ESTIMATES FOR "NONSINGULAR" MULTIPLICATIVE CHARACTER SUMS

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1. Introduction, and the basic setting

Let k be a finite field of characteristic p and cardinality q, ψ a nontrivial $\mathbb{C}^\times$-valued additive character of k, and χ a nontrivial $\mathbb{C}^\times$-valued multiplicative character of $k^\times$. We extend χ to a function on all of k by defining $\chi(0) := 0$.

Recall that, given an integer n, a number $\alpha \in \mathbb{C}$ is said to be pure of weight n (relative to q) if it and all its $Aut(\mathbb{C}/\mathbb{Q})$-conjugates have absolute value $\sqrt{q^n}$. Such an α is necessarily algebraic over $\mathbb{Q}$.

Fix a polynomial $f(X)$ in $k[X]$ of degree $d \geq 1$. If d is prime to p, one has the estimate

$$|\sum_{x \in k} \psi(f(x))| \leq (d-1)\sqrt{q}.$$

More precisely, the associated L-function is a polynomial of degree $d-1$ which is pure of weight one, i.e., all its reciprocal zeroes have absolute value $\sqrt{q}$.

The analogue for multiplicative character sums in one variable is this. Whatever the value of d, if f has d distinct zeroes (in some overfield of k), then one has the estimate

$$|\sum_{x \in k} \chi(f(x))| \leq (d-1)\sqrt{q}.$$

The associated L-function is again a polynomial of degree $d-1$, but in general it is mixed of weights one and zero. It is pure of weight one if and only if *in addition* χ^d is nontrivial, otherwise it has $d-2$ reciprocal roots which are pure of weight one, and one reciprocal root which is pure of weight zero.

It was known to Davenport and Hasse (cf. [Dav-Ha], [Ha], [We-OSES], and [We-NS, footnote on page 498]) in 1934 that both these results followed from the then unproven Riemann Hypothesis for curves over finite fields (in the first case for the curve of affine equation $Y^q - Y = f(X)$, in the second case for the curve of affine equation $Y^N = f(X)$, for N the order of χ).

What happens to these results in several variables? Fix an integer $n \geq 1$, and a polynomial $f(X) := f(X_1, \ldots, X_n)$ in $k[X_1, \ldots, X_n]$ of degree $d \geq 1$, say

$$f = f_d + f_{d-1} + \ldots + f_0$$

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with f_i homogeneous of degree i. If d is prime to p and if the equation $f_d = 0$ defines a smooth, degree d hypersurface in $\mathbb{P}^{n-1}$ (for $n = 1$, this means only that f_d is not identically 0), then Deligne proved [De-Weil I, Theorem 8.4] the estimate

$$|\sum_{x \in k^n} \psi(f(x))| \leq (d-1)^n \sqrt{q^n}.$$

Moreover, he proved that the associated L function is a polynomial (for n odd) or a reciprocal polynomial (for n even) of degree $(d-1)^n$ which is pure of weight n.

We will establish n-variable multiplicative character sum analogues of Deligne's n-variable additive character sum results (and of the mild generalizations of Deligne's results given in [Ka-SE, 5.1.1 and 5.4.1]). With these results, the n-variable situation almost perfectly mirrors the one-variable situation, except that in several variables our method sometimes requires the assumption that d be prime to p, an assumption which was never necessary in the one-variable case. One striking aspect of this mirroring is that the degrees of the L functions for the multiplicative character sums we consider coincide with the degrees for the corresponding additive character sums.

2. Statement of the two main theorems

Fix a finite field k of characteristic p and cardinality q. Let χ be a nontrivial $\mathbb{C}^\times$-valued multiplicative character of $k^\times$. Fix an integer $n \geq 1$, and a polynomial $f(X) := f(X_1, \ldots, X_n)$ in $k[X_1, \ldots, X_n]$ of degree $d \geq 1$, say

$$f = f_d + f_{d-1} + \ldots + f_0$$

with f_i homogeneous of degree i. Suppose that

(a) the equation $f_d = 0$ defines a smooth, degree d hypersurface in $\mathbb{P}^{n-1}_k$,
(b) the equation $f = 0$ defines a smooth hypersurface in $\mathbb{A}^n_k$.

Theorem 1. *In the above situation, if d is prime to p and if χ^d is nontrivial, we have the estimate*

$$|\sum_{x \in k^n} \chi(f(x))| \leq (d-1)^n \sqrt{q^n}.$$

The associated L function is a polynomial $P(T)$ (for n odd) or a reciprocal polynomial $1/P(T)$(for n even) of degree $(d-1)^n$, and $P(T)$ is pure of weight n.

Theorem 2. *In the above situation, If χ^d is trivial, we have the estimate*

$$|\sum_{x \in k^n} \chi(f(x))| \leq (d-1)^n \sqrt{q^n}.$$

The associated L function is a polynomial $P(T)$ (for n odd) or a reciprocal polynomial $1/P(T)$(for n even) of degree $(d-1)^n$, and $P(T)$ is mixed of weights n and $n-1$: it has $((d-1)^{n+1} - (-1)^{n+1})/d$ reciprocal roots which are pure of weight n, and it has $((d-1)^n - (-1)^n)/d$ reciprocal roots which are pure of weight $n-1$.

3. Statement of a second version of the main theorems

In this section, we give a generalization which is analogous to our additive character generalization [Ka-SE, 5.1.1] of [De-Weil I, Theorem 8.4]. Fix a finite field k of characteristic p and cardinality q. Let χ be a nontrivial $\mathbb{C}^\times$-valued multiplicative character of $k^\times$. Fix an integer $n \geq 1$.Let X/k be a projective, smooth, and geometrically connected k-scheme of dimension $n \geq 1$, given with a projective embedding $X \hookrightarrow \mathbb{P}^N_k := \mathbb{P}$. Fix a strictly positive integer d. Denote by L the class of $\mathcal{O}_\mathbb{P}(1)$, and by $c(X)$ the total Chern class of X. Define non-negative integers C_0, C_1, and C by

$$C_0 := (-1)^n \int_X \frac{c(X)}{(1+dL)},$$
$$C_1 := (-1)^{n-1} \int_X \frac{Lc(X)}{(1+L)(1+dL)},$$
$$C := (-1)^n \int_X \frac{c(X)}{(1+L)(1+dL)}.$$

Notice that

$$C = C_0 + C_1.$$

Let $Z \in H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(1))$ define a hyperplane (itself denoted Z) in $\mathbb{P}$. Let $H \in H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(d))$ define a hypersurface (itself denoted H) in $\mathbb{P}$. Assume that

(a) $X \cap Z$ is lisse of codimension 1 in X.
(b) $X \cap H$ is lisse of codimension 1 in X.
(c) $X \cap H \cap Z$ is lisse of codimension 2 in X.

To this data, we attach the smooth affine k-scheme

$$V := X - X \cap Z,$$

,and the function

$$f := H/Z^d : V \to \mathbb{A}^1_k.$$

Theorem 3. *Hypotheses and notations as above, suppose that d is prime to p, and that χ^d is nontrivial. Then we have the estimate*

$$|\sum_{x \in V(k)} \chi(f(x))| \leq C\sqrt{q^n}.$$

The associated L function is a polynomial $P(T)$ (for n odd) or a reciprocal polynomial $1/P(T)$(for n even) of degree C, and $P(T)$ is pure of weight n: all its reciprocal roots are pure of weight n.

Theorem 4. *Hypotheses and notations as above, suppose that χ^d is trivial. Then we have the estimate*

$$|\sum_{x \in V(k)} \chi(f(x))| \leq C\sqrt{q^n}.$$

The associated L function is a polynomial $P(T)$ (for n odd) or a reciprocal polynomial $1/P(T)$(for n even) of degree C, and $P(T)$ is mixed of weights n and $n-1$: it has C_0 reciprocal roots which are pure of weight n, and it has C_1 reciprocal roots which are pure of weight $n-1$.

4. Statement of a third version of the main theorems

In this section, we give a generalization which is analogous to our additive character generalization [Ka-SE, 5.4.1] of [De-Weil I, Theorem 8.4]. Fix a finite field k of characteristic p and cardinality q. Let χ be a nontrivial $\mathbb{C}^\times$-valued multiplicative character of $k^\times$. Fix an integer $n \geq 1$.Let X/k be a projective, smooth, and geometrically connected k-scheme of dimension $n \geq 1$, given with a projective embedding $X \hookrightarrow \mathbb{P}^N_k := \mathbb{P}$. Fix an integer $r \geq 1$, and two r-tuples $(d_1, \ldots, d_r)$ and $(b_1, \ldots, b_r)$ of strictly positive integers. Define

$$d := \sum_{i=1}^{r} b_i d_i.$$

Define

$$b := lcm(b_1, \ldots, b_r).$$

Denote by L the class of $\mathcal{O}_{\mathbb{P}}(1)$, and by $c(X)$ the total Chern class of X. Define a non-negative integer C by

$$C := (-1)^n \int_X \frac{c(X)}{(1+dL)\Pi_{i=1}^r (1+d_i L)}.$$

Suppose given

$$Z_i \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d_i))$$

for each i, which defines a hypersurface (also denoted Z_i) in $\mathbb{P}$. Suppose given

$$H \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d))$$

which defines a hypersurface (also denoted H) in $\mathbb{P}$. Assume that

(a) All the b_i are prime to p.
(b) For every subset $I \subset \{1, 2, \ldots, r\}$, $X \cap \bigcap_{i \in I} Z_i$ is lisse of codimension $\#I$ in X.
(c) For every subset $I \subset \{1, 2, \ldots, r\}$, $X \cap H \cap \bigcap_{i \in I} Z_i$ is lisse of codimension $1+\#I$ in X.

To this data, we attach the smooth affine k-scheme

$$V := X - X \cap \bigcup_{i \in I} Z_i,$$

and the function

$$f := H/\Pi_{i=1}^r Z_i^{b_i} : V \to \mathbb{A}^1_k.$$

Theorem 5. *Hypotheses and notations as above, suppose that χ^b is nontrivial. Then we have the estimate*

$$|\sum_{x \in V(k)} \chi(f(x))| \le C\sqrt{q^n}.$$

The associated L function is a polynomial $P(T)$ (for n odd) or a reciprocal polynomial $1/P(T)$(for n even) of degree C, and $P(T)$ is pure of weight n: all its reciprocal roots are pure of weight n.

Theorem 6. *Hypotheses and notations as above, suppose that χ^b is trivial. Then we have the estimate*

$$|\sum_{x \in V(k)} \chi(f(x))| \le C\sqrt{q^n}.$$

The associated L function is a polynomial $P(T)$ (for n odd) or a reciprocal polynomial $1/P(T)$(for n even) of degree C, and $P(T)$ is mixed of weight $\le n$: each of its reciprocal roots is pure of some non-negative integer weight $w \le n$.

5. Proof of the theorems: first steps

Fix a prime number l invertible in k, an algebraic closure $\bar{\mathbb{Q}}_l$ of $\mathbb{Q}_l$, and a field isomorphism ι of $\bar{\mathbb{Q}}_l$ with $\mathbb{C}$. By means of ι, we view χ as a nontrivial $\bar{\mathbb{Q}}_l^\times$-valued character of $k^\times$. We denote by $\mathcal{L}_\chi$ on $\mathbb{A}^1_k$ the extension by zero across the origin of the lisse rank one Kummer sheaf [De-AFT] on the multiplicative group $\mathbb{G}_{m,k}$. For any scheme X and any morphism $f : X \to \mathbb{A}^1_k$, we define the constructible $\bar{\mathbb{Q}}_l$-sheaf $\mathcal{L}_{\chi(f)} := f^\star\mathcal{L}_\chi$ on X: it is lisse of rank one on $X[1/f]$, and extended by zero to all of X.

Theorem 1 is the special case of Theorem 3 in which X is $\mathbb{P}^n_k := Proj(k[X_0, \dots, X_n])$, Z is X_0, and $H := \sum_{i=0}^d X_0^{d-i} f_i$. See [Ka-ESES, Introduction] for the verification that the constant C of Theorem 3 is $(d-1)^n$ in this special case. Similarly, Theorem 2 is a special case of Theorem 4. Theorem 3 is visibly a special case of Theorem 5.

So it suffices to prove Theorems 4, 5, and 6. All of these concern the exponential sum

$$\sum_{x \in V(k)} \chi(f(x)) = \sum_{x \in V[1/f](k)} \chi(f(x)).$$

Let us denote

$$\bar{V} := V \times_k \bar{k}.$$

By the Lefschetz trace formula, we know that

$$\sum_{x \in V[1/f](k)} \chi(f(x)) = \sum_i (-1)^i Trace(F_k | H^i_c(\bar{V}[1/f], \mathcal{L}_{\chi(f)})).$$

Theorem 5 results in the standard way (compare [Ka-SE, 5.4.1]) from Deligne's Weil II estimate [De-Weil II, 3.3.3 and 3.3.4] and the following l-adic theorem.

Theorem 7. *Hypotheses and notations as in Theorem 5, we have*

(1) *The natural "forget supports" map is an isomorphism*

$$H_c^\star(\bar{V}[1/f], \mathcal{L}_{\chi(f)}) \cong H^\star(\bar{V}[1/f], \mathcal{L}_{\chi(f)}).$$

(2) *The compact Euler characteristic is given by*

$$\chi_c(\bar{V}[1/f], \mathcal{L}_{\chi(f)}) = (-1)^n C.$$

(3) *For $i \neq n$, $H_c^i(\bar{V}[1/f], \mathcal{L}_{\chi(f)}) = 0$.*

(4) *The group $H_c^n(\bar{V}[1/f], \mathcal{L}_{\chi(f)})$ has dimension C, and is pure of weight n.*

(5) *For $\bar{\chi}$ the inverse character to χ, the cup-product pairing*

$$H_c^n(\bar{V}[1/f], \mathcal{L}_{\chi(f)}) \times H_c^n(\bar{V}[1/f], \mathcal{L}_{\bar{\chi}(f)}) \to H_c^{2n}(\bar{V}[1/f], \bar{\mathbb{Q}}_l) \cong \bar{\mathbb{Q}}_l(-n)$$

is a $Gal(\bar{k}/k)$-equivariant perfect pairing. In particular, if χ has order two, then $H_c^n(\bar{V}[1/f], \mathcal{L}_{\chi(f)})$ is $Gal(\bar{k}/k)$-equivariantly self-dual toward $\bar{\mathbb{Q}}_l(-n)$, by a pairing which is orthogonal for n even and symplectic for n odd.

Because $V[1/f]/k$ is smooth and affine, and $\mathcal{L}_{\chi(f)}$ is lisse and pure of weight zero, parts (3) and (4) result from parts (1) and (2), cf. [Se-MSE]. Part (5) is not needed for Theorem 5, but may be useful in other contexts. It results from part (1) and Poincaré duality. The sign of the pairing is as asserted because the coefficients $\mathcal{L}_{\chi(f)}$ are themselves orthogonally self-dual when χ has order two.

Similarly, Theorem 6 results from [De-Weil II, 3.3.3 and 3.3.4] and the following l-adic theorem.

Theorem 8. *Hypotheses and notations as in Theorem 6, the groups $H_c^i(\bar{V}[1/f], \mathcal{L}_{\chi(f)})$ vanish for $i \neq n$, and the compact Euler characteristic is given by*

$$\chi_c(\bar{V}[1/f], \mathcal{L}_{\chi(f)}) = (-1)^n C.$$

Thereom 6 asserts that in addition the weights are non-negative integers $\leq n$. Because $\mathcal{L}_{\chi(f)}$ is pure of weight zero, we know that $H_c^n(\bar{V}[1/f], \mathcal{L}_{\chi(f)})$ is mixed of integer weights $\leq n$. The trace function of $\mathcal{L}_{\chi(f)}$ has values which are algebraic integers, so the L function as a power series has algebraic integer coeffiecients, and constant term 1. But by the theorem we have

$$L(T)^{(-1)^{n+1}} = det(1 - TF_k | H_c^n(\bar{V}[1/f], \mathcal{L}_{\chi(f)}).$$

Therefore all the eigenvalues of F_k on $H_c^n(\bar{V}[1/f], \mathcal{L}_{\chi(f)})$ are algebraic integers. Hence their weights must be non-negative.

6. Further reduction steps, via the incidence variety

We now pass from V to the incidence variety $\tilde{X}_{\mathbb{A}^1} \subset X \times \mathbb{A}^1$ defined as the locus of points $(x \in X, \lambda \in \mathbb{A}^1)$ where $H - \lambda \Pi_i Z_i^{b_i}$ vanishes. This incidence variety $\tilde{X}_{\mathbb{A}^1}$ is smooth over k, cf. [Ka-SE, pp. 173-174]. The second projection defines a proper flat k-morphism

$$\tilde{f} : \tilde{X}_{\mathbb{A}^1} \to \mathbb{A}^1$$

with

$$\tilde{f}^{-1}(\lambda) = X \cap (H = \lambda \Pi_i Z_i^{b_i}).$$

We denote by $\tilde{X}_{\mathbb{G}_m} \subset X \times \mathbb{G}_m$ the inverse image of $\mathbb{G}_m$ under $\tilde{f}$. The variety V sits in $\tilde{X}_{\mathbb{A}^1}$ as an open set, with closed complement the product $(X \cap H \cap \bigcup_i Z_i) \times \mathbb{A}^1$. The function $\tilde{f}$ agrees with f on V, and is the second projection on the complement. Similarly, $V[1/f]$ sits in $\tilde{X}_{\mathbb{G}_m}$ as an open set, with closed complement the product $(X \cap H \cap \bigcup_i Z_i) \times \mathbb{G}_m$. The function $\tilde{f}$ agrees with f on $V[1/f]$, and is the second projection on the complement.

Because χ is nontrivial, we have $H_c^\star(\mathbb{G}_{m,\bar{k}}, \mathcal{L}_\chi) = 0$. By Kunneth we get

$$H_c^\star(((X \cap H \cap \bigcup_i Z_i) \times \mathbb{G}_m)_{\bar{k}}, \mathcal{L}_{\chi(\tilde{f})}) = 0.$$

So the excision sequence for $\mathcal{L}_{\chi(\tilde{f})}$ and

$$V[1/f] \subset \tilde{X}_{\mathbb{G}_m} \supset (X \cap H \cap \bigcup_i Z_i) \times \mathbb{G}_m$$

gives

$$H_c^\star(\bar{V}[1/f], \mathcal{L}_{\chi(f)}) \cong H_c^\star(\tilde{X}_{\mathbb{G}_{m,\bar{k}}}, \mathcal{L}_{\chi(\tilde{f})}).$$

The Poincaré dual of this isomorphism, applied with the inverse character $\bar{\chi}$, is an isomorphism

$$H^\star(\tilde{X}_{\mathbb{G}_{m,\bar{k}}}, \mathcal{L}_{\chi(\tilde{f})}) \cong H^\star(\bar{V}[1/f], \mathcal{L}_{\chi(f)}).$$

Moreover, if we surround the "forget supports" map on $\tilde{X}_{\mathbb{G}_{m,\bar{k}}}$ with these isomorphisms,

$$H_c^\star(\bar{V}[1/f], \mathcal{L}_{\chi(f)}) \cong H_c^\star(\tilde{X}_{\mathbb{G}_{m,\bar{k}}}, \mathcal{L}_{\chi(\tilde{f})}) \to H^\star(\tilde{X}_{\mathbb{G}_{m,\bar{k}}}, \mathcal{L}_{\chi(\tilde{f})}) \cong H^\star(\bar{V}[1/f], \mathcal{L}_{\chi(f)}),$$

we get the "forget supports" map on $\bar{V}[1/f]$. Thus Theorem 7 results from

Theorem 9. *Hypotheses and notations as in Theorem* 5*, we have*

(1) *The natural "forget supports" map is an isomorphism*

$$H_c^\star(\tilde{X}_{\mathbb{G}_{m,\bar{k}}}, \mathcal{L}_{\chi(\tilde{f})}) \cong H^\star(\tilde{X}_{\mathbb{G}_{m,\bar{k}}}, \mathcal{L}_{\chi(\tilde{f})}).$$

(2) *The compact Euler characteristic is given by*

$$\chi_c(\tilde{X}_{\mathbb{G}_{m,\bar{k}}}, \mathcal{L}_{\chi(\tilde{f})}) = (-1)^n C.$$

Similarly, Theorem 8 results from

Theorem 10. *Hypotheses and notations as in Theorem 6, the groups* $H^i_c(\tilde{X}_{\mathbb{G}_{m,\bar{k}}}, \mathcal{L}_{\chi(\tilde{f})})$ *vanish for* $i \neq n$*, and the compact Euler characteristic is given by*

$$\chi_c(\tilde{X}_{\mathbb{G}_{m,\bar{k}}}, \mathcal{L}_{\chi(\tilde{f})}) = (-1)^n C.$$

7. Proof of Theorems 9 and 10, by the method of pencils

We now turn to a study of the proper flat k-morphism

$$\tilde{f} : \tilde{X}_{\mathbb{A}^1} \to \mathbb{A}^1,$$

and the higher direct image sheaves (remember $\tilde{f}$ is proper)

$$R^i \tilde{f}_! \bar{\mathbb{Q}}_l \cong R^i \tilde{f}_\star \bar{\mathbb{Q}}_l$$

on $\mathbb{A}^1$.

Proposition 11. *The morphism*

$$\tilde{f} : \tilde{X}_{\mathbb{A}^1} \to \mathbb{A}^1$$

is proper and smooth of relative dimension $n-1$ *over a Zariski open neighborhood of the origin* 0 *in* $\mathbb{A}^1$*. Its singular fibres have at worst isolated singularities. The sheaves* $R^i \tilde{f}_\star \bar{\mathbb{Q}}_l$ *are all lisse in a Zariski open neighborhood of the origin in* $\mathbb{A}^1$*.*

Proof. The fibre over a point λ is $X \cap (H = \lambda \Pi_i Z_i^{b_i})$, which is smooth for $\lambda = 0$, hence smooth for λ in a Zariski open neighborhood of 0. By [SGA4, Exposé XV, 2.1 and Exposé XVI, 2.1], the sheaves $R^i \tilde{f}_\star \bar{\mathbb{Q}}_l$ are all lisse in a Zariski open neighborhood of the origin in $\mathbb{A}^1$. Whatever the point λ, the intersection $\tilde{f}^{-1}(\lambda) \cap Z_1$ is always the same, namely $X \cap H \cap Z_1$, which by hypothesis is smooth of codimension 2 in X. Therefore Z_1 does not meet the singular locus of $\tilde{f}^{-1}(\lambda)$. But Z_1 is the intersection with X of a hypersurface in the ambient $\mathbb{P}$, so it meets every closed subscheme of X of strictly positive dimension. Therefore the singular locus of $\tilde{f}^{-1}(\lambda)$ has dimension at most 0. ☐

Corollary 12. *For* $i \geq n+1$*, the sheaf* $R^i \tilde{f}_\star \bar{\mathbb{Q}}_l$ *is lisse on* $\mathbb{A}^1$*. For* $i = n$*, denote by* $j : U \hookrightarrow \mathbb{A}^1$ *the inclusion of a dense open set on which* $R^n \tilde{f}_\star \bar{\mathbb{Q}}_l$ *is lisse. Then we have a short exact sequence of sheaves on* $\mathbb{A}^1$*,*

$$0 \to (punctual) \to R^n \tilde{f}_\star \bar{\mathbb{Q}}_l \to j_\star j^\star R^n \tilde{f}_\star \bar{\mathbb{Q}}_l \to 0$$

and the sheaf $j_\star j^\star R^n \tilde{f}_\star \bar{\mathbb{Q}}_l$ *is lisse on* $\mathbb{A}^1$*.*

Proof. This is immediate from [SGA7, Exposé I, Cor. 4.3] and the preceding result, cf. [Ka-ESES, proof of Theorem 13]. ☐

Proposition 13. *The sheaves* $R^i \tilde{f}_\star \bar{\mathbb{Q}}_l$ *on* $\mathbb{A}^1$ *are all tamely ramified at* ∞*, and in their semisimplifications as representations of the inertia group* $I(\infty)$ *only characters of order dividing* b *occur.*

Proof. This is proven in [Ka-SE, 5.4.2]. □

Corollary 14. *For $i \geq n+1$, the sheaf $R^i\tilde{f}_\star\bar{\mathbb{Q}}_l$ is geometrically constant on $\mathbb{A}^1$. For $i = n$, the sheaf $j_\star j^\star R^n\tilde{f}_\star\bar{\mathbb{Q}}_l$ is geometrically constant on $\mathbb{A}^1$.*

Proof. These sheaves are lisse on $\mathbb{A}^1$ and tamely ramified at ∞, so are geometrically constant. □

Proposition 15. *For* any *nontrivial χ, the groups $H^i_c(\tilde{X}_{\mathbb{G}_{m,\bar{k}}}, \mathcal{L}_{\chi(\tilde{f})})$ vanish for $i \neq n$.*

Proof. For $i \leq n-1$, use the isomorphism

$$H^\star_c(\bar{V}[1/f], \mathcal{L}_{\chi(f)}) \cong H^\star_c(\tilde{X}_{\mathbb{G}_{m,\bar{k}}}, \mathcal{L}_{\chi(\tilde{f})}).$$

Since $V[1/f]$ is smooth and *affine* of dimension n, and the coefficients are lisse, these groups vanish for $i \leq n-1$ by the Poincaré dual of the Lefschetz affine theorem.

For $i \geq n+1$, use the Leray spectral sequence for compact cohomology for

$$\tilde{f} : \tilde{X}_{\mathbb{G}_m} \to \mathbb{G}_m$$

and the sheaf $\mathcal{L}_{\chi(\tilde{f})} := \tilde{f}^\star\mathcal{L}_\chi$,

$$E_2^{a,b} = H^a_c(\mathbb{G}_{m,\bar{k}}, R^b\tilde{f}_\star\mathcal{L}_{\chi(\tilde{f})}) \cong H^a_c(\mathbb{G}_{m,\bar{k}}, R^b\tilde{f}_\star\bar{\mathbb{Q}}_l \otimes \mathcal{L}_\chi) \Rightarrow H^{a+b}_c(\tilde{X}_{\mathbb{G}_{m,\bar{k}}}, \mathcal{L}_{\chi(\tilde{f})}).$$

The only possibly nonvanishing $E_2^{a,b}$ terms have $a \in [0,2]$. It suffices to show that $E_2^{a,b} = 0$ for $a+b \geq n+1$. They vanish for $b \geq n+1$, because $R^b\tilde{f}_\star\bar{\mathbb{Q}}_l$ is geometrically constant, and $H^\star_c(\mathbb{G}_{m,\bar{k}}, \mathcal{L}_\chi)$ vanishes. For $b = n$, $R^n\tilde{f}_\star\bar{\mathbb{Q}}_l$ sits in a short exact sequence

$$0 \to (punctual) \to R^n\tilde{f}_\star\bar{\mathbb{Q}}_l \to (geometrically\ constant) \to 0,$$

so we get $E_2^{a,n} = 0$ for $a \geq 1$. It remains to show that $E_2^{2,n-1} = 0$. For this we use the $i = n-1$ case of the fact (Proposition 11 above) that all the sheaves $R\tilde{f}_\star\bar{\mathbb{Q}}_l$ on $\mathbb{A}^1$ are lisse near the origin. The required vanishing then results from the first part of the following lemma. □

Lemma 16. *Let $\mathcal{F}$ be a constructible $\bar{\mathbb{Q}}_l$-sheaf on $\mathbb{A}^1_{\bar{k}}$, χ a nontrivial multiplicative character of $k^\times$, and ψ a nontrivial additive character of k. Then we have the following results.*

(1) *If $\mathcal{F}$ is lisse on a Zariski open neighborhood of the origin, then*

$$H^2_c(\mathbb{G}_{m,\bar{k}}, \mathcal{F} \otimes \mathcal{L}_\chi) = 0.$$

(2) *If $\mathcal{F}$ is lisse on a Zariski open neighborhood of the origin and is tamely ramified at ∞, then we have an equality of Euler characteristics*

$$\chi_c(\mathbb{G}_{m,\bar{k}}, \mathcal{F} \otimes \mathcal{L}_\chi) = \chi_c(\mathbb{A}^1_{\bar{k}}, \mathcal{F} \otimes \mathcal{L}_\psi) = \chi_c(\mathbb{A}^1_{\bar{k}}, \mathcal{F}) - rank(\mathcal{F}_{\bar{\eta}}).$$

(3) *Suppose $\mathcal{F}$ is lisse on a Zariski open neighborhood of the origin and tamely ramified at ∞, and that the only characters of I_∞ which occur have order dividing b. If χ^b is nontrivial, then the "forget supports" map is an isomorphism*

$$H^\star_c(\mathbb{G}_{m,\bar{k}}, \mathcal{F} \otimes \mathcal{L}_\chi) \cong H^\star(\mathbb{G}_{m,\bar{k}}, \mathcal{F} \otimes \mathcal{L}_\chi).$$

Proof. To prove (1), use the fact that H^2_c is a birational invariant, i.e. for any dense open set $U \subset \mathbb{G}_{m,\bar{k}}$ we have

$$H^2_c(U, \mathcal{F} \otimes \mathcal{L}_\chi) \cong H^2_c(\mathbb{G}_{m,\bar{k}}, \mathcal{F} \otimes \mathcal{L}_\chi).$$

Take for U a dense open set where $\mathcal{F} \otimes \mathcal{L}_\chi$ is lisse. Then $H^2_c(U, \mathcal{F} \otimes \mathcal{L}_\chi)$ is the (Tate-twisted) coinvariants of $\pi_1(U)$ acting on $(\mathcal{F} \otimes \mathcal{L}_\chi)_{\bar{\eta}}$. These coinvariants are themselves a quotient of the coinvariants under the inertia group I_0. But the I_0-coinvarinats vanish, because as I_0-representation we have is simply a direct sum of several copies of the nontrivial character of I_0 given by χ.

To prove (2), we simply apply the Grothendieck-Neron-Ogg-Shafarevic Euler- Poincaré formula. Because $\mathcal{L}_\chi$ is lisse of rank 1 on $\mathbb{G}_{m,\bar{k}}$ and tame at both 0 and ∞, for any constructible $\bar{\mathbb{Q}}_l$-sheaf $\mathcal{F}$ on on $\mathbb{A}^1_{\bar{k}}$ we have

$$\chi_c(\mathbb{G}_{m,\bar{k}}, \mathcal{F} \otimes \mathcal{L}_\chi) = \chi_c(\mathbb{G}_{m,\bar{k}}, \mathcal{F}).$$

For $\mathcal{F}$ lisse near 0, we have

$$\chi_c(\mathbb{G}_{m,\bar{k}}, \mathcal{F}) = \chi_c(\mathbb{A}^1_{\bar{k}}, \mathcal{F}) - rank(\mathcal{F}_{\bar{\eta}}).$$

For $\mathcal{F}$ tame at ∞, we have

$$\chi_c(\mathbb{A}^1_{\bar{k}}, \mathcal{F} \otimes \mathcal{L}_\psi) = \chi_c(\mathbb{A}^1_{\bar{k}}, \mathcal{F}) - Swan_\infty(\mathcal{F}) = \chi_c(\mathbb{A}^1_{\bar{k}}, \mathcal{F}) - rank(\mathcal{F}_{\bar{\eta}}).$$

To prove (3), denote by $j : \mathbb{G}_{m,\bar{k}} \to \mathbb{A}^1_{\bar{k}}$ the inclusion. We must show that the natural "forget supports" map

$$j_!(\mathcal{F} \otimes \mathcal{L}_\chi) \to Rj_\star(\mathcal{F} \otimes \mathcal{L}_\chi)$$

is an isomorphism. At 0, $\mathcal{F}$ is lisse, and at ∞ its inertia representation involves only characters of I_∞ of order dividing b. So at both 0 and at ∞, $\mathcal{F} \otimes \mathcal{L}_\chi$ is a successive extension of nontrivial characters of the inertia group, so we indeed get

$$j_!(\mathcal{F} \otimes \mathcal{L}_\chi) \cong Rj_\star(\mathcal{F} \otimes \mathcal{L}_\chi).$$

□

Corollary 17. *We have an equality of Euler characteristics*

$$\chi_c(\tilde{X}_{\mathbb{G}_{m,\bar{k}}}, \mathcal{L}_{\chi(\tilde{f})}) = \chi_c(\tilde{X}_{\mathbb{A}^1_{\bar{k}}}, \mathcal{L}_{\psi(\tilde{f})}) = (-1)^n C.$$

Proof. To see that the χ and ψ Euler characteristics are equal, compute each using the Leray spectral sequence as the alternating sum of the dimensions of the E_2 terms. By Propositions 11 and 13, we may apply part (2) of the previous result to the E_2 terms. That the ψ Euler characteristic is equal to $(-1)^n C$ is proven in [Ka-SE, 5.4.1]. □

Thus Theorem 10 is now completely proven. To complete the proof of Theorem 9, we must show that the "forget supports" map

$$H_c^\star(\bar{V}[1/f], \mathcal{L}_{\chi(f)}) \to H^\star(\bar{V}[1/f], \mathcal{L}_{\chi(f)})$$

is an isomorphism. To see this, compare the Leray spectral sequences for both compact and ordinary cohomology,

$$E_2^{a,b} = H_c^a(\mathbb{G}_{m,\bar{k}}, R^b\tilde{f}_\star\mathcal{L}_{\chi(\tilde{f})}) \cong H_c^a(\mathbb{G}_{m,\bar{k}}, R^b\tilde{f}_\star\bar{\mathbb{Q}}_l \otimes \mathcal{L}_\chi) \Rightarrow H_c^{a+b}(\tilde{X}_{\mathbb{G}_{m,\bar{k}}}, \mathcal{L}_{\chi(\tilde{f})})$$

and

$$E_2^{a,b} = H^a(\mathbb{G}_{m,\bar{k}}, R^b\tilde{f}_\star\mathcal{L}_{\chi(\tilde{f})}) \cong H^a(\mathbb{G}_{m,\bar{k}}, R^b\tilde{f}_\star\bar{\mathbb{Q}}_l \otimes \mathcal{L}_\chi) \Rightarrow H^{a+b}(\tilde{X}_{\mathbb{G}_{m,\bar{k}}}, \mathcal{L}_{\chi(\tilde{f})}).$$

It suffices to show that the "forget supports" map on the E_2 terms

$$H_c^a(\mathbb{G}_{m,\bar{k}}, R^b\tilde{f}_\star\bar{\mathbb{Q}}_l \otimes \mathcal{L}_\chi) \to H^a(\mathbb{G}_{m,\bar{k}}, R^b\tilde{f}_\star\bar{\mathbb{Q}}_l \otimes \mathcal{L}_\chi)$$

is an isomorphism for all (a, b). This is given by part (3) of the above lemma.

Thus we have proven Theorems 9 and 10, and with them Theorems 1, 3, 5, 6, 7, and 8. It remains only to prove Theorem 4 (of which Theorem 2 was a special case).

8. Proof of Theorem 4

Let us recall the situation. We are given a positive integer d (which is not necessarily prime to p) and a nontrivial multiplicative character χ with χ^d trivial. We have X/k which is projective, smooth, and geometrically connected, of dimension n, sitting in $\mathbb{P}$. We are given a linear form Z which defines a hyperplane Z transverse to X, and a degree d form H which defines a hypersurface which is transverse both to X and to $X \cap Z$. On $V := X[1/Z]$ we have the function $f := H/Z^d$.

Because H has degree d and χ^d is trivial, it makes sense to speak of the lisse, rank one Kummer sheaf $\mathcal{L}_{\chi(H)}$ on $X[1/H]$. Concretely, for any linear form L, we can form the usual Kummer sheaf $\mathcal{L}_{\chi(H/L^d)}$ on $X[1/LH]$. For any two linear forms L_1 and L_2, the Kummer sheaf $\mathcal{L}_{\chi(L_1^d/L_2^d)}$ on $X[1/L_1L_2]$ is canonically trivial. This allows us to patch together $\mathcal{L}_{\chi(H/L^d)}$ on $X[1/LH]$ for variable L to get the desired lisse, rank one Kummer sheaf $\mathcal{L}_{\chi(H)}$ on $X[1/H]$. This Kummer sheaf agrees with $\mathcal{L}_{\chi(f)}$ on $V[1/f] = X[1/ZH]$. So we have (writing $\bar{X}$ for $X_{\bar{k}}$)

$$H_c^\star(\bar{V}[1/f], \mathcal{L}_{\chi(f)}) \cong H_c^\star(\bar{X}[1/HZ], \mathcal{L}_{\chi(H)}).$$

Now look at the excision sequence for

$$\bar{X}[1/HZ] = \bar{V}[1/f] \subset \bar{X}[1/H] \supset \bar{X} \cap \bar{Z}[1/H],$$

$$\cdots \to H_c^i(\bar{X}[1/HZ], \mathcal{L}_{\chi(H)}) \to H_c^i(\bar{X}[1/H], \mathcal{L}_{\chi(H)}) \to H_c^i(\bar{X} \cap \bar{Z}[1/H], \mathcal{L}_{\chi(H)}) \to \ldots,$$

which we may rewrite as

$$\cdots \to H_c^{i-1}(\bar{X} \cap \bar{Z}[1/H], \mathcal{L}_{\chi(H)}) \to H_c^i(\bar{V}[1/f], \mathcal{L}_{\chi(f)}) \to H_c^i(\bar{X}[1/H], \mathcal{L}_{\chi(H)}) \to \ldots.$$

Using this excision sequence, we see that Theorem 4 results immediately from the following theorem.

Theorem 18. *Let d be a positive integer, χ a nontrivial multiplicative character with χ^d trivial, and X/k projective, smooth, and geometrically connected, of dimension n, sitting in $\mathbb{P}$.*

(1) *Suppose given a degree d form H which defines a hypersurface which is transverse to X. Then the "forget supports" map defines an isomorphism*

$$H_c^\star(\bar{X}[1/H], \mathcal{L}_{\chi(H)}) \cong H^\star(\bar{X}[1/H], \mathcal{L}_{\chi(H)}).$$

The groups $H_c^i(\bar{X}[1/H], \mathcal{L}_{\chi(H)})$ vanish for $i \neq n$. The remaining group H_c^n is pure of weight n, and has dimension $C_0 := (-1)^n \int_X \frac{c(X)}{(1+dL)}$.

(2) *Suppose that we are given in addition a linear form Z on X which is transverse both to X and to $X \cap H$. Then the "forget supports" map defines an isomorphism*

$$H_c^\star(\bar{X} \cap \bar{Z}[1/H], \mathcal{L}_{\chi(H)}) \cong H^\star(\bar{X} \cap \bar{Z}[1/H], \mathcal{L}_{\chi(H)}).$$

The groups $H_c^i(\bar{X} \cap \bar{Z}[1/H], \mathcal{L}_{\chi(H)})$ vanish for $i \neq n-1$. The remaining group H_c^{n-1} is pure of weight $n-1$, and has dimension $C_1 := (-1)^{n-1} \int_X \frac{Lc(X)}{(1+L)(1+dL)}$.

Proof. Because H is transverse to X, and χ is nontrivial, if we denote by

$$j : X[1/H] \hookrightarrow X$$

the inclusion, we have

$$j_!\mathcal{L}_{\chi(H)} \cong Rj_\star\mathcal{L}_{\chi(H)}.$$

[Indeed, at a point of $X \cap H$ the situation is etale over that at the origin in $\mathbb{A}^n$ of

$$(j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1) \times \mathbb{A}^{n-1}$$

and the sheaf $pr_1^{-1}\mathcal{L}_\chi$ on the source.] Hence

$$H_c^\star(\bar{X}[1/H], \mathcal{L}_{\chi(H)}) \cong H^\star(\bar{X}[1/H], \mathcal{L}_{\chi(H)}).$$

Since $\bar{X}[1/H]$ is smooth and affine of dimension n, and the coefficients are lisse and pure of weight zero, we find that $H_c^i(\bar{X}[1/H], \mathcal{L}_{\chi(H)})$ vanishes for $i \neq n$ and that H_c^n is pure of weight n.

Because $\mathcal{L}_{\chi(H)}$ is lisse of rank one and trivialized by a finite etale covering of degree prime to p (namely of degree the order of χ), we have [Ka-SE, Cor. 1 of 5.5.2] the equalities ("Hurwitz's formula")

$$\chi_c(\bar{X}[1/H], \mathcal{L}_{\chi(H)}) = \chi_c(\bar{X}[1/H], \bar{\mathbb{Q}}_l) = \chi_c(\bar{X}, \bar{\mathbb{Q}}_l) - \chi_c(\bar{X} \cap \bar{H}, \bar{\mathbb{Q}}_l).$$

Because H is transverse to $X \cap Z$, this same argument shows that the "forget supports" map defines an isomorphism

$$H_c^\star(\bar{X} \cap \bar{Z}[1/H], \mathcal{L}_{\chi(H)}) \cong H^\star(\bar{X} \cap \bar{Z}[1/H], \mathcal{L}_{\chi(H)}),$$

that $H^i_c(\bar{X} \cap \bar{Z}[1/H], \mathcal{L}_{\chi(H)})$ vanishes for $i \neq n-1$, and that H^{n-1}_c is pure of weight $n-1$. And just as above, we have the equalities

$$\chi_c(\bar{X} \cap \bar{Z}[1/H], \mathcal{L}_{\chi(H)}) = \chi_c(\bar{X} \cap \bar{Z}[1/H], \bar{\mathbb{Q}}_l) = \chi_c(\bar{X} \cap \bar{Z}, \bar{\mathbb{Q}}_l) - \chi_c(\bar{X} \cap \bar{Z} \cap \bar{H}, \bar{\mathbb{Q}}_l).$$

We recall now the Chern class formulas [SGA7, Exposé XVII, 5.7.5] [Ka-SE, page 163] for Euler characteristic:

$$\chi_c(\bar{X}, \bar{\mathbb{Q}}_l) = \int_X c(X),$$

$$\chi_c(\bar{X} \cap \bar{H}, \bar{\mathbb{Q}}_l) = \int_X \frac{dLc(X)}{1+dL},$$

$$\chi_c(\bar{X} \cap \bar{Z}, \bar{\mathbb{Q}}_l) = \int_X \frac{Lc(X)}{1+L},$$

$$\chi_c(\bar{X} \cap \bar{Z} \cap \bar{H}, \bar{\mathbb{Q}}_l) = \int_X \frac{dL^2 c(X)}{(1+L)(1+dL)}.$$

Recall the constants

$$C_0 := (-1)^n \int_X \frac{c(X)}{(1+dL)},$$

$$C_1 := (-1)^{n-1} \int_X \frac{Lc(X)}{(1+L)(1+dL)}.$$

We leave to the reader the elementary verification that we have

$$(-1)^{n-1} C_1 = \chi_c(\bar{X} \cap \bar{Z}, \bar{\mathbb{Q}}_l) - \chi_c(\bar{X} \cap \bar{Z} \cap \bar{H}, \bar{\mathbb{Q}}_l),$$

and

$$(-1)^n C_0 = \chi_c(\bar{X}, \bar{\mathbb{Q}}_l) - \chi_c(\bar{X} \cap \bar{H}, \bar{\mathbb{Q}}_l).$$

□

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