}$$

the coherent sheaf with integrable connection on $S_{\mathbf{C}}$

(5.1.1.1) $$(\mathbf{R}^n f_{\mathbf{C}*}(\Omega^{\bullet}_{X_{\mathbf{C}}/S_{\mathbf{C}}}(\log D)), \nabla$$

becomes trivial on a finite étale covering $\varphi: S' \to S_{\mathbf{C}}$, *i.e., becomes isomorphic to* $((\mathcal{O}_{S'})^{b_n}, d)$, *where* $b_n =$ *the rank of* $\mathbf{R}^n f_*(\Omega^{\bullet}_{X/S}(\log D))$ *over the generic point of* S.

Proof. By (4.3.3), it suffices to prove that the Hodge filtration F on $\mathbf{R}^n f_{\mathbf{C}*}(\Omega^{\bullet}_{X_{\mathbf{C}}/S_{\mathbf{C}}}(\log D_{\mathbf{C}}))$ is horizontal for the Gauss-Manin connection. For this it suffices to prove that the Hodge filtration is horizontal over $\mathscr{U}_{\mathbf{C}}$, since the *obstruction* to the horizontality is the $\mathcal{O}_{S_{\mathbf{C}}}$-linear Kodaira-Spencer mapping between locally free $\mathcal{O}_{S_{\mathbf{C}}}$-modules (cf. (1.4.1.8))

(5.1.2.0)
$$\begin{aligned}
&\bigoplus_{p+q=n} R^q f_{\mathbf{C}*}(\Omega^p_{X_{\mathbf{C}}/S_{\mathbf{C}}}(\log D_{\mathbf{C}})) \\
&\qquad \xrightarrow{\ \rho\ } \bigoplus_{p+q=n} \Omega^1_{S_{\mathbf{C}}/\mathbf{C}} \otimes R^{q+1} f_{\mathbf{C}*}(\Omega^{p-1}_{X_{\mathbf{C}}/S_{\mathbf{C}}}(\log D_{\mathbf{C}}))
\end{aligned}$$

which cannot vanish over the open set $\mathscr{U}_{\mathbf{C}}$ without vanishing on $S_{\mathbf{C}}$. We will in fact prove that the Hodge filtration is horizontal on $\mathbf{R}^n f_*(\Omega^{\bullet}_{X/S}(\log D))|\mathscr{U}$, or equivalently that for any section D of $Der(\mathscr{U}/T)$, the $\mathcal{O}_{\mathscr{U}}$-linear Kodaira-Spencer mapping between free $\mathcal{O}_{\mathscr{U}}$-modules

(5.1.2.1) $$\bigoplus_{p+q=n} R^q f_*(\Omega^p_{X/S}(\log D))|\mathscr{U} \xrightarrow{\rho(D)} \bigoplus_{p+q=n} R^{q+1}(\Omega^{p-1}_{X/S}(\log D))|\mathscr{U}$$

vanishes.

Let $\mathscr{U} = \mathrm{Spec}(B)$. By (3.3.1), the hypothesis (5.1.0) implies that for every point of T with values in (the spectrum of) a finite field $\mathbf{F}_q$ of characteristic $p \in \Sigma$, the inverse image on $B \otimes_A \mathbf{F}_q$ of the mapping $\rho(D)$ is zero. This implies that for every maximal ideal $\mathfrak{m}$ of B such that $B/\mathfrak{m}B$ has characteristic $p \in \Sigma$, the inverse image on $B/\mathfrak{m}B$ of the mapping $\rho(D)$ vanishes, i.e., all the matrix coefficients of $\rho(D)$ lie in $\mathfrak{m}$. Because B is a finitely generated integral domain whose fraction field is of characteristic zero, it follows immediately from Noether's normalization theorem that for any infinite set Σ of prime numbers, the intersection of all maximal ideals of B with residue characteristic in Σ is reduced to zero. Thus the Kodaira-Spencer mapping (5.1.2.1) vanishes. This concludes the proof.

5.2. A Global Situation with a Group of Operators

(5.2.0) Hypotheses as in (5.0.0), let G be a finite group, which acts as a group of S-automorphisms of X which are stable on D. Let g be the least common multiple of the orders of the elements of G, and let $A_0 = \mathbf{Z}[1/g, \xi_g]$, where ξ_g is a primitive g-th root of unity. We suppose $A \supset A_0$.

(5.2.1) As is well-known (cf. [38]), *every* irreducible representation of G in a finite dimensional complex vector space may be obtained by extending scalars from a representation of G in a locally free A_0-module of finite rank.

(5.2.2) By (3.2.2 bis), if χ is any representation of G in a locally free A_0-module which is irreducible over $\mathbf{C}$, its "reduction" modulo any maximal $\mathfrak{m}$ of A_0, $\chi(\mathfrak{m})$, is absolutely irreducible over $A_0/\mathfrak{m}$. Letting σ denote any automorphism of $\mathbf{C}$ which extends the Frobenius automorphism corresponding to $\mathfrak{m}$ of A_0, we have $\chi^\sigma(\mathfrak{m}) = \chi(\mathfrak{m})^{(p)}$ (cf. (3.2.1)). In particular, $\chi(\mathfrak{m})$ and $\chi(\mathfrak{m})^{(p)}$ are obtained from irreducible representations in the same $\mathbf{Q}$-conjugacy class (cf. (4.4.0.1)). Replacing the reference (4.3.3) by (4.4.2), the proof of yields:

(5.3) Theorem. *Hypotheses as in* (5.2.0), *let* $n \geqq 0$ *be an integer, and let* Δ *be a* $\mathbf{Q}$*-conjugacy class of irreducible* $\mathbf{C}$*-representations of* G*. For each* $\chi \in \Delta$*, let* $P(\chi)$ *denote the corresponding projector in* $A_0[G]$ (*cf.* (4.4.0), (4.4.0.1)). *Suppose that there is a non-void affine open set* $\mathscr{U} \subset S$ *satisfying* (5.0.1), *and an infinite set* Σ *of prime numbers, such that the following condition holds.*

(5.3.0) *For every point of* T *with values in (the spectrum of) a finite field* $\mathbf{F}_q$ *of characteristic* $p \in \Sigma$*, after the base change* $\mathrm{Spec}(\mathbf{F}_q) \to T$ (*cf. the diagram* (5.1.0.0)), *the p-curvature of each of the free* $\mathcal{O}_{\mathscr{U}} \otimes_A \mathbf{F}_q$ *modules with integrable connection*

$$(P(\chi)\, \mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)) | \mathscr{U}, \nabla) \otimes \mathbf{F}_q, \quad \chi \in \Delta \tag{5.3.0.1}$$

is zero.

Then after the change of base $\mathrm{Spec}(\mathbf{C}) \to T$ (*cf. the diagram* (5.1.1.0)) *each of the coherent sheaves with integrable connection on* $S_{\mathbf{C}}$

$$(P(\chi)\,\mathbf{R}^n f_{\mathbf{C}*}(\Omega^\bullet_{X_{\mathbf{C}}/S_{\mathbf{C}}}(\log D_{\mathbf{C}}))\,\nabla), \qquad \chi \in \Delta$$

becomes trivial on a finite étale covering $\varphi: S' \to S_{\mathbf{C}}$, *i.e., becomes isomorphic to* $((\mathcal{O}_{S'})^{b_n(\chi)}, d)$, *where* $b_n(\chi)$ *is the rank of* $P(\chi)\,\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D))$ *over the generic point of* S.

5.4. The Birational Point of View

(5.4.0) Let $S_{\mathbf{C}}$ be a connected smooth $\mathbf{C}$-scheme, and $(M_{\mathbf{C}}, \nabla_{\mathbf{C}})$ a locally free sheaf of finite rank with integrable connection on $S_{\mathbf{C}}$.

(5.4.1) *Remark.* In order that there exist a finite étale covering of $S_{\mathbf{C}}$ on which $(M_{\mathbf{C}}, \nabla_{\mathbf{C}})$ becomes trivial (as locally free sheaf with integrable connection), it is necessary and sufficient that there exist a finite extension K of the function field L of $S_{\mathbf{C}}$ such that $(M_{\mathbf{C}}, \nabla_{\mathbf{C}}) \otimes K$ is trivial (i.e., so that $M_{\mathbf{C}} \otimes K$ is spanned over K by horizontal sections).

(5.4.2) *Proof.* The necessity is obvious. For sufficiency, notice that by hypotheses $(M_{\mathbf{C}}, \nabla_{\mathbf{C}})$ becomes trivial on a finite étale covering $\mathcal{U}'_{\mathbf{C}}$ of some nonvoid affine open set $\mathcal{U}_{\mathbf{C}} \subset S_{\mathbf{C}}$. Thus $(M_{\mathbf{C}}, \nabla_{\mathbf{C}})$ on $\mathcal{U}'_{\mathbf{C}}$ has regular singular points, hence on $\mathcal{U}_{\mathbf{C}}$, hence on $S_{\mathbf{C}}$ (cf. [7], 4.1). So it suffices to prove that the associated monodromy representation of $\pi_1(S_{\mathbf{C}}, s)$ on the $\mathbf{C}$-space M_s of germs of holomorphic horizontal sections at s factors through a finite group. Taking $s \in \mathcal{U}_{\mathbf{C}}$, we have by hypothesis that the composite

$$\pi_1(\mathcal{U}_{\mathbf{C}}, s) \to \pi_1(S_{\mathbf{C}}, s) \to \mathrm{Aut}(M_s)$$

factors through a finite group. As $\pi_1(\mathcal{U}_{\mathbf{C}}, s) \to \pi_1(S_{\mathbf{C}}, s)$ is *surjective*, it follows that the monodromy representation of $\pi_1(S_{\mathbf{C}}, s)$ factors through a finite group. Q.E.D.

(5.4.2.1) When either of the equivalent conditions of (5.4.1) holds, we say that $(M_{\mathbf{C}}, \nabla_{\mathbf{C}})$ *has a full set of algebraic solutions.*

(5.4.3) Fix an infinite set Σ of prime numbers. Let A be a finitely generated (over $\mathbf{Z}$) subring of $\mathbf{C}$, $T = \mathrm{Spec}(A)$, $\mathcal{U} \to T$ an affine smooth morphism with geometrically connected fibres whose "complex fibre" $\mathcal{U}_{\mathbf{C}}$ "is" an affine open subset of $S_{\mathbf{C}}$, and (M, ∇) a free $\mathcal{O}_{\mathcal{U}}$-module of finite rank with integrable connection relative to T, whose inverse image on $\mathcal{U}_{\mathbf{C}}$ "is" $(M_{\mathbf{C}}, \nabla_{\mathbf{C}})|\mathcal{U}_{\mathbf{C}}$.

Consider the following property:

(5.4.3.1) There is an affine open set $\mathcal{V} \subset \mathcal{U}$ such that for every point of T with values in (the spectrum of) a finite field $\mathbf{F}_q$ of characteristic $p \in \Sigma$, after the base change $\mathrm{Spec}(\mathbf{F}_q) \to T$

$$\begin{tikzcd} V \otimes \mathbf{F}_q \arrow[r] \arrow[d] & \mathscr{V} \subset \mathscr{U} \arrow[d] \\ \mathrm{Spec}(\mathbf{F}_q) \arrow[r] & T \end{tikzcd}$$

the p-curvature of the free $\mathcal{O}_{\mathscr{V}} \otimes \mathbf{F}_q$-module with integrable connection

$$((M, \nabla)|\mathscr{V}) \otimes \mathbf{F}_q$$

is zero.

(5.4.3.2) A standard "passage to the limit" shows that if the property (5.4.3.1) holds for one set of choices

$$(A, \mathscr{U}, (M, \nabla))$$

then it holds for *every choice*. It is thus an intrinsic property of the germ of $(M_{\mathbf{C}}, \nabla_{\mathbf{C}})$ at the generic point of $S_{\mathbf{C}}$ which we call *having p-curvature zero for almost all primes* $p \in \Sigma$.

Thus, given a function field $K/\mathbf{C}$ (i.e., a finitely generated field extension of $\mathbf{C}$), a differential equation (N, ∇) over K (i.e., a finite dimensional K-space N together with an integrable connection ∇: $N \to \Omega^1_{K/\mathbf{C}} \otimes_K N$), and an infinite set Σ of prime numbers, it makes sense to say that (N, ∇) has or has not p-curvature zero for almost all primes $p \in \Sigma$.

(5.4.3.3) Because p-curvature is a differential invariant, its vanishing is of a local nature for the étale topology. It follows that (N, ∇) has p-curvature zero for almost all $p \in \Sigma$ if and only if there exists a finite algebraic extension L/K such that $(N, \nabla) \underset{K}{\otimes} L$ has p-curvature zero for almost all $p \in \Sigma$.

Putting together (5.4.1) and (5.4.3.3), we have the obvious

(5.4.4) **Proposition.** *Let (N, ∇) be a differential equation over a function field $K/\mathbf{C}$ (5.4.3.2). If (N, ∇) has a full set of algebraic solutions, then it has p-curvature zero for almost all primes p.*

(5.4.5) Let $K/\mathbf{C}$ be a function field, and let U_K be a smooth quasi-projective K-variety. By Hironaka ([18]), we can find a projective and smooth K-variety X_K which contains U_K as an open set, such that the complement $D_K = X_K - U_K$ is a union of smooth divisors $D_{i,K}$ in X_K which have normal crossings. By "general nonsense", we can find a finitely generated subring A of $\mathbf{C}$, a smooth $\mathrm{Spec}(A)$-scheme S with geometrically connected fibres such that K is the function field of its complex fibre $S_{\mathbf{C}}$, a projective and smooth S-scheme f: $X \to S$ and divisors D_i in X which are smooth over S and cross normally relative to S, such that the fibre of X (respectively of D_i) over the generic point of $S^{\mathbf{C}}$ is X_K (resp. $D_{i,K}$).

(5.4.5.1) The stalk over the generic point of $S_{\mathbf{C}}$ of the coherent $\mathcal{O}_S$-module with integrable connection relative to T

$$(\mathbf{R}^n f_*(\Omega^\bullet_{X/S}(\log D)), \nabla) \tag{5.4.5.2}$$

is the differential equation over K

$$(H^n_{DR}(U_K/K), \nabla). \tag{5.4.5.3}$$

Combining (5.4.1), (5.4.3.2), (5.4.4) and (5.4.5.1), we may restate Theorem 5.5 birationally:

(5.5) Theorem ($=5.1$ bis). *Let K be a function field over $\mathbf{C}$, U a smooth quasi-projective K-variety, and $n \geqq 0$ an integer. Then the following conditions are equivalent.*

(5.5.1) *$(H^n_{DR}(U/K), \nabla)$ has a full set of algebraic solutions*

(5.5.2) *There exists an infinite set Σ of prime numbers such that $(H^n_{DR}(U/K), \nabla)$ has p-curvature zero for almost all $p \in \Sigma$.*

(5.5.3) *$(H^n_{DR}(U/K), \nabla)$ has p-curvature zero for almost all primes p.*

5.6. In the situation of (5.4.5), suppose that a finite group G acts as a group of K-automorphisms of U_K. By Hironaka ([18, 19, 41]) we can find a compactification X_K of U_K as in (5.4.5) on which G acts as a group of K-automorphisms, so that $U_K \hookrightarrow X_K$ is a G-morphism. By "general nonsense", we can find A, S, X, D_i as in (5.4.5) such that G acts as S-automorphisms of X preserving $D = \bigcup D_i$, in a way which "gives back" the action of G on X_K by passage to the generic point of $S_{\mathbf{C}}$. Thus we may restate 5.3 with a group of operators in a birational way.

5.7. Theorem ($=5.3$ bis). *Let K be a function field over $\mathbf{C}$, U a smooth quasi-projective K-variety, G a finite group which acts as a group of K-automorphisms of U, $n \geqq 0$ and integer, and Δ a $\mathbf{Q}$-conjugacy class* (4.4.0.1) *of irreducible complex representations of G. Then the following conditions are equivalent:*

(5.7.1) *For each irreducible representation $\chi \in \Delta$, the χ-parts of the de Rham cohomology $(P(\chi)\, H^n_{DR}(U/K), \nabla)$ has a full set of algebraic solutions.*

(5.7.2) *There exists an infinite set Σ of prime numbers such that for every $\chi \in \Delta$, $(P(\chi)\, H^n_{DR}(U/K), \nabla)$ has p-curvature zero for almost all primes $p \in \Sigma$.*

(5.7.3) *For every $\chi \in \Delta$, $(P(\chi)\, H^n_{DR}(U/K), \nabla)$ has p-curvature zero for almost all primes p.*

6. Applications to the Hypergeometric Equation

6.0. Relations with Ordinary Differential Equations

(6.0.0) Let T a scheme, S a smooth T-scheme, and (M, ∇) a locally free $\mathcal{O}_S$-module of finite rank with an integrable T-connection. Recall that the *dual* of (M, ∇), noted $(\check{M}, \check{\nabla})$, is defined by

(6.0.0.1) $$\check{M} = Hom_{\mathcal{O}_S}(M, \mathcal{O}_S)$$

with the connection $\check{\nabla}$ defined by requiring that for local section m of M, $\check{m}$ of $\check{M}$, and D of $Der(S/T)$, we have

(6.0.0.2) $$\langle \nabla(D)(m), \check{m}\rangle + \langle m, \check{\nabla}(D)(\check{m})\rangle = D(\langle m, \check{m}\rangle).$$

Iteration of (6.0.0.2) gives, for every integer $n \geqq 1$,

(6.0.0.3) $$D^n(\langle m, \check{m}\rangle) = \sum_{i=0}^{m} \binom{n}{i} \langle (\nabla(D))^{n-i}(m), (\check{\nabla}(D))^i(\check{m})\rangle.$$

Taking $n = p$ in characteristic p, we find

(6.0.1) *Remark.* Hypotheses as in (6.0.0), if T is a scheme of characteristic p, then the p-curvature (cf. [24], 5.2) of $(\check{M}, \check{\nabla})$ is the *negative* of the transpose of the p-curvature of (M, ∇):

(6.0.1.0) $$\langle \psi_\nabla(D)(m), \check{m}\rangle + \langle m, \psi_{\check{\nabla}}(D)(\check{m})\rangle = 0.$$

(6.0.2) **Proposition.** *Hypotheses as in* (6.0.0), *suppose T is a reduced and irreducible scheme of characteristic p. Let k denote its function field. Suppose S is a smooth T-scheme which is irreducible, and denote by K its function field. The following condition on (M, ∇) are equivalent.*

(6.0.2.0) *(M, ∇) has p-curvature zero.*

(6.0.2.1) *$(\check{M}, \check{\nabla})$ has p-curvature zero.*

(6.0.2.2) *The dimension over $k \cdot K^p$ of $(M \otimes K)^\nabla$ is the rank of M.*

(6.0.2.3) *The dimension over $k \cdot K^p$ of $(\check{M} \otimes K)^{\check{\nabla}}$ is the rank of M.*

Proof. By (6.0.1), (6.0.2.0) $\Leftrightarrow$ (6.0.2.1), so it suffices to show that (6.0.2.0) $\Leftrightarrow$ (6.0.2.2). For this, we form the cartesian diagram

(6.0.2.4) $$\begin{array}{ccc} S^{(p)} & \xrightarrow{\sigma} & S \\ \downarrow{\scriptstyle g^{(p)}} & & \downarrow{\scriptstyle g} \\ T & \xrightarrow{F_{\mathrm{abs}}} & T \end{array}$$

and "decorate" it with the relative Frobenius $F: S \to S^{(p)}$

(6.0.2.5)
$$\begin{tikzcd} S \arrow[r, "F"] \arrow[dr, "g"'] & S^{(p)} \arrow[r, "\sigma"] \arrow[d, "g^{(p)}"] & S \arrow[r, "F"] \arrow[d, "g"] & S^{(p)} \arrow[dl, "g^{(p)}"] \\ & T \arrow[r, "F_{\mathrm{abs}}"] & T & \end{tikzcd}$$

The corresponding diagram of function fields is

(6.0.2.6)
$$\begin{tikzcd} K & k \cdot K^p \arrow[l, hook'] & K \arrow[l, "x^p \leftarrow x"'] & k \cdot K^p \arrow[l, hook'] \\ & k \arrow[ul, hook] \arrow[u] & k \arrow[l, "x^p \leftarrow x"'] \arrow[u] \arrow[ur, hook] & \end{tikzcd}$$

By Cartier's theorem (cf. [24], 5.1), (M, ∇) has p-curvature zero if and only if the canonical morphism

(6.0.2.7)
$$F^*(F_*(M^\nabla)) \to M$$

is an isomorphism. On the other hand, since the p-curvature may be interpreted as an $\mathcal{O}_S$-homomorphism

(6.0.2.8)
$$\psi: M \to F_{\mathrm{abs}}^*(\Omega^1_{S/T}) \otimes_{\mathcal{O}_S} M$$

between locally free modules, it vanishes if and only if it vanishes *over the generic point of* S. Using Cartier's theorem over the generic point, the p-curvature vanishes there if and only if the canonical map of K-vector spaces

(6.0.2.9)
$$(M \otimes K)^\nabla \underset{kK^p}{\otimes} K \to M$$

is an isomorphism. Thus it remains to prove only that (6.0.2.9) is an isomorphism if and only if its source and target have the same dimension. This is indeed the case, because (6.0.2.9) is *always* injective; in fact we have the apparently more general

(6.0.3) **Proposition.** *Let T be a scheme of characteristic p, S a smooth T-scheme, and (M, ∇) a quasi-coherent $\mathcal{O}_S$-module with integrable T-connection. The canonical mapping (cf. the diagram* (6.0.2.7))

(6.0.3.0)
$$F_*(M^\nabla) \otimes_{\mathcal{O}_{S^{(p)}}} \mathcal{O}_S = F^*(F_*(M^\nabla)) \to M$$

is always injective.

Proof. The question is local on S, so we may assume S is étale over $\mathbf{A}^n_T$ via $X_1, \ldots, X_n$. Then $\mathcal{O}_S$ is a free $F^{-1}(\mathcal{O}_{S^{(p)}})$-module with basis the monomials $X^W \overset{\text{def'n}}{=} X_1^{W_1} \ldots X_n^{W_n}$ having $0 \leqq W_i \leqq p-1$. Thus we must show that whenever *horizontal* sections m_W, one for each exponent-system W as above verify

(6.0.3.1) $$\sum_W X^W m_W = 0 \quad \text{in } M$$

then all $m_W = 0$.

Let $|W| = \sum W_i \in \mathbf{Z}$, and put $D^W = \prod_i \frac{\left(\frac{\partial}{\partial X_i}\right)^{W_i}}{W_i!}$; this has a meaning because all $W_i \leqq p-1$. For any two exponent V, W as above, we have

(6.0.3.2) $$\begin{aligned} &D^V(X^W) = 0 \quad \text{unless } W_i \geqq V_i \quad \text{for all } i \\ &D^V(X^V) = 1 \\ &D^V(X^W) = \left(\prod_i \binom{W_i}{V_i}\right) X^{W-V} \quad \text{if } W_i \geqq V_i \text{ for all } i. \end{aligned}$$

Suppose that (6.0.3.1) holds, but that not all $m_W = 0$. Among exponent systems W having $m_W \neq 0$, let V have a maximal weight $|V|$. Let's apply $\nabla(D^V)$ to (6.0.3.1):

(6.0.3.3) $$\sum_W \nabla(D^V)(X^W m_W) = 0.$$

Because all the m_W are *horizontal*, $\nabla(D^V)(X^W m_W) = D^V(X^W) \cdot m_W$, whence (6.0.3.3) becomes

(6.0.3.4) $$\sum_W D^V(X^W) \cdot m_W = 0.$$

By construction, $|W| > |V|$ implies $m_W = 0$, while $|W| < |V|$ implies $D^V(X^W) = 0$, and $|V| = |W|$, $V \neq W$ implies $D^V(X^W) = 0$, as follows from (6.0.3.2). Thus (6.0.3.4) reduces to

$$D^V(X^V) m_V = 0$$

whence $m_V = 0$, a contradiction. Q.E.D.

(6.0.4) Let T be any scheme, and S a scheme étale over $\mathbf{A}^1_T$ via a section X of $\mathcal{O}_S$. Let (M, ∇) be a free $\mathcal{O}_S$-module of rank n with integrable T-connection, such that there exists a basis $e_0, \ldots, e_{n-1}$ of M in terms of which the connection takes the form

(6.0.4.0) $$\begin{aligned} &\nabla\left(\frac{d}{dX}\right)(e_i) = e_{i+1} \quad \text{for } 0 \leqq i \leqq n-2 \\ &\nabla\left(\frac{d}{dX}\right)(e_{n-1}) = \sum_{i=0}^{n-1} a_i e_i; \quad a_i \in \Gamma(S, \mathcal{O}_S). \end{aligned}$$

Let $\check{e}_0, \ldots, \check{e}_{n-1}$ denote the *dual* basis of the dual module $\check{M}$. A section

(6.0.4.1) $$\sum_{i=0}^{n-1} f_i \check{e}_i, \quad f_i \text{ local sections of } \mathcal{O}_S$$

is horizontal for the dual connection $\check{\nabla}$ on $\check{M}$ *if and only if* its coefficients satisfy

(6.0.4.2) $$\begin{aligned} \frac{dfi}{dX} &= f_{i+1} \quad \text{for } 0 \leqq i \leqq n-2 \\ \frac{dfn-1}{dX} &= \sum_{i=0}^{n-1} a_i f_i. \end{aligned}$$

Thus

(6.0.4.3) The projection $\check{M} \to \mathcal{O}_S$ defined by $\sum f_i \check{e}_i \to f_0$ induces an *isomorphism* between $(\check{M})^{\check{\nabla}}$, the sheaf of germs of horizontal sections of $\check{M}$, and the sheaf of germs of sections of $\mathcal{O}_S$ which are annihilated by

$$\left(\frac{d}{dX}\right)^n - \sum_{i=0}^{n-1} a_i \left(\frac{d}{dX}\right)^i.$$

If T has characteristic p, this isomorphism is $F^{-1}(\mathcal{O}_{S^{(p)}})$-linear. Combining (6.0.4.3) and (6.0.2), we find

(6.0.5) **Proposition.** *Let T be a reduced and irreducible scheme of characteristic p, S an irreducible scheme which is étale over $\mathbf{A}^1_T$ by means of a section X of $\mathcal{O}_S$. Let k (resp. K) denote the function field of T (resp. S). Let (M, ∇) be as in* (6.0.4). *Then (M, ∇) has p-curvature zero if and only if the field K contains $n = \operatorname{rank} M$ solutions of the ordinary differential equation*

(6.0.5.1) $$\left(\frac{d}{dX}\right)^n (f) = \sum_{i=0}^{n-1} a_i \left(\frac{d}{dX}\right)^i (f)$$

which are linearly independent over the subfield $k \cdot K^p$.

(6.0.6) Suppose further that k is a perfect field (then $kK^p = K^p$). For any discrete valuation v of K, let v_0 denote the induced valuation of K^p. Clearly the ramification index $e(v/v_0)$ (= the index of the value groups) is p. As K is a p-dimensional vector space over K^p, it follows that if t is a uniformizing parameter in K vor v, the elements $1, t, \ldots, t^{p-1}$ form a basis of K over K^p, and thus provide an isomorphism of K^p-vector spaces

(6.0.6.0) $$K \simeq \underbrace{K^p \oplus \cdots \oplus K^p}_{p \text{ times}}.$$

In terms of this isomorphism, the v-adic topology on K is just the p-fold product of the v_0-adic topology on K^p;

$$\text{(6.0.6.1)}\qquad \operatorname{ord}_v\left(\sum_{i=0}^{p-1} f_i^p t^i\right) = \min_i\left(\operatorname{ord}_v(f_i^p t^i)\right) = \min_i\left(i + p \operatorname{ord}_{v_0}(f_i^p)\right).$$

Let K_v (resp. $(K^p)_{v_0}$) denote the completion of K (resp. K^p) with respect to the valuation v (resp. v_0). Then (6.0.6.0) gives

$$\text{(6.0.6.2)}\qquad K_v \simeq \underbrace{(K^p)_{v_0} \oplus \cdots \oplus (K^p)_{v_0}}_{p\,\text{times}} \simeq K \underset{K^p}{\otimes} (K^p)_{v_0}.$$

Since $(K^p)_{v_0} = K_v^p$, we have

$$\text{(6.0.6.3)}\qquad K_v = K \underset{K^p}{\otimes} K_v^p.$$

Let $\operatorname{Soln}(K)$ (resp. $\operatorname{Soln}(K_v)$) denote the K^p (resp. K_v^p) vector space of solutions of the differential equation (6.0.5.1) which lie in K (resp. in K_v). Then

$$\text{(6.0.6.4)}\qquad \operatorname{Soln}(K_v) \xleftarrow{\sim} \operatorname{Soln}(K) \underset{K_p}{\otimes} K_v^p.$$

This is because the differential operator

$$\text{(6.0.6.5)}\qquad \left(\frac{d}{dX}\right)^n - \sum_{i=0}^{n-1} a_i \left(\frac{d}{dX}\right)^i : K \to K$$

is K^p-linear, and the differential operator

$$\text{(6.0.6.6)}\qquad \left(\frac{d}{dX}\right)^n - \sum_{i=0}^{n-1} a_i \left(\frac{d}{dX}\right)^i : K_v \to K_v$$

is deduced from (6.0.6.5) by the (flat!) extension of scalars $K^p \hookrightarrow K_v^p$.

(6.0.6.7) Because we can "clear denominators" by multiplying by p-th powers, the spaces $\operatorname{Soln}(K)$ and $\operatorname{Soln}(K_v)$ are in fact *spanned* by those solutions which lies in the valuation rings of K and K_v respectively, or indeed by solutions which lie in *any* subring of K (resp. of K_v) whose fraction field is all of K (resp. K_v).

Putting this together with (6.0.5), we find

(6.0.7) **Corollary.** *Hypotheses as in* (6.0.5), *suppose k perfect, and let v be any discrete valuation of K. Then (M, ∇) has p-curvature zero if and only if the complete field K_v, or its valuation ring $\mathcal{O}_v$, contains $n = \operatorname{rank} M$ solutions of* (6.0.5.1) *linearly independent over K_v^p.*

(6.0.8) *Remark.* The assumption that k be perfect is used to insure that for any discrete valuation v of K, noting v_0 its restriction to kK^p, we have ramification $e(v/v_0) = p$. When k is not perfect, it can happen that $e(v/v_0) = 1$ (cf. Bourbaki, XXX-Algebra Commutative, chpt. 6, excercise 3b) to § 8, p. 187).

6.1. The Hypergeometric Equation-Definition

(6.1.0) For any scheme T, we will denote by λ the standard coordinate on $\mathbf{A}^1_T$, and by S_T the open subset of $\mathbf{A}^1_T$ where the section $\lambda(1-\lambda)\in \Gamma(\mathbf{A}^1_T, \mathcal{O}_{\mathbf{A}^1_T})$ is invertible.

For any sections $a, b, c\in\Gamma(T, \mathcal{O}_T)$, we define the hypergeometric module $E(a, b, c)$ on S_T to be the free $\mathcal{O}_{S_T}$-module of rank two with base e_0, e_1, and integrable T-connection

(6.1.1)
$$\begin{aligned} &\nabla\left(\frac{d}{d\lambda}\right)(e_0)=e_1\\ &\nabla\left(\frac{d}{d\lambda}\right)(e_1)=-\frac{\big(c-(a+b+1)\lambda\big)}{\lambda(1-\lambda)}e_1+\frac{ab}{\lambda(1-\lambda)}e_0. \end{aligned}$$

As explained in (6.0.4.3), the horizontal sections of the *dual* of $E(a, b, c)$ over an open set $\mathcal{U}$ of S_T "are" the sections $f\in\Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$ which satisfy the hypergeometric equation with parameters a, b, c

(6.1.2)
$$\lambda(1-\lambda)\left(\frac{d}{d\lambda}\right)^2(f)+\big(c-(a+b+1)\lambda\big)\frac{df}{d\lambda}-abf=0.$$

6.2. Theorem. *Let $T=\operatorname{Spec}(A)$, with A a finitely generated subring of $\mathbf{C}$, and let $a, b, c\in A$. The following conditions are equivalent:*

(6.2.1) *The hypergeometric module $E(a, b, c)$ on $S_{\mathbf{C}}$ becomes trivial on a finite étale covering of $S_{\mathbf{C}}$.*

(6.2.2) *There exists a finite extension of the field $\mathbf{C}(\lambda)$ over which $E(a, b, c)$ becomes trivial.*

(6.2.3) *The hypergeometric equation* (6.1.2) *with parameters a, b, c has a full set of algebraic solutions* (cf. (5.4.2.1)).

(6.2.4) *The hypergeometric equation* (6.1.2) *with parameters a, b, c, has a finite monodromy group.*

(6.2.5) *The inverse image of $E(a, b, c)$ on $\mathbf{C}(\lambda)/\mathbf{C}$ has p-curvature zero for almost all primes p* (cf. (5.4.3.2)).

(6.2.6) *The parameters a, b, c are all rational numbers, and for almost all primes p, the reduction modulo p of the hypergeometric equation* (6.1.2) *with parameters a, b, c has two solutions in $\mathbf{F}_p[\lambda]$ (resp. $\mathbf{F}_p(\lambda)$, resp. $\mathbf{F}_p[[\lambda]]$, resp. $\mathbf{F}_p((\lambda))$) which are linearly independent over $\mathbf{F}_p(\lambda^p)$ (resp. $\mathbf{F}(\lambda^p)$, resp. $\mathbf{F}_p(\lambda^p)$, resp. $\mathbf{F}_p((\lambda^p))$).*

Proof. By (5.4.1), (6.2.1)$\Leftrightarrow$(6.2.2), and by (6.0.4.3), (6.2.2)$\Leftrightarrow$(6.2.3). Because the hypergeometric equation has regular singular points, (6.2.3)$\Leftrightarrow$(6.2.4). By (5.4.4), we have (6.2.2)$\Rightarrow$(6.2.5). By ([24], Theorem 1.3.0), (6.2.5) implies that $E(a, b, c)$ has regular singular points on $S_{\mathbf{C}}$, and that its *local monodromy* around each singular point $0, 1, \infty$ is of finite order. In particular, (6.2.5) implies that $E(a, b, c)$ on $S_{\mathbf{C}}$ has *rational* exponents at each of $0, 1, \infty$, or, what is the same, that the hypergeometric equation (6.1.2) with parameters a, b, c has rational exponents at $0, 1, \infty$. As the exponents are 0 and $1-c$ at 0, 0 and $c-a-b$ at 1, and a and b at ∞, (6.2.5) implies that $a, b, c \in \mathbf{Q}$. The rest of the implication (6.2.5)$\Rightarrow$(6.2.6) results from the fact that $E(a, b, c)$ "comes from" T=an open subset of $\operatorname{Spec}(\mathbf{Z})$, so that we may test for "p-curvature zero for almost all p" by seeing if the reduction mod p on $E(a, b, c)$ on $S_{\mathbf{F}_p}$ has p-curvature zero for almost all p (and perform this latter test prime by prime availing ourselves of (6.0.6.7) and (6.0.7)).

The implication (6.2.6)$\Rightarrow$(6.2.5) follows by (6.0.6.7) and (6.0.7).

To conclude the proof, we must show that (6.2.6) implies one of the equivalent conditions (6.2.4) or (6.2.2). This will occupy the rest of this chapter.

6.3. Conclusion of the Proof of 6.2 in a Special Case

(6.3.0) **Proposition.** *Suppose that one of the "exponent differences"*

(6.3.0.0) $$1-c, \quad c-a-b, \quad a-b$$

lies in $\mathbf{Z}$. *Then* (6.2.6) *implies* (6.2.4).

Proof. By ([24], Theorem 1.3.0), the hypothesis (6.2.6) implies that the *local* monodromy of the hypergeometric equation with parameters a, b, c around each singular point $0, 1, \infty$ is of *finite order*. The (topological) fundamental group of $\mathbf{P}^1_{\mathbf{C}} - \{0, 1, \infty\}$ is (the free group) generated by *any two* of the elements $\gamma_0, \gamma_1, \gamma_\infty$ (where γ_s="turning once around s", $s=0, 1, \infty$). Thus the monodromy group of the hypergeometric equation is generated by the local monodromy around *any two* of the three singular points $0, 1, \infty$. As each of local monodromy transformation is of finite order, it suffices to show that any one of them commutes with any other of them (then the monodromy group is an abelian group, generated by two elements of finite order, hence is finite!).

The integrality of one of the exponent differences means that one of the local monodromy transformations has both eigenvalues equal. As it is of finite order, hence semisimple, this local monodromy transformation is necessarily *scalar*, and hence commutes with either of the other two local monodromy transformations.

6.4. Solutions of the Hypergeometric Equation in Characteristic p

(6.4.0) **Proposition** (*compare* [23]). *Let a, b, c be integers contained in $\{0, 1, \ldots, p-1\}$. In order that the hypergeometric equation with parameters $-a, -b, -c$ admit two "$\operatorname{mod} p$" solutions in $\mathbf{F}_p[\lambda]$ which are linearly independent over $\mathbf{F}_p(\lambda)$, it is necessary and sufficient that either $b > c \geqq a$ or $a > c \geqq b$.*

Proof. Let $\ell = \ell(-a, -b, -c)$ denote the hypergeometric differential operator with parameters $-a, -b, -c$;

$$\begin{aligned} \ell = \ell(-a, -b, -c) = \lambda(1-\lambda)\left(\frac{d}{d\lambda}\right)^2 \\ +[-c-(1-a-b)\lambda]\frac{d}{d\lambda} - ab. \end{aligned} \tag{6.4.0.1}$$

ℓ is an $\mathbf{F}_p[\lambda^p]$-linear endomorphism of $\mathbf{F}_p[\lambda]$, which maps the $\mathbf{F}_p$-module N of polynomials of degree at most $p-1$ to itself (in an $\mathbf{F}_p$-linear manner). Because the canonical map

$$\mathbf{F}_p[\lambda] \xleftarrow{\sim} \mathbf{F}_p[\lambda^p] \otimes_{\mathbf{F}_p} N \tag{6.4.0.2}$$

is an isomorphism of $\mathbf{F}_p[\lambda^p]$-modules, we have ($\mathbf{F}_p[\lambda^p]$ being flat over $\mathbf{F}_p$!) an isomorphism of $\mathbf{F}_p[\lambda^p]$-module

$$\operatorname{Ker}\ell \ \text{ in } \ \mathbf{F}_p[\lambda] \xleftarrow{\sim} \mathbf{F}_p[\lambda^p] \otimes_{\mathbf{F}_p} (\operatorname{Ker}\ell \text{ in } N). \tag{6.4.0.3}$$

Similarly, we obtain isomorphisms

$$\operatorname{Ker}\ell \ \text{ in } \ \mathbf{F}_p(\lambda) \xleftarrow{\sim} \mathbf{F}_p(\lambda^p) \otimes_{\mathbf{F}_p} (\operatorname{Ker}\ell \text{ in } N), \tag{6.4.0.4}$$

$$\operatorname{Ker}\ell \ \text{ in } \ \mathbf{F}_p[[\lambda]] \xleftarrow{\sim} \mathbf{F}_p[[\lambda^p]] \otimes_{\mathbf{F}_p} (\operatorname{Ker}\ell \text{ in } N), \tag{6.4.0.5}$$

$$\operatorname{Ker}\ell \ \text{ in } \ \mathbf{F}_p((\lambda)) \xleftarrow{\sim} \mathbf{F}_p((\lambda^p)) \otimes_{\mathbf{F}_p} (\operatorname{Ker}\ell \text{ in } N). \tag{6.4.0.6}$$

We next calculate the matrix of $\ell(-a, -b, -c)$ acting on N, in the basis $1, \lambda, \ldots, \lambda^{p-1}$ of N: for any integer n, we have

$$\ell(-a, -b, -c)(\lambda^n) = n(n-1-c)\lambda^{n-1} - (n-a)(n-b)\lambda^n. \tag{6.4.0.7}$$

Define polynomials P, Q by

$$\begin{aligned} P(X) &= -(X-a)(X-b) \\ Q(X) &= (X+1)(X-c). \end{aligned} \tag{6.4.0.8}$$

Then the matrix of $\ell(-a, -b, -c)$ on N is

$$(6.4.0.9)\qquad \begin{pmatrix} P(0) & Q(0) & & & & \\ & P(1) & Q(1) & & \bigcirc & \\ & & P(2) & \ddots & & \\ & \bigcirc & & \ddots & \ddots & \\ & & & & \ddots & Q(p-2) \\ & & & & & P(p-1) \end{pmatrix}$$

It's proper values are thus its diagonal terms. If $a=b$, then all but one of these diagonal terms is non-zero, and hence zero is a proper value of multiplicity one, which implies that the kernel of $\ell(-a, -b, -c)$ in N is one dimensional. Thus we may assume $a \neq b$, and that $a<b$, since the problem is symmetric in a and b.

As $a \neq b$, the matrix (6.4.0.9) has precisely two diagonal terms which vanish, namely $P(a)$ and $P(v)$. The subspace $N(\leqq b)$ of N consisting of polynomials of degree $\leqq b$ is stable under ℓ, and ℓ is invertible on $N/N(\leqq b)$. Since the formation of the maximal nilpotent subspace is an left exact functor, it follows that the inclusion of $N(\leqq b)$ in N induces an isomorphism

$$(6.4.0.10)\qquad \operatorname{Ker}\ell \ \text{ in } \ N(\leqq b) \xrightarrow{\sim} \operatorname{Ker}\ell \ \text{ in } \ N.$$

Furthermore, the subspace $N(\leqq a-1)$ of $N(\leqq b)$ consisting of polynomials of degree $\leqq a-1$ is stable under ℓ, and ℓ is *invertible* on it. It follows that the projection of $N(\leqq b)$ onto $N(\leqq b)/N(\leqq a-1)$ induces an isomorphism

$$(6.4.0.11)\qquad \operatorname{Ker}\ell \ \text{ in } \ N(\leqq b) \xrightarrow{\sim} \operatorname{Ker}\ell \ \text{ in } \ N(\leqq b)/N(\leqq a-1).$$

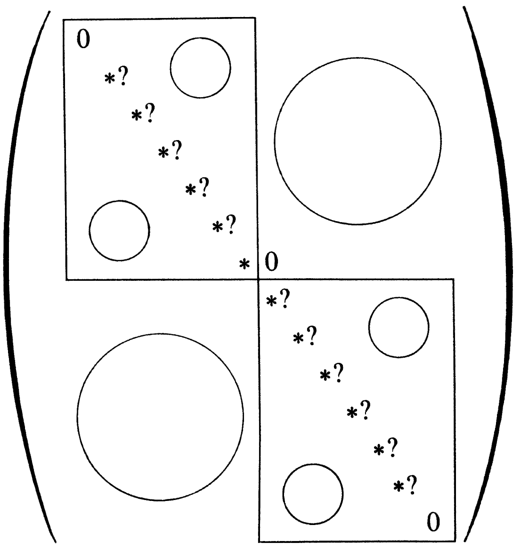
The matrix of $\ell(-a, -b, -c)$ on $N(\leqq b)/N(\leqq a-1)$ is

$$(6.4.0.12)\qquad \begin{pmatrix} P(a) & Q(a) & & & & \\ & P(a+1) & Q(a+1) & & \bigcirc & \\ & & P(a+2) & \ddots & & \\ & \bigcirc & & \ddots & \ddots & \\ & & & & \ddots & Q(b-1) \\ & & & & & P(b) \end{pmatrix}$$

which has the shape

$$\text{(6.4.0.13)} \qquad \begin{pmatrix} 0 & Q(a) & & & \\ & * & Q(a+1) & & \\ & & * & \ddots & \\ & & & * & Q(b-1) \\ & & & & 0 \end{pmatrix}$$

the $*$'s indicating non-zero diagonal terms. If *none* of the off-diagonal terms vanishes, this matrix obviously has as *image* the subspace $N(\leqq b-1)/N(\leqq a-1)$ of $N(\leqq b)/N(\leqq a-1)$, and hence has one-dimensional kernel. Conversely, if one of the off-diagonal terms vanishes, then the matrix (6.4.0.13) is in block form

(6.4.0.14) 

and clearly has two-dimensional kernel. But the off-diagonal terms are $Q(a)$, $Q(a+1), \ldots, Q(b-1)$. As the only possible zero of $Q(X)$ in $\{0, \ldots, p-2\}$ is c, it follows that an off-diagonal term of (6.3.0.13) vanishes if and only if $c \in \{a, a+1, \ldots, b-1\}$. Q.E.D.

6.5. The Calculus of Fractional Parts ([44])

(6.5.0) For any real number x, its fractional part $\langle x \rangle$ is the unique real number satisfying

(6.5.0.1)
$$\begin{aligned} &0 \leqq \langle x \rangle < 1 \\ &x \equiv \langle x \rangle \text{ modulo } \mathbf{Z}. \end{aligned}$$

As real valued function, it satisfies

(6.5.0.2)
$$\begin{aligned} &\langle x \rangle = 0 \Leftrightarrow x \in \mathbf{Z} \\ &\langle \langle x \rangle + \langle x \rangle \rangle = \langle x + y \rangle \\ &\langle x \rangle = \langle y \rangle \Leftrightarrow x \equiv y \bmod \mathbf{Z} \\ &\langle x \rangle + \langle -x \rangle = 1 \quad \text{if} \quad x \in \mathbf{Z}. \end{aligned}$$

(6.5.1) For any prime number p, and any $\alpha \in \mathbf{Q} \cap \mathbf{Z}_p$ (i.e., any rational number with denominator prime to p), we define $R_p(\alpha)$ to be the unique *integer* such that

(6.5.1.0)
$$\begin{aligned} &0 \leqq R_p(\alpha) \leqq p-1 \\ &\alpha \equiv R_p(\alpha) \text{ modulo } p(\mathbf{Q} \cap \mathbf{Z}_p). \end{aligned}$$

As function from $\mathbf{Q} \cap \mathbf{Z}_p$ to $\{0, 1, \ldots, p-1\} \subset \mathbf{Z}$, it satisfies

(6.5.1.1)
$$R_p(\alpha) = R_p(\beta) \Leftrightarrow \alpha \equiv \beta \bmod p(\mathbf{Q} \cap \mathbf{Z}_p).$$

(6.5.2) **Lemma.** *Let $\alpha \in \mathbf{Z}$, $N \in \mathbf{Z}$, and suppose $p \nmid N$. Choose integers B and Δ such that $p\Delta = 1 + NB$. Then*

(6.5.2.0)
$$\frac{1}{p} R_p\left(\frac{-\alpha}{N}\right) = \left\langle \left\langle \frac{\alpha \Delta}{N} \right\rangle - \frac{\alpha}{pN} \right\rangle.$$

Proof.
$$\begin{aligned} R_p\left(\frac{-\alpha}{N}\right) &= R_p\left(\frac{-\alpha B}{NB}\right) = R_p\left(\frac{\alpha B}{1-p\Delta}\right) = R_p(\alpha B) = p\left\langle \frac{\alpha B}{\alpha} \right\rangle \\ &= p\left\langle \frac{\alpha}{p}\left(\frac{p\Delta - 1}{N}\right) \right\rangle = p\left\langle \frac{\alpha \Delta}{N} - \frac{\alpha}{pN} \right\rangle \\ &= p\left\langle \left\langle \frac{\alpha \Delta}{N} \right\rangle - \frac{\alpha}{pN} \right\rangle. \quad \text{Q.E.D.} \end{aligned}$$

(6.5.2.1) **Corollary.** *If $\alpha, N \in \mathbf{Z}$, $\frac{\alpha}{N} \notin \mathbf{Z}$, and $p \nmid N$, then if $p > |\alpha|$, we have*

(6.5.2.1.0)
$$\frac{1}{p} R_p\left(\frac{-\alpha}{N}\right) = \left\langle \frac{\alpha \Delta}{N} \right\rangle - \frac{\alpha}{pN}.$$

Proof. As $\frac{\alpha}{N} \notin \mathbf{Z}$, and Δ is invertible modulo N, $\frac{\alpha \Delta}{N} \notin \mathbf{Z}$, and hence

(6.5.2.1.1)
$$\left\langle \frac{\alpha \Delta}{N} \right\rangle \in \left\{ \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N} \right\}.$$

As $p>|\alpha|$, $\left|\frac{\alpha}{pN}\right|<\frac{1}{N}$, and hence

$$0\leqq\left\langle\frac{\alpha\Delta}{N}\right\rangle-\frac{\alpha}{pN}<1 \tag{6.5.2.1.2}$$

whence

$$\left\langle\frac{\alpha\Delta}{N}\right\rangle-\frac{\alpha}{pN}=\left\langle\left\langle\frac{\alpha\Delta}{N}\right\rangle-\frac{\alpha}{pN}\right\rangle. \quad \text{Q.E.D.} \tag{6.5.2.1.3}$$

(6.5.3) **Corollary.** *Let $\alpha\in\mathbf{Z}$, $N\in\mathbf{Z}$ with $N\neq 0$. For each invertible element Δ in $\mathbf{Z}/N\mathbf{Z}$, we have the limit formula*

$$\lim_{\substack{p\to\infty\\ p\Delta\equiv 1(N)}}\frac{1}{p}R_p\left(\frac{-\alpha}{N}\right)=\begin{cases}\left\langle\frac{\alpha\Delta}{N}\right\rangle & \text{if } \frac{\alpha}{N}\notin\mathbf{Z}\\ 0 & \text{if } \frac{\alpha}{N}\in\mathbf{Z},\ \frac{\alpha}{N}\leqq 0\\ 1 & \text{if } \frac{\alpha}{N}\in\mathbf{Z},\ \frac{\alpha}{N}>0.\end{cases} \tag{6.5.3.1}$$

Proof. If $\alpha/N\notin\mathbf{Z}$, this follows immediately from (6.5.2.1). If $\alpha/N\in\mathbf{Z}$, then for all primes $p>|\alpha|$,

$$\frac{1}{p}R_p\left(\frac{-\alpha}{N}\right)=\left\langle\frac{-\alpha}{Np}\right\rangle=\begin{cases}\frac{-\alpha}{Np} & \text{if } \frac{\alpha}{N}\leqq 0\\ 1-\frac{\alpha}{Np} & \text{if } \frac{\alpha}{N}>0.\end{cases}$$

6.6.0. Proposition. *Let $a, b, c\in\mathbf{Q}$, and suppose that none of the "exponent differences" $1-c$, $c-a-b$, $a-b$ lies in $\mathbf{Z}$. In order that (6.2.6) be verified, it is necessary and sufficient that both of the following conditions be verified.*

(6.6.0.1) *None of the numbers $a, b, c-a, c-b$ lies in $\mathbf{Z}$.*

(6.6.0.2) *Let N be a common denominator for a, b, c. For every $\Delta\in\mathbf{Z}$ invertible in $\mathbf{Z}/N\mathbf{Z}$, we have*

$$\textit{either}\quad 1>\langle a\Delta\rangle>\langle c\Delta\rangle>\langle b\Delta\rangle>0$$
$$\textit{or}\quad 1>\langle b\Delta\rangle>\langle c\Delta\rangle>\langle a\Delta\rangle>0.$$

Proof. Let's first prove sufficiency. Fix $a\Delta$ invertible in $\mathbf{Z}/N\mathbf{Z}$, and suppose, to fix ideas, that $1>\langle b\Delta\rangle>\langle c\Delta\rangle>\langle a\Delta\rangle>0$. Our limit formula (6.5.3.1) then implies that for all sufficiently large primes p such that $p\Delta\equiv 1(N)$, we have

$$1>\frac{1}{p}R_p(-b)>\frac{1}{p}R_p(-c)>\frac{1}{p}R_p(-a)>0. \tag{6.6.0.3}$$

By (6.4.0), this implies that the hypergeometric equation with parameters a, b, c has "two" mod p solutions for all sufficiently large primes p satisfying $p\Delta \equiv 1\ (N)$. As there are only a finite number of Δ's to consider ($\mathbf{Z}/N\mathbf{Z}$ being finite), (6.2.6) follows. Now for the necessity. Choose a Δ invertible in $\mathbf{Z}/N\mathbf{Z}$. By hypothesis, for every sufficiently large prime p with $p\Delta \equiv 1(N)$, we have either

$$\frac{1}{p} R_p(-b) > \frac{1}{p} R_p(-c) \geqq \frac{1}{p} R_p(-a) \tag{6.6.0.4}$$

or

$$\frac{1}{p} R_p(-a) > \frac{1}{p} R_p(-c) \geqq \frac{1}{p} R_p(-b). \tag{6.6.0.5}$$

As there are infinitely many primes with $p\Delta \equiv 1(N)$, either (6.6.0.4) or (6.6.0.5) must hold for infinitely many such primes. As the roles of a and b are symmetric, we may suppose (6.6.0.4) holds infinitely often for such primes. We may then apply the limit formula (6.5.3.1) to (6.6.0.4), to obtain one of the following inequalities

$$1 > \langle b\Delta \rangle \geqq \langle c\Delta \rangle \geqq \langle a\Delta \rangle > 0 \quad \text{if } a \notin \mathbf{Z}, \quad b \notin \mathbf{Z}, \tag{6.6.0.6}$$

$$1 > \langle b\Delta \rangle \geqq \langle c\Delta \rangle > 0 \quad \text{if } b \notin \mathbf{Z}, \quad a \in \mathbf{Z} \quad \text{and} \quad a \leqq 0, \tag{6.6.0.7}$$

$$1 > \langle c\Delta \rangle \geqq \langle a\Delta \rangle > 0 \quad \text{if } a \notin \mathbf{Z}, \quad b \in \mathbf{Z} \quad \text{and} \quad b > 0, \tag{6.6.0.8}$$

$$1 > \langle c\Delta \rangle > 0 \quad \text{if } a, b \in \mathbf{Z}, \quad b > 0 \quad \text{and} \quad a \leqq 0. \tag{6.6.0.9}$$

We now use the hypothesis that *none* of the "exponent differences" $1-c$, $c-a-b$, $a-b$ lie in $\mathbf{Z}$ to eliminate the last three cases. Since $a-b \notin \mathbf{Z}$, (6.6.0.9) is impossible, i.e., not *both* a and b are integers.

If $b \in \mathbf{Z}$, $b > 0$, then $\frac{1}{p} R_p(b) \sim 1$ for *all* p sufficiently large, so for *every* Δ which is invertible modulo N, the limit formula gives

$$1 > \langle c\Delta \rangle \geqq \langle a\Delta \rangle > 0. \tag{6.6.0.10}$$

In particular, replacing Δ by $-\Delta$,

$$1 > \langle -\Delta c \rangle \geqq \langle -\Delta a \rangle > 0. \tag{6.6.0.11}$$

But $\langle -\Delta c \rangle = 1 - \langle \Delta c \rangle$, $\langle -\Delta a \rangle = 1 - \langle \Delta a \rangle$, so that (6.6.0.11) is the opposite inequality to (6.6.0.10), and we conclude $\langle c\Delta \rangle = \langle a\Delta \rangle$ for *all* Δ invertible mod N. Hence $\langle c \rangle = \langle a \rangle$, whence $c-a$ lies in $\mathbf{Z}$, whence (as $b \in \mathbf{Z}$), $c-a-b$ lies in $\mathbf{Z}$, contrary to hypothesis. Thus (6.6.0.8) does not occur.

If $a\in\mathbf{Z}$, $a\leqq 0$, then $\frac{1}{p}R_p(-a)\sim 1$ for *all* p sufficiently large. Then for any Δ invertible mod N, the limit formula gives

$$1>\langle b\Delta\rangle\geqq\langle c\Delta\rangle>0. \tag{6.6.0.12}$$

Just as above, replacing Δ by $-\Delta$ shows $\langle b\Delta\rangle=\langle c\Delta\rangle$ for all invertible Δ modulo N. Hence $c-b$ lies in $\mathbf{Z}$, hence (as $a\in\mathbf{Z}$), $c-b-a$ lies in $\mathbf{Z}$, another contradiction, so (6.6.0.7) does not occur.

Thus we have shown that $a\notin\mathbf{Z}$, $b\notin\mathbf{Z}$, and that for every Δ invertible mod N, we have either

$$1>\langle b\Delta\rangle\geqq\langle c\Delta\rangle\geqq\langle a\Delta\rangle>0 \tag{6.6.0.13}$$

or

$$1>\langle a\Delta\rangle\geqq\langle c\Delta\rangle\geqq\langle b\Delta\rangle>0. \tag{6.6.0.14}$$

We next show that $c-a\notin\mathbf{Z}$ and $c-b\notin\mathbf{Z}$. As by hypothesis $a-b\notin\mathbf{Z}$, we cannot have both $c-a\in\mathbf{Z}$ *and* $c-b\in\mathbf{Z}$. Suppose that $c-a\in\mathbf{Z}$. Then $c-b\notin\mathbf{Z}$, hence for all Δ invertible mod N, $\Delta c-\Delta b\notin\mathbf{Z}$, whence we have either

$$1>\langle b\Delta\rangle>\langle c\Delta\rangle=\langle a\Delta\rangle>0 \tag{6.6.0.15}$$

or

$$1>\langle a\Delta\rangle=\langle c\Delta\rangle>\langle b\Delta\rangle>0. \tag{6.6.0.16}$$

But (6.6.0.15) cannot hold for both Δ *and* $-\Delta$. Hence there exists a Δ invertible modulo N for which (6.6.0.16) holds. Rewriting (6.6.0.1) for *this* Δ via the limit formula (6.5.3.1), we get

$$1>\lim_{\substack{p\to\infty\\ p\Delta\equiv(N)}}\frac{1}{p}R_p(-a)=\lim_{\substack{p\to\infty\\ p\Delta\equiv 1(N)}}\frac{1}{p}R_p(-c)>\lim_{\substack{p\to\infty\\ p\Delta\equiv 1(N)}}\frac{1}{p}R_p(-b)>0. \tag{6.6.0.17}$$

Thus for all p sufficiently large with $p\Delta\equiv 1(N)$, we have

$$\frac{1}{p}R_p(-c)>\frac{1}{p}R_p(-b)>0. \tag{6.6.0.18}$$

This is incompatible with the first of the two following inequalities, one or the other of which holds for any sufficiently large prime in virtue of (6.4.0)

$$\frac{1}{p}R_p(-b)>\frac{1}{p}R_p(-c)\geqq\frac{1}{p}R_p(-a), \tag{6.6.0.19}$$

$$\frac{1}{p}R_p(-a)>\frac{1}{p}R_p(-c)\geqq\frac{1}{p}R_p(-b). \tag{6.6.0.20}$$

Hence (6.6.0.20) holds, and for all sufficiently large p with $p\Delta \equiv 1\,(N)$, we have

$$\frac{1}{p} R_p(-a) > \frac{1}{p} R_p(-c) > \frac{1}{p} R_p(-b). \tag{6.6.0.21}$$

Substituting via (6.5.2.1), (6.6.0.21) may be rewritten

$$\langle a\Delta\rangle - \frac{a}{b} > \langle c\Delta\rangle - \frac{c}{p} > \langle b\Delta\rangle - \frac{b}{p}. \tag{6.6.0.22}$$

Because $c-a \in \mathbf{Z}$, $\langle a\Delta\rangle = \langle c\Delta\rangle$, and hence

$$\frac{-a}{p} > \frac{-c}{p}. \tag{6.6.0.23}$$

Because (6.6.0.16) holds for Δ, (6.6.0.15) must hold for $-\Delta$. Using the limit formula, (6.6.0.15) may be rewritten

$$\begin{aligned} 1 > \lim_{\substack{p\to\infty \\ p\Delta \equiv -1(N)}} \frac{1}{p} R_p(-b) &> \lim_{\substack{p\to\infty \\ p\Delta \equiv -1(N)}} \frac{1}{p} R_p(-c) \\ &= \lim_{\substack{p\to\infty \\ p\Delta \equiv -1(N)}} \frac{1}{p} R_p(-a) > 0. \end{aligned} \tag{6.6.0.24}$$

In particular, for all sufficiently large p with $p\Delta \equiv -1 \bmod (N)$, we have

$$\frac{1}{p} R_p(-b) > \frac{1}{p} R_p(-c). \tag{6.6.0.25}$$

As this is incompatible with (6.6.0.20), (6.6.0.19) holds for such p, and in particular, (6.6.0) gives

$$\frac{1}{p} R_p(-c) \geqq \frac{1}{p} R_p(-a). \tag{6.6.0.26}$$

Substituting via (6.5.2.1), this gives

$$\langle -c\Delta\rangle - \frac{c}{p} \geqq \langle -a\Delta\rangle - \frac{a}{p}. \tag{6.6.0.27}$$

As $c-a \in \mathbf{Z}$, $\langle -c\Delta\rangle = \langle -a\Delta\rangle$, hence

$$\frac{-c}{p} \geqq \frac{-a}{p}, \tag{6.6.0.28}$$

which contradicts (6.6.0.23), and concludes the proof that $c-a \notin \mathbf{Z}$. By symmetry we conclude that $c-b \notin \mathbf{Z}$.

To conclude, we simply observe that as $c-a\notin \mathbf{Z}$, $c-b\notin \mathbf{Z}$, the inequalities in (6.6.0.15) and (6.6.0.16), already established, are necessarily *strict*. This concludes the proof of (6.6.0).

(6.6.1) Combining 6.3 and 6.6.0, we are "reduced" to proving the implication (6.2.6) $\Rightarrow$ (6.2.4) under the additional hypothesis that the rational numbers a, b, c satisfy

$$\text{(6.6.1.0)} \quad a\notin \mathbf{Z},\ b\notin \mathbf{Z},\ c\notin \mathbf{Z},\ c-a\notin \mathbf{Z},\ c-b\notin \mathbf{Z},\ a-b\notin \mathbf{Z},\ c-a-b\notin \mathbf{Z}.$$

(6.6.2) **Corollary.** *In order that rational numbers a, b, c satisfy* (6.2.6) *and* (6.6.1.0), *it is necessary and sufficient that for any $\Delta\in \mathbf{Z}$ which is invertible modulo N for a common denominator N of a, b, c, the rational numbers Δa, Δb, Δc satisfy* (6.2.6) *and* (6.6.1.0).

Proof. The sufficiency is clear; take $\Delta=1$. For the necessity, (6.6.1.0) will still hold because Δ is invertible mod N, and the condition (6.6.0.2) is obviously invariant under $(a, b, c)\to(\Delta a, \Delta b, \Delta c)$ for such Δ. By (6.6.0), (6.6.0.2) and (6.6.1.0) imply (6.2.6).

(6.6.3) **Corollary.** *In order that rational numbers a, b, c satisfy* (6.2.6) *and* (6.6.1.0), *it is necessary and sufficient that for any integers r, s, t, the rational numbers $a+r$, $b+s$, $c+t$ satisfy* (6.2.6) *and* (6.6.1.0).

Proof. Sufficiency is clear, necessity follows from (6.6.0).

6.7. In view of (6.6.3), we must show that if a, b, c satisfy (6.6.1.0), then if (6.2.4) holds for a, b, c, it holds for $a+r$, $b+s$, $c+t$ whenever $r, s, t\in \mathbf{Z}$. In fact, we have the more precise

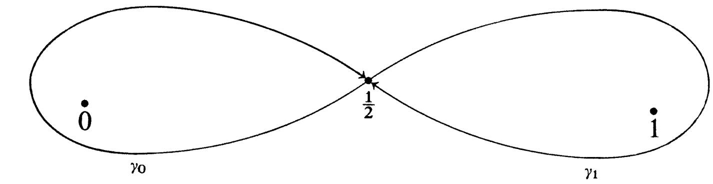
(6.7.1) **Proposition.** *Let a, b, c be complex numbers satisfying* (6.6.1.0). *The monodromy group of the hypergeometric equation with parameters a, b, c depends only on the classes of a, b, and c modulo $\mathbf{Z}$. More precisely, the equivalence class of the two-dimensional complex representation of $\pi_1(\mathbf{P}^1_{\mathbf{C}}-\{0, 1, \infty\}, \frac{1}{2})$ on the space of solutions at $\frac{1}{2}$ of the hypergeometric equation with parameters a, b, c does not change of we replace a, b, c by $a+r$, $b+s$, $c+t$, where r, s, t are arbitrary integers.*

Proof. The proof is by means of the explicit formulas given in Bateman ([1], pp. 93–94). Given complex numbers a, b, c satisfying (6.6.1.0), a basis of the space of germs of solutions at $\frac{1}{2}$ is provided by the functions

$$\text{(6.7.1.0)} \qquad \begin{aligned} u_1 &= F(a, b, c; Z) \\ u_2 &= Z^{1-c} F(a-c+1, b-c+1, 2-c; Z) \end{aligned}$$

Let γ_0 (resp. γ_1) denote the class in $\pi_1(\mathbf{P}^1_{\mathbf{C}}-\{0, 1, \infty\}; \frac{1}{2})$ of the loop "turning once counterclockwise aroung 0 (resp. 1)".

(6.7.1.1)



Let $C_0(a,b,c)$ (resp. $C_1(a,b,c)$) be the matrix of "analytic continuation along γ_0 (resp. γ_1)" of the space of germs of solutions at $\frac{1}{2}$, in terms of the basis (6.7.1.0). Then (cf. [1], p. 93–94),

(6.7.1.2) $$C_0(a,b,c)=\begin{pmatrix}1 & 0\\ 0 & \exp(-2\pi i c)\end{pmatrix}$$

(6.7.1.3) $$C_1(a,b,c)=\begin{pmatrix}B_{1,1}(a,b,c) & B_{2,1}(a,b,c)\\ B_{1,2}(a,b,c) & B_{2,2}(a,b,c)\end{pmatrix}$$

where the coefficients $B_{i,j}(a,b,c)$ are given by

(6.7.1.4) $$B_{1,1}(a,b,c)=1-2i\exp(\pi i(c-a-b))\frac{\sin(\pi a)\sin(\pi b)}{\sin(\pi c)},$$

(6.7.1.5) $$B_{2,2}(a,b,c)=1+2i\exp(\pi i(c-a-b))\frac{\sin(\pi(c-a))\sin(\pi(c-b))}{\sin(\pi c)},$$

(6.7.1.6) $$B_{1,2}(a,b,c)=-2\pi i\exp(\pi i(c-a-b))\frac{\Gamma(c)\,\Gamma(c-1)}{\Gamma(c-a)\,\Gamma(c-b)\,\Gamma(b)\,\Gamma(a)},$$

(6.7.1.7) $$\begin{aligned}&B_{2,1}(a,b,c)\\ &=2\pi i\exp(\pi i(c-a-b))\frac{\Gamma(2-c)\,\Gamma(1-c)}{\Gamma(1-a)\,\Gamma(b-a)\,\Gamma(1+a-c)\,\Gamma(1+b-c)}.\end{aligned}$$

Clearly, the matrix $C_0(a,b,c)$ depends only on the classes of a, b, and c modulo $\mathbf{Z}$. Because the functions $\exp(i\pi X)$, $\sin(\pi X)$, and $\Gamma(X)\cdot\Gamma(1-X)$ all satisfy the function equation

(6.7.1.8) $$f(X+n)=(-1)^n f(X), \qquad \text{for } n\in\mathbf{Z}$$

it follows immediately that each of

(6.7.1.9) $$\begin{gathered}B_{1,1}(a,b,c)\\ B_{2,2}(a,b,c)\\ B_{1,2}(a,b,c)\,B_{2,1}(a,b,c)\end{gathered}$$

depends only on the classes of a, b, and c modulo $\mathbf{Z}$.

From this invariance, together with the fact that neither $B_{1,2}(a,b,c)$ nor $B_{2,1}(a,b,c)$ vanishes (because a,b,c satisfy (6.6.0.1)!), it follows that for any integers r,s,t, the matrices $C_1(a,b,c)$ and $C_1(a+r,b+s,c+t)$

are conjugate (by a *diagonal* matrix) in $GL(2, \mathbf{C})$. As already noted, $C_0(a, b, c) = C_0(a+r, b+s, c+t)$, and as $\pi_1(\mathbf{P}^1_{\mathbf{C}} - \{0, 1, \infty\}; \frac{1}{2})$ is the free group on γ_0 and γ_1, the representations in question are indeed conjugate. Q.E.D.

6.8.0. In this section, we study the relation between the hypergeometric equation and curves of the form $y^n = x^a(x-1)^b(x-\lambda)^c$. This relation was known to Euler, in the form of his integral representation (cf. [1], p. 115, (6), or [45], p. 293)

$$F(\alpha, \beta, \gamma; \lambda) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)} \int_1^\infty x^{\alpha-\gamma}(x-1)^{\gamma-\beta-1}(x-\lambda)^{-\alpha}\,dx.$$

It was reconsidere recently by Messing ([32]), from the conception of whose manuscript we have borrowed heavily.

(6.8.1) Let n, a, b, c be strictly positive integers. We denote by $X(n; a, b, c)$ the spectrum of the smooth $\mathbf{C}(\lambda)$-algebra

(6.8.1.0) $A(n; a, b, c) = \mathbf{C}(\lambda)[x, y, 1/y]/(y^n - x^a(x-1)^b(x-\lambda)^c)\mathbf{C}(\lambda)[x, y, 1/y]$,

which is finite and étale over

(6.8.1.1) $$B = \mathbf{C}(\lambda)[x][1/x(x-1)(x-\lambda)],$$

with basis

(6.8.1.2) $$1, 1/y, 1/y^2, \ldots, 1/y^{n-1}.$$

To any n-th root of unity $\xi \in \boldsymbol{\mu}_n$, we assosciate the $\mathbf{C}(\lambda)$-automorphism of $X(n; a, b, c)$ given by

(6.8.1.3) $$\begin{aligned} x &\mapsto x \\ y &\mapsto \xi y. \end{aligned}$$

This defines an action of $\boldsymbol{\mu}_n$ on $X(n; a, b, c)$. Let χ denote the "identical" character of $\boldsymbol{\mu}_n$, i.e., $\chi(\xi) = \xi$, and for any integer ℓ, let $\chi(\ell)$ denote the character of $\boldsymbol{\mu}_n$ given by

(6.8.1.4) $$\chi(\ell)(\xi) = \xi^\ell.$$

This basis (6.8.1.2) of $A(n; a, b, c)$ over B is just its "isotypique" decomposition according to the characters of $\boldsymbol{\mu}_n$:

(6.8.1.5) $$P(\chi(-\ell))\,A(n; a, b, c) = y^{-\ell} B = \lim_{r \to \infty} y^{-\ell - nr}\,\mathbf{C}(\lambda)[x].$$

The de Rham cohomology group $H^1_{DR}(X(n; a, b, c)/\mathbf{C}(\lambda))$ is the cokernel of the exterior differention map

(6.8.1.6) $$A(n; a, b, c) \xrightarrow{d} \Omega^1_{A(n; a, b, c)/\mathbf{C}(\lambda)}$$

because $X(n; a, b, c)$ is affine and smooth of relative dimension one over $\mathbf{C}(\lambda)$. The module $\Omega^1_{A(n; a, b, c)/\mathbf{C}(\lambda)}$ is free over $A(n; a, b, c)$ with basis dx.

The isotypique decomposition of the complex (6.8.1.6) with respect to $\boldsymbol{\mu}_n$ is given by

$$\begin{array}{ccc} P(\chi(-\ell))(A(n;a,b,c)) & \xrightarrow{d} & P(\chi(-\ell))(A(n;a,b,c))\cdot dx \\ \| & & \| \\ y^{-\ell}B & \xrightarrow{d} & y^{-\ell}B\cdot dx = \varinjlim_{r\to\infty} y^{-\ell-rn}\cdot \mathbf{C}(\lambda)[x]\cdot dx. \end{array} \tag{6.8.1.7}$$

Thus

$$P(\chi(-\ell))H^1_{DR}(X(n;a,b,c)/\mathbf{C}(\lambda)) = \text{cokernel of } y^{-\ell}B \xrightarrow{d} y^{-\ell}B\cdot dx. \tag{6.8.1.8}$$

The gauss-Manin connection $\nabla(d/d\lambda)$ on $H^1_{DR}(X(n;a,b,c)/\mathbf{C}(\lambda))$ is deduced from the endomorphism of the complex (6.8.1.6) given by $\mathrm{Lie}(\partial/\partial\lambda)$, where $\partial/\partial\lambda$ denotes the unique derivation of $A(n;a,b,c)$ which kills x and extends the derivation $d/d\lambda$ of $\mathbf{C}(\lambda)$. Because $\partial/\partial\lambda$ commutes with the action of $\boldsymbol{\mu}_n$ on $A(n;a,b,c)$, $\mathrm{Lie}(\partial/\partial\lambda)$ induces an endomorphism of each of the complexes

$$y^{-\ell}\cdot B \xrightarrow{d} y^{-\ell}B\cdot dx$$

which induces the Gauss-Manin connection on the cokernel.

(6.8.2) **Proposition.** *Let n, a, b, c be strictly positive integers. Then for every integer $\ell \geqq 1$, there is a horizontal morphism of $\mathbf{C}(\lambda)$-modules with connection (cf. (6.1.0))*

$$\begin{aligned} &E\left(\frac{\ell c}{n}, \frac{\ell a+\ell b+\ell c}{n}-1, \frac{\ell a+\ell c}{n}\right)\Big|\mathbf{C}(\lambda) \\ &\quad\to P(\chi(-\ell))\,H^1_{DR}(X(n;a,b,c)/\mathbf{C}(\lambda)) \end{aligned} \tag{6.8.2.0}$$

defined by

$$\begin{aligned} &e_0 \to \text{the class of } \frac{dx}{y^\ell} \\ &e_1 \to \nabla\left(\frac{d}{d\lambda}\right)\left(\text{the class of } \frac{dx}{y^\ell}\right). \end{aligned} \tag{6.8.2.1}$$

Proof. In concrete terms, the assertion is that the operator

$$\begin{aligned} &\lambda(1-\lambda)\,\nabla\left(\frac{d}{d\lambda}\right)^2 + \left[\frac{\ell a+\ell c}{n} - \left(\frac{\ell a+\ell b+2\ell c}{n}\right)\lambda\right]\nabla\left(\frac{d}{d\lambda}\right) \\ &\quad -\frac{\ell c}{n}\left(\frac{\ell a+\ell b+\ell c}{n}-1\right) \end{aligned} \tag{6.8.2.2}$$

annihilates the class of dx/y^ℓ in $H^1_{DR}(X(n;a,b,c)/\mathbf{C}(\lambda))$. In fact, direct calculation (cf. [1], p. 60, (11)) shows that in $\Omega^1_{A(n;a,b,c)/\mathbf{C}(\lambda)}$ we have

$$
\begin{aligned}
& d\left(\frac{\ell c}{n}\cdot\frac{x(x-1)}{y(x-\lambda)}\right) \\
(6.8.2.3)\qquad & =\left\{\lambda(1-\lambda)\left(\mathrm{Lie}\left(\frac{\partial}{\partial\lambda}\right)\right)^2+\left[\frac{\ell a+\ell b}{n}-\left(\frac{\ell a+\ell b+2\ell c}{n}\right)\lambda\right]\right. \\
& \quad\left.\cdot\,\mathrm{Lie}\left(\frac{\partial}{\partial\lambda}\right)-\frac{\ell c}{n}\left(\frac{\ell a+\ell b+\ell c}{n}-1\right)\right\}\left(\frac{dx}{y^{\ell}}\right). \quad \text{Q.E.D.}
\end{aligned}
$$

(6.8.3) **Proposition.** *Hypotheses as in* (6.8.2), *if n does not divide $a+b+c$, then for every integer $\ell \geqq 1$ which is invertible modulo n, we have*

$$
(6.8.3.0)\qquad \mathbf{C}(\lambda)[x]\,\frac{dx}{y^{\ell}}\subset\mathbf{C}(\lambda)\,\frac{dx}{y^{\ell}}+\mathbf{C}(\lambda)\,\frac{x\,dx}{y^{\ell}}+d\left(\mathbf{C}(\lambda)[x]\,\frac{dx}{y^{\ell}}\right).
$$

Proof. For every integer $m\geqq 2$, we readity calculate

$$
\begin{aligned}
d\left(\frac{x^{m-1}(x-1)(x-\lambda)}{y^{\ell}}\right) &= \frac{d(x^{m-1}(x-1)(x-\lambda))}{y^{\ell}} \\
&\quad -\frac{\ell\,x^{m-1}(x-1)(x-\lambda)}{y^{\ell}}\cdot\frac{dy}{y} \\
(6.8.3.1)\qquad &= \frac{d(x^{m+1}-(1+\lambda)\,x^{m}+\lambda\,x^{m-1})}{y^{\ell}} \\
&\quad -\frac{\ell\,x^{m-2}\cdot x(x-1)(x-\lambda)}{y^{\ell}}\left\{\frac{a/n}{x}+\frac{b/n}{x-1}+\frac{c/n}{x-\lambda}\right\} \\
&= \left(m+1-\frac{\ell a+\ell b+\ell c}{n}\right)x^{m}\,\frac{dx}{y^{\ell}}+P(x)\,\frac{dx}{y^{\ell}}
\end{aligned}
$$

where $P(x)\in\mathbf{C}(\lambda)[x]$ is a polynomial of degree at most $m-1$. Because n does not divide $a+b+c$, and ℓ is invertible modulo n, n does not divide $\ell a+\ell b+\ell c$, whence $m+1\neq\left(\frac{\ell a+\ell b+\ell c}{n}\right)$. Q.E.D.

(6.8.4) **Proposition.** *Hypotheses as in* (6.8.2), *if n does not divide $a+b+c$, then for every integer $\ell\geqq 1$ which is invertible modulo n, the image of the mapping* (6.8.2.0) *contains the classes of dx/y^{ℓ} and $x\,dx/y^{\ell}$.*

Proof. Direct calculation shows that in $\Omega^1_{A(n;\,a,b,c)/\mathbf{C}(\lambda)}$ we have

$$
\begin{aligned}
d\left(\frac{x(x-1)}{y^{\ell}}\right) &= \left[2-\left(\frac{\ell a+\ell b+\ell c}{n}\right]\frac{x\,dx}{y^{\ell}}\right. \\
(6.8.4.0)\qquad &\quad +\left(\frac{\ell a+\ell c}{n}-1-\frac{\ell c\lambda}{n}\right)\frac{dx}{y^{\ell}} \\
&\quad +\lambda(1-\lambda)\,\mathrm{Lie}\left(\frac{\partial}{\partial\lambda}\right)\left(\frac{dx}{y^{\ell}}\right).
\end{aligned}
$$

As n does not divide $a+b+c$, and ℓ is invertible modulo n, n does not divide $\ell a+\ell b+\ell c$, whence $2 \nmid \left(\frac{\ell a+\ell b+\ell c}{n}\right)$. Q.E.D.

(6.8.5) **Proposition.** *Hypotheses as in* (6.8.2), *suppose that n does not divide $a+b+c$. For each integer $1 \leqq \ell \leqq n-1$ which is invertible modulo n, consider the sequence of integers $\ell, \ell+n, \ell+2n, \ldots$. For all r sufficiently large, the mapping* (6.8.2.0) *for $\ell+rn$*

$$E(\ell+rn)\frac{c}{n}, (\ell+rn)\left(\frac{a+b+c}{n}\right)-1, (\ell+rn)\left(\frac{a+c}{n}\right) \tag{6.8.5.0}$$

$$\to P(\chi(-\ell))\, H^1_{DR}(X(n;\, a, b, c)/\mathbf{C}(\lambda))$$

is surjective.

Proof. By (6.8.1.8), any finite number of elements in

$$P(\chi(-\ell))\, H^1_{DR}(X(n;\, a, b, c)/\mathbf{C}(\lambda))$$

may be represented by differentials lying in

$$\mathbf{C}(\lambda)[x]\frac{dx}{y^{\ell+rn}}$$

for a suitably large r. By (6.8.3), these elements are linearly dependent upon the classes of

$$\frac{dx}{y^{\ell+rn}}, \quad \frac{x\,dx}{y^{\ell+rn}}. \tag{6.8.5.1}$$

Thus any *finite* number of elements in $P(\chi(-\ell))\, H^1_{DR}(X(n;\, a, b, c)/\mathbf{C}(\lambda))$ span a subspace having dimension at most two. Hence the space $P(\chi(-\ell))\, H^1_{DR}(X(n;\, a, b, c)/\mathbf{C}(\lambda))$ has dimension at most two, and for all sufficiently large integers r, the two classes (6.8.5.1) span. The result now follows from (6.8.4).

(6.8.6) **Proposition.** *Hypotheses as in* (6.8.2), *suppose that n does not divide a, b, c or $a+b+c$. Then for any integer $\ell \geqq 1$ which is invertible modulo n, the mapping* (6.8.2.0) *is an isomorphism:*

$$E\left(\frac{\ell c}{n}, \frac{\ell a+\ell b+\ell c}{n}-1, \frac{\ell a+\ell c}{n}\right)\Big|\mathbf{C}(\lambda) \tag{6.8.6.0}$$

$$\xrightarrow{\sim} P(\chi(-\ell))\, H^1_{DR}(X(n;\, a, b, c)/\mathbf{C}(\lambda)).$$

Proof. By (6.8.5), the target of (6.8.6.0) has dimension at most two, while its source has dimension two. If we can exhibit two linearly independent elements of the image, it will follows that the target has dimension *precisely* two, and that (6.8.6.0) is surjective, and hence an isomorphism (source and target having the same dimension). By (6.8.4), the image contains the classes

(6.8.6.1) $$\frac{dx}{y^{\ell}}, \quad \frac{x\,dx}{y^{\ell}}$$

whence it suffices to prove that they are linearly independent in $P(\chi(-\ell))\, H^1_{DR}(X(n;\, a, b, c)/\mathbf{C}(\lambda))$. Suppose the contrary. Then there exist (by (6.8.1.8))

(6.8.6.2) $$\begin{gathered} \alpha, \beta \in \mathbf{C}(\lambda), \quad \text{not both zero} \\ R(x) \in B = \mathbf{C}(\lambda)[x][1/x(x-1)(x-\lambda)] \end{gathered}$$

such that

(6.8.6.3) $$d\left(\frac{R(x)}{y^{\ell}}\right) = (\alpha + \beta x)\frac{dx}{y^{\ell}} \quad \text{in} \quad \Omega^1_{A(n;\, a, b, c)/\mathbf{C}(\lambda)}.$$

This implies that on the open set where $R(x)$ is invertible, we have

(6.8.6.4) $$\frac{dR(x)}{R(x)} - \ell\frac{dy}{y} = \frac{(\alpha+\beta x)}{R(x)}\,dx$$

or, equivalently

(6.8.6.5) $$\frac{dR(x)}{R(x)} - \ell\left(\frac{a/n}{x} + \frac{b/n}{x-1} + \frac{c/n}{x-\lambda}\right)dx = \frac{(\alpha+\beta x)}{R(x)}\,dx \quad \text{in} \quad \Omega^1_{\mathbf{C}(\lambda, x)/\mathbf{C}(\lambda)}.$$

We first remark that $R(x)$ is necessarily a *polynomial* in x. If not, it has a pole at one of the three points $x=0, 1$ or λ (because $R(x) \in B$), in which case

(6.8.6.6) $$\frac{(\alpha+\beta x)}{R(x)}\,dx$$

would be holomorphic at one of the points $x=0$, 1, or λ and in particular, would have zero *residue* at one of the points $x=0, 1, \lambda$. But

(6.8.6.7) $$\frac{dR(x)}{R(x)} - \ell\left(\frac{a/n}{x} + \frac{b/n}{x-1} + \frac{c/n}{x-\lambda}\right)dx$$

necessarily has residues at $x=0, 1, \lambda$ which are congruent modulo $\mathbf{Z}$ to $-\ell a/n$, $-\ell b/n$, $-\ell c/n$, which by hypothesis are *not* integers. Hence (6.8.6.7) has non-zero (even non-integral) residues at each point $x=0, 1, \lambda$. Thus $R(x)$ is a polynomial. We further claim that the polynomial $R(x)$ has zeroes at $x=0$, 1, and λ. For if not, $R(x)$ would be a unit at one of the points $x=0, 1, \lambda$, and hence

$$\frac{(\alpha+\beta x)\,dx}{R(x)}-\frac{dR(x)}{R(x)} \tag{6.8.6.8}$$

would be holomorphic at one of the points $x=0, 1, \lambda$. In view of (6.8.6.5), this is absurd, hence $R(x)$ *is a polynomial of degree at least three.*

Because $R(x)$ is a polynomial of degree at least three, it follows that

$$\frac{(\alpha+\beta x)\,dx}{R(x)} \tag{6.8.6.9}$$

is *holomorphic* at $x=\infty$, hence has zero residue there. Clearly, the residue at $x=\infty$ of

$$\begin{aligned}\ell\left(\frac{a/n}{x}+\frac{b/n}{x-1}+\frac{c/n}{x-\lambda}\right)dx&=\ell\, a/n\cdot\frac{dx}{x}+\ell\, b/n\cdot\frac{d(x-1)}{x-1}\\&\quad+\ell\, c/n\,\frac{d(x-\lambda)}{x-\lambda}\end{aligned} \tag{6.8.6.10}$$

is just $-\ell\, a/n-\ell\, b/n-\ell\, c/n$, while the residue at $x=\infty$ of

$$\frac{dR(x)}{R(x)} \tag{6.8.6.11}$$

is $-\text{degree}\,(R(x))$. By (6.8.6.5), we must have

$$-\text{degree}\,R(x)+\ell\, a/n+\ell\, b/n+\ell\, c/n=0 \tag{6.8.6.12}$$

which is absurd, as $\ell\left(\frac{a+b+c}{n}\right)$ is not an integer. Q.E.D.

6.9. Conclusion of the Proof of 6.2

(6.9.0) Let a, b, c, n be integers, $n\geqq 1$, and suppose that the hypergeometric equation with parameters a/n, b/n, c/n has two "mod p" solutions for almost all p (i.e., suppose a/n, b/n, c/n verify (6.2.6)). We must prove that $E(a/n, b/n, c/n)$ has a full set of algebraic solutions (cf. (5.4.2.1)), in order to conclude the proof of 6.2.

(6.9.1) As we saw in 6.3, this is the case if any of the exponent differences

$$1-c/n, \quad c/n-a/n-b/n, \quad a/n-b/n \tag{6.9.1.1}$$

is an integer.

(6.9.2) We thus assume that none of the exponent differences is an integer; then 6.6.0 implies that none of the numbers

$$a/n, \quad b/n, \quad c/n-a/n, \quad c/n-b/n \tag{6.9.2.1}$$

is an integer, and that for every integer $\ell \geqq 1$ which is invertible modulo n, the numbers $\ell a/n, \ell b/n, \ell c/n$ also satisfy (6.9.0) (cf. (6.6.2)).

By (6.6.3) and (6.7.1), the hypothesis and conclusion of the alleged implication (6.2.6) $\Rightarrow$ (6.2.4) depend only on the classes modulo $\mathbf{Z}$ of a/n, b/n, c/n. Thus we may and will suppose in addition that

(6.9.2.2) $$\frac{a}{n}>0, \quad \frac{c}{n}-\frac{a}{n}>0, \quad 1+b/n-c/n>0.$$

We define strictly positive integers A, B, C by

(6.9.2.3) $$\frac{A}{n}=\frac{c-a}{n}, \frac{B}{n}=1+\frac{b-c}{n}, \frac{C}{n}=\frac{a}{n}.$$

The non-integrality of $\dfrac{c-a}{n}, \dfrac{c-b}{n}, \dfrac{a}{n}, \dfrac{b}{n}$ (cf. (6.9.1.1), (6.9.2.1)) implies that n does not divide any of the integers $A, B, C, A+B+C$. Then we may apply (6.8.6):

(6.9.3) For any integer $\ell \geqq 1$ which is invertible modulo n,

(6.9.3.0) $$E(\ell a/n, \ell b/n, \ell c/n)|\mathbf{C}(\lambda) \xrightarrow{\sim} P(\chi(-\ell))\, H^1_{DR}(X(n; A, B, C)/\mathbf{C}(\lambda)).$$

Thus, for every integer $1 \leqq \ell \leqq n-1$ which is invertible modulo n,

(6.9.3.1) $$P(\chi(-\ell))\, H^1_{DR}(X(n; A, B, C)/\mathbf{C}(\lambda))$$

with its Gauss-Manin connection has p-curvature zero for almost all prime p (cf. (5.4.3.3)). As ℓ varies over the integers $1 \leqq \ell \leqq n-1$, the characters $\chi(-\ell)$ run over a $\mathbf{Q}$-conjugacy class of irreducible representations of μ_n (namely, the faithful ones). Thus we may apply Theorem 5.7 to deduce that for every integer $\ell \geqq 1$ invertible modulo n,

(6.9.3.2) $$P(\chi(-\ell))\, H^1_{DR}(X(n; A, B, C))$$

with the Gauss-Manin connection has a full set of algebraic solutions. By (6.9.3.0), it follows that for every integer $\ell \geqq 1$ which is invertible modulo n, in particular for $\ell = 1$, the hypergeometric module

(6.9.3.3) $$E(\ell a/n, \ell b/n, \ell c/n)\,|\,\mathbf{C}(\lambda)$$

has a full set of algebraic solutions. Q.E.D.

(6.9.4) **Corollary.** *Let a, b, c, n be integers, $n \geqq 1$, and suppose none of the exponent difference*

(6.9.4.0) $$1-c/n, \quad c/n-a/n-b/n, \quad a/n-b/n$$

is an integer. Then the hypergeometric equation with parameters a/n, b/n, c/n has two algebraic solutions if and only if, for every integer $1 \leqq \ell \leqq n-1$

which is invertible modulo n, we have either

$$1 > \langle \ell a \rangle > \langle \ell c \rangle > \langle \ell b \rangle > 0 \quad \text{or} \quad 1 > \langle \ell b \rangle > \langle \ell c \rangle > \langle \ell a \rangle > 0. \tag{6.9.4.1}$$

Proof. This follows from 6.2 and 6.6.0.

(6.9.5) *Remark.* (Interpretation of (6.9.4)). Given three distinct roots of unity ξ_1, ξ_2, ξ_3 in $\mathbf{C}$ all distinct from 1, we say that ξ_1 and ξ_2 *separate* ξ_3 and 1 if, in marching counterclockwise around the unit circle, starting at 1, we encounter either ξ_1 or ξ_2 but not both before we encounter ξ_3. Let

$$\begin{aligned} \xi_{a/n} &= \exp\left(2\pi i \frac{a}{n}\right) \\ \xi_{b/n} &= \exp\left(2\pi i \frac{b}{n}\right) \\ \xi_{c/n} &= \exp\left(2\pi i \frac{c}{n}\right). \end{aligned} \tag{6.9.5.0}$$

The condition (6.9.4.1) is "simply" that these three roots of unity are distinct from each other and from 1, and that for *every* automorphism σ of $\mathbf{C}$,

$$(\xi_{a/n})^\sigma \ \text{and} \ (\xi_{b/n})^\sigma \ \text{separate} \ (\xi_{c/n})^\sigma \ \text{and} \ 1. \tag{6.9.5.1}$$

7. *p*-Curvature and the Cartier Operation; a Problem on Elliptic Curves

7.0. Let T be a scheme, and S a smooth T-scheme. A connection on $\mathcal{O}_S$ relative to T is necessarily of the form

$$\begin{aligned} &\nabla: \mathcal{O}_S \to \Omega^1_{S/T} \\ &\nabla(f) = df + f\omega \end{aligned} \tag{7.0.1}$$

where $\omega = \nabla(1) \in \Gamma(S, \Omega^1_{S/T})$. Denoting this connection by ∇_ω, we have $(\mathcal{O}_S, \nabla_\omega) \otimes (\mathcal{O}_S, \nabla_{\omega'}) = (\mathcal{O}_S, \nabla_{\omega+\omega'})$, and the mapping $\omega \to \nabla_\omega$ establishes an isomorphism between the group $\Gamma(S, \Omega^1_{S/T})$ and the group of connection on $\mathcal{O}_S$. The subgroup of *closed* one-forms corresponds to the subgroup of integrable connections: indeed, the curvature K_ω of ∇_ω is the mapping

$$\begin{aligned} &K_\omega: \mathcal{O}_S \to \Omega^2_{S/T} \\ &K_\omega(f) = f \cdot d\omega. \end{aligned} \tag{7.0.2}$$

7.1. Suppose further that T is a scheme of characteristic p. Then if $\omega \in \Gamma(S, \Omega^1_{S/T})$ is a closed one-form, the p-curvature, noted ψ_ω, of the integrable T-connection ∇_ω on $\mathcal{O}_S$ is a p-linear mapping

(7.1.0) $$\psi_\omega\colon Der(S/T) \to \mathrm{End}_{\mathcal{O}_S}(\mathcal{O}_S) \simeq \mathcal{O}_S$$

which is *additive* in the variable closed one-form ω.

(7.1.1) Recall from (6.0.2.5) the commutative diagram (in which $g\colon S \to T$ is the structural morphism)

(7.1.1.0)
$$\begin{tikzcd} S \arrow[r, "F"] \arrow[dr, "g"'] & S^{(p)} \arrow[r, "\sigma"] \arrow[d, "g^{(p)}"] & S \arrow[r, "F"] \arrow[d, "g"] & S^{(p)} \arrow[dl, "g^{(p)}"] \\ & T \arrow[r, "F_{\mathrm{abs}}"] & T & \end{tikzcd}$$

the middle square of which is cartesian.

(7.1.2) **Proposition.** *The p-curvature ψ_ω of ∇_ω is given by the formula*

(7.1.2.0) $$\psi_\omega(D) = F^*(\langle \sigma^*(\omega) - \mathscr{C}(\omega), \sigma^*(D)\rangle)$$

where $\mathscr{C}$ is Cartier's isomorphism (2.2.1) $\mathscr{H}^1(F_*\Omega^\bullet_{S/T}) \xrightarrow{\sim} \Omega^1_{S^{(p)}/T}$ *(applied to the class of the closed form ω), $\sigma^*(D)$ is the T-derivation of $S^{(p)}$ defined by*

(7.1.2.1) $\sigma^*(D)(\sigma^*(f)) = \sigma^*(D(f))$, *$f$ a local section of $\mathcal{O}_S$,*

and $\langle\,,\rangle$ is the canonical pairing of $\mathcal{O}_{S^{(p)}}$-modules

(7.1.2.1) $$\Omega^1_{S^{(p)}/T} \otimes Der(S^{(p)}/T) \to \mathcal{O}_{S^{(p)}}.$$

Proof. The proof depends "only" on Hochschild's identity (3.5.2.6), and the definitions.

The question is local on S, which we may suppose étale over $\mathbf{A}^n_T$ via coordinates $s_1, \ldots, s_n$. We first calculate $\psi_\omega(\mathrm{D})$.

(7.1.2.2) $$\psi_\omega(D) = (\nabla_\omega(D))^p - \nabla_\omega(D^p) = (D + \langle\omega, D\rangle)^p - D^p - \langle\omega, D^p\rangle.$$

By Jacobson's formula, we have

(7.1.2.3) $$(D + \langle\omega, D\rangle)^p = D^p + D^{p-1}(\langle\omega, D\rangle) + (\langle\omega, D\rangle)^p.$$

As

(7.1.2.4) $$(\langle\omega, D\rangle)^p = F^*\sigma^*(\langle\omega, D\rangle) = F^*(\langle\sigma^*\omega, \sigma^*D\rangle),$$

we have finally the formula

(7.1.2.5) $$\psi_\omega(D) = D^{p-1}(\langle\omega, D\rangle) + F^*(\langle\sigma^*\omega, \sigma^*D\rangle) - \langle\omega, D^p\rangle.$$

Comparing (7.1.2.5) with the asserted formula (7.1.2.0), we see that it remains to show that

(7.1.2.6) $$F^*(\langle\mathscr{C}\omega, \sigma^*D\rangle) = \langle\omega, D^p\rangle - D^{p-1}(\langle\omega, D\rangle).$$

As ω is *closed*, it may be written (cf. (2.2.1))

(7.1.2.7) $\quad \omega = \sum F^*(a_i)\, s_i^{p-1}\, ds_i + dg; \qquad a_1, \ldots, a_n, g \in \Gamma(S, \mathcal{O}_S).$

As both sides of the asserted equality (7.1.2.6), regarded as function of the closed one-form ω, are $F^{-1}(\mathcal{O}_{S^{(p)}})$-linear and annihilate exact one-forms, we are reduced to the case

(7.1.2.8) $$\omega = s_i^{p-1}\, ds_i.$$

Then (cf. (2.2.1))

(7.1.2.9) $$\mathscr{C}(\omega) = d\big(\sigma^*(s_i)\big),$$

and (7.1.2.6) becomes

(7.1.2.10) $\quad F^*\big(\langle d\sigma^*(s_i), \sigma^*(D)\rangle\big) = \langle s_i^{p-1}\, ds_i, D^p\rangle - D^{p-1}(\langle s_i^{p-1}\, ds_i, D\rangle),$

or equivalently

(7.1.2.11) $$\big(D(s_i)\big)^p = s_i^{p-1}\, D^p(s_i) - D^{p-1}\big(s_i^{p-1}\, D(s_i)\big)$$

which is none other than Hochschild's identity (3.5.2.6). Q.E.D.

(7.1.3) **Corollary.** *Hypotheses as in* 7.1, *the following conditions on a closed one-form* $\omega \in \Gamma(S, \Omega^1_{S/T})$ *are equivalent.*

(7.1.3.1) *The connection* ∇_ω *on* $\mathcal{O}_S$ *has p-curvature zero*

(7.1.3.2) $$\mathscr{C}(\omega) = \sigma^*(\omega)$$

(7.1.3.3) ω *is locally logarithmic, i.e., locally on S there exists an invertible function g with* $\omega = dg/g$.

Proof. (7.1.3.1) $\Leftrightarrow$ (7.1.3.2) by (7.1.2.0). To see that (7.1.3.1) $\Leftrightarrow$ (7.1.3.3), recall that by Cartier's theorem (cf. (6.0.3) and [24], Theorem 5.1), $(\mathcal{O}_S, \nabla_\omega)$ has p-curvature zero if and only if $\mathcal{O}_S$ is spanned as $\mathcal{O}_S$-module by the subsheaf of germs of horizontal functions. Thus (7.1.3.1) is true if and only if there exists locally on S an *invertible* section of $\mathcal{O}_S$, f, with $0 = \nabla_\omega(f) = df + f\omega$, or equivalently (taking $g = f^{-1}$), if and only if ω is locally logarithmic. Q.E.D.

(7.1.4) *Remark.* In Cartier's original "operator" (cf. [2]), the absolute Frobenius $F_{\mathrm{abs}}\colon T \to T$ was an isomorphism; T was, in fact, the spectrum of a perfect field. Then $\sigma\colon S^{(p)} \to S$ was also an isomorphism, and the *original* Cartier operation $\mathscr{C}_{\mathrm{original}}$ was defined as an additive isomorphism

(7.1.4.0) $$\mathscr{C}_{\mathrm{original}}\colon \mathscr{H}^i(\Omega^\bullet_{S/T}) \xrightarrow{\sim} \Omega^i_{S/T}$$

which satisfied

(7.1.4.1) $$\mathscr{C}_{\mathrm{original}}(f^p\,\omega) = f; \qquad \mathscr{C}_{\mathrm{original}}(\omega).$$

In our notation, the relation between $\mathscr{C}_{\text{original}}$ and $\mathscr{C}$ is just

$$\mathscr{C}_{\text{original}} = (\sigma^{-1})^* \circ \mathscr{C}. \tag{7.1.4.2}$$

7.2. Let T be an arbitrary scheme, S a smooth T-scheme. Let $\mathscr{L}$ be an invertible sheaf on S, given by transition function f_{ij} with respect to an open covering $\mathscr{U}_i$ of S. Then giving a T-connection ∇ on $\mathscr{L}$ is equivalent to giving, for each $\mathscr{U}_i$, a one form $\omega_i \in \Gamma(\mathscr{U}_i, \Omega^1_{S/T})$ subject to the compatibility

$$\omega_i - \omega_j = df_{ij}/f_{ij}. \tag{7.2.0.0}$$

The connection is integrable if and only if each ω_i is closed. If $(\mathscr{L}, \nabla)$ and $(\mathscr{L}', \nabla')$ are two invertible sheaves with connection, both given with respect to the same open covering $\mathscr{U}_i$ of S by data (ω_i, f_{ij}) and (ω'_i, f'_{ij}) as above, an isomorphism between $(\mathscr{L}, \nabla)$ and $(\mathscr{L}', \nabla')$ is just the giving of invertible functions $g_i \in \Gamma(\mathscr{U}_i, \mathscr{O}_S^*)$ subject to the conditions

$$\begin{aligned} f_{ij} &= f'_{ij}(g_i/g_j) && \text{on } \mathscr{U}_i \cap \mathscr{U}_j \\ \omega_i - \omega'_i &= dg_i/g_i && \text{on } \mathscr{U}_i. \end{aligned} \tag{7.2.0.1}$$

Thus we have:

(7.2.1) **Proposition.** *The group of isomorphism classes of invertible sheaves on S with T-connections* $(\mathscr{L}, \nabla)$ *(under tensor product) is*

$$\mathbf{H}^1(S, \mathscr{O}_S^* \xrightarrow{d\log} \Omega^1_{S/T}).$$

The group of isomorphism classes of invertible sheaves on S with integrable T-connection is $\mathbf{H}^1(S, \Omega^*_{S/T})$ *where* $\Omega^*_{S/T}$ *denotes the multiplicative de Rham complex*

$$\mathscr{O}_S^* \xrightarrow{d\log} \Omega^1_{S/T} \xrightarrow{d} \Omega^2_{S/T} \xrightarrow{d} \cdots. \tag{7.2.1.0}$$

As a corollary of (7.1.2), we have

(7.2.2) **Proposition.** *If T has characteristic p, and* $(\mathscr{L}, \nabla)$ *is an invertible* $\mathscr{O}_S$*-module with integrable T-connection, given by the data* (ω_i, f_{ij}) *on an open covering* $\mathscr{U}_i$ *of S, its p-curvature*

$$\psi: Der(S/T) \to \operatorname{End}_{\mathscr{O}_S}(\mathscr{L}) \simeq \mathscr{O}_S \tag{7.2.2.0}$$

is the p-linear mapping given locally by

$$\psi(D) = F^*(\langle \sigma^*(\omega_i) - \mathscr{C}(\omega_i), \sigma^*(D)\rangle) \quad \text{over } \mathscr{U}_i. \tag{7.2.2.1}$$

[This formula has a global meaning, because, by (7.1.3.3),

$$\begin{aligned} &\sigma^*(\omega_i) - \mathscr{C}(\omega_i) - \sigma^*(\omega_j) + \mathscr{C}(\omega_j) \\ &\quad = \sigma^*(df_{ij}/f_{ij}) - \mathscr{C}(df_{ij}/f_{ij}) = 0]. \end{aligned} \tag{7.2.2.2}$$

(7.2.3) *Remark.* ψ depends *additively* upon the invertible sheaf with integrable connection $(\mathscr{L}, \nabla)$.

7.3. When T is (the spectrum of) a field k of characteristic zero, the group $\mathbf{H}^1(S, \Omega^*_{S/T})$ has a classical interpretation. Recall that a meromorphic closed one-form ω on S is said to be a *differential of the third kind* if there exists an open covering $\{\mathscr{U}_i\}$ of S, and on each $\mathscr{U}_i$ a closed holomorphic one form $\omega_i \in \Gamma(\mathscr{U}_i, \Omega^1_{S/T})$ and a non-zero *meromorphic* function g_i on $\mathscr{U}_i$, such that

(7.3.0.1) $$\omega = \omega_i + dg_i/g_i \quad \text{on } \mathscr{U}_i, \quad d\omega_i = 0.$$

Now define $f_{ij} = g_j/g_i$, a meromorphic function on $\mathscr{U}_i \cap \mathscr{U}_j$. In fact, $f_{ij} \in \Gamma(\mathscr{U}_i \cap \mathscr{U}_j, \mathcal{O}_S^*)$, because on $\mathscr{U}_i \cap \mathscr{U}_j$

(7.3.0.2) $$df_{ij}/f_{ij} = dg_j/g_j - dg_i/g_i = \omega_i - \omega_j$$

so that df_{ij}/f_{ij} is holomorphic on $\mathscr{U}_i \cap \mathscr{U}_j$. Because we are in characteristic zero, this implies that f_{ij} and $1/f_{ij}$ are holomorphic on $\mathscr{U}_i \cap \mathscr{U}_j$.

Thus we have assosciated to a differential of the third kind ω on S a Cech 1-cocycle (ω_i, f_{ij}) for the complex $\Omega^*_{S/T}$. If, on the same covering $\mathscr{U}_i$, we choose different closed holomorphic one forms ω'_i on $\mathscr{U}_i$ and different meromorphic functions g'_i on $\mathscr{U}_i$, such that

(7.3.0.3) $$\omega = \omega_i + dg_i/dg_i = \omega'_i + dg'_i/g'_i \quad \text{on } \mathscr{U}_i,$$

then $h_i = g'_i/g_i$ lies in $\Gamma(\mathscr{U}_i, \mathcal{O}_S^*)$, because $\omega_i - \omega'_i = dh_i/h_i$ is holomorphic on $\mathscr{U}_i$, and hence (ω_i, f_{ij}) and (ω'_i, f'_{ij}) differ by the coboundary $(dh_i/h_i, h_i/h_j)$.

Thus we have a well-defined mapping from the group of differentials of the third kind on S to the group $\mathbf{H}^1(S, \Omega^*_{S/T})$.

(7.3.1) **Proposition.** *Let S be a smooth connected scheme over a field k of characteristic zero, and let K denote the function field of S. Then the above constructed mapping sits in the exact sequence, in which* d.t.k. (S) *denotes the group of differentials of the third kind on S:*

(7.3.1.0) $$0 \to k^* \to K^* \xrightarrow{d\log} \text{d.t.k.}(S) \to \mathbf{H}^1(S, \Omega^*_{S/k}) \to 0.$$

Proof. To prove exactness at $\mathbf{H}^1(S, \Omega^*_{S/k})$, let (ω_i, f_{ij}) be a Cech 1-cocycle for some covering $\mathscr{U}_i$ of S, and consider the meromorphic differential $\omega = \omega_1$. By the cocycle condition, $d\omega = 0$, and

(7.3.1.1) $$\omega = \omega_i + df_{1i}/f_{1i} \quad \text{on } \mathscr{U}_i,$$

so ω is a d.t.k., and from the data 7.3.1.0, the procedure of (7.3.0) just reconstructs the cocycle (ω_i, f_{ij}).

To prove exactness at d.t.k.(S), notice first that if $\omega = dg/g$, in the construction (7.3.0) we may take all $\omega_i = 0$, all $g_i = g$, whence we construct the zero cocycle (0, 1). If a d.t.k. ω dies in $\mathbf{H}^1(S, \Omega^*_{S/k})$, there exists an open covering $\mathcal{U}_i$ of S, and on each $\mathcal{U}_i$ there exists

$$\begin{aligned} &\text{a closed } \omega_i \in \Gamma(\mathcal{U}_i, \Omega^1_{S/T}) \\ &\text{a meromorphic function } g_i \in K^* \\ &\text{an invertible function } h_i \in \Gamma(\mathcal{U}_i, \mathcal{O}^*_S) \end{aligned} \tag{7.3.1.2}$$

such that

$$\begin{aligned} \omega &= \omega_i + dg_i/g_i && \text{on } \mathcal{U}_i \\ \omega_i &= dh_i/h_i && \text{on } \mathcal{U}_i \\ g_j/g_i &= h_i/h_j && \text{on } \mathcal{U}_i \cap \mathcal{U}_j. \end{aligned} \tag{7.3.1.3}$$

Then $\omega = d(g_i h_i)/g_i h_i$, and $g_i h_i = g_j h_j$ is a global meromorphic function, so ω is logarithmic. Q.E.D.

7.4. The Case of Curves

(7.4.0) Let T be affine, say $T = \operatorname{Spec}(A)$, and S a projective and smooth curve over T with geometrically connected fibres. Then $\Omega^*_{S/T}$ is just the two-term complex

$$\mathcal{O}^*_S \xrightarrow{d\log} \Omega^1_{S/T}. \tag{7.4.0.0}$$

Passing to hypercohomology, we have a long exact sequence

$$\begin{aligned} &0 \to \Gamma(S, \Omega^1_{S/T}) \to \mathbf{H}^1(S, \Omega^*_{S/T}) \\ &\quad \to H^1(S, \mathcal{O}^*_S) \xrightarrow{d\log} H^1(S, \Omega^1_{S/T}) \end{aligned} \tag{7.4.0.1}$$

(the left hand zero because $\Gamma(S, \mathcal{O}^*_S) = \Gamma(T, \mathcal{O}^*_T)$ is annihilated by $d\log$). The trace morphism (cf. [15, 42]) defines a functorial isomorphism

$$H^1(S, \Omega^1_{S/T}) \xrightarrow{\sim} A = \Gamma(T, \mathcal{O}_T). \tag{7.4.0.2}$$

(7.4.0.3) If P is a *section* of S/T, its image $|P|$ is a divisor in S which is smooth over T. The inverse of its sheaf of ideals is an invertible sheaf on S, noted classically $\mathcal{O}_S([P])$. If $P_1, \ldots, P_r$ are sections, and $n_1, \ldots, n_r$ integers, we define the invertible sheaf

$$\mathcal{O}_S(\Sigma\, n_i[P_i]) = \mathcal{O}_S([P_1])^{\otimes n_1} \otimes \cdots \otimes \mathcal{O}_S([P_r])^{\otimes n_r}. \tag{7.4.0.4}$$

The composite mapping

$$H^1(S, \mathcal{O}^*_S) \xrightarrow{d\log} H^1(S, \Omega^1_{S/T}) \xrightarrow[\sim]{\text{trace}} A \tag{7.4.0.5}$$

allows us to attach to the *class* in $H^1(S, \mathcal{O}^*_S)$ of $\mathcal{O}_S(\Sigma\, n_i[P_i])$ an element of A, which is none other than the image in A of the integer $\Sigma\, n_i$. Thus

(7.4.0.6) *the necessary and sufficient condition for the invertible sheaf* $\mathcal{O}_S(\Sigma n_i[P_i])$ *to admit an integrable T-connection is that* $\Sigma n_i = 0$ *in A.*

(7.4.1.0) Suppose further that A is a principal ideal domain, with fraction field k, and that S admits a *section* over A. Because S/A is projective, and A is principal, any k-valued point of S_k extends to a unique section of S over A, so it is the same to assume that S_k has a rational point.

(7.4.1.1) Then the Picard scheme $\mathrm{Pic}_{S/A}$ exists, is an extension of $\mathbf{Z}_A$ by the abelian subscheme $\mathrm{Pic}^0_{S/A}$, and its formation commutes with arbitrary change of base $A \to A'$. Because $\mathrm{Pic}^0_{S/A}$ is *projective*, and A is principal, any k-valued point of $\mathrm{Pic}^0_{S_{k/k}}$ = the Jacobian of S_k extends to a unique section of $\mathrm{Pic}^0_{S/A}$ over A.

(7.4.1.2) Because A is principal and S/A has a section, we have

(7.4.1.3) $$H^1(S, \mathcal{O}_S^*) \xrightarrow{\sim} \mathrm{Pic}_{S/A}(A).$$

We define $\mathbf{H}^1(S, \Omega^*_{S/T})_0$ to be the inverse image of $\mathrm{Pic}^0_{S/A}(A)$ under the canonical mapping

(7.4.1.4) $$\mathbf{H}^1(S, \Omega^*_{S/T}) \to H^1(S, \mathcal{O}_S^*) \simeq \mathrm{Pic}_{S/A}(A).$$

The long cohomology sequence (7.4.0.1) gives a short exact sequence

(7.4.1.5) $$0 \to \Gamma(S, \Omega^1_{S/T}) \to \mathbf{H}^1(S, \Omega^*_{S/T})_0 \to \mathrm{Pic}^0_{S/A}(A) \to 0$$

which the specialist will recognize as the exact sequence of T-valued points of the "universal extension of the abelian scheme $\mathrm{Pic}^0_{S/A}$ by a vector group" (cf. [33, 33a, 40]). We will make no use of this interpretation.

(7.4.2) *Remark.* If the fraction field k of A has characteristic zero, it follows from (7.4.0.5) that

(7.4.2.1) $$\mathbf{H}^1(S, \Omega^*_{S/T})_0 = \mathbf{H}^1(S, \Omega^*_{S/T}).$$

If $A = k$ is a field of characteristic zero, we can combine (7.4.2.1) with (7.3.1.0), and (7.4.1.5) becomes the classical exact sequence (K denoting the function field of S)

(7.4.2.2) $$0 \to \Gamma(S, \Omega^1_{S/k}) \to \frac{\mathrm{d.t.k.}(S)}{d\log(K^*)} \to \mathrm{Pic}^0_{S/k}(k) \to 0,$$

in which a d.t.k. ω on S is mapped to the class of its residue-divisor Σ residue $_P(\omega)[P]$ (compare [33a]). [Remember that on a *curve*, ω is a d.t.k. $\Leftrightarrow \omega$ has at worst first order poles, and *integer* residues.]

(7.4.3) *The Birational Point of View: Again*

(7.4.3.0) Let K/k be a function field in one variable, k a field of characteristic zero and let $\omega \in \Omega^1_{K/k}$ be a (necessarily closed) one-form. We recall the following facts:

(7.4.3.1) In order for the rank-one differential equation (K, ∇_ω) on K to become trivial on a finite extension of K, it is necessary and sufficient that there exist an integer $n \geqq 1$ and a function $g \in K^*$ such that $n\omega = dg/g$. (If $n\omega = dg/g$, then $f = g^{-1/n}$ is a non-zero algebraic solution of the equation $\nabla_\omega(f) = 0$. Conversely, if f is a non-zero algebraic solution, let n be the degree of $K(f)/K$. Then $n\omega = -\mathrm{trace}(df/f) = dg/g$, $g = \mathrm{Norm}(1/f)$.)

(7.4.3.2) If (K, ∇_ω) has p-curvature zero for almost all primes p (in the sense of (5.4.3.2)) (i.e., if ω is "locally logarithmic mod p" for almost all primes p in the same sense (cf. (7.1.3.3)), then ω has at worst first order poles, and rational residues (i.e., there is an integer $n \geqq 1$ such that $n\omega$ is a differential of the third kind on the complete non-singular model of K/k).

(For by ([24], Theorem 13.0), the differential equation (K, ∇_ω) has regular singular points, and rational exponents.)

(7.4.4) **Proposition.** *Assumptions as in* (7.4.3.0), *suppose* $K = k(x)$ *is the rational function field in one variable. If* $\omega \in \Omega^1_{K/k}$, *then the following conditions are equivalent.*

(7.4.4.0) *The differential equation of rank one* (K, ∇_ω) *on* K *becomes trivial on a finite extension* L *of* K.

(7.4.4.1) *The differential equation* (K, ∇_ω) *has p-curvature zero for almost all primes p* (*cf.* (5.4.3.2)).

Proof. (7.4.4.0) $\Rightarrow$ (7.4.4.1) by (5.4.4). By (7.4.3.2), (7.4.4.1) implies that an integral multiple $n\omega$ ($n \geqq 1$) is a differential of the third kind on $\mathbf{P}^1_k$, hence (cf. (7.4.2.2)) is logarithmic. By (7.4.3.1), this implies (7.4.4.0). Alternate proof if $k = \mathbf{C}$: again by ([24], 13.0), the differential equation (K, ∇_ω) has all of its local monodromy transformations of *finite order*. Because the rank is *one*, the global monodromy group of the equation is *abelian*. Because we are working on an open set in $\mathbf{P}^1_{\mathbf{C}}$, the global monodromy group is generated by the (finitely many non-trivial) local ones. Hence the equation has a finite monodromy group. By ([24], 13.0), it has regular singular points, so by the fundamental comparison theorem ([7], Theorem 5.9), it becomes trivial on a finite extension L of K. Q.E.D.

7.5. The Case of Elliptic Curves

(7.5.0) We do not know whether or not the analogue of (7.4.4) is true for elliptic function fields. We would like to explain how this analogue is equivalent, in certain special cases, to a rather striking diophantine statement, which we view as an (unproved!) arithmetic version of Manin's "theorem of the Kernel" (cf. [29]).

(7.5.1) Let E be an elliptic curve (= abelian scheme of relative dimension one) over $T = \mathrm{Spec}(\mathbf{Z}[1/n])$. We *suppose* that the generic fibre $E_{\mathbf{Q}}$ has a non-trivial rational point of order two, for reasons which will become clear.

For each prime p which does not divide n, we put $E_p = E \times_T (\operatorname{Spec} \mathbf{F}_p))$, the elliptic curve over $\mathbf{F}_p$ obtained by reduction modulo p.

The fact that $\mathbf{Z}[1/n]$ is a principal ideal domain, and the projectivity of E over it, gives

$$E(T) \xrightarrow{\sim} E_{\mathbf{Q}}(\mathbf{Q}) \tag{7.5.1.0}$$

and permits the definition of the homomorphism of "reduction modulo p" for each p not dividing n

$$E_{\mathbf{Q}}(\mathbf{Q}) \xrightarrow{\sim} E(T) \longrightarrow E_p(\mathbf{F}_p). \tag{7.5.1.1}$$

This homomorphism is injective on the subgroup of $E_{\mathbf{Q}}(\mathbf{Q})$ consisting of torsion elements of order prime to p. If $p \neq 2$ and p is prime to n, the finite group $E_p(\mathbf{F}_p)$ contains a non-trivial element of order two (this being true of $E_{\mathbf{Q}}(\mathbf{Q})$ by hypothesis), hence has an even number of elements.

(7.5.2) **Lemma.** *Hypotheses as in* (7.5.1), *for all primes $p \geqq 7$ prime to n, the finite group $E_p(\mathbf{F}_p)$ has order prime to p.*

Proof. If not, it has even order divisible by p, hence has $\geqq 2p$ elements. For $p \geqq 7$, this contradicts the Reimann hypothesis

$$|\# E_p(\mathbf{F}_p) - p - 1| \leqq 2\sqrt{p}. \quad \text{Q.E.D.}$$

(7.5.3) **Lemma.** *Hypotheses as in* (7.5.1), *let $p \geqq 7$, $p \nmid n$, and suppose ω is a non-zero differential of the first kind on E_p, i.e., $\omega \in \Gamma(E_p, \Omega^1_{E_p/\mathbf{F}_p})$, and $\omega \neq 0$. Then ω is not locally logarithmic on E_p.*

Proof. If ω is locally logarithmic, $\mathscr{C}(\omega) = \omega$, and by duality the p-th power mapping induces 1 on $H^1(E_p, \mathcal{O}_{E_p})$; i.e., the Hass invariant $H \in \mathbf{F}_p$ of E_p is 1. But (cf. [25]),

$$\# E_p(\mathbf{F}_p) \equiv 1 - H \text{ modulo } p \tag{7.5.3.1}$$

which contradicts 7.5.2 if $H = 1$. Q.E.D.

(7.5.4) Let K denote the function field of $E_{\mathbf{Q}}$, and let $\omega \in \Omega^1_{K/\mathbf{Q}}$ be a differential such that (K, ∇_ω) has p-curvature zero for almost all primes p. We hope that for some integer $n \geqq 1$, $n\omega$ will be *logarithmic*; we will make some "reductions" which are permissible with respect to this hope.

(7.5.5) According to (7.4.3.2), we may, replacing ω by an integral multiple $n\omega$, $n \geqq 1$, suppose that ω is a differential of the third on $E_{\mathbf{Q}}$. By (7.2.1.0), (7.3.1), and (7.4.2.1), a differential of the third kind ω on $E_{\mathbf{Q}}$, taken modulo logarithmic differentials, *is* an isomorphism class of invertible sheaves on $E_{\mathbf{Q}}$ of degree zero with connection $(\mathscr{L}_{\mathbf{Q}}, \nabla_{\mathbf{Q}})$. Because $\operatorname{Pic}^0_{E/T}$ is projective (it's just E, in fact!) and $\mathbf{Z}[1/n]$ is principal, the invertible sheaf $\mathscr{L}_{\mathbf{Q}}$ on $E_{\mathbf{Q}}$ extends uniquely (up to isomorphism) to an invertible sheaf $\mathscr{L}$ on E which fibre by fibre has degree zero (in fact, $\mathscr{L}_{\mathbf{Q}} \sim \mathscr{O}_{E_{\mathbf{Q}}}([P_{\mathbf{Q}}]-[0])$, for a unique $P_{\mathbf{Q}} \in E_{\mathbf{Q}}(\mathbf{Q})$, and, denoting by P its unique prolongation to a section of E over T, we have $\mathscr{L} = \mathscr{O}_E([P]-[0])$ (cf. (7.4.0.3)).

(7.5.6) At the expense of enlarging the integer n, i.e. localizing on $\mathrm{Spec}(\mathbf{Z}[1/n])$, we may suppose that the conncetion $\nabla_{\mathbf{Q}}$ on $\mathscr{L}_{\mathbf{Q}}$ extends to a T-connection ∇ on $\mathscr{L}$. The hypothesis that the original d.t.k. ω on $E_{\mathbf{Q}}$ was locally logarithmic mod p for almost all p is equivalent to the hypothesis that, for almost all p, the inverse image of $(\mathscr{L}, \nabla)$ on E_p has p-curvature zero.

(7.5.7) Suppose we view $(\mathscr{L}, \nabla)$ as an element in $\mathbf{H}^1(E, \Omega^*_{E/T})_0$, which sits in the short exact sequence (7.4.1.5)

$$0 \to \Gamma(E, \Omega^1_{E/T}) \to \mathbf{H}^1(E, \Omega^*_{E/T})_0 \to E(T) \to 0. \tag{7.5.7.1}$$

Reducing modulo p, we find an element $(\mathscr{L}_p, \nabla_p)$ in $\mathbf{H}^1(E_p, \Omega^*_{E_p/\mathbf{F}_p})_0$ which sits in the short exact sequence

$$0 \to \Gamma(E_p, \Omega^1_{E_p/\mathbf{F}_p}) \to \mathbf{H}^1(E_p, \Omega^*_{E_p/\mathbf{F}_p})_0 \to E_p(\mathbf{F}_p) \to 0. \tag{7.5.7.2}$$

For $p \geqq 7$ prime to n, the sequence (7.5.7.2) is an extension of a group of order prime to p (namely $E_p(\mathbf{F}_p)$, cf. (7.5.3)) by a group killed by p (namely the one-dimensional $\mathbf{F}_p$-space $\Gamma(E_p, \Omega^1_{E_p/\mathbf{F}_p})$), hence has a unique section

$$0 \to \Gamma(E_p, \Omega^1_{E_p/\mathbf{F}_p}) \to \mathbf{H}^1(E_p, \Omega^*_{E_p/\mathbf{F}_p})_0 \rightleftarrows E_p(\mathbf{F}_p) \to 0. \tag{7.5.7.3}$$

(7.5.7.4) **Lemma.** *Let $\mathscr{L}_p$ be an invertible sheaf of degree zero on E_p, and let $(\mathscr{L}_p, \nabla(p\text{-can}))$ be its image under the section* (7.5.7.3)*. The connection $\nabla(p\text{-can})$ on $\mathscr{L}_p$ is the unique connection on $\mathscr{L}_p$ which has p-curvature zero.*

Proof. Let m be an integer prime to p such that $(\mathscr{L}_p)^{\otimes m} \simeq \mathscr{O}_{E_p}$. Then $(\mathscr{L}_p, \nabla(p\text{-can}))^{\otimes m} \simeq (\mathscr{O}_{E_p}, d)$, the zero element of $\mathbf{H}^1(E_p, \Omega^*_{E_p/\mathbf{F}_p})$, because $\mathscr{L}_p \mapsto (\mathscr{L}_p, \nabla(p\text{-can}))$ is a homomorphism. Thus $(\mathscr{L}_p, \nabla(p\text{-can}))^{\otimes m}$ has p-curvature zero, but its p-curvature is just m times the p-curvature of $(\mathscr{L}_p, \nabla(p\text{-can}))$ (viewing both as p-linear mappings from $\mathit{Der}(E_p/\mathbf{F}_p)$ to $\mathscr{O}_{E_p}$), and, as m is invertible mod p, it follows that $(\mathscr{L}_p, \nabla(p\text{-can}))$ has p-curvature zero. As the difference between two connections of p-curvature zero on $\mathscr{L}_p$ is a connection of p-curvature zero on $\mathscr{O}_{E_p}$, it follows from (7.5.3) and (7.1.3) that $\mathscr{L}_p$ admits at most one connection of p-curvature zero. Q.E.D.

(7.5.8) **Construction.** Given $(\mathscr{L}, \nabla) \in \mathbf{H}^1(E, \Omega^*_{E/T})_0$, for each prime $p \geqq 7$ prime to n, consider the difference

$$(\mathscr{L}_p, \nabla_p) - (\mathscr{L}_p, \nabla(p\text{-can})) \in \Gamma(E_p, \Omega^1_{E_p/\mathbf{F}_p}).$$

Taking this difference simultaneously for all primes $\geqq 7$, $p \nmid n$, we get a group homomorphism

$$\mathbf{H}^1(E, \Omega^*_{E/T})_0 \to \prod_{\substack{p \geqq 7 \\ p \nmid n}} \Gamma(E_p, \Omega^1_{E_p/\mathbf{F}_p}). \tag{7.5.8.1}$$

(7.5.8.2) **Proposition.** *The kernel of this homomorphism consists precisely of those* $(\mathscr{L}, \nabla)$ *which have p-curvature zero for all* $p \geqq 7$, $p \nmid n$. *The inverse image of the torsion subgroup of the target (those "tuples" having almost all components zero) consists precisely of those* $(\mathscr{L}, \nabla)$ *which have p-curvature zero for almost all primes p.*

Proof. This follows from (7.5.7.4) and the definition of (7.5.8.1). Q.E.D.

(7.5.9) The restriction of the homomorphism (7.5.8.1) to the subgroup $\Gamma(E, \Omega^1_{E/T})$ of T-connections on the structural sheaf $\mathcal{O}_E$ is just the diagonal embedding via simultaneous reduction mod p:

$$\Gamma(E, \Omega^1_{E/T}) \to \prod_{\substack{p \geqq 7 \\ p \nmid n}} \Gamma(E_p, \Omega^1_{E_p/\mathbf{F}_p}). \tag{7.5.9.0}$$

Passing to the quotient, (7.5.8.1) induces a homomorphism

$$E_{\mathbf{Q}}(\mathbf{Q}) = E(T) \to \frac{\prod_{p \geqq 7, p \nmid n} \Gamma(E_p, \Omega^1_{E_p/\mathbf{F}_p})}{\Gamma(E, \Omega^1_{E/T})}. \tag{7.5.9.1}$$

(7.5.9.2) **Proposition.** *A point* $P_{\mathbf{Q}} \in E_{\mathbf{Q}}(\mathbf{Q})$ *lies in the kernel of* (7.5.9.1) *if and only if there exists on the invertible sheaf* $\mathscr{L} = \mathcal{O}_E([P]-[0])$ *on E a T-connection* ∇ *which has p-curvature zero for all* $p \geqq 7$, $p \nmid n$. *If such a connection exists, it is unique.*

Proof. This follows formally from (7.5.8.2).

(7.5.9.3) **Proposition.** *A point* $P_{\mathbf{Q}} \in E_{\mathbf{Q}}(\mathbf{Q})$ *lies in the inverse image of the torsion subgroup if and only if there exists a non-void open set* $\mathscr{U} = \operatorname{Spec}(\mathbf{Z}[1/nm])$ *in* $T = \operatorname{Spec}(\mathbf{Z}[1/n])$ *over which* $\mathscr{L}_{\mathscr{U}} = \mathcal{O}_{E_{\mathscr{U}}}([P]-[0])$ *admits a connection which has p-curvature zero for all primes* $p \geqq 7$ *not dividing n m.*

Proof. If $m P_{\mathbf{Q}}$ lies in the kernel, $\mathscr{L}^{\otimes m}$ admits a unique connection ∇ of p-curvature zero for all $p \geqq 7$ not dividing n. Let $\mathscr{U}$ be the open subset of T where m is invertible. Because m is invertible on $\mathscr{U}$, $\Gamma(E_{\mathscr{U}}, \Omega^1_{E_{\mathscr{U}}/\mathscr{U}})$ is uniquely divisible by m, hence there is a unique connection ∇' on $\mathscr{L}_{\mathscr{U}}$ xuch that $(\mathscr{L}_{\mathscr{U}}, \nabla')^{\otimes m} \sim (\mathscr{L}^{\otimes m}, \nabla)$ on $E_{\mathscr{U}}$. For all primes $p \geqq 7$ not dividing $n m$, $(\mathscr{L}_{\mathscr{U}}, \nabla')$ has p-curvature zero, because its m-th power does, and $p \nmid n$ (compare the proof of (7.5.7.4)). Conversely, if $\mathscr{L}_{\mathscr{U}}$ admits such a connection ∇', let's show that for M a high power of m, $(\mathscr{L}_{\mathscr{U}}, \nabla')^{\otimes M}$ extends to all of E. Since $\mathscr{L}_{\mathscr{U}}$ extends to $\mathscr{L}$, take a finite covering of E by affine open sets V_i which trivialize $\mathscr{L}$, and suppose $\mathscr{L}$ is given by transition functions f_{ij}. Then ∇' is given by 1-forms ω_i holomorphic on $V_i \cap E_{\mathscr{U}} = V_i[1/m]$. By the quasicoherence of $\Omega^1_{E/T}$, there is an power M of m such that $M\omega_i$ is holomorphic on all of V_i. As there are only finitely many V_i, a common M works for all. Now $(\mathscr{L}_{\mathscr{U}}, \nabla')^{\otimes M}$ is given by transition functions $(M\omega_i, f_{ij}^M)$, which *do* extend. Q.E.D.

Combining (7.5.9.3) with the reduction steps (7.5.4)–(7.5.6), we find

(7.5.10) **Tautology.** Let E be an elliptic curve over $T = \mathbf{Z}[1/n]$, with a (non-trivial) rational point of order two. Then the truth of the analogue of (7.4.4) for the function field $K/\mathbf{Q}$ of $E_{\mathbf{Q}}$ is equivalent to the following conjecture:

(7.5.11) **Conjecture.** *The kernel of the homomorphism* (7.5.9.1)

$$E_{\mathbf{Q}}(\mathbf{Q}) = E(T) \rightarrow \frac{\prod_{p \geq 7,\, p \nmid n} \Gamma(E_p, \Omega^1_{E_p/\mathbf{F}_p})}{\Gamma(E, \Omega^1_{E/T})}$$

is contained in the torsion subgroup of $E_{\mathbf{Q}}(\mathbf{Q})$.

Appendix

Riemann's Existence Theorem

An extremely useful form of the theorem is an easy consequence of GAGA, resolution, and the theory of differential equations with regular singular points (cf. [39, 18, 7, 36]).

Theorem. *Let S be a smooth connected* $\mathbf{C}$*-scheme, and let S^{an} denote the corresponding complex manifold. Denote by* Etale(S) (*resp.* Etale(S^{an})) *the category of finite étale coverings of S (resp. S^{an}). Then the natural functor* Etale$(S) \rightarrow$ Etale(S^{an}) *is an equivalence of categories.*

Proof. We will explicitly construct an inverse functor. Let $f: \mathscr{X} \rightarrow S^{\mathrm{an}}$ be finite and étale, and put $\mathscr{F} = f_* \mathcal{O}_{\mathscr{X}}$. Because f is finite and flat, $\mathscr{F}$ is a locally free sheaf of algebras. Because f is étale, it is endowed with a canonical integrable connection ∇, the direct image of the standard connection "exterior differentiation" on $\mathcal{O}_{\mathscr{X}}$, for which the algebra structure is horizontal. By resolution, there is a proper and smooth $\mathbf{C}$-scheme $\bar{S}$ containing S as an open set, such that complement of S in $\bar{S}$ is a divisor D with normal crossings. By the theory of differential equations with regular singular points, there exists for any locally free sheaf with integrable connection $(\mathscr{G}, \nabla)$ on S^{an} a unique pair $(\mathscr{G}_{\mathrm{can}}, \nabla_{\mathrm{can}})$ consisting of a locally free sheaf $\mathscr{G}_{\mathrm{can}}$ on $\bar{S}^{\mathrm{an}}$ which prolongs $\mathscr{G}$, and a "connection with logarithmic singularities along D",

$$\nabla_{\mathrm{can}}: \mathscr{G}_{\mathrm{can}} \rightarrow \Omega^1_{\bar{S}^{\mathrm{an}}}(\log D) \otimes \mathscr{G}_{\mathrm{can}}$$

which extends ∇, whose *exponents* along each local irreducible branch D_i of D lie in the strip $0 \leqq Re(z) < 1$. The pair $(\mathcal{G}_{\mathrm{can}}, \nabla_{\mathrm{can}})$ is called the *quasi*-canonical extension of $(\mathcal{G}, \nabla)$.

Although functorial in $(\mathcal{G}, \nabla)$, the formation of the quasi-canonical extension does not commute with tensor product. However, given $(\mathcal{G}, \nabla)$ and $(\mathcal{H}, \nabla')$ there is a unique *horizontal* morphism

$$\mathcal{G}_{\mathrm{can}} \otimes \mathcal{H}_{\mathrm{can}} \to (\mathcal{G} \otimes \mathcal{H})_{\mathrm{can}}$$

which prolongs the identity. Applying this with $\mathcal{F} = \mathcal{G} = \mathcal{H}$, we find a horizontal morphism

$$\mathcal{F}_{\mathrm{can}} \otimes \mathcal{F}_{\mathrm{can}} \to (\mathcal{F} \otimes \mathcal{F})_{\mathrm{can}}.$$

By functoriality, the algebra structure on $\mathcal{F}$ prolongs to a horizontal morphism

$$(\mathcal{F} \otimes \mathcal{F})_{\mathrm{can}} \to \mathcal{F}_{\mathrm{can}}.$$

Composing these last two maps, we obtain a horizontal multiplication on $\mathcal{F}_{\mathrm{can}}$ which extends the given algebra structure on $\mathcal{F}$.

The locally free sheaf of algebras $\mathcal{F}_{\mathrm{can}}$ on $\bar{S}^{\mathrm{an}}$ corresponds to a finite flat morphism of analytic spaces

$$\bar{f}: \bar{\mathcal{X}} \to \bar{S}^{\mathrm{an}}$$

which prolongs $f: \mathcal{X} \to S^{\mathrm{an}}$. By GAGA, the morphism $\bar{f}$ comes from a unique finite flat morphism of proper **C**-schemes $\bar{X} \to \bar{S}$, and $\bar{X} | S \to S$ is a finite étale covering of S.

This construction is the desired inverse functor Etale $(S^{\mathrm{an}}) \to$ Etale (S). Q.E.D.

References

0. Atiyah, M., Hodge, W.: Integrals of the second kind on an algebraic variety. Annals of Math. **62**, 56–91 (1955).
1. Bateman Manuscript Project (Erdelyi Ed.): Higher transcendental functions, vol. 1. New-York: McGraw-Hill 1953.
2. Cartier, P.: Une nouvelle opération sur les formes différentielles. C. R. Acad. Sci. Paris **244**, 426–428 (1957).
3. Cartier, P.: Questions de rationalité des diviseurs en géométrie algébrique. Bull. Soc. Math. France **86**, 177–251 (1958).
4. Clemens, H.: Picard-Lefschetz theorem for families of nonsingular algebraic varieties acquiring ordinary singularities. Trans. Amer. Math. Soc. **136**, 93–108 (1969).
5. Deligne, P.: Théorème de Lefschetz et critères de dégénérescence de suites spectrales. Publ. Math. I. H. E. S. **35**, 107–126 (1969).

5a. Deligne, P.: Cohomologie des intersections complètes. Exposé XI, SGA 7, 1969. Multigraph available from I. H. E. S, 91-Bures-sur-Yvette, France.

6. Deligne, P.: Travaux de Griffiths. Exposé 376, Séminaire N. Bourbaki, 1969/1970. Lectures Notes in Mathematics **180**. Berlin-Heidelberg-New York: Springer 1971.

7. Deligne, P.: Equations différentielles à points singuliers réguliers. Lecture Notes in Mathematics **163**. Berlin-Heidelberg-New York: Springer 1970.
8. Deligne, P.: Théorie de Hodge, Publ. Math. I. H. E. S. **40** (1971).
8a. Dwork, B.: P-adic cycles. Publ. Math. I. H. E. S. **37**, 27–115 (1970).
9. Goursat, E.: L'Equation d'Euler et de Gauss. Paris: Hermann 1936.
10. Goursat, E.: Intégrales Algébriques. Paris: Hermann 1938.
11. Griffiths, P.: Periods of integrals on algebraic manifolds; summary of main results and discussion of open problems. Bull. Amer. Math. Soc. **75**, 2, 228–296 (1970).
12. Grothendieck, A., Dieudonne, J.: Eléments de géométrie algébrique, Publ. Math. I. H. E. S. **4**, **8**, **11**, **17**, **20**, **24**, **28**, **32**.
13. Grothendieck, A.: Fondements de la géométrie algébrique. Secrétariat mathématique, 11, rue Pierre Curie, Paris 5°, 1962.
14. Grothendieck, A.: On the de Rham cohomology of algebraic varieties. Publ. Math. I. H. E. S. **29** (1966).
15. Hartshorne, R.: Residues and duality. Lecture Notes in Mathematics **20**. Berlin-Heidelberg-New York: Springer 1966.
16. Hasse, H.: Existenz separabler zyklischer unverzweigter Erweiterungskörper vom Primzahlgrade p über elliptischen Funktionenkörpern der Charakteristik p. J. Reine angew. Math. **172**, 77–85 (1934).
17. Hasse, H., Witt, E.: Zyklische unverzweigte Erweiterungskörper vom Primzahlgrade p über einem algebraischen Funktionenkörper der Charakteristik p. Monatsh. für Math. u. Phys. **43**, 477–492 (1936).
18. Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero I, II. Annals of Math. **79**, 109–326 (1964).
19. Hironaka, H.: Bimeromorphic smoothing of a complex-analytic space. Preprint available from Mathematics Institute, University of Warwick (1971).
20. Hochschild, G.: Simple algebras with purely inseparable splitting fields of exponent one. Trans. Amer. Math. Soc. **79**, 477–489 (1955).
21. Honda, T.: Differential equations and formal groups. (Preprint.) Presented at the 1971 U.S-Japan Number Theory conference.
22. Igusa, J.: Class number of a definite quaternion with prime discriminant. Proc. Nat'l. Acad. Sci **44**, 312–314 (1958).
23. Ihara, Y.: Schwarzian equations I. Preprint, 1971.
23a. Katz, N.: On the differential equations satisfied by period matrices. Publ. Math. I. H. E. S. **35** (1968).
24. Katz, N.: Nilpotent connections and the monodromy theorem; application of a result of Turrittin. Publ. Math. I. H. E. S. **39**, 355–232 (1970).
25. Katz, N.: Une formule de congruence pour la fonction ξ. Exposé XXII, SGA 7, 1969, multigraph available from I. H. E. S. 91-Bures-sur-Yvette, France.
26. Kodaira, K., Spencer, D.C.: On deformations of complex structures, I, II, Annals of Math. **67**, 328–466 (1958).
27. Manin, Ju.: Algebraic curves over fields with differentiation. AMS Translations (2), **37**, 59–78 (1964).
28. Manin, Ju.: The Hasse-Witt matrix of an algebraic curve. AMS Translations (2), **45**, 245–264 (1965).
29. Manin, Ju.: Rational points of algebraic curves over function fields, AMS Translations (2), **50**, 189–234 (1966).
30. Mazur, B.: Frobenius and the Hodge filtration, Preprint available form Mathematics Institute, University of Warwick (1971).
31. Mazur, B.: Frobenius and the Hodge filtration. (To appear.)
32. Messing, W.: On the nilpotence of the hypergeometric equation. (To appear.)
33. Messing, W.: The crystals associated to Barsoth-Tate groups; with applications to abelian schemes. Thesis, Princeton, 1971. (To appear.)

33a. Messing, W., Mazur, B.: Cristalline cohomology and the universal extension of an abelian scheme. (To appear.)
34. Mumford, D.: Abelian varieties. Bombay: Oxford University Press 1971.
35. Oda, T., Katz, N.: On the differentiation of de Rham cohomology classes with respect to parameters. J. Math. Kyoto Univer. **8**, 199–213 (1968).
36. Raynaud, M.: Géométrie algébrique et géométrie analytique. Exposé XII, SGA 1. Lecture Notes in Mathematics **224**. Berlin-Heidelberg-New York: Springer 1971.
37. Serre, J. P.: Représentations linéaires des groupes finis. Paris: Hermann 1967.
38. Serre, J. P.: Sur la topologie des variétés algébriques en caractéristique *p*. Symposio International de Topologia Algebraica. Mexico, 1958.
39. Serre, J. P.: Géométrie algébrique et géométrie analytique. Ann Inst. Fourier. Grenoble **6**, 1–42 (1956).
40. Tate, J.: *W-C* groups over *p*-adic fields. Exposé 156. Séminaire Bourbaki 1957/1958. New-York: W. A. Benjamin 1966.
41. Tessier, B., Lejeune, M.: Quelques calculs utiles pour la résolution des singularités. Multigraph available from Centre de Mathématiques, Ecole Polytechnique, 17, rue Descartes, Paris 5°, (1971).
42. Verdier, J. L.: Base change for twisted inverse image of coherent sheaves. Algebraic geometry. Bombay: Oxford University Press 1968.
43. Weil, A.: Variétés Kählériennes. Act. Sci. et Ind. 1267, Paris: Hermann 1958.
44. Weil, A.: Jacobi sums as "Größencharaktere". Trans. Amer. Math. Soc. **73**, 487–495 (1952).
45. Whittaker, E. T., Watson, G. N.: A course of modern analy̆sis. Cambridge: Cambridge University Press 1962.

Nicholas M. Katz
Fine Hall
Department of Mathematics
Princeton University
Princeton, N. J. 08540
USA

(Received May 31, 1972)