A SEMICONTINUITY RESULT FOR MONODROMY UNDER DEGENERATION

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1. Introduction

We fix a a prime number l. We denote by E_λ a finite extension of $\mathbb{Q}_l$ inside a chosen algebraic closure $\bar{\mathbb{Q}}_l$ of $\mathbb{Q}_l$, by $\mathcal{O}_\lambda$ the ring of integers in E_λ, by $\mathbb{F}_\lambda$ its residue field, and by $\bar{\mathbb{F}}_\lambda$ an algebraic closure of $\mathbb{F}_\lambda$. We take as coefficient field A one of the fields on the following list: $\mathbb{F}_\lambda$, $\bar{\mathbb{F}}_\lambda$, E_λ, or $\bar{\mathbb{Q}}_l$.

We work over a field k in which l is invertible. We are given a smooth connected k-scheme S/k, separated and of finite type, of dimension $r \geq 1$. In S, we are given a reduced and irreducible closed subscheme Z, of some dimension $d \geq 0$. We assume that an open dense set $V_1 \subset Z$ is smooth over k (a condition which is automatic if the ground field k is perfect).

On S, we are given a constructible A-sheaf $\mathcal{F}$. Because $\mathcal{F}$ is constructible, its restriction to $S - Z$ is constructible, so there exists a dense open set U in $S - Z$ on which $\mathcal{F}$ is lisse. Similarly, the restriction of $\mathcal{F}$ to V_1 is lisse, so there exists a dense open set V in V_1 on which $\mathcal{F}$ is lisse. Let us denote by j the inclusion of U into S, and by i the inclusion of V into S. Thus we have a lisse A-sheaf $j^\star\mathcal{F}$ on U, and a lisse A-sheaf $i^\star\mathcal{F}$ on V.

In this generality, there is absolutely nothing one can say relating the monodromy of the lisse A-sheaf $i^\star\mathcal{F}$ on V to the monodomy of the lisse A-sheaf $j^\star\mathcal{F}$ on U. However, there is a class of constructible A-sheaves $\mathcal{F}$ on S for which these monodromies are related, namely those "of perverse origin".

We say that a constructible A-sheaf $\mathcal{F}$ on S is of perverse origin if there exists a perverse A-sheaf M on S such that

$$\mathcal{F} \cong \mathcal{H}^{-r}(M).$$

We say that any such perverse sheaf M gives rise to $\mathcal{F}$.

The geometric interest of this notion is that, as we shall recall below (cf Corollaries 5 and 6), for any affine morphism $f : X \to S$, and for any perverse A-sheaf M on X, the constructible A-sheaf

$$R^{-r}f_!M := \mathcal{H}^{-r}(Rf_!(M))$$

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on S is of perverse origin. In particular, suppose X/k is a local complete intersection, everywhere of dimension $n+r$. Then for any lisse A-sheaf $\mathcal{G}$ on X,

$$M := \mathcal{G}[dim(X)] = \mathcal{G}[r+n]$$

is perverse on X. Hence for any affine morphism $f : X \to S$, the constructible A-sheaf

$$R^n f_! \mathcal{G} = \mathcal{H}^{-r}(Rf_!(M))$$

on S is of perverse origin .

Our main result is that for $\mathcal{F}$ of perverse origin, the monodromy of the lisse A-sheaf $i^\star\mathcal{F}$ on V is *"smaller"* than the monodromy of the lisse A-sheaf $j^\star\mathcal{F}$ on U.

To make this precise, let us pick geometric points u of U and v of V. We have monodromy homomorphisms

$$\rho_U : \pi_1(U,u) \to Aut_A(\mathcal{F}_u)$$

and

$$\rho_V : \pi_1(V,v) \to Aut_A(\mathcal{F}_v)$$

attached to $j^\star\mathcal{F}$ on U and to $i^\star\mathcal{F}$ on V respectively. We define compact subgroups

$$\Gamma_U := Image(\rho_U) \subset Aut_A(\mathcal{F}_u)$$

and

$$\Gamma_V := Image(\rho_V) \subset Aut_A(\mathcal{F}_v).$$

Theorem 1. *For S/k smooth and connected of dimension $r \geq 1$ and for $\mathcal{F}$ a constructible A-sheaf on S of perverse origin, the group Γ_V is isomorphic to a subquotient of the group Γ_U. More precisely, there exists a compact group D, a continuous group homomorphism*

$$D \to \Gamma_U,$$

a closed normal *subgroup*

$$I \lhd D,$$

and an A-linear embedding

$$\mathcal{F}_v \subset \mathcal{F}_u^I$$

with the following property: if we view $\mathcal{F}_u^I$ as a representation of D/I, then the subspace

$$\mathcal{F}_v \subset \mathcal{F}_u^I$$

is D/I-stable, and under the induced action of D/I on $\mathcal{F}_v$, the image of D/I in $Aut_A(\mathcal{F}_v)$ is the group Γ_V.

Before giving the proof of the theorem, we must develop some basic properties of sheaves of perverse origin.

2. Basic properties of sheaves of perverse origin

Throughout this section, S/k is smooth and connected, separated and of finite type, of dimension $r \geq 1$, and $\mathcal{F}$ is a constructible A-sheaf on S of perverse origin.

Recall that for M a perverse A-sheaf on S, its ordinary cohomology sheaves $\mathcal{H}^i(M)$ vanish for i outside the interval [-r, 0]. This is obvious for simple objects from their explicit description as middle extensions, and it follows for the general case because any perverse sheaf is a successive extension of finitely many simple objects, cf. [BBD, 2.1.11 and 4.3.1].

Recall that attached to any object K in $D^b_c(S, A)$ are its perverse cohomology sheaves ${}^p\mathcal{H}^i(K)$: these are perverse A-sheaves on S, all but finitely many of which vanish. Their behavior under shifts is given by

$$ {}^p\mathcal{H}^i(K[j]) = {}^p\mathcal{H}^{i+j}(K). $$

A distinguished triangle gives rise to a long exact sequence of perverse cohomology sheaves.

Given two integers $a \leq b$, an object K in $D^b_c(S, A)$ is said to lie in ${}^pD^{[a,b]}$ if its perverse cohomology sheaves ${}^p\mathcal{H}^i(K)$ vanish for i outside the closed interval $[a, b]$. Similarly, an object K in $D^b_c(S, A)$ is said to lie in ${}^pD^{\geq a}$ (respectively in ${}^pD^{\leq b}$) if its perverse cohomology sheaves ${}^p\mathcal{H}^i(K)$ vanish for $i < a$ (respectively for $i > b$). An object of ${}^pD^{[a,a]}$ is precisely an object of the form $M[-a]$ with M perverse. For $a < b$, any object K of ${}^pD^{[a,b]}$ is a successive extension of its shifted perverse cohomology sheaves ${}^p\mathcal{H}^i(K)[-i]$, $i \in [a, b]$. More precisely, any object K of ${}^pD^{[a,b]}$ sits in a distinguished triangle

$$ {}^p\mathcal{H}^a(K)[-a] \to K \to {}^p\tau_{\geq a+1}(K) \to $$

with the third term ${}^p\tau_{\geq a+1}(K)$ in ${}^pD^{[a+1,b]}$.

Lemma 2. *Let K be an object of ${}^pD^{[a,b]}$. Then its ordinary cohomology sheaves $\mathcal{H}^i(K)$ vanish for i outside the interval $[a - r, b]$.*

Proof. We proceed by induction on $b - a$. If K lies in ${}^pD^{[a,a]}$, then K is $M[-a]$ with M perverse. The ordinary cohomology sheaves of M, $\mathcal{H}^i(M)$, vanish for i outside the interval $[-r, 0]$. So those of $M[-a]$ vanish outside $[a - r, a]$. To do the induction step, use the distinguished triangle

$$ {}^p\mathcal{H}^a(K)[-a] \to K \to {}^p\tau_{\geq a+1}(K) \to $$

above, and its long exact cohomology sequence of ordinary cohomology sheaves. □

Corollary 3. *Let K be an object of ${}^pD^{\geq a}$. Then its ordinary cohomology sheaves $\mathcal{H}^i(K)$ vanish for $i < a - r$.*

Proposition 4. *Let K be an object of ${}^pD^{\geq 0}$ on S. Then its $-r$'th ordinary cohomology sheaf $\mathcal{H}^{-r}(K)$ is of perverse origin.*

Proof. We have a distinguished triangle

$$M \to K \to {}^p\tau_{\geq 1}(K) \to$$

whose first term $M :={}^p \mathcal{H}^0(K)$ is perverse, and whose last term ${}^p\tau_{\geq 1}(K)$ lies in ${}^pD^{\geq 1}$. From the long exact cohomology sequence for ordinary cohomology sheaves, we find

$$\mathcal{H}^{-r}(M) \cong \mathcal{H}^{-r}(K).$$

□

Corollary 5. *For any affine morphism $f : X \to S$, and for any perverse A-sheaf M on X, the constructible A-sheaf*

$$R^{-r}f_!M := \mathcal{H}^{-r}(Rf_!(M))$$

on S is of perverse origin.

Proof. Indeed, one knows [BBD, 4.1.1] that for an affine morphism f, $Rf_\star$ maps ${}^pD^{\leq 0}$ on X to ${}^pD^{\leq 0}$ on S. Dually, $Rf_!$ maps ${}^pD^{\geq 0}$ on X to ${}^pD^{\geq 0}$ on S [BBD, 4.1.2]. So for M perverse on X, $Rf_!M$ lies in ${}^pD^{\geq 0}$ on S, and we apply to it the previous result. □

Corollary 6. *Suppose X/k is a local complete intersection, everywhere of dimension $n + r$. For any lisse A-sheaf $\mathcal{G}$ on X, and any affine morphism $f : X \to S$, the constructible A-sheaf $R^nf_!\mathcal{G}$ on S is of perverse origin .*

Proof. Because X/k is a local complete intersection, everywhere of dimension $n + r$, given any lisse A-sheaf $\mathcal{G}$ on X, the object

$$M := \mathcal{G}[dim(X)] = \mathcal{G}[r + n]$$

is perverse on X. [See [Ka-PES II, Lemma 2.1] for the case when $\mathcal{G}$ is the constant sheaf A, and reduce to this case by observing that if K is perverse on X and $\mathcal{G}$ is lisse on X, then $\mathcal{G} \otimes_A K$ is perverse on X.] Now apply the previous result to M. □

Proposition 7. *Let $\mathcal{F}$ be of perverse origin on S. For any connected smooth k-scheme T/k, and for any k-morphism $f : T \to S$, the pullback $f^\star\mathcal{F}$ is of perverse origin on T.*

Proof. We factor f as the closed immersion i of T into $T \times_k S$ by (id, f), followed by the projection pr_2 of $T \times_k S$ onto S. So it suffices to treat separately the case when f is smooth, everywhere of some relative dimension a, and the case when f is a regular closed immersion, everywhere of some codimension b. Pick M perverse on S giving rise to $\mathcal{F}$. In the first case, $K := f^\star M[a]$ is perverse on T [BBD, paragraph above 4.2.5], $dim(T) = r + a$, and

$$f^\star\mathcal{F} = f^\star\mathcal{H}^{-r}(M) = \mathcal{H}^{-r-a}(f^\star M[a]) = \mathcal{H}^{-r-a}(K)$$

is thus of perverse origin on T. In the second case, $K := f^\star M[-b]$ lies in ${}^pD^{[0,b]}$ (apply [BBD, 4.1.10(ii)] b times Zariski locally on T, and observe that the property of lying in ${}^pD^{[0,b]}$ can be checked Zariski locally, since it amounts to the vanishing of certain perverse cohomology sheaves), $dim(T) = r - b$, and

$$f^\star\mathcal{F} = f^\star\mathcal{H}^{-r}(M) = \mathcal{H}^{b-r}(f^\star M[-b]) = \mathcal{H}^{b-r}(K)$$

is thus, by the previous Proposition, of perverse origin on T. $\square$

Proposition 8. *Let $\mathcal{F}$ be of perverse origin on S. Given a connected smooth k-scheme T/k of dimension a, with function field $k(T)$ and generic point $\eta := Spec(k(T))$, and given a smooth k-morphism $f : S \to T$ with generic fibre $S_\eta/k(T)$, the restriction $\mathcal{F}_\eta := \mathcal{F}|S_\eta$ is of perverse origin on $S_\eta/k(T)$.*

Proof. Indeed, for M perverse on S giving rise to $\mathcal{F}$, $M_\eta[-a]$ is perverse on $S_\eta/k(T)$ and gives rise to $\mathcal{F}_\eta$. $\square$

Proposition 9. *Let $\mathcal{F}$ be of perverse origin on S. For $j : U \to S$ the inclusion of any dense open set on which $\mathcal{F}$ is lisse, the canonical map*

$$\mathcal{F} \to j_\star j^\star \mathcal{F}$$

is injective.

Proof. Let M be a perverse A-sheaf on S which gives rise to $\mathcal{F}$. We know [BBD, 4.3.1] that the category of perverse A-sheaves on S is an abelian category which is both artinian and noetherian, so every object is a successive extension of finitely many simple objects. We proceed by induction on the length of M.

If M is simple, then in fact we have $\mathcal{F} \cong j_\star j^\star\mathcal{F}$. To see this, we distinguish two cases. The first case is that M is supported in an irreducible closed subscheme W of S with $dim(W) \leq r - 1$. In this case its ordinary cohomology sheaves $\mathcal{H}^i(M)$ vanish for i outside the closed interval $[1 - r, 0]$. Thus $\mathcal{F} = 0$ in this case, so the assertion trivially holds. The second case is that M is the middle extension of its restriction to any dense open set on which it is lisse. In this case M is $j_{\star !}(j^\star\mathcal{F}[r])$, and from the explicit description [BBD, 2.1.11] of middle extension we see that

$$\mathcal{H}^{-r}(M) = j_\star j^\star \mathcal{F}.$$

In the general case, we pick a simple subobject M_1 of M, and denote

$$M_2 := M/M_1.$$

We put

$$\mathcal{F}_i := \mathcal{H}^{-r}(M_i)$$

for $i = 1, 2$. Then the short exact sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

leads to a left exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2.$$

By induction, we know that

$$\mathcal{F}_i \hookrightarrow j_\star j^\star \mathcal{F}_i$$

for $i = 1, 2$. A simple diagram chase shows that $\mathcal{F} \hookrightarrow j_\star j^\star \mathcal{F}$, as required. □

3. Proof of the theorem

Proof. If the theorem is true for one choice of geometric points u of U and v of V, it is true for any other choice. So we may assume that u lies over the generic point of U, and that v lies over the generic point of V. By Proposition 7, we may at will shrink S to any dense open set $S' \subset S$ which meets Z, then replace U and V by their intersections with S'. This changes neither Γ_U nor Γ_V.

Denote by $Z_1 \subset Z$ the closed subset $Z - V$. Shrinking S to $S - Z_1$ we reduce to the case when Z is smooth in S, and $\mathcal{F}$ is lisse on Z. Pulling $\mathcal{F}$ back to the blowup of S along Z, allowable by Proposition 7, changes neither Γ_U nor Γ_V, and reduces us to the case where Z is a connected smooth divisor in S.

We now focus on the relative dimension r of S/k. We first treat the case $r = 1$. In this case, S is a smooth connected curve over k, and Z is a closed connected subscheme of S which is etale over k. Thus Z is a closed point $Spec(L)$ of S, with L/k a finite separable extension. Deleting from S the finitely many closed points other than Z at which $\mathcal{F}$ is not lisse, we may further assume that $\mathcal{F}$ is a sheaf of perverse origin on S which is lisse on $S - Z$.

We now come to the essential point, that denoting by j the inclusion $S - Z \subset S$, the canonical map

$$\mathcal{F} \to j_\star j^\star \mathcal{F}$$

is injective (by Proposition 9).

Let us spell out what this mean concretely (compare [Mil, II.3.12 and II.3.16]). Denote by K the function field of S, by

$$\eta : Spec(K) \to S$$

the (inclusion of the) generic point of S, by $\bar{K}$ an algebraic closure of K, by

$$\bar{\eta} : Spec(\bar{K}) \to S$$

the (inclusion of the) corresponding geometric generic point of S, and by $K^{sep} \subset \bar{K}$ the separable closure of K inside $\bar{K}$. The stalk $\mathcal{F}_{\bar{\eta}}$ is the representation of $Gal(K^{sep}/K)$ obtained from viewing $\mathcal{F}|S - Z$ as a representation of $\pi_1(S - Z, \bar{\eta})$ and composing with the canonical surjection

$$Gal(K^{sep}/K) \to \pi_1(S - Z, \bar{\eta}).$$

View the closed point Z as a discrete valuation v of K, and extend v to a valuation $\bar{v}$ of $\bar{K}$, with valuation ring $\mathcal{O}_{\bar{v}} \subset \bar{K}$. The residue field of $\mathcal{O}_{\bar{v}}$ is an algebraic closure of L, so a geometric generic point $\bar{z}$ of Z. Inside $Gal(K^{sep}/K)$, we have the corresponding decomposition group $D := D_{\bar{v}}$, which contains as a normal subgroup the inertia group $I := I_{\bar{v}}$. We have a short exact sequence of groups

$$0 \to I \to D \to Gal(L^{sep}/L) \to 0.$$

The stalk $\mathcal{F}_{\bar{z}}$ of $\mathcal{F}$ at $\bar{z}$ is the representation of $D/I \cong Gal(L^{sep}/L) \cong \pi_1(Z, \bar{z})$ given by $\mathcal{F}|Z$. The stalk $(j_\star j^\star \mathcal{F})_{\bar{z}}$ of $j_\star j^\star \mathcal{F}$ at $\bar{z}$ is the representation of $D/I \cong Gal(L^{sep}/L) \cong \pi_1(Z, \bar{z})$ on the subspace of I-invariants in the restriction to D of the representation of $Gal(K^{sep}/K)$ on $\mathcal{F}_{\bar{\eta}}$:

$$(j_\star j^\star \mathcal{F})_{\bar{z}} \cong (\mathcal{F}_{\bar{\eta}})^I$$

For $\mathcal{F}$ any constructible sheaf on S which is lisse on $S - Z$, the injectivity of

$$\mathcal{F} \hookrightarrow j_\star j^\star \mathcal{F}$$

means precisely that at the single point Z we have an A-linear D-equivariant inclusion

$$\mathcal{F}_{\bar{z}} \hookrightarrow (j_\star j^\star \mathcal{F})_{\bar{z}} \cong (\mathcal{F}_{\bar{\eta}})^I.$$

So to conclude the proof of the theorem in the case $r = 1$, we have only to take $I \lhd D$ as our groups, and use the composite homomorphism

$$D \hookrightarrow Gal(K^{sep}/K) \to \pi_1(S - Z, \bar{\eta}) \to Aut_A(\mathcal{F}_{\bar{\eta}})$$

to map D to

$$\Gamma_U := Image(\pi_1(S - Z, \bar{\eta}) \to Aut_A(\mathcal{F}_{\bar{\eta}})).$$

For general $r \geq 1$, we argue as follows. Recall that $\mathcal{F}$ is lisse on an open dense set $U \subset S - Z$, and is lisse on Z. Pick a closed point z in Z. Shrinking S to a Zariski open neighborhood of z in S, we reduce to the case where there exist r functions $s_i, i = 1, \ldots, r$ on S which define an etale k-morphism $S \to \mathbb{A}_k^r$ and such that Z is defined in S by the single equation $s_r = 0$. Then the map $S \to \mathbb{A}_k^{r-1}$ defined by $s_i, i = 1, \ldots, r-1$ is smooth of relative dimension 1, and makes Z etale over $\mathbb{A}_k^{r-1}$. Denote by F the function field of $\mathbb{A}_k^{r-1}$, and make the base change of our situation $(S, Z)/\mathbb{A}_k^{r-1}$ from $\mathbb{A}_k^{r-1}$ to $Spec(F)$. This is allowable by Proposition 8. We obtain a situation $(S_F, Z_F)/F$ in which S_F is a connected smooth curve over F, and Z_F is a nonvoid connected closed subscheme of S_F which is etale over F. Thus Z_F is a closed point $Spec(L)$ of S_F, with L a finite separable extension of F. The $r = 1$ case of the theorem applies to this situation over F. Its truth here gives the theorem for our situation $(S, Z)/\mathbb{A}_k^{r-1}$. Indeed, the connected normal schemes U and U_F have the same function fields, and, being normal, their fundamental groups are both quotients of the absolute galois group of their common function field. So the groups Γ_U and Γ_{U_F} coincide. Similarly for $V = Z$, the groups Γ_V and Γ_{V_F} coincide. $\square$

4. Application to Zariski closures of monodromy groups

As an immediate corollary of the theorem, we obtain:

Corollary 10. *Hypotheses and notations as in the theorem, denote by N_U the rank of the lisse A-sheaf $j^\star\mathcal{F}$ on U, and by N_V the rank of the lisse A-sheaf $i^\star\mathcal{F}$ on V.*

(1) *We have the inequality of ranks*

$$N_V \leq N_U.$$

(2) *Suppose in addition that A is $\bar{\mathbb{Q}}_l$. Denote by G_U the $\bar{\mathbb{Q}}_l$- algebraic group which is the Zariski closure of Γ_U in $Aut_{\bar{\mathbb{Q}}_l}(\mathcal{F}_u) \cong GL(N_U, \bar{\mathbb{Q}}_l)$, and denote by G_V the $\bar{\mathbb{Q}}_l$-algebraic group which is the Zariski closure of Γ_V in $Aut_{\bar{\mathbb{Q}}_l}(\mathcal{F}_v) \cong GL(N_V, \bar{\mathbb{Q}}_l)$. Then the algebraic group G_V is a subquotient of G_U.*

In particular, we have

(2a) *if G_U is finite (or equivalently if Γ_U is finite) then G_V is finite (or equivalently Γ_V is finite),*
(2b) *$dim(G_V) \leq dim(G_U)$,*
(2c) *$rank(G_V) \leq rank(G_U)$.*

5. Appendix: when and where is a sheaf of perverse origin lisse?

Proposition 11. *Hypotheses and notations as in the theorem, the sheaf $\mathcal{F}$ of perverse origin on S is lisse, say of rank N, if and only if its stalks $\mathcal{F}_s$ at all geometric points s of S have constant rank N.*

Proof. It is trivial that if $\mathcal{F}$ is lisse on S, then its stalks have constant rank. Suppose now that $\mathcal{F}$ on S is of perverse origin, and that all its stalks have constant rank N. We must show that $\mathcal{F}$ is lisse on S.

It suffices to show that $\mathcal{F}$ is lisse on an open set $V \subset S$ whose complement $S - V$ has codimension ≥ 2 in S. Indeed, by Zariski-Nagata purity, if we denote by $j : V \to S$ the inclusion, the lisse sheaf $j^\star\mathcal{F}$ on V extends uniquely to a lisse sheaf $\mathcal{E}$ on S. For any lisse sheaf $\mathcal{E}$ on S, and any dense open set $V \subset S$, we have $\mathcal{E} \cong j_\star j^\star\mathcal{E}$. But $j^\star\mathcal{E} \cong j^\star\mathcal{F}$, so we find that $\mathcal{E} \cong j_\star j^\star\mathcal{F}$. In particular, $j_\star j^\star\mathcal{F}$ is lisse on S, and hence all its stalks have constant rank N. The injective (by Proposition 9) map

$$\mathcal{F} \hookrightarrow j_\star j^\star\mathcal{F}$$

must be an isomorphism, because at each geometric point the stalks of both source and target have rank N. Thus we find

$$\mathcal{F} \cong j_\star j^\star\mathcal{F} \cong \mathcal{E},$$

which shows that $\mathcal{F}$ is lisse on S.

We now show that an $\mathcal{F}$ of perverse origin on S which has constant rank N must be lisse. Thanks to the above discussion, we may remove from S any closed set of codimension 2 or more. Thus we may assume that $\mathcal{F}$ is lisse on an open set

$U \subset S$, inclusion denoted $j : U \to S$, and that that the complement $S - U$ is a disjoint union of finitely many irreducible divisors Z_i. Denote by η the generic point of S. At the generic point z_i of Z_i, the local ring $\mathcal{O}_{S,z_i}$ is a discrete valuation ring. For suitable geometric points $\bar{\eta}$ and $\bar{z}_i$ lying over η and z_i respectively, we have the inertia and decomposition groups I_i and D_i. We have an injective (by Propostion 9) D_i-equivariant map

$$\mathcal{F}_{\bar{z}_i} \hookrightarrow (j_\star j^\star \mathcal{F})_{\bar{z}_i} \cong (\mathcal{F}_{\bar{\eta}})^{I_i} \subset \mathcal{F}_{\bar{\eta}}.$$

As both $\mathcal{F}_{\bar{z}_i}$ and $\mathcal{F}_{\bar{\eta}}$ have the same rank N, all the displayed maps must be isomorphisms. Therefore I_i acts trivially on $\mathcal{F}_{\bar{\eta}}$. Thus $j^\star \mathcal{F}$ is a lisse sheaf on U which is unramified at the generic point of each Z_i. So by Zariski-Nagata purity, $j^\star \mathcal{F}$ extends to a lisse sheaf $\mathcal{E}$ on S. Exactly as in the paragraph above, we see that

$$\mathcal{F} \cong j_\star j^\star \mathcal{F} \cong \mathcal{E},$$

which shows that $\mathcal{F}$ is lisse on S. □

Proposition 12. *Hypotheses and notations as in the theorem, let $\mathcal{F}$ be of perverse origin on S. The integer-valued function on S given by*

$$s \mapsto rank(\mathcal{F}_s)$$

is lower semicontinuous, i.e., for every integer $r \geq 0$, there exists a reduced closed subscheme $S_{\leq r} \subset S$ such that a geometric point s of S lies in $S_{\leq r}$ if and only if the stalk $\mathcal{F}_s$ has rank $\leq r$. If we denote by N the generic rank of $\mathcal{F}$, then $S = S_{\leq N}$, and $S - S_{\leq N-1}$ is the largest open set on which $\mathcal{F}$ is lisse.

Proof. Once we show the lower semicontinuity of the rank, the second assertion is immediate from the preceeding proposition.

To show the lower semicontinuity, we first reduce to the case when k is perfect. Indeed, for k^{per} the perfection of k, and $S_1 := S \otimes_k k^{per}$, the natural map $\pi : S_1 \to S$ is a universal homeomorphism, S_1/k^{per} is smooth and connected, and $\pi^\star \mathcal{F}$ is of perverse origin on S_1/k^{per}. Thus it suffices to treat the case when k is perfect.

Because $\mathcal{F}$ is a constructible sheaf, its rank function is constructible. So to show lower semicontinuity, it suffices to show that the rank decreases under specialization. Thus let $Z \subset S$ be an irreducible reduced closed subscheme, with geometric generic point $\bar{\eta}_Z$. We must show that at any geometric point $z \in Z$, we have

$$rank(\mathcal{F}_{\bar{\eta}_Z}) \leq rank(\mathcal{F}_z).$$

Because the field k is perfect, we may, by de Jong [de Jong, Thm. 3.1], find a smooth connected k-scheme Z_1 and a proper surjective k-morphism $f : Z_1 \to Z$. By Proposition 7, $f^\star \mathcal{F}$ is of perverse origin on Z_1. By Corollary 10, (1), applied on Z_1, we get the asserted inequality of ranks. □

Acknowledgement This work began as a diophantine proof of Parts (2a) and (2c) of the Corollary, in the special case when the ground field k is finite, and when the

sheaf $\mathcal{F}$ of perverse origin on S is $R^n f_! \mathcal{G}$ for $f : X \to S$ a smooth affine morphism everywhere of relative dimension n, with $\mathcal{G}$ a lisse sheaf on X. I owe to Deligne both the idea of formulating the theorem in terms of subquotients , and the idea that it applied to sheaves of perverse origin.

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