Monodromy and the Tate conjecture: Picard numbers and Mordell−Weil ranks in families

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Introduction We use results of Deligne on ℓ−adic monodromy and equidistribution, combined with elementary facts about the eigenvalues of elements in the orthogonal group, to give upper bounds for the average "middle Picard number" in various equicharacteristic families of even dimensional hypersurfaces, cf. 6.11, 6.12, 6.14, 7.6, 8.12. We also give upper bounds for the average Mordell−Weil rank of the Jacobian of the generic fibre in various equicharacteristic families of surfaces fibred over $\mathbb{P}^1$, cf. 9.7, 9.8. If the relevant Tate Conjecture holds, each upper bound we find for an average is in fact equal to that average

The paper is organized as follows:

1.0 Review of the Tate Conjecture

1.1 Let us begin by recalling the general Tate Conjectures about algebraic cycles on varieties over finitely generated ground fields, cf. Tate's articles [Tate−Alg] and [Tate−Conj]. We start with a field k, a separable closure $\overline{k}$ of k, and $\mathrm{Gal}(\overline{k}/k)$ its absolute galois group. We consider a projective, smooth, geometrically connected variety X/k of dimension $\dim(X) \geq 1$. For each integer i with $0 \leq i \leq \dim(X)$, we denote by $\mathcal{Z}^i(X)$ the free abelian group generated by the irreducible subvarieties on X of codimension i. For each prime number ℓ invertible in k, we have the ℓ−adic cohomology group $H^{2i}(X\otimes_k\overline{k}, \mathbb{Q}_\ell)$, on which $\mathrm{Gal}(\overline{k}/k)$ acts continuously, and its Tate−twisted variant $H^{2i}(X\otimes_k\overline{k}, \mathbb{Q}_\ell(i)) = H^{2i}(X\otimes_k\overline{k}, \mathbb{Q}_\ell)(i)$, cf. [Tate−Alg]. An element Z in $\mathcal{Z}^i(X)$ has a cohomology class cl(Z) in $H^{2i}(X\otimes_k\overline{k}, \mathbb{Q}_\ell(i))$ which is known to be invariant under $\mathrm{Gal}(\overline{k}/k)$, so we may view the formation of this cycle class as a map of abelian groups

1.1.1 $$\mathcal{Z}^i(X) \to H^{2i}(X\otimes_k\overline{k}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}/k)},$$

which extends to a $\mathbb{Q}_\ell$−linear map

1.1.2 $$\mathcal{Z}^i(X)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell \to H^{2i}(X\otimes_k\overline{k}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}/k)}.$$

1.2 The Tate conjecture asserts that if the field k is finitely generated over its prime field, then this last map is surjective.

1.3 For any field k, let us denote by $\mathrm{AlgH}^{2i}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(i))$ the subspace of $H^{2i}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(i))$ spanned by codimension i algebraic cycles on X, i.e., $\mathrm{AlgH}^{2i}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(i))$ is the image of $\mathcal{Z}^i(X)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell$ under the cycle class map. Let us denote by $\rho_{i,\ell}(X)$ the dimension of this subspace:

1.3.1 $$\rho_{i,\ell}(X/k) := \dim_{\mathbb{Q}_\ell} \mathrm{AlgH}^{2i}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(i)).$$

We will refer to $\rho_{i,\ell}(X/k)$ as the ℓ−adic i'th Picard number of X/k. This Picard number is not known in general to be independent of ℓ: it is independent of ℓ if numerical equivalence on X is equal to ℓ−adic homological equivalence for every ℓ invertible in k, cf. [Kl−SC, page 17], where this conjecture is called D(X) and [Ta−Alg, page 72], where it is called E(X). It will also be convenient to denote

1.3.2 $$\rho_{i,\ell,\mathrm{geom}}(X/k) := \rho_{i,\ell}(X\otimes_k\bar{k}/\bar{k}),$$

the geometric ℓ−adic i'th Picard number of X/k.

2.0 The Tate Conjecture over a finite field

2.1 Now let us specialize to the case in which the field k is a finite field, of cardinality denoted q. In this case, the group $\mathrm{Gal}(\bar{k}/k)$ has a standard generator, the automorphism $x \mapsto x^q$, whose **inverse** is called the geometric Frobenius element and denoted F_q, or F_k, or just F if no confusion is likely. Since F is itself a generator of $\mathrm{Gal}(\bar{k}/k)$, we have

2.1.1 $$H^{2i}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\bar{k}/k)} = H^{2i}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(i))^{F=1}$$
$$:= \mathrm{Ker}(F-1 \mid H^{2i}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(i))) \cong \mathrm{Ker}(F-q^i \mid H^{2i}(X\otimes_k\bar{k}, \mathbb{Q}_\ell)).$$

Now it is unknown in general that F acts semisimply on $H^{2i}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(i))$, so a priori we have an inclusion (which is conjecturally an equality)

2.1.2 $$H^{2i}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(i))^{F=1} \subset H^{2i}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(i))^{F-1 \text{ nilpotent}},$$

i.e.,

$$H^{2i}(X\otimes_k\bar{k}, \mathbb{Q}_\ell)^{F=q^i} \subset H^{2i}(X\otimes_k\bar{k}, \mathbb{Q}_\ell)^{F-q^i \text{ nilpotent}}$$

The cycle class map thus sits in a commutative diagram

2.1.3 $$\begin{array}{ccc} \mathcal{Z}^i(X)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell & \to & H^{2i}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(i))^{F=1} \\ & \searrow & \cap \\ & & H^{2i}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(i))^{F-1 \text{ nilpotent}}, \end{array}$$

and a stronger form of the Tate Conjecture asserts that the diagonal arrow

2.1.4 $$\mathcal{Z}^i(X)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell \to H^{2i}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(i))^{F-1 \text{ nilpotent}}$$

is surjective.

2.2 This last conjecture is equivalent to the following numerical statement, in terms of various flavors of "Picard numbers". We have defined above $\rho_{i,\ell}(X/k)$, the ℓ−adic i'th Picard number of

X/k.

2.3 Let us denote by $\rho_{i,\ell,an}(X/k)$ the multiplicity of 1 as a zero of the characteristic polynomial of F on $H^{2i}(X\otimes_k\overline{k}, \mathbb{Q}_\ell(i))$, or equivalently the multiplicity of q^i as a zero of the characteristic polynomial of F on $H^{2i}(X\otimes_k\overline{k}, \mathbb{Q}_\ell)$. We know by Deligne [Del−Weil I, 1.6] that these characteristic polynomials are independent of ℓ, so we may write simply $\rho_{i,an}(X/k)$ for $\rho_{i,\ell,an}(X/k)$. We will refer to $\rho_{i,an}(X/k)$ as the analytic i'th Picard number of X/k. Then the above cited stronger version of the Tate Conjecture is equivalent to the equality

2.3.1 $$\rho_{i,\ell}(X/k) = \rho_{i,an}(X/k).$$

Notice that in this case of a finite ground field, we have an a priori inequality

2.3.2 $$\rho_{i,\ell}(X/k) \le \rho_{i,an}(X/k).$$

[In the case when k is $\mathbb{Q}$ or a number field, there is a Tate conjecture which asserts that $\rho_{i,\ell}(X/k)$ is the order of pole at s=i+1 of the L−function built on $H^{2i}(X\otimes_k\overline{k}, \mathbb{Q}_\ell))$ viewed as a representation of $Gal(\overline{k}/k)$. But in that case there is, as yet, no a priori inequality (in either direction!) between $\rho_{i,\ell}(X/k)$ and the order of zero at s=i+1. (The Euler product defining the L−function converges in $Re(s) > i + 1$, so it makes unconditional sense to speak of the order of pole, as the largest integer r such $(s-1-i)^r L(s)$ has a nonzero limit as $s \to i+1$ from the right.)]

2.4 Over a finite field k, there is a unique extension k_n/k of any given degree n. If we start with X/k and apply the Tate conjecture to $X\otimes_k k_n/k_n$, it becomes

2.4.1 $$\rho_{i,\ell}(X\otimes_k k_n/k_n) = \rho_{i,an}(X\otimes_k k_n/k_n),$$

where $\rho_{i,an}(X\otimes_k k_n/k_n)$ is the total of the multiplicities of all n'th roots of unity as eigenvalues of F on $H^{2i}(X\otimes_k\overline{k}, \mathbb{Q}_\ell(i))$. If we pass to $\overline{k}$, viewed as the increasing union of the fields $k_{n!}$, then the Tate conjecture predicts

2.4.2 $$\rho_{i,\ell,geom}(X/k) = \rho_{i,an,geom}(X/k),$$

where

2.4.3 $\rho_{i,an,geom}(X/k)$:= the total of the multiplicities of all roots of unity as eigenvalues of F on $H^{2i}(X\otimes_k\overline{k}, \mathbb{Q}_\ell(i))$.

3.0 Middle−dimensional cohomology

3.1 Continuing over a finite field k, suppose X/k, still projective, smooth, and geometrically connected, has even dimension 2d, and let us take i=d in the above discussion. The cup−product pairing on middle dimensional cohomology

$$H^{2d}(X\otimes_k\overline{k}, \mathbb{Q}_\ell(d)) \times H^{2d}(X\otimes_k\overline{k}, \mathbb{Q}_\ell(d)) \to H^{4d}(X\otimes_k\overline{k}, \mathbb{Q}_\ell(2d)) \cong \mathbb{Q}_\ell$$

is an **orthogonal** (because X is even−dimensional) autoduality on $H^{2d}(X\otimes_k\overline{k}, \mathbb{Q}_\ell(d))$ which is $Gal(\overline{k}/k)$−equivariant. In particular, the cup−product is F−equivariant. As F, and all of $Gal(\overline{k}/k)$, acts trivially on $\mathbb{Q}_\ell$, this equivariance means that under cup−product, we have

3.1.3 $$(Fx, Fy) = (x, y)$$

for any two elements x, y in $H^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$. If we denote by O the orthogonal group $\mathrm{Aut}(H^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$, cup product), then what we are observing is that F lies in O.

4.0 Hypersurface sections of a fixed ambient variety

4.1 Suppose further that we are given a projective, smooth, geometrically connected Y/k of odd dimension 2d+1, together with a very ample invertible sheaf $\mathcal{L}$ on Y. For example, Y might be $\mathbb{P}^{2d+1}$ with $\mathcal{L}$ taken to be $\mathcal{O}_{\mathbb{P}^{2d+1}}(D)$ for some positive integer D, or Y might be $\mathbb{P}^1\times\mathbb{P}^{2d}$ with $\mathcal{L}$ taken to be $\mathcal{O}_{\mathbb{P}^1}(a)\boxtimes\mathcal{O}_{\mathbb{P}^{2d}}(b)$ for positive integers a and b. Denote by L_Y in $H^2(Y\otimes_k\bar{k}, \mathbb{Q}_\ell(1))$ the class of $\mathcal{L}$.

4.2 Suppose that our X is a closed subscheme of Y, defined in Y by the vanishing of a global section of $\mathcal{L}$. Denote by $i : X \to Y$ the inclusion. Denote by L in $H^2(X\otimes_k\bar{k}, \mathbb{Q}_\ell(1))$ the restriction $i^*(L_Y)$ of the class L_Y. The restriction map

4.2.1 $$i^* : H^{2n}(Y\otimes_k\bar{k}, \mathbb{Q}_\ell(n)) \to H^{2n}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(n))$$

is bijective for n < d, and is injective for n=d (by the weak Lefschetz theorem, [SGA 5, VII, 7.1], or, in dual form, [SGA 4, XIV, 3,3]). For n=d it sits in a commutative diagram

4.2.2
$$\begin{tikzcd}
H^{2d}(Y\otimes_k\bar{k}, \mathbb{Q}_\ell(d)) \arrow[r, "i^*"] \arrow[dr, "L_Y"'] & H^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d)) \arrow[d, "i_* \text{ (Gysin map)}"] \\
 & H^{2d+2}(Y\otimes_k\bar{k}, \mathbb{Q}_\ell(d+1))
\end{tikzcd}$$

in which the slanted map, multiplication by L_Y, is an isomorphism (by the hard Lefschetz theorem, proven by Deligne [De–Weil II, 4.1.1]). Thus we may view $H^{2d}(Y\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$ as a subspace of $H^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$, and on this subspace the intersection form (i.e., cup product on X) is nondegenerate. The orthogonal of this subspace is denoted $\mathrm{Ev}^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$, "ev" for évanescante, because in a Lefschetz pencil setting it is the subspace spanned by all the vanishing cycles, cf. [De–Weil I, 5.8]. [The notation should strictly speaking be something like $\mathrm{Ev}_Y{}^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$, since the space in question depends crucially on the ambient Y.]

4.3 So we have an orthogonal direct sum "vanishing" decomposition

4.3.1 $$H^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d)) = \mathrm{Ev}^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d)) \oplus H^{2d}(Y\otimes_k\bar{k}, \mathbb{Q}_\ell(d)).$$

We can also characterize $\mathrm{Ev}^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$ as the kernel of the Gysin map

4.3.2 $$i_* : H^{2d}(X\otimes_k \bar{k}, \mathbb{Q}_\ell(d)) \to H^{2d+2}(Y\otimes_k \bar{k}, \mathbb{Q}_\ell(d+1)).$$

We can describe the above decomposition as follows. Start with a class x in $H^{2d}(X\otimes_k \bar{k}, \mathbb{Q}_\ell(d))$, and write i_*x as $L_Y b$ for a unique b in $H^{2d}(Y\otimes_k \bar{k}, \mathbb{Q}_\ell(d))$. Then $a := x - i^*b$ lies in $\mathrm{Ker}(i_*) = \mathrm{Ev}^{2d}(X\otimes_k \bar{k}, \mathbb{Q}_\ell(d))$, and $x = a + i^*b$ is the desired decomposition.

Remark 4.4 If we start with an algebraic cohomology class x in $\mathrm{AlgH}^{2d}(X\otimes_k \bar{k}, \mathbb{Q}_\ell(d))$, the class i_*x lies in $\mathrm{AlgH}^{2d+2}(Y\otimes_k \bar{k}, \mathbb{Q}_\ell(d+1))$, but we do **not** know in general that the unique class b in $H^{2d}(Y\otimes_k \bar{k}, \mathbb{Q}_\ell(d))$ with $L_Y b = i_*x$ is algebraic. In other words, we do not know in general that the map

$$L_Y : \mathrm{AlgH}^{2d}(Y\otimes_k \bar{k}, \mathbb{Q}_\ell(d)) \to \mathrm{AlgH}^{2d+2}(Y\otimes_k \bar{k}, \mathbb{Q}_\ell(d+1))$$

is bijective, a condition we could name $A(Y, L, 2d, \ell)$ a la [Kl−Alg] or [Kl−SC]. [The interest of knowing this is that once b is algebraic, then i^*b is algebraic, and hence $a = x - i^*b$ is algebraic.] One important case when we do know $A(Y, L, 2d, \ell)$, albeit for a trivial reason, is when all of $H^{2d}(Y\otimes_k \bar{k}, \mathbb{Q}_\ell(d))$ is algebraic. For example, a smooth hypersurface or complete intersection (of any dimension, odd or even) in an ambient space all of whose cohomology is algebraic, such as a projective space or a Grassmannian or any product of these, will have the property that all of its cohomology outside the middle dimension is algebraic (use the weak Lefschetz theorem to get the cohomology strictly below the middle dimension from the ambient space, and then the hard Lefschetz theorem to get the cohomology strictly above the middle dimension from that strictly below).

4.5 Let us denote by

4.5.1 $$\mathrm{AlgEv}^{2d}(X\otimes_k \bar{k}, \mathbb{Q}_\ell(d)) \subset \mathrm{AlgH}^{2d}(X\otimes_k \bar{k}, \mathbb{Q}_\ell(d))$$

the intersection

4.5.2 $$\mathrm{AlgEv}^{2d}(X\otimes_k \bar{k}, \mathbb{Q}_\ell(d)) = \mathrm{Ev}^{2d}(X\otimes_k \bar{k}, \mathbb{Q}_\ell(d)) \cap \mathrm{AlgH}^{2d}(X\otimes_k \bar{k}, \mathbb{Q}_\ell(d))$$

inside H^{2d}. [If $A(Y, L, 2d, \ell)$ holds, e.g., if $H^{2d}(Y\otimes_k \bar{k}, \mathbb{Q}_\ell(d))$ is entirely algebraic, we can also describe $\mathrm{AlgEv}^{2d}(X\otimes_k \bar{k}, \mathbb{Q}_\ell(d))$ as the image of $\mathrm{AlgH}^{2d}(X\otimes_k \bar{k}, \mathbb{Q}_\ell(d))$ in $\mathrm{Ev}^{2d}(X\otimes_k \bar{k}, \mathbb{Q}_\ell(d))$ under the "vanishing" decomposition. If $A(Y, L, 2d, \ell)$ is false, this image might be strictly larger than $\mathrm{AlgEv}^{2d}(X\otimes_k \bar{k}, \mathbb{Q}_\ell(d))$.]

4.6 Let us denote by $\rho_{d,\ell,ev}(X)$ the dimension of this subspace:

4.6.1 $$\rho_{d,\ell,ev}(X/k) := \dim_{\mathbb{Q}_\ell} \mathrm{AlgEv}^{2d}(X\otimes_k \bar{k}, \mathbb{Q}_\ell(d)).$$

We will refer to $\rho_{d,\ell,ev}(X/k)$ as the ℓ−adic middle vanishing Picard number of X/k.

4.7 Let us denote by $\rho_{d,\ell,an,ev}(X/k)$ the multiplicity of 1 as a zero of the characteristic polynomial of F on $\mathrm{Ev}^{2d}(X\otimes_k \bar{k}, \mathbb{Q}_\ell(d))$), or equivalently the multiplicity of q^d as a zero of the characteristic polynomial of F on $Ev^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell)$). This analytic vanishing Picard number is independent of the choice of ℓ invertible in k. [To see this, recall that we have an injective map

4.7.1 $$i^* : H^{2d}(Y\otimes_k\bar{k}, \mathbb{Q}_\ell(d)) \hookrightarrow H^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d)),$$

so we have

4.7.2 $$\det(1 - TF \mid Ev^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d)))$$
$$=\det(1 - TF \mid H^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d)))/\det(1 - TF \mid H^{2d}(Y\otimes_k\bar{k}, \mathbb{Q}_\ell(d))),$$

and in this last expression, both numerator and denominator are independent of the choice of ℓ invertible in k. Taking degrees shows that the dimension of Ev^{2d} is independent of ℓ as well.]

4.8 Thus we will write

4.8.1 $$\rho_{d,an,ev}(X/k) := \rho_{d,\ell,an,ev}(X/k),$$

and we will refer to $\rho_{d,an,ev}(X/k)$ as the analytic middle vanishing Picard number of X/k. Then the Tate Conjecture for H^{2d} in the strong form $\rho_{d,\ell}(X/k) = \rho_{d,an}(X/k)$ implies the equality

4.8.2 $$\rho_{d,\ell,ev}(X/k) = \rho_{d,an,ev}(X/k).$$

Just as above, we have the a priori inequality

4.8.3 $$\rho_{d,\ell,ev}(X/k) \le \rho_{d,an,ev}(X/k).$$

4.9 The space $Ev^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$ is orthogonally self–dual under the $Gal(\bar{k}/k)$–equivariant cup product pairing

4.9.1 $$Ev^{2d}\times Ev^{2d} \to H^{4d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(2d)) \cong \mathbb{Q}_\ell.$$

Then F (or any element of $Gal(\bar{k}/k)$ acting on $Ev^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$) lies in O_{ev}, the orthogonal group $Aut(Ev^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d)), \text{cup prod.})$.

4.10 Now let us recall some standard facts about orthogonal groups O(N) and special orthogonal groups SO(N) over fields of odd characteristic. We denote by $O_-(N) \subset O(N)$ the set of elements of determinant –1. First of all, if N is odd, then every element in SO(N) has an eigenvalue 1, and every element in $O_-(N)$ has an eigenvalue –1. The remaining N–1 eigenvalues can be grouped into (N–1)/2 pairs of inverses $(\alpha, 1/\alpha)$. If N is even, every element in $O_-(N)$ has both 1 and –1 as eigenvalues, and the remaining N–2 eigenvalues can be grouped into (N–2)/2 pairs of inverses. For even N, the eigenvalues of an element of SO(N) can be grouped into N/2 pairs of inverses. For later use, let us observe that we can summarize the information about the automatic occurence of 1 as an eigenvalue independently of the parity of N as follows: If A in O(N) has $\det(-A) = -1$, then A has an eigenvalue 1. [Here is a mnemonic to rememeber this, based on ideas which have become widespread in the context of the Birch Swinnerton Dyer conjecture: det(–A) is the sign in the functional equation of det(1–TA), namely $\det(1 - T^{-1}A) = T^{-N}\det(-A)\det(1-TA)$ and we get forced vanishing of det(1–TA) at T=1 when the sign in the functional equation is odd.]

4.11 Now let us apply these standard facts to middle analytic Picard numbers. Let k be a finite field, X and Y as above. Denote by $\mathrm{ev}^{2d}(X\otimes_k\bar{k})$ the middle vanishing Betti number:

4.11.1 $$\mathrm{ev}^{2d}(X\otimes_k\bar{k}) := \dim \mathrm{Ev}^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$$
$$= \dim H^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d)) - \dim H^{2d}(Y\otimes_k\bar{k}, \mathbb{Q}_\ell(d)).$$

4.12 Suppose first $\mathrm{ev}^{2d}(X\otimes_k\bar{k})$ is odd. Either F lies in SO_{ev}, and has 1 an eigenvalue, whence $\rho_{d,an,ev}(X/k) \geq 1$, or F lies in $O_{-,ev}$, so has -1 as an eigenvalue, whence F^2 has 1 as eigenvalue, so over the quadratic extension k_2 of k, we have $\rho_{d,an,ev}(X\otimes_k k_2/k_2) \geq 1$.

4.13 If $\mathrm{ev}^{2d}(X\otimes_k\bar{k})$ is even, then we get no conclusion if F lies in SO_{ev}, but if F lies in $O_{-,ev}$, then F has both 1 and -1 as eigenvalues, and so we get two inequalities

4.13.1 $$\rho_{d,an,ev}(X/k) \geq 1, \rho_{d,an,v}(X\otimes_k k_2/k_2) \geq 2.$$

5.0 Smooth hypersurfaces in projective space

5.1 Let us take for X a smooth hypersurface of degree D in $\mathbb{P}^{n+1}$. Then X has dimension n, and $\mathrm{ev}^n(X\otimes_k\bar{k})$ is given by

5.1.1 $$\mathrm{ev}^n(X\otimes_k\bar{k}) = (D-1)((D-1)^{n+1} - (-1)^{n+1})/D,$$

cf. [Dw, page 5]. In the case when n=2d is even, this becomes

5.1.2 $$\mathrm{ev}^{2d}(X\otimes_k\bar{k}) = (D-1)((D-1)^{2d+1} +1)/D.$$

The vanishing decomposition is simply

5.1.3 $$H^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d)) = \mathrm{Ev}^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d)) \oplus \mathbb{Q}_\ell L^d.$$

Lemma 5.2 The integers $\mathrm{ev}^{2d}(X\otimes_k\bar{k})$ and D have opposite parities.

proof The ratio $((D-1)^{2d+1} +1)/D$ is an integer–coefficient polynomial in D with odd constant term 2d+1. So if D is even, both terms D−1 and the ratio $((D-1)^{2d+1} +1)/D$ are odd. If D is odd, then D−1 is even, and hence ev^{2d}, is even. QED

5.3 Thus the Tate conjecture implies that for any smooth projective hypersurface X/k of even dimension $2d \geq 2$ and of even degree D over a finite field k, if we pass to the quadratic extension k_2 of k, we always have $\rho_{d,\ell,ev}(X\otimes_k k_2/k_2) \geq 1$. This striking observation was already made by Shioda nearly twenty years ago, cf. [Sh–Pic, 7.5] and [Sh–Alg, 5.2]. Equivalently, we can look at the middle Picard number instead of the middle vanishing one, the two being related for hypersurfaces by

5.3.1 $$\rho_{d,\ell}(X\otimes_k k_2/k_2) = 1 + \rho_{d,\ell,ev}(X\otimes_k k_2/k_2),$$

as is immediate from the particular shape 5.1.3 of the vanishing decomposition. In terms of the middle Picard number, the Tate conjecture predicts that we always have $\rho_{d,\ell}(X\otimes_k k_2/k_2) \geq 2$. In particular, any smooth, even dimensional hypersurface X of even degree X over $\bar{k}$ is supposed to have middle vanishing Picard number at least 1, and middle Picard number at least 2. How can one exhibit a priori a nonzero algebraic class in Ev^{2d}?

5.4 The situation over a finite field thus seems to be in striking contrast with that of Noether's theorem, according to which in the universal family of smooth hypersurfaces in $\mathbb{P}^{2d+1}$ of any degree D with $2d(D-2) \geq 4$, in any given characteristic, the geometric generic fibre (in the universal family) has middle Picard number $\rho_{d,\ell,geom} = 1$. But the two situations are in fact closely related. In the universal family, the image of $\pi_1{}^{geom}$ is Zariski dense in the full orthogonal group O_{ev}, cf. 6.2 below. For Noether's theorem, one needs only the absolute irreducibility of the action of the image of $\pi_1{}^{geom}$, not the exact determination of its Zariski closure as O_{ev}, cf. [SGA 7, Exposé XIX, 1.3 and 1.4]. Our finite field results depend on its exact determination as O_{ev}. The "paradox" is that although O(odd) is irreducible in its standard representation (the phenomenon underlying Noether's theorem), every element in O(odd) has 1 or −1 as an eigenvalue (the phenomenon underlying our finite field predictions in even degree D).

5.5 Terasoma has observed [Ter] that by combining the proof of Noether's theorem with a clever use of Hilbert irreducibility, one gets the existence of examples over $\mathbb{Q}$ of smooth hypersurfaces in $\mathbb{P}^{2d+1}$ of any given degree D with $2d(D-2) \geq 4$ which over $\mathbb{C}$ have $\rho_{d,\ell,geom} =1$. [We should also mention in passing that Shioda [Shi−Alg] has constructed beautiful explicit examples over $\mathbb{Q}$ of smooth surfaces in $\mathbb{P}^3$ of any degree $m \geq 5$ prime to 6 which over $\mathbb{C}$ have $\rho_{d,\ell,geom} = 1$, but as their degree is odd, these examples are not strictly germane to the present discussion.] . Therefore it would seem there can be no "universal" construction of the "extra" algebraic cycle which is to exist in even degree over the algebraic closure of a finite field (indeed, it is to exist already over at worst the quadratic extension over which we begin). The situation is perhaps reminiscent of the situation regarding the self−product E×E of an elliptic curve with itself. Over the algebraic closure of a finite field, ρ is either 4 (if E is ordinary) or 6 (if E is supersingular), whereas "in general" ρ for E×E is only 3. What happens is that ρ is 2 + rank of End(E), and over a field which is not algebraic over a finite field, "most" elliptic curves have End(E) = $\mathbb{Z}$. Over the algebraic closure of a finite field, all but finitely many curves are ordinary, and for these it is the Frobenius which provides the "extra" element of End(E).

6.0 Families of smooth hypersurfaces in projective space

6.1 Let us fix an even dimension $2d \geq 2$, and a degree $D \geq 3$. The universal family of smooth hypersurfaces of degree D in $\mathbb{P}^{2d+1}$ is parameterized by the open set Hyp(2d, D) in the (giant) projective space (with homogeneous coordinates the coefficients) of all homogeneous forms of degree D in 2d+2 variables where the discriminant is invertible, cf. [Ka−Sar, 11.4.4], where Hyp(2d, D) is denoted $\mathcal{H}_{2d,\, D}$. If we invert a prime ℓ, then over Hyp(2d, D)[$1/\ell$] the spaces $Ev^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$ attached to the various hypersurfaces fit together to form a lisse $\mathbb{Q}_\ell$−sheaf Ev^{2d}. The orthogonal autodualities on each $Ev^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$ fit together in an orthogonal autoduality

6.1.1 $\mathrm{Ev}^{2d} \times \mathrm{Ev}^{2d} \to \mathbb{Q}_\ell$

of lisse sheaves on Hyp(2d, D)[1/ℓ], and thus the corresponding monodromy representation

6.1.2 $\pi_1(\mathrm{Hyp}(2d, D)[1/\ell], \text{base point}) \to \mathrm{Aut}(\mathrm{Ev}^{2d}, \text{cup product})$

lands in the orthogonal group O_{ev}:

6.1.3 $\pi_1(\mathrm{Hyp}(2d, D)[1/\ell], \text{base point}) \to O_{ev}(\mathbb{Q}_\ell)$.

Theorem 6.2 ([De–Weil II, 4.4.1], cf. also [Ka–Sar, 11.4.9]) Fix $2d \geq 2$ and $D \geq 3$ as above. If $2d = 2$, suppose further that $D \geq 4$. Fix a prime number $p \neq \ell$, and consider the restriction of the lisse $\mathbb{Q}_\ell$–sheaf Ev^{2d} to the spaces $\mathrm{Hyp}(2d, D)\otimes\mathbb{F}_p$ and $\mathrm{Hyp}(2d, D)\otimes\overline{\mathbb{F}}_p$. Under the monodromy representation, the group

$$\pi_1^{\mathrm{arith}} := \pi_1(\mathrm{Hyp}(2d, D)\otimes\mathbb{F}_p, \text{any base point } \xi)$$

and its subgroup

$$\pi_1^{\mathrm{geom}} := \pi_1(\mathrm{Hyp}(2d, D)\otimes\overline{\mathbb{F}}_p, \text{same base point } \xi)$$

both land in $O_{ev}(\mathbb{Q}_\ell)$. The Zariski closure G_{geom} of the image of π_1^{geom} in $O_{ev}(\mathbb{Q}_\ell)$ is the entire group O_{ev}.

6.3 We now use this together with Deligne's equidistribution theorem [De–Weil II, 3..5.3] in the form given in [Ka–GKM, 3.6] and in [Ka–Sar, 9.2.6, in part 5) of whose statement "g" should be "Λ"]. We denote

6.3.1 $N := \text{rank of } \mathrm{Ev}^{2d} = \dim \mathrm{Ev}^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$

for X any particular smooth hypersurface in $\mathbb{P}^{2d+1}$ of degree D. We denote by $O(N)_{\mathbb{R}}$ the classical compact orthogonal group of size N: intrinsically, $O(N)_{\mathbb{R}}$ is a maximal compact subgroup of the complex orthogonal group $O(N)(\mathbb{C})$. A key fact is that on $O(N)_{\mathbb{R}}$ the function "reversed characteristic polynomial", $A \mapsto \det(1 - TA)$, separates conjugacy classes.

6.4 Given X/k a smooth hypersurface in $\mathbb{P}^{2d+1}$ of degree D over a finite field k, we now recall the construction of its ("vanishing") Frobenius conjugacy class $\theta(X/k)$ in the classical group $O(N)_{\mathbb{R}}$. Pick a prime ℓ invertible in k, and form the $\mathbb{Q}_\ell$–coefficient polynomial

6.4.1 $\det(1 - T F_k \mid \mathrm{Ev}^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d)))$.

This polynomial has $\mathbb{Q}$–coefficients, independent of the choice of ℓ invertible in k, all of its complex roots lie on the unit circle, and viewed ℓ–adically it is the reversed characteristic polynomial of an element in $O_{ev}(\mathbb{Q}_\ell)$. As explained in [Ka–Sar, 11.4.1], it results from [De–Weil I] that $\det(1 - TF_k \mid \mathrm{Ev}^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d)))$ is the reversed characteristic polynomial $\det(1-T\theta(X/k))$ of a unique conjugacy class $\theta(X/k)$ in the classical group $O(N)_{\mathbb{R}}$.

6.5 According to Deligne's equidistribution theorem, for a large finite field k, as X/k runs over all the smooth hypersurfaces in $\mathbb{P}^{2d+1}$ of degree D over k, the conjugacy classes are approximately equidistributed in the space $O(N)_{\mathbb{R}}{}^{\#}$ of conjugacy classes of $O(N)_{\mathbb{R}}$ for Haar measure. More precisely, for any $\mathbb{C}$−valued continuous central function $A \mapsto f(A)$ on $O(N)_{\mathbb{R}}$, and for dA the total mass one Haar measure on $O(N)_{\mathbb{R}}$, we have the limit formula

6.5.1 $$\int_{O(N)_{\mathbb{R}}} f(A)dA = \lim_{\#k\to\infty} (1/\#\mathrm{Hyp}(2d, D)(k))\Sigma_{X/k \text{ in } \mathrm{Hyp}(2d, D)(k)} f(\theta(X/k)).$$

6.6 There is a standard extension of this result, to more general functions f, cf. [Ka−Sar, AD11.4], which will be useful for us below. Let Z be any closed subset of $O(N)_{\mathbb{R}}$ of Haar measure zero which is stable by $O(N)_{\mathbb{R}}$−conjugation, and let f be a bounded, $\mathbb{C}$−valued central function on $O(N)_{\mathbb{R}}$ whose restriction to $O(N)_{\mathbb{R}} - Z$ is continuous. For such an f the limit formula 6.5.1 remains valid.

6.7 Let us explain the set Z we have in mind. Suppose first that N is odd. Then for A in O(N), det(A) is an eigenvalue of A, and we can form the "reduced" characteristic polynomial

6.7.1 $$\mathrm{Rdet}(1-TA) := \det(1-TA)/(1-T\det(A)).$$

If N is even, then for A in O_(N) both ±1 are eigenvalues of A, and we define the "reduced" characteristic polynomial to be

6.7.2 $$\mathrm{Rdet}(1-TA) := \det(1-TA)/(1-T^2).$$

For N even and A in SO(N), we define the "reduced" characteristic polynomial to be

6.7.3 $$\mathrm{Rdet}(1-TA) := \det(1-TA).$$

6.8 Whatever the parity of N, $A \mapsto \mathrm{Rdet}(1-TA)$ is a continuous central function on O(N) with values in the space of real polynomials of degree at most N. The idea is that the zeroes of Rdet(1−TA) are those zeroes of det(1−TA) that are in addition to the ones automatically imposed by A's being in O(N) and having given determinant.

6.9 For any complex number α, we denote by $Z(\alpha) \subset O(N)_{\mathbb{R}}$ the set

$$Z(\alpha) := \{A \text{ in } O(N)_{\mathbb{R}} \text{ with } \mathrm{Rdet}(1-\alpha A) = 0\}.$$

This set is empty unless α lies on the unit circle. For any α, is it a closed subset of $O(N)_{\mathbb{R}}$, stable by $O(N)_{\mathbb{R}}$−conjugation, and of Haar measure zero. For given N, the roots of unity ζ in $\mathbb{C}$ which satisfy an equation over $\mathbb{Q}$ of degree at most N form a finite set, say $\mu(\deg\le N)$. We will take

6.9.1 $$Z := \cup_{\zeta \text{ in } \mu(\deg\le N)} Z(\zeta) \subset O(N)_{\mathbb{R}}.$$

Lemma 6.10 Given a finite field k and a smooth hypersurface X/k in $\mathbb{P}^{2d+1}$ of degree D, consider the conjugacy class $\theta(X/k)$. If any root of unity β is a zero of the polynomial $\mathrm{Rdet}(1-T\theta(X/k))$, then β lies in $\mu(\deg\le N)$, and $\theta(X/k)$ lies in Z.

proof The polynomial $\det(1-T\theta(X/k))$ has $\mathbb{Q}$−coefficients and degree N, and hence the polynomial $\mathrm{Rdet}(1-T\theta(X/k))$ has $\mathbb{Q}$−coefficients and degree at most N. QED

Theorem 6.11 Fix an even integer 2d ≥ 2, and an integer D ≥ 3. If d=1, assume D ≥ 4. Put N :=

$ev(2d, D) = (D-1)((D-1)^{2d+1} + 1)/D$. Given a finite field k, a prime number ℓ invertible in k, and a smooth hypersurface X/k in $\mathbb{P}^{2d+1}$ of degree D, denote by $\theta(X/k)$ the conjugacy class in the classical group $O(N)_{\mathbb{R}}$ given by the action of F on $Ev^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$.
1) Denote by Hyp(2d, D)(k,+) and Hyp(2d, D)(k,−) the subsets of Hyp(2d, D)(k) consisting of those X/k for which $\det(-\theta(X/k))$, the sign in the functional equation, has the indicated sign. The fraction of those X/k in Hyp(2d, D)(k) which lie in Hyp(2d, D)(k,+) (respectively in Hyp(2d, D)(k,−)) tends to 1/2 as $\#k \to \infty$.
2) Suppose N is odd, or equivalently that D is even.
2a)The fraction of those X/k in Hyp(2d, D)(k,−) whose $\theta(X/k)$ has 1 as an eigenvalue of multiplicity one and has no other eigenvalue which is a root of unity, tends to 1 as $\#k \to \infty$.
2b)The fraction of those X/k in Hyp(2d, D)(k,+) whose $\theta(X/k)$ has −1 as an eigenvalue of multiplicity one and has no other eigenvalue which is a root of unity, tends to 1 as $\#k \to \infty$.
3) Suppose that N is even, or equivalently that D is odd.
3a) The fraction of those X/k in Hyp(2d, D)(k,−) whose $\theta(X/k)$ has both ±1 as eigenvalues of multiplicity one and has no other eigenvalue which is a root of unity, tends to 1 as $\#k \to \infty$.
3b) The fraction of those X/k in Hyp(2d, D)(k,+) whose $\theta(X/k)$ has no eigenvalues which are roots of unity, tends to 1 as $\#k \to \infty$.

proof Assertion 1) is immediate from Chebotarev and the fact that the ±1−valued character of $\pi_1(\mathrm{Hyp}(2d, D)\otimes\mathbb{F}_p)$ given by $\det(Ev^{2d})$ is nontrivial on $\pi_1^{geom} := \pi_1(\mathrm{Hyp}(2d, D)\otimes\bar{\mathbb{F}}_p)$. This nontriviality is itself immediate from the fact that π_1 lands in O(N) and the deep fact that the image of π_1^{geom} is Zariski dense in O(N)). Once we have 1), the rest of the assertions follow by applying Deligne's equidistribution theorem to the following functions on $O(N)_{\mathbb{R}}$, all of which are bounded, central, and continuous on $O(N)_{\mathbb{R}} - Z$:

for 2a) (char. fct. of $SO(N)_{\mathbb{R}}$)×(char. fct. of $O(N)_{\mathbb{R}} - Z$).
for 2b) (char. fct. of $O_{-}(N)_{\mathbb{R}}$)×(char. fct. of $O(N)_{\mathbb{R}} - Z$).
for 3a) (char. fct. of $O_{-}(N)_{\mathbb{R}}$)×(char. fct. of $O(N)_{\mathbb{R}} - Z$).
for 3b) (char. fct. of $SO(N)_{\mathbb{R}}$)×(char. fct. of $O(N)_{\mathbb{R}} - Z$). QED

Corollary 6.12 Hypotheses and notations as in Theorem 6.11, recall (from 4.7 and 4.8.1) the definition

6.12.1 $\rho_{d,an,ev}(X/k)$:= the multiplicity of 1 as eigenvalue of F on $Ev^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$, and define

6.12.2 $\rho_{d,an,ev,quad}(X/k)$:= the total of the multiplicities of ±1 as eigenvalues of F on $Ev^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$, 6.12.3 $\rho_{d,an,ev,geom}(X/k)$:= the total of the multiplicities of all roots of unity as eigenvalues of F on $Ev^{2d}(X\otimes_k\overline{k}, \mathbb{Q}_\ell(d))$.

Recall (from 4.6.1) the definition

6.12.4 $\rho_{d,\ell,ev}(X/k)$:= dimension of $AlgEv^{2d}(X\otimes_k\overline{k}, \mathbb{Q}_\ell(d))$,

and define

6.12.5 $\rho_{d,\ell,ev,quad}(X/k) = \rho_{d,\ell,ev}(X\otimes_k k_2/k_2)$,

6.12.6 $\rho_{d,\ell,ev,geom}(X/k) = \rho_{d,\ell,ev}(X\otimes_k\overline{k}/\overline{k})$.

The limit as $\#k \to \infty$ of the averages over Hyp(2d, D)(k) and Hyp(2d, D)(k, ±) of the non–negative integer–valued functions $\rho_{d,an,ev}(X/k)$, $\rho_{d,an,prim,ev}(X/k)$, and $\rho_{d,an,ev,geom}(X/k)$ are given by the following tables:

6.12.7

N odd	Hyp(2d, D)(k)	Hyp(2d, D)(k, –)	Hyp(2d, D)(k, +)
$\rho_{d,an,ev}(X/k)$	1/2	1	0
$\rho_{d,an,ev,quad}(X/k)$	1	1	1
$\rho_{d,an,ev,geom}(X/k)$	1	1	1

6.12.8

N even	Hyp(2d, D)(k)	Hyp(2d, D)(k, –)	Hyp(2d, D)(k, +)
$\rho_{d,an,ev}(X/k)$	1/2	1	0
$\rho_{d,an,ev,quad}(X/k)$	1	2	0
$\rho_{d,an,ev,geom}(X/k)$	1	2	0

proof By part 1) of Theorem 6.11, the first column of the tables is the average of the second and third. To compute the second and third columns, we argue as follows. Outside the points of Hyp(2d, D)(k, ±) whose $\theta(X/k)$'s lie in the closed set Z of measure zero, the functions being averaged are constant, with the values given in the table. The functions being averaged are all bounded (by N), so their values on points in Z does not matter for the average, because the fraction of points of Hyp(2d, D)(k, ±) whose $\theta(X/k)$'s lie in Z tends to zero as $\#k \to \infty$. QED

6.13 If we combine these results on analytic Picard numbers together with the trivial inequalities

(cf. 4.8.3)

6.13.1 $0 \le \rho_{d,\ell,ev}(X/k) \le \rho_{d,an,ev}(X/k)$,

6.13.2 $0 \le \rho_{d,\ell,ev,geom}(X/k) \le \rho_{d,an,ev,geom}(X/k)$,

we find the following corollary of Theorem 6.11.

Corollary 6.14 Hypotheses and notations as in Theorem 6.11, we have the following unconditional results about actual Picard numbers.

1) The fraction of X/k in Hyp(2d, D)(k, +) with $\rho_{d,\ell,ev}(X/k) = 0$ tends to 1 as $\#k \to \infty$.

1a) If N is even, the fraction of X/k in Hyp(2d, D)(k, +) with $\rho_{d,\ell,ev,geom}(X/k) = 0$ tends to 1 as $\#k \to \infty$.

1b) If N is odd, the fraction of X/k in Hyp(2d, D)(k, +) with $\rho_{d,\ell,ev,quad}(X/k) \le 1$ tends to 1 as $\#k \to \infty$.

1c) If N is odd, the fraction of X/k in Hyp(2d, D)(k, +) with $\rho_{d,\ell,ev,geom}(X/k) \le 1$ tends to 1 as $\#k \to \infty$.

2) The fraction of X/k in Hyp(2d, D)(k, –) with $\rho_{d,\ell,ev}(X/k) \le 1$ tends to 1 as $\#k \to \infty$.

2a) If N is odd, the fraction of X/k in Hyp(2d, D)(k, –) with $\rho_{d,\ell,ev,geom}(X/k) \le 1$ tends to 1 as $\#k \to \infty$.

2b) If N is even, the fraction of X/k in Hyp(2d, D)(k, –) with $\rho_{d,\ell,ev,quad}(X/k) \le 2$ tends to 1 as $\#k \to \infty$.

2c) If N is even, the fraction of X/k in Hyp(2d, D)(k, –) with $\rho_{d,\ell,ev,geom}(X/k) \le 2$ tends to 1 as $\#k \to \infty$.

Question 6.15 Fix an even integer $2d \ge 2$ and an odd integer D, with $D \ge 5$ if $2d = 2$, and $D \ge 3$ otherwise. According to part 1a), if we take a large finite field k, then at least 49 percent the smooth hypersurfaces X/k of degree D in $\mathbb{P}^{2d+1}$ have $\rho_{d,\ell,ev,geom}(X/k) = 0$. Shioda has constructed explicit such examples for degree D prime to 6(2d+1) over every prime field $\mathbb{F}_p$ with $p \equiv 1 \bmod (D-1)^{2d+1} + 1$. Are there examples of every predicted odd degree D and even dimension 2d over every prime field? Is there some a priori reason this cannot be true?

7.0 Families of smooth hypersurfaces in products of projective spaces

7.1 Let us now fix an integer $r \ge 2$, and two r–tuples of positive integers

7.1.1 $$\mathcal{N} = (n_1, n_2, ..., n_r) \text{ and } \mathcal{D} = (D_1, D_2, ..., D_r).$$

We take as ambient variety Y the product of projective spaces

7.1.2 $$Y := \prod_{i=1 \text{ to } r} \mathbb{P}^{n_i},$$

on which we have the very ample line bundle

7.1.3 $$\mathcal{L} := \mathbb{Q}_{i=1 \text{ to } r}\, \mathcal{O}_{\mathbb{P}^{n_i}}(D_i).$$

7.2 We suppose that $\sum n_i$ is odd, say

7.2.1 $$\sum n_i = 2d+1.$$

7.3 Given a finite field k, a prime ℓ invertible in k, and a smooth, geometrically connected X/k which is defined inside $Y\otimes_{\mathbb{Z}}k$ by the vanishing of a multi–homogeneous form of multi–degree $(D_1, D_2, ..., D_r)$, i.e., by the vanishing of a global section of $\mathcal{L}$, we have its middle "vanishing" cohomology group $Ev^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$.

7.4 The universal family of smooth hypersurfaces of multi–degree $(D_1, D_2, ..., D_r)$ in $Y := \prod_{i=1 \text{ to } r}\mathbb{P}^{n_i}$, is parameterized by the open set

7.4.1 $$\mathrm{Hyp}(2d, \mathcal{N}, \mathcal{D}) := \mathrm{Hyp}(2d, (n_1, n_2, ..., n_r), (D_1, D_2, ..., D_r))$$

in the (giant) projective space (with homogeneous coordinates the coefficients) of all multi–homogeneous forms of multi–degree $(D_1, D_2, ..., D_r)$ where the discriminant (:= equation of the dual variety) is invertible. As soon as we invert a prime ℓ, then on the parameter space $\mathrm{Hyp}(2d, \mathcal{N}, \mathcal{D})[1/\ell]$ the groups $Ev^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$ attached to the various hypersurfaces fit together to form a lisse $\mathbb{Q}_\ell$–sheaf Ev^{2d}. The orthogonal autodualities on each $Ev^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$ fit together in an orthogonal autoduality

7.4.2 $$Ev^{2d} \times Ev^{2d} \to \mathbb{Q}_\ell$$

of lisse sheaves on $\mathrm{Hyp}(2d, \mathcal{N}, \mathcal{D})[1/\ell]$, and thus the corresponding monodromy representation

7.4.3 $$\pi_1(\mathrm{Hyp}(2d, \mathcal{N}, \mathcal{D})[1/\ell], \text{base point}) \to \mathrm{Aut}(Ev^{2d}, \text{cup prod.})$$

lands in the orthogonal group O_{ev}:

7.4.4 $$\pi_1(\mathrm{Hyp}(2d, \mathcal{N}, \mathcal{D})[1/\ell], \text{base point}) \to O_{ev}(\mathbb{Q}_\ell).$$

Theorem 7.5 ([De–Weil II, 4.4.1]) Fix an even integer $2d \geq 2$, an integer $r \geq 2$, and r–tuples of positive integers

$$(n_1, n_2, ..., n_r) \text{ and } (D_1, D_2, ..., D_r),$$

with

$$\sum n_i = 2d+1.$$

Suppose that for each i = 1 to r, we have

$$D_i \geq 1 + n_i.$$

Fix an odd prime number $p \neq \ell$, and consider the restriction of the lisse $\mathbb{Q}_\ell$–sheaf Ev^{2d} to $\mathrm{Hyp}(2d, \mathcal{N}, \mathcal{D})\otimes\mathbb{F}_p$ and $\mathrm{Hyp}(2d, \mathcal{N}, \mathcal{D})\otimes\bar{\mathbb{F}}_p$. Under the monodromy representation, the group

$$\pi_1{}^{arith} := \pi_1(\mathrm{Hyp}(2d, \mathcal{N}, \mathcal{D})\otimes\mathbb{F}_p, \text{any base point } \xi)$$

and its subgroup

$$\pi_1{}^{geom} := \pi_1(\mathrm{Hyp}(2d, \mathcal{N}, \mathcal{D})\otimes\bar{\mathbb{F}}_p, \text{same base point } \xi)$$

both land in $O_{ev}(\mathbb{Q}_\ell)$. The Zariski closure G_{geom} of the image of $\pi_1{}^{geom}$ in $O_{ev}(\mathbb{Q}_\ell)$ is the entire group O_{ev}.

proof We first prove that over any algebraically closed field k of odd characteristic, there exist Lefschetz pencils on $Y := \prod_{i=1 \text{ to } r} \mathbb{P}^{n_i}$ of hypersurface sections of multidegree $(D_1, D_2, ..., D_r)$, provided that for all i we have $D_i \geq 2$. [This statement is presumably true in characteristic two as well, by some adaptation to the multihomogeneous case of Deligne's argument as given in [SGA 7, Expose XVII, section 4], but we will not pursue this question here.]

Because we are in odd characteristic, it suffices,by [SGA 7, XVII, 3.7], to show that given any k−point x_0 of Y, there is a hypersurface of multidegree $(D_1, D_2, ..., D_r)$ in Y/k which has an ordinary double point at x_0. By a suitable choice of coordinates, we may assume the point x_0 in Y $:= \prod_{i=1 \text{ to } r} \mathbb{P}^{n_i}$ is the product of the points with homogeneous coordininates $X_{i,j}$, j=0 to n_i, in the the i'th factor given by (1, 0,..., 0). Take affine coordinates $x_{i,j} := X_{i,j}/X_{i,0}$ for j =1 to n_i on the i'th factor. Then we want to write down an equation in the $x_{i,j}$ which has an ordinary double point at the origin, and each monomial of which has, for each i, total degree at most D_i in the $x_{i,*}$ variables. Because for all i we have $D_i \geq 2$, we may take the equation

$$\Sigma_{i,j} (x_{i,j})^2 = 0.$$

Once we know Lefschetz pencils exist, we argue as follows. If we restrict the lisse sheaf Ev^{2d} to a general line in Hyp(2d, $\mathcal{N}$, $\mathcal{D}$)$\otimes\overline{\mathbb{F}}_p$, we do not change the image of π_1^{geom}. Thus we are reduced to showing that a sufficiently general Lefschetz pencil has geometric monodromy Zariski dense in O_{ev}. By Deligne [De−Weil II, 4.4.1], for Lefschetz pencils with even fibre dimension, either the geometric monodromy Zariski dense in O_{ev}, or the geometric monodromy is a finite and absolutely irreducible subgroup of O_{even}.

So what we must rule out is that on the entire parameter space Hyp(2d, $\mathcal{N}$, $\mathcal{D}$)$\otimes\overline{\mathbb{F}}_p$, the geometric monodromy of Ev^{2d} is a finite and absolutely irreducible subgroup, say G, of O_{even}. Suppose it were. Since π_1^{geom} is normal in π_1^{arith}, each Frobenius $F_{E,x}$ attached to a point x of Hyp(2d, $\mathcal{N}$, $\mathcal{D}$)$\otimes\mathbb{F}_p$ with values in a finite extension E of $\mathbb{F}_p$ normalizes G. But G is finite, so Aut(G) is finite, so a fixed power of $F_{E,x}$ commutes with G, hence is scalar. Since the only scalars in O_{ev} are ±1, a fixed power of every Frobenius is 1. Therefore for every finite field k of characteristic p, and every smooth hypersurface X/k in Y/k of multidegree $(D_1, D_2, ..., D_r)$, every eigenvalue of F on $Ev^{2d}(X\otimes_k\overline{k}, \mathbb{Q}_\ell(d))$ is a root of unity. Now all the cohomology of the ambient Y is algebraic, so we find

1) every eigenvalue of F on $H^{2d}(X\otimes_k\overline{k}, \mathbb{Q}_\ell(d))$ is a root of unity,

2) all the cohomology of $X\otimes_k\overline{k}$ outside its middle dimension is algebraic.

Therefore the reduction mod p of the zeta function of every X/k as above is 1/(1−T):

$$\mathrm{Zeta}(X/k, T) \equiv 1/(1-T) \bmod p\mathbb{Z}[[T]].$$

On the other hand, by the congruence formula for the zeta function [SGA 7 Expose XXII, 3.1], we have

$$\mathrm{Zeta}(X/k, T) \equiv \prod_{i=0 \text{ to } 2d} \det(1-TF|H^i(X, \mathcal{O}_X)^{(-1)^{i+1}},$$

valid for **any** proper X/k of dimension at most 2d.

Now for any hypersurface in Y/k of any multi−degree, we have

$$H^0(X, \mathcal{O}_X) = k, \text{ F acts as the identity,}$$

$$H^i(X, \mathcal{O}_X) = 0 \text{ for } 0 < i < 2d$$

Thus we find that, if we have finite monodromy, then

$$\det(1-TF|H^{2d}(X, \mathcal{O}_X)) = 1 \text{ in } k[T],$$

for every **smooth** hypersurface X/k in Y/k of multidegree $(D_1, D_2, ..., D_r)$. This means precisely that F on $H^{2d}(X, \mathcal{O}_X)$ is nilpotent. If we denote by F_{abs} the p−th power map, it induces a p−linear endomorphism of $H^{2d}(X, \mathcal{O}_X)$, whose $\deg(k/\mathbb{F}_p)$'th power is F. Thus finite monodromy implies that for every **smooth** hypersurface X/k in Y/k of multidegree $(D_1, D_2, ..., D_r)$, we have

$$F_{abs} \text{ on } H^{2d}(X, \mathcal{O}_X) \text{ is nilpotent.}$$

Thus if we denote by g_a the classical "arithmetic genus" of X,

$$g_a := \dim H^{2d}(X, \mathcal{O}_X) = \prod_{i=1 \text{ to } r} \dim H^{n_i}(\mathbb{P}^{n_i}, \mathcal{O}(-D_i)),$$

we find that

$$(F_{abs})^{g_a} = 0 \text{ on } H^{2d}(X, \mathcal{O}_X).$$

From this it follows that for every hypersurface X/k in Y/k of multidegree $(D_1, D_2, ..., D_r)$, smooth or not, we have

$$(F_{abs})^{g_a} = 0 \text{ on } H^{2d}(X, \mathcal{O}_X).$$

[The point is that if we denote by $f : \mathcal{X} \to \mathbb{P}^{giant}$ the universal family of all hypersurfaces in Y of given multidegree $(D_1, D_2, ..., D_r)$, the coherent higher direct images $R^if_*\mathcal{O}_{\mathcal{X}}$ on $\mathbb{P}^{giant}$ are locally free $\mathcal{O}_{\mathbb{P}}$ modules of formation compatible with arbitrary change of base, which vanish for i not 0 or 2d, and which have $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathbb{P}}$. Since $\mathbb{P}^{giant}\otimes_{\mathbb{Z}}\mathbb{F}_p$ is reduced and the open set $\mathrm{Hyp}(2d, \mathcal{N}, \mathcal{D})\otimes\mathbb{F}_p$ of $\mathbb{P}^{giant}\otimes_{\mathbb{Z}}\mathbb{F}_p$ is dense, we get the vanishing of $(F_{abs})^{g_a}$ first on $R^{2d}f_*\mathcal{O}_{\mathcal{X}} | \mathrm{Hyp}(2d, \mathcal{N}, \mathcal{D})\otimes\mathbb{F}_p$, then on $R^{2d}f_*\mathcal{O}_{\mathcal{X}}$ on all of $\mathbb{P}^{giant}$, then on the individual fibres $H^{2d}(X, \mathcal{O}_X)$.]

Once we have this nilpotence of F_{abs}, we return to the congruence formula to infer that, if we have finite monodromy, then we have the congruence

$$\#X(k) \equiv 1 \bmod p$$

for every finite extension k of $\mathbb{F}_p$ and for every hypersurface X/k in Y/k of multidegree $(D_1, D_2, ..., D_r)$.

It is this last statement which we will show to be false as soon as each $D_i \geq 1 + n_i$ for each i = 1 to r. Indeed, we will exhibit an X/k for which #X(k) = 0. For this, we simply take an equation which is the product over i =1 to r of forms F_i over k, with F_i of degree D_i in the i'th set of variables, say $\prod_i F_i(X_{i,0}, X_{i,1}, ..., X_{i,n_i})$, with the property that F_i has no nontrivial zeroes in $(k)^{1+n_i}$. The easiest way to do this is to take the extension field E/k of degree D_i, pick a set of $1 + n_i$ elements $e_0, e_1, ..., e_{n_i}$ in E which are linearly independent over k (possible because $1 + n_i \leq D_i$), and to take F_i to be the norm form

$$F_i(X_{i,0}, X_{i,1}, ..., X_{i,n_i}) := \mathrm{Norm}_{E/k}(\Sigma_{j=0 \text{ to } n_i} e_j X_{i,j}).\text{ QED}$$

Once we have this result, we get exactly the same results, in odd characteristic, that we had in the case of hypersurfaces in a single projective space. We state them briefly.

Theorem 7.6 Fix an even integer $2d \geq 2$, an integer $r \geq 2$, and r−tuples of positive integers

$$(n_1, n_2, ..., n_r) \text{ and } (D_1, D_2, ..., D_r),$$

with

$$\Sigma n_i = 2d+1.$$

Suppose that for each i = 1 to r, we have

$$D_i \geq 1 + n_i.$$

Denote by N the common middle "vanishing" Betti number

$$N := \dim \mathrm{Ev}^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d)),\ \ell \text{ invertible in k},$$

of smooth hypersurfaces over fields X/k in $\prod_{i=1 \text{ to } r} \mathbb{P}^{n_i}$ of given multidegree $(D_1, D_2, ..., D_r)$. Given a finite field k, denote by θ(X/k) the conjugacy class in the classical group $O(N)_{\mathbb{R}}$ given by the action of F on $\mathrm{Ev}^{2d}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(d))$. Then the conclusions of Theorem 6.11, and of Corollaries 6.12 and 6.14 remain valid, provided that in their statements we systematically replace "Hyp(2d, D)" by "Hyp(2d, $\mathcal{N}$, $\mathcal{D}$)[1/2]", and restrict the variable finite field k to run only over those of odd characteristic.

8.0 Hypersurfaces in $\mathbb{P}^1\times\mathbb{P}^n$ as families over $\mathbb{P}^1$

Proposition 8.1 In $\mathbb{P}^1\times\mathbb{P}^n$, take a smooth hypersurface X of any bidegree (d, D) over a field k. The first projection $pr_1 : X \to \mathbb{P}^1$ is smooth over the generic point of $\mathbb{P}^1$, provided char(k) = 0 or char(k) > Max(D, $(D-1)^n$).

proof The generic fibre X_η of this projection is a regular scheme over the function field $K := k(\mathbb{P}^1)$, and at the same time it is a degree D hypersurface in $\mathbb{P}^n$, defined over K. If K has characteristic zero, the notions "smooth" and "regular" agree for schemes of finite type over K. If char(K) = p > 0, we apply the following elementary lemma.

Lemma 8.2 Over an infinite field K, suppose $X \subset \mathbb{P}^n$ is a hypersurface of degree D, say of equation $F(X_i\text{'s})=0$, which is a regular scheme. If char(K) = p > Max(D, $(D-1)^n$), then X/K is smooth.

proof For any separable algebraic extension L of K, $X\otimes_K L$ remains a regular scheme. So it suffices to treat the case when the field K is separably closed. Consider Sing(X/K), the subscheme of $\mathbb{P}^n$ defined by the vanishing of F and all the $\partial F/\partial X_i$. Since p > D, Sing(X/K) is defined by the vanishing just of all the $\partial F/\partial X_i$, which all have the same degree D–1. Because X is regular, Sing(X/K) has no K–rational points. Let us temporarily admit the truth of the following elementary but useful sublemma.

SubLemma 8.3 Over an infinite field K, suppose we are given a closed subscheme S in $\mathbb{P}^n$ which is defined by the vanishing of some collection of homogeneous forms, all of the same degree d. Either the scheme S is empty, or there exists a field extension L/K of degree at most d^n (in fact, of degree at most $d^{n-\dim(S)}$) such that S(L) is nonempty.

8.4 We apply the sublemma to S = Sing(X/K), which is defined in $\mathbb{P}^n$ by the vanishing of forms of degree D–1. Then either S is empty, in which case X/K is smooth, or S(L) is nonempty for some field extension of K of degree at most $(D-1)^n$. But $p > (D-1)^n$, so L/K is separable, and hence, K being separably closed, L = K, and S(K) is nonempty, which contradicts the regularity of X. QED modulo the sublemma.

proof of Sublemma 8.3 If S is empty, there is nothing to prove. If S is nonempty, denote by $\delta \geq 0$ its dimension: $\delta := \dim(S)$. Because the field K is infinite, there exists a K–rational linear subspace $H \cong \mathbb{P}^{n-\delta}$ of codimension δ in $\mathbb{P}^n$ whose scheme–theoretic intersection with S is finite over K. Replacing S by $H\cap S$, which is defined in H by the vanishing of forms of degree d, we reduce to treating universally the case in which S/K is finite.

In this case, we argue as follows. S is defined in $\mathbb{P}^n$ by the vanishing of some forms F_i of degree d, so by finitely many, say $F_1, F_2, ..., F_r$. Because the field K is infinite, and S has dimension zero, there exist n sufficiently general K–linear combinations of the F_i's, say $G_1, ..., G_n$, such that $(G_1, ..., G_n)$ defines a complete intersection in $\mathbb{P}^n$, call it Z, necessarily of dimension zero (cf. [Eis–St] for a discussion of how to find such linear combinations G_i effectively). Then $S^{red} \subset S \subset Z$, and Z/K is finite of degree d^n. Therefore S^{red}/K is finite of degree at most d^n. But S^{red} is then a disjoint union of spectra of fields, $S^{red} = \coprod \mathrm{Spec}(L_i)$, with

$$d^n \geq \deg(S^{red}/K) = \sum \deg(L_i/K).$$

Thus S^{red} and hence S itself, have points with values in fields (namely the L_i) of degree at most d^n over K. QED

8.5 We now explore the situation in arbitrary characteristic.

Proposition 8.6 Fix an integer $n \geq 1$ and a bidegree (d, D), both d, $D \geq 1$. There exists an open set

$$\mathrm{SGFHyp}(n, (1,n), (d, D)) \subset \mathrm{Hyp}(n, (1,n), (d, D))$$

with the following property: for any field k and any k−valued point h in Hyp(n, (1,n), (d, D))(k), corresponding to a smooth hypersurface X/k of bidegree (d, D) in $\mathbb{P}^1 \times \mathbb{P}^n$, h lies in SGFHyp(n, (1,n), (d, D)) if and only if the first projection

$$pr_1 : X \to \mathbb{P}^1$$

has smooth generic fibre (SGF) over $\mathbb{P}^1$.

proof Write $\mathcal{H}$ for Hyp(n, (1,n), (d, D)), and consider the universal smooth hypersurface $F_{univ} = 0$ in $\mathcal{H} \times \mathbb{P}^1 \times \mathbb{P}^n$ of the given bidegree, say $\mathcal{X}/\mathcal{H} \times \mathbb{P}^1$. Denote by $\mathrm{Sing}(\mathcal{X}/\mathcal{H} \times \mathbb{P}^1)$ its singular locus, defined in $\mathcal{H} \times \mathbb{P}^1 \times \mathbb{P}^n$ by the vanishing of F_{univ} and its partial derivatives with respect to the homogeneous coordinates of $\mathbb{P}^n$. Then $\mathrm{Sing}(\mathcal{X}/\mathcal{H} \times \mathbb{P}^1)$ is proper over $\mathcal{H} \times \mathbb{P}^1$, so its image in $\mathcal{H} \times \mathbb{P}^1$ is closed. Denote by $S \subset \mathcal{H} \times \mathbb{P}^1$ the reduced closed subscheme which is the image of $\mathrm{Sing}(\mathcal{X}/\mathcal{H} \times \mathbb{P}^1)$ with its reduced structure. Then a k−valued point h in $\mathcal{H}(k)$ lies in $\mathrm{SGF}\mathcal{H}(k)$ if and only if $S \cap (h \times \mathbb{P}^1)$, the closed subscheme of $\mathbb{P}^1/k$ over which $pr_1 : X \to \mathbb{P}^1$ has singular fibres, is finite. So our result is a special case of the following lemma (itself a special case of [EGA IV, Part 3, 13.1.5]).

Lemma 8.7 Let H be scheme, and S a closed subscheme of $H \times \mathbb{P}^1$. There exists an open set $U \subset H$ with the following property: for any field k, a point h in H(k) lies in U(k) if and only if the intersection $S \cap (h \times \mathbb{P}^1)$ in $\mathbb{P}^1/k$ is finite.

proof Pick homogeneous coordinates X, Y in $\mathbb{P}^1$. Fix a field k, and a k−valued point h in H(k). Because $S \cap (h \times \mathbb{P}^1)$ in $\mathbb{P}^1/k$ is closed, it is either finite or it is all of $\mathbb{P}^1/k$. It is finite if and only if it is defined in $\mathbb{P}^1/k$ by the vanishing of some nonzero homogeneous form G(X,Y) over k. Given any nonzero form G over k, we claim there is a homogeneous form K(X,Y) with $\mathbb{Z}$ coefficients having no common factor (i.e., K is primitive over $\mathbb{Z}$) such that over k, G and K have no common zero. To see this, we use the fact that over the prime field k_0 of k (i.e., k_0 is either $\mathbb{Q}$ or $\mathbb{F}_p$), there are irreducible polynomials $k_m(x)$ in one variable of every degree $m \geq 1$. Take for each $m \geq 1$ a primitive $K_m(X, Y)$ over $\mathbb{Z}$ of degree m whose image over k_0 is $k_0^\times$−proportional to $Y^m k_m(X/Y)$. Then the K_m for two distinct m have no common zero. Therefore G can have a common zero with K_m for at most degree(G) distinct values of m.

So either $S\cap(h\times\mathbb{P}^1)$ is finite, in which case there exists a primitive form $K(X, Y)$ over $\mathbb{Z}$ such that $S\cap(h\times\mathbb{P}^1)\cap(K=0)$ is empty, or $S\cap(h\times\mathbb{P}^1)$ is $\mathbb{P}^1/k$, in which case $S\cap(h\times\mathbb{P}^1)\cap(K=0)$ is nonempty for every primitive K.

Now for each $\mathbb{Z}$−primitive form K, consider the closed subscheme $S\cap(K=0)$ in $H\times\mathbb{P}^1$, which is automatically **proper** over H. The complement in H of its image is therefore an **open** set U_K in H. A k−valued point of U_K is precisely a point h in H(k) for which $S\cap(h\times\mathbb{P}^1)\cap(K=0)$ is empty. Therefore the open set of H given by

$$U := \cup_{\mathbb{Z}\text{−primitive forms } K} U_K$$

does the job. QED

Proposition 8.8 The open set

$$SGFHyp(n, (1,n), (d, D)) \subset Hyp(n, (1,n), (d, D))$$

has non−void intersection with every fibre of Hyp(n, (1,n), (d, D)) over Spec($\mathbb{Z}$), i.e., the open set

$$SGFHyp(n, (1,n), (d, D))\otimes\mathbb{F}_p \subset Hyp(n, (1,n), (d, D))\otimes\mathbb{F}_p$$

is nonempty for every prime p.

proof Given p, we must show there exists a smooth X/k over some field k of characteristic p, whose generic fibre over $\mathbb{P}^1$ is smooth. If d=1, then X/k is of bidegree (1, D), which means precisely that X/k is a pencil of hypersurface sections of degree D in $\mathbb{P}^n$. Since Lefschetz pencils of hypersurfaces in $\mathbb{P}^n$ of any degree D exist (trivially for D=1, when any pencil is Lefschetz, by [SGA 7, Expose XVII, 2.5.1] for $D \geq 2$]), we have only to take X to be a Lefschetz pencil in this case. Once we have an X/k of bidegree (1, D) which is a Lefschetz pencil, we take a map $f: \mathbb{P}^1 \to \mathbb{P}^1$ of degree d which is finite etale over each point of the target $\mathbb{P}^1$ over which our Lefschetz pencil has a singular fibre.

Then the fibre product of $pr_1 : X \to \mathbb{P}^1$ with $f: \mathbb{P}^1 \to \mathbb{P}^1$ is the desired smooth hypersurface of bidegree (d, D). In terms of homogeneous coordinates (λ, μ) on $\mathbb{P}^1$ and (X_i) on $\mathbb{P}^n$, the original Lefschetz pencil has an equation of the form

$$\lambda F(X) = \mu G(X),$$

the map f has the form

$$f : (\lambda, \mu) \mapsto (P(\lambda, \mu), Q(\lambda, \mu)),$$

with P and Q forms of degree d, and the fibre product has equation

$$P(\lambda, \mu)F(X) = Q(\lambda, \mu)G(X). \qquad \text{QED}$$

Proposition 8.9 Fix integers $n \geq 2$, $d \geq 1$, $D \geq 2$.

1) In the projective space P := AllHyp(n, (1,n), (d, D)) of all (not necessarily smooth) hypersurfaces in $\mathbb{P}^1\times\mathbb{P}^n$ of bidegree (d, D), there is an open set $AFH \subset P$ with the following property: for any field k, a k−valued point p of P lies in AFH if and only if the corresponding

hypersurface X/k of bidegree (d, D) in $\mathbb{P}^1\times\mathbb{P}^n$, viewed as fibred over $\mathbb{P}^1$ by pr_1, has all its fibres hypersurfaces ("AFH") in $\mathbb{P}^n$.
2) In the open set AFH of P, there is an open set AFGI with the following property: for any field k, a k−valued point p of AFH lies in AFGI if and only if the corresponding hypersurface X/k of bidegree (d, D) in $\mathbb{P}^1\times\mathbb{P}^n$, viewed as fibred over $\mathbb{P}^1$ by pr_1, has all its fibres geometrically irreducible ("AFGI").

proof 1) A hypersurface X in $\mathbb{P}^1\times\mathbb{P}^n$ of bidegree (d, D) is defined by the vanishing of a bihomogeneous form F of bidegree (d, D) on $\mathbb{P}^1\times\mathbb{P}^n$, i.e., by the vanishing of a nonzero global section F of $\mathcal{O}_{\mathbb{P}^1}(d)\otimes\mathcal{O}_{\mathbb{P}^n}(D)$. When we view X as fibred over $\mathbb{P}^1$, the fibre over a point α in $\mathbb{P}^1$ is the zero locus of the restriction of F to $\alpha\times\mathbb{P}^n$. This fibre fails to be a hypersurface if and only it the the restriction of F to $\alpha\times\mathbb{P}^d$ is identically zero, and this in turn happens if and only if we can write F as LG, where L is the linear form on $\mathbb{P}^1$ whose vanishing defines α, and where G is some bihomogeneous form on $\mathbb{P}^1\times\mathbb{P}^n$ of bidegree (d−1, D). So if we denote by $P_{d-1,D}$ the projective space of all bihomogeneous forms on $\mathbb{P}^1\times\mathbb{P}^n$ of bidegree (d−1, D), then the points in P which fail to be AFH are the image in P of the multiplication map

$$(\mathbb{P}^1)^\vee \times P_{d-1,D} \to P$$
$$(L, G) \mapsto LG.$$

This map is proper, so its image is closed. Its complement is the desired opens set AFH in P.
2) View AFH as the space of degree d maps from $\mathbb{P}^1$ to the projective space T := AllHyp(n, D) of all degree D hypersurfaces in $\mathbb{P}^n$. In T, the geometrically reducible hypersurfaces form a closed set Red $\subset$ T, namely Red is the union of the images of the (necessarily proper) "multiplication of homogeneous forms" maps

$$\mathrm{AllHyp}(n, A)\times\mathrm{AllHyp}(n, D-A) \to \mathrm{AllHyp}(n, D).$$

for A =1 to D−1.

From the interpretation of AFH as the space of degree d maps of $\mathbb{P}^1$to T, we have a tautological map τ : AFH$\times\mathbb{P}^1$ $\to$ AllHyp(n, D). We denote by Z $\subset$ AFH$\times\mathbb{P}^1$ the closed subset τ^{-1}(Red). Thus, for any field k, a k−valued point (p, x) in AFH$\times\mathbb{P}^1$ lies in Z if and only if the hypersurface X/k in $\mathbb{P}^1\times\mathbb{P}^n$ corresponding to p has its fibre $pr_1^{-1}(x)$ over x geometrically reducible. Because $\mathbb{P}^1$ is proper over Spec($\mathbb{Z}$), the image of Z in AFH is a closed set, say W $\subset$ AFH. The complementary open set AFH − W is the desired open set AFGI. QED

Proposition 8.10 Suppose n ≥ 2, d ≥ 1, and D ≥ 2. If n=2, suppose D ≥ 3. Then the open set AFGI in P := AllHyp(n, (1,n), (d, D)) meets every fibre of P over Spec($\mathbb{Z}$), i.e., AFGI$\otimes\mathbb{F}_p$ $\subset$

$P \otimes \mathbb{F}_p$ is nonempty for every prime p.

proof We claim that the examples we used to prove 8.8 all lie in AFGI. These examples have, as geometric fibres, hypersurfaces that are smooth except for, at worst, one ordinary double point. Any reducible hypersurface has a singular locus of codimension at most one, so if $n \geq 3$ any hypersurface in $\mathbb{P}^n$ with at worst isolated singularities is irreducible, and if n=2 any smooth plane curve is irreducible. We must show that if a reducible plane curve of degree $D \geq 2$ is smooth outside a single point x_0, and x_0 is an ordinary double point, then D=2. [Of course this happens for D=2, as the example XY=0 in $\mathbb{P}^2$ shows.] If we factor the equation F of our curve into irreducible factors $F_1F_2...F_r$, then any point where any two F_i intersect is singular, so x_0 must be the unique point of interesection of every two of the F_i. In local coordinates x, y at x_0, each equation F_i is $f_i(x,y)$, $f_i(0,0) = 0$. Thus $\prod_{i=1 \text{ to } r} f_i(x,y)$ starts only in degree r, so we must have r=2 if x_0 is to be an ordinary double point. Thus $F = F_1F_2$, and x_0 is the unique point of intersection of F_1 and F_2. Because x_0 is an ordinary double point, the curve in the formal neighborhood of x_0 has an equation xy=0. Therefore if the curve in the formal neighborhood of x_0 also has an equation $f_1f_2 = 0$ with both f_i in the maximal ideal, then f_1 and f_2 intersect transversely at x_0. But if the degrees of F_1 and F_2 are A and B respectively, and if x_0 is their unique point of intersection, then the intersection multiplicity of F_1 and F_2 at x_0 is AB. As the intersection is transverse, AB=1, so A = B =1, and D = A + B = 2. QED

8.11 We now specialize the general multihomogeneous case to the special case of Hyp(2n, (1, 2n), (d, D)). The arguments work just as well over either of the spaces

SGFHyp(2n, (1, 2n), (d, D))[1/2] or

AFGI∩SGFHyp(2n, (1, 2n), (d, D))[1/2]

as over the bigger space Hyp(2n, (1, 2n), (d, D))[1/2], and we get the following.

Theorem 8.12 Fix an even integer $2n \geq 2$, a bidegree (d, D) with $d \geq 2$ and $D \geq 2n+1$. Denote by N the common middle "vanishing" Betti number

$$N := \dim \mathrm{Ev}^{2n}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(n)),\ \ell \text{ invertible in k},$$

of SGF smooth hypersurfaces over fields X/k in $\mathbb{P}^1\times\mathbb{P}^{2n}$ of given bidegree (d, D). Given a finite field k, denote by θ(X/k) the conjugacy class in the classical group $O(N)_\mathbb{R}$ given by the action of F on $\mathrm{Ev}^{2n}(X\otimes_k\bar{k}, \mathbb{Q}_\ell(n))$. Then the conclusions of Theorem 7.6 remain valid, provided that in their statements we systematically replace "Hyp(2n, (1, 2n), (d, D))[1/2]" by "SGFHyp(2n, (1, 2n), (d, D))[1/2]", or by "AFGI∩SGFHyp(2n, (1, 2n), (d, D))[1/2]".

9.0 Mordell–Weil rank in families of Jacobians

9.1 We now specialize to a smooth (hyper)surface C/k in $\mathbb{P}^1\times\mathbb{P}^2$ of some bidegree (d, D), with

$d \geq 2$ and $D \geq 3$, over a field k. We will think of $pr_1 : C \to \mathbb{P}^1$ as a family of plane curves of degree D, and we will assume that the generic fibre C_η is smooth over the rational function field $k(\mathbb{P}^1)$. As we have seen above, this SGF assumption is automatic in characteristic $p > (D-1)^2$. We denote by $r(C/k)$ the Mordell–Weil rank of the Jacobian J_η of C_η, viewed as an abelian variety over the rational function field $k(\mathbb{P}^1)$:

9.1.1 $$r(C/k) := \text{Mordell Weil rank of } \mathrm{Jac}(C_\eta/k(\mathbb{P}^1)).$$

We will study how this rank $r(C/k)$ varies as C ranges over all the smooth SGF hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^2$ of the given bidegree (d, D) over a large finite field k of odd characteristic.

9.2 We will also be concerned with the "geometric" rank

9.2.1 $$r_{geom}(C/k) := r(C \otimes_k \overline{k}/\overline{k}),$$

and, when k is finite, with unique quadratic extension k_2, with the "quadratic" rank

9.2.2 $$r_{quad}(C/k) := r(C \otimes_k k_2/k_2).$$

9.3 Let us recall the connection between the Mordell–Weil rank $r(C/k)$ and the classical Picard number $\rho(C/k)$ of C viewed as surface over a perfect field k. For divisors on a surface, we have the Hodge index theorem, so torsion algebraic equivalence coincides with numerical equivalence, and both coincide with ℓ–adic homological equivalence for any ℓ invertible in k. We write

9.3.1 $$\rho(C/k) := \rho_{1,\ell}(C/k),$$

9.3.2 $$\rho_{ev}(C/k) := \rho_{1,\ell,ev}(C/k).$$

The ambient space $\mathbb{P}^1 \times \mathbb{P}^2$ has all of its cohomology algebraic. Its H^2 is of rank two, so we have

9.3.3 $$\rho_{ev}(C/k) = \rho(C/k) - 2.$$

9.4 The Mordell–Weil rank $r(C/k)$ and the Picard number $\rho(C/k)$ are related by

9.4.1 $$\rho(C/k) = r(C/k) + 2 + \sum_{\text{closed points P of } \mathbb{P}^1} (m_P - 1),$$

where, at each closed point P of $\mathbb{P}^1/k$, the integer m_P is the number of irreducible components of the fibre over P, viewed as scheme over the residue field k(P). cf [Tate–BSD, 4.5 and discussion immediately above]. For our C/k, we may rewrite this as

9.4.2 $$\rho_{ev}(C/k) = r(C/k) + \sum_{\text{closed points P}} (m_P - 1).$$

We extract from this the inequality

9.4.3 $$r(C/k) \leq \rho_{ev}(C/k).$$

We remark that for C/k in the open dense set AFGI∩SGF (cf. 8.9) all the fibres are geometrically irreducible, so each $m_P = 1$. Thus

9.4.4 $$r(C/k) = \rho_{ev}(C/k) \text{ for } C/k \text{ in the open dense set AFGI} \cap \text{SGF}.$$

9.5 When k is finite, we write

9.5.1 $\rho_{\text{an, ev}}(C/k) := \rho_{1,\text{an, ev}}(C/k)$

:= the multiplicity of 1 as eigenvalue of F on $Ev^2(C\otimes_k\bar{k}, \mathbb{Q}_\ell(1))$.

So over a finite field we have the chain of inequalities

9.5.2 $0 \le r(C/k) \le \rho_{\text{ev}}(C/k) \le \rho_{\text{an, ev}}(C/k)$.

9.3 Now we bring to bear the results we have already obtained about the behavior of $\rho_{\text{an, ev}}(C/k)$ as k varies over large finite fields of odd characteristic, and C/k varies over smooth hypersurfaces of bidegree (d, D) in $\mathbb{P}^1\times\mathbb{P}^2$.

9.4 Before stating the result, we need to calculate the middle "vanishing" Betti number of a smooth surface in $\mathbb{P}^1\times\mathbb{P}^2$.

Lemma 9.5 Fix integers $d \ge 1$ and $D \ge 1$. Over an algebraically closed field of characteristic not ℓ, any smooth surface X in $\mathbb{P}^1\times\mathbb{P}^2$ of bidegree (d, D) has ℓ–adic Euler characteristic $\chi(X, \mathbb{Q}_\ell)$ and $\dim Ev^2(X, \mathbb{Q}_\ell(1))$ given by the explicit formula

$$\chi(X, \mathbb{Q}_\ell) = 4 + \dim Ev^2(X, \mathbb{Q}_\ell(1)) = 2D(3-D) + 3d(D-1)^2.$$

proof For the first equality, we argue as follows. For a smooth surface X in $\mathbb{P}^1\times\mathbb{P}^2$, weak Lefschetz gives $h^1(X) = 0$, so by Poincare duality $h^3(X) = 0$. We have $h^0(X) = h^4(X) = 1$, so all in all $\chi(X) = h^2(X) + 2 = ev^2(X) + 4$.

We now turn to the numerical evaluation of $\chi(X)$. Suppose first that d=1. Then we may compute by taking X to be the total space of a Lefschetz pencil of plane curves of degree D. Thus X is the blowup of $\mathbb{P}^2$ at the D^2 points of intersection of any two members of the pencil, and hence

$$\chi(X, \mathbb{Q}_\ell) = \chi_c(\mathbb{P}^2 - (D^2 \text{ points})) + D^2\chi(\mathbb{P}^1) = D^2 + 3.$$

If we think of this X as fibred over $\mathbb{P}^1$ by plane curves of degree D, which are smooth except over a finite set of points S in $\mathbb{P}^1$, over each of which the fibre is smooth except for one ordinary double point, and remember that for a Lefschetz pencil of odd fibre dimension the $R^if_*\mathbb{Q}_\ell$ are tame (local monodromies are unipotent, by the Picard–Lefschetz formula), we get

$$\chi(X, \mathbb{Q}_\ell) = \chi(X - \text{singular fibres}, \mathbb{Q}_\ell) + (\#S)\chi(\text{a singular fibre})$$
$$= \chi(\mathbb{P}^1 - S)\chi(\text{a smooth fibre}) + (\#S)\chi(\text{a singular fibre}).$$

But for plane curves, an ordinary double point increases the Euler characteristic by 1, so we find

$$\chi(X, \mathbb{Q}_\ell) = (2 - \#S)\chi(\text{a smooth fibre}) +(\#S)(1 + \chi(\text{a smooth fibre}))$$
$$= 2\chi(\text{a smooth fibre}) + \#S = 2D(3-D) + \#S.$$

This allows us to solve for #S, the number of singular fibres in a Lefschetz pencil of plane curves of degree D:

$$\#S = D^2 + 3 + 2D(D-3) = 3(D-1)^2.$$

To do the general case of bidegree (d, D), we may compute for the pullback, call it X_d, of the Lefschetz pencil by a self−map of degree d of $\mathbb{P}^1$ which is finite etale over the points where the pencil has singular fibres. Now we have d#S singular fibres, each smooth except for an ordinary double point, so by the fibration calculation method above we now find

$\chi(X_d, \mathbb{Q}_\ell)$
$= (2 - d\#S)\chi(\text{a smooth fibre}) + (d\#S)(1 + \chi(\text{a smooth fibre}))$
$= 2\chi(\text{a smooth fibre}) + d\#S$
$= 2D(3-D) + d\#S$
$= 2D(3-D) + 3d(D-1)^2$. QED

9.6 Combining this numerical lemma with the inequalities (9.5.2)

9.6.1 $$0 \le r(\mathcal{C}/k) \le \rho_{ev}(\mathcal{C}/k) \le \rho_{an,\, ev}(\mathcal{C}/k)$$

and the bihomogeneous AFGI∩SGF and SGF variants 8.12 of 7.6, we find the following result.

Theorem 9.7 Fix integers $d \ge 2$ and $D \ge 3$, and consider smooth surfaces $\mathcal{C}/k$ in $\mathbb{P}^1 \times \mathbb{P}^2$ of bidegree (d, D), over variable finite fields k of odd characteristic. Denote by N the rank of Ev^2 for any such $\mathcal{C}/k$:

$$N := \dim Ev^2(\mathcal{C}\otimes_k \bar{k}, \mathbb{Q}_\ell(1)) = 2D(3-D) + 3d(D-1)^2 - 4.$$

Denote by SGF$\mathcal{H}$ the parameter space

$$\mathrm{SGF}\mathcal{H} := \mathrm{SGFHyp}(2, (1, 2), (d, D)) \subset \mathrm{Hyp}(2, (1, 2), (d, D)).$$

0) The fraction of those $\mathcal{C}/k$ in SGF$\mathcal{H}$(k) which lie in SGF$\mathcal{H}$(k, +) (resp. SGF$\mathcal{H}$(k), −)) tends to 1/2 as #k → ∞.

1) The fraction of $\mathcal{C}/k$ in SGF$\mathcal{H}$(k, +) with $r(\mathcal{C}/k) = 0$ tends to 1 as #k → ∞.

1a) If N is even, the fraction of $\mathcal{C}/k$ in SGF$\mathcal{H}$(k, +) with $r_{geom}(\mathcal{C}/k) = 0$ tends to 1 as #k → ∞.

1b) If N is odd, the fraction of $\mathcal{C}/k$ in SGF$\mathcal{H}$(k, +) with $r_{quad}(\mathcal{C}/k) \le 1$ tends to 1 as #k → ∞.

1c) If N is odd, the fraction of $\mathcal{C}/k$ in SGF$\mathcal{H}$(k, +) with $r_{geom}(\mathcal{C}/k) \le 1$ tends to 1 as #k → ∞.

2) The fraction of $\mathcal{C}/k$ in SGF$\mathcal{H}$(k, −) with $r(\mathcal{C}/k) \le 1$ tends to 1 as #k → ∞.

2a) If N is odd, the fraction of $\mathcal{C}/k$ in SGF$\mathcal{H}$(k, −) with $r_{geom}(\mathcal{C}/k) \le 1$ tends to 1 as #k → ∞.

2b) If N is even, the fraction of $\mathcal{C}/k$ in SGF$\mathcal{H}$(k, −) with $r_{quad}(\mathcal{C}/k) \le 2$ tends to 1 as #k → ∞.

2c) If N is even, the fraction of $\mathcal{C}/k$ in SGF$\mathcal{H}$(k, −) with $r_{geom}(\mathcal{C}/k) \le 2$ tends to 1 as #k → ∞.

3) All statements 0) through 2c) above remain valid if we replace "SGF$\mathcal{H}$" by "AFGI∩SGF$\mathcal{H}$" throughout.

Corollary on average ranks 9.8 Hypotheses and notations as in Theorem 9.7:
1) We have the following ("unconditional") upper bounds:

9.8.1.1 $\limsup_{\mathrm{odd}\ \#k \to \infty}$(average over SGF$\mathcal{H}$(k) of r($C$/k)) ≤ 1/2,

9.8.1.2 $\limsup_{\mathrm{odd}\ \#k \to \infty}$(average over SGF$\mathcal{H}$(k) of r_{quad}(C/k)) ≤ 1,

9.8.1.3 $\limsup_{\mathrm{odd}\ \#k \to \infty}$(average over SGF$\mathcal{H}$(k) of r_{geom}(C/k)) ≤ 1.
2) The above statements 9.8.1.1−3 remain valid if we replace SGF$\mathcal{H}$ by AFGI∩SGF$\mathcal{H}$ throughout.
3) **If** the Tate conjecture holds for all the surfaces C/k in AFGI∩SGF$\mathcal{H}$, these inequalities are equalities:
9.8.3.1 $\lim_{\mathrm{odd}\ \#k \to \infty}$(average over AFGI∩SGF$\mathcal{H}$(k) of r($C$/k)) = 1/2,

9.8.3.2 $\lim_{\mathrm{odd}\ \#k \to \infty}$(average over AFGI∩SGF$\mathcal{H}$(k) of r_{quad}(C/k))=1,

9.8.3.3 $\lim_{\mathrm{odd}\ \#k \to \infty}$(average over AFGI∩SGF$\mathcal{H}$(k) of r_{geom}(C/k)) = 1.
4) **If** the Tate conjecture holds for all the surfaces C/k in AFGI∩SGF$\mathcal{H}$, then we also have the equalities
9.8.4.1 $\lim_{\mathrm{odd}\ \#k \to \infty}$(average over SGF$\mathcal{H}$(k) of r($C$/k)) = 1/2,

9.8.4.2 $\lim_{\mathrm{odd}\ \#k \to \infty}$(average over SGF$\mathcal{H}$(k) of r_{quad}(C/k))=1,

9.8.4.3 $\lim_{\mathrm{odd}\ \#k \to \infty}$(average over SGF$\mathcal{H}$(k) of r_{geom}(C/k)) = 1.
proof The only point that needs to be explained is how part 4) follows from part 3), i.e, why the points in SGF$\mathcal{H}$(k) not in AFGI∩SGF$\mathcal{H}$(k) make a contribution to the average which goes to zero as #k → ∞. This is immediate from the following two facts:
1) The ratio #AFGI∩SGF$\mathcal{H}$(k)/#SGF$\mathcal{H}$(k) → 1 as #k → ∞.
2) The function r_{geom}(C/k) is uniformly bounded (indeed we have

$$r_{\mathrm{geom}}(C/k) \le \rho_{\mathrm{ev,geom}}(C/k) \le \mathrm{ev}^2 = 2D(3-D) + 3d(D-1)^2 - 4). \text{ QED}$$

Question 9.9 What, if any, are the number field analogues of the quantities r_{quad}(C/k) and r_{geom}(C/k)?

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