Advances in Mathematics 358 (2019) 106859

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Local systems and finite unitary and symplectic groups

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ARTICLE INFO

Article history:
Received 4 June 2019
Received in revised form 10 October 2019
Accepted 12 October 2019
Available online xxxx
Communicated by Zhiwei Yun

Keywords:
Local systems
Monodromy groups
Weil representations
Finite symplectic groups
Finite unitary groups

ABSTRACT

For powers q of any odd prime p and any integer $n \geq 2$, we exhibit explicit local systems, on the affine line $\mathbb{A}^1$ in characteristic $p > 0$ if $2|n$ and on the affine plane $\mathbb{A}^2$ if $2 \nmid n$, whose geometric monodromy groups are the finite symplectic groups $\mathrm{Sp}_{2n}(q)$. When $n \geq 3$ is odd, we show that the explicit rigid local systems on the affine line in characteristic $p > 0$ constructed in [11] do have the special unitary groups $\mathrm{SU}_n(q)$ as their geometric monodromy groups as conjectured therein, and also prove another conjecture of [11] that predicted their arithmetic monodromy groups.

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[1] The second author gratefully acknowledges the support of the NSF (grants DMS-1839351 and DMS-1840702), and the Joshua Barlaz Chair in Mathematics. The paper is partially based upon work supported by the NSF under grant DMS-1440140 while the second author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2018 semester. It is a pleasure to thank the Institute for support, hospitality, and stimulating environments.

[2] The authors are grateful to the referee for careful reading and helpful comments on the paper.

https://doi.org/10.1016/j.aim.2019.106859

Contents

1. Overall introduction

We first recall from [12, §1] the underlined motivation for this work. The solution [15] (see also [14]) of Abhyankar's Conjecture for the affine line in characteristic $p > 0$ tells us that any finite group G which is generated by its Sylow p-subgroups occurs as a quotient of the geometric fundamental group $\pi_1(\mathbb{A}^1/\overline{\mathbb{F}_p})$ of the affine line $\mathbb{A}^1/\overline{\mathbb{F}_p}$ over $\overline{\mathbb{F}_p}$. In a series of papers (see e.g. [1]), Abhyankar has written down explicit equations which realize many finite groups of Lie type as such quotients.

Suppose we are given such a finite group G (i.e., one which is generated by its Sylow p-subgroups), together with a faithful representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$. Because G is finite, there is always some number field K such that the image of ρ lands in $\mathrm{GL}_n(K)$. If we now choose a prime number ℓ and an embedding of K into $\overline{\mathbb{Q}_\ell}$, we can view ρ as a representation $\rho : G \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$. Since G is a quotient of $\pi_1(\mathbb{A}^1/\overline{\mathbb{F}_p})$, we can compose

$$\pi_1(\mathbb{A}^1/\overline{\mathbb{F}_p}) \twoheadrightarrow G \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell}),$$

to get a continuous ℓ-adic representation of $\pi_1(\mathbb{A}^1/\overline{\mathbb{F}_p})$, i.e., an ℓ-adic **local system** on $\mathbb{A}^1/\overline{\mathbb{F}_p}$, whose image is the finite group G.

There are a plethora of local systems on the affine line attached to families of exponential sums. In the ideal world, we would be able, given the data (G, ρ) and any $\ell \neq p$, to write down a "simple to remember" family of exponential sums incarnating a local system which gives (G, ρ). Needless to say, we are far from being in the ideal world.

In our earlier paper [12, Theorem 3.2], we gave explicit rigid local systems on the affine line $\mathbb{A}^1$ in characteristic $p > 2$ whose geometric monodromy groups were proved to be the finite symplectic groups $\mathrm{Sp}_{2n}(q)$, so long as[3] $n \geq 2$ was itself prime to p and so long as q was a power p^a of p such that the exponent a was prime to p.

Here we develop some new ideas which, when $n \geq 3$ is odd, give us rigid local systems incarnating all the $q+1$ irreducible representations of $\mathrm{SU}_n(q)$ of degree either $\frac{q^n+1}{q+1}$ (q of

[3] The case $n = 1$ of $\mathrm{SL}_2(q)$ was done in [11].

these) or $\frac{q^n-q}{q+1}$ (one of these). It turns out a posteriori that these rigid local systems are precisely those occurring in [11, Conjecture 9.2], where they were conjectured to have these monodromy groups for all odd $n \geq 3$ but only proven [11, Theorem 19.1] to have them when n was 3, with the additional proviso that 3 not divide $q+1$. As a result, we are able to determine the arithmetic and geometric monodromy groups of these local systems and also prove Conjecture 9.2 of [11].

These ideas also lead us to local systems on $\mathbb{A}^2$ whose geometric monodromy groups are the finite symplectic groups $\mathrm{Sp}_{2n}(q)$ for every odd $n \geq 3$ and every power q of the odd prime p. In contrast to [12], there are no "prime to p" hypotheses on either n or on $\log_p(q)$.

Another chain of ideas leads us to local systems on $\mathbb{A}^1$ whose geometric monodromy groups are the finite symplectic groups $\mathrm{Sp}_{2n}(q)$ for every **even** n and every power q of the odd prime p. Again here there are no "prime to p" hypotheses on either n or on $\log_p(q)$.

In both of the n even and n odd cases, a key new idea is to study certain two-parameter local systems and their monodromy groups first, and then descend to our target one-parameter systems by specialization.

2. Introduction to the n odd case

For an **odd** integer $n \geq 3$, and a prime power $q \geq 2$, the irreducible representations (over $\mathbb{C}$) of lowest degree after the trivial representation of the group $\mathrm{SU}_n(q)$ are a symplectic representation of dimension $\frac{q^n+1}{q+1} - 1 = \frac{q^n-q}{q+1}$, and q representations of dimension $\frac{q^n+1}{q+1}$. When q is odd, exactly one of these q representations is orthogonal, otherwise none is. The direct sum of these $q+1$ representations is called the big Weil representation of $\mathrm{SU}_n(q)$.

In the paper [11], we wrote down $q+1$ rigid local systems on the affine line $\mathbb{A}^1/\overline{\mathbb{F}_p}$ whose geometric monodromy groups we conjectured to be the images of $\mathrm{SU}_n(q)$ in these $q+1$ representations. We were able to prove this only in the case when $n = 3$ and $\gcd(n, q+1) = 1$ (the condition that $\mathrm{SU}_n(q) = \mathrm{PSU}_n(q)$), where we made use of the results of Dick Gross [6]. In this paper, we use a completely different method, which also starts[4] with results of Gross, to prove these conjectures for any odd $n \geq 3$ and for any odd prime power q, see Theorem 4.4.

The method used here, which requires that q be odd, is based on a striking group-theoretic relation between the Weil representations of $\mathrm{SU}_n(q)$ and $\mathrm{Sp}_{2n}(q)$, and on the determination of those subgroups of $\mathrm{Sp}_{2n}(q)$ to which the Weil representation restricts "as though" it were the Weil representation of $\mathrm{SU}_n(q)$, cf. Theorem 3.4. We are able to apply this result to our local systems, in Section 3, by invoking results of [12], which was devoted to questions around $\mathrm{Sp}_{2n}(q)$. Furthermore, our Theorem 4.3 also improves the

[4] The results here use the results of [12], which in turn uses the results of [11] for $\mathrm{SL}_2(q)$, and those use [6] in an essential way.

main results Theorems 3.1 and 6.8 of [12] in the case $2 \nmid n$, by removing the condition that $p \nmid n \cdot \log_p(q)$ for the prime $p|q$.

The main results in the n odd case are Theorems 4.2, 4.3, 4.4, 5.1, and 5.2 that determine the arithmetic and geometric monodromy groups of the constructed local systems, and Theorem 5.3 that establishes Conjecture 9.2 of [11].

3. Unitary-type subgroups of finite symplectic groups

Let $q = p^a$ be any power of a prime p and $n \geq 2$. It is well known, see e.g. [5, Theorem 4.9.2], that the function

$$\zeta_{n,q} = \zeta_n : g \mapsto (-1)^n (-q)^{\dim_{\mathbb{F}_{q^2}} \mathrm{Ker}(g - 1_W)}$$

defines a complex character, called the (reducible) *Weil character*, of the general unitary group $\mathrm{GU}_n(q) = \mathrm{GU}(W)$, where $W = \mathbb{F}_{q^2}^n$ is a non-degenerate Hermitian space with Hermitian product $\circ$. Note that the $\mathbb{F}_q$-bilinear form

$$(u|v) = \mathrm{Trace}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\theta u \circ v)$$

on W, for a fixed $\theta \in \mathbb{F}_{q^2}^\times$ with $\theta^{q-1} = -1$, is non-degenerate symplectic. This leads to an embedding

$$\tilde{G} := \mathrm{GU}_n(q) \hookrightarrow \mathrm{Sp}_{2n}(q).$$

Similarly, the function

$$\tau_{n,q} = \tau_n : g \mapsto q^{\dim_{\mathbb{F}_q} \mathrm{Ker}(g - 1_U)}$$

defines a complex character, called the (reducible) *Weil character*, of the general linear group $\mathrm{GL}_n(q) = \mathrm{GL}(U)$, where $U = \mathbb{F}_q^n$, see e.g. [5, Corollary 1.4]. Again we can embed $\mathrm{GL}_n(q)$ into $\mathrm{Sp}_{2n}(q)$ so that $\mathrm{GL}_n(q)$ stabilizes a complementary pair (U, U') of maximal totally isotropic subspaces of the symplectic space $\mathbb{F}_q^{2n}$. For the reader's convenience, we record the following statement, which follows from [5, Theorem 2.4(c)] in the GL-case, and from [5, Theorem 3.3] in the GU-case:

Theorem 3.1. *Let q be an odd prime power and let $n \in \mathbb{Z}_{\geq 1}$. Then the following statements hold.*

(i) *Let $\tilde{\chi}_2$ denote the unique complex character of degree 1 and of order 2 of $\mathrm{GU}_n(q)$. Then the restriction of any of the two big Weil characters (of degree q^n, and denoted* Weil_1 *and* Weil_2 *in [12, §2]) of $\mathrm{Sp}_{2n}(q)$ to $\mathrm{GU}_n(q)$ is $\tilde{\zeta}_n := \tilde{\chi}_2 \zeta_n$.*

(ii) *Let* χ_2 *denote the unique complex character of degree* 1 *and of order* 2 *of* $\mathrm{GL}_n(q)$*. Then the restriction of any of the two big Weil characters* Weil_1*,* Weil_2 *of* $\mathrm{Sp}_{2n}(q)$ *to* $\mathrm{GL}_n(q)$ *is* $\chi_2\tau_n$*.*

Fix a generator σ of $\mathbb{F}_{q^2}^\times$ and set $\rho := \sigma^{q-1}$. We also fix a primitive (q^2-1)th root of unity $\boldsymbol{\sigma} \in \mathbb{C}^\times$ and let $\boldsymbol{\rho} = \boldsymbol{\sigma}^{q-1}$. By Theorem 3.1(i),

$$(\mathsf{Weil}_1)|_{\tilde{G}} = \tilde{\zeta}_n = \sum_{i=0}^{q} \tilde{\zeta}_{i,n} \tag{3.1.1}$$

decomposes as the sum of $q+1$ characters of $\tilde{G}$, where

$$\tilde{\zeta}_{i,n}(g) = \frac{(-1)^n \tilde{\chi}_2(g)}{q+1} \sum_{l=0}^{q} \boldsymbol{\rho}^{il} (-q)^{\dim \mathrm{Ker}(g-\rho^l \cdot 1_W)}; \tag{3.1.2}$$

this formula is obtained by applying [17, Lemma 4.1] to the "untwisted" character ζ_n. In particular, $\tilde{\zeta}_{i,n}$ has degree $(q^n-(-1)^n)/(q+1)$ if $i>0$ and $(q^n+(-1)^n q)/(q+1)$ if $i=0$. (Note that formula (3.1.2) also holds for $2|q$, where we define $\tilde{\zeta}_n = \zeta_n$, $\tilde{\chi}_2 = 1_{\tilde{G}}$ in that case.)

We will let $\zeta_{i,n}$ denote the restriction of $\tilde{\zeta}_{i,n}$ to $G = \mathrm{SU}_n(q)$, for $0 \le i \le q$. If $n \ge 3$, then these $q+1$ characters are all irreducible and distinct. If $n=2$, then $\zeta_{i,n}$ is irreducible, unless q is odd and $i=(q+1)/2$, in which case it is a sum of two irreducible characters of degree $(q-1)/2$, see [17, Lemma 4.7]. Formula (3.1.2) implies that Weil characters $\zeta_{i,n}$ enjoy the following branching rule while restricting to the natural subgroup $H := \mathrm{Stab}_G(w) \cong \mathrm{SU}_{n-1}(q)$ ($w \in W$ any anisotropic vector):

$$\zeta_{i,n}|_H = \sum_{j=0,\ j\neq i}^{q} \zeta_{j,n-1}. \tag{3.1.3}$$

Furthermore, complex conjugation fixes $\tilde{\zeta}_{0,n}$ and sends $\tilde{\zeta}_{j,n}$ to $\tilde{\zeta}_{q+1-j,n}$ when $1 \le j \le q$. As $n \ge 3$ is odd, it is also known that $\tilde{\zeta}_{0,n}$ is of symplectic type; let

$$\Psi_0 : \tilde{G} \to \mathrm{Sp}(V)$$

be a complex representation affording this character. If $2 \nmid q$, then $\tilde{\zeta}_{(q+1)/2,n}$ is of orthogonal type; let

$$\Psi_{(q+1)/2} : \tilde{G} \to \mathrm{O}(V)$$

be a complex representation affording this character. In the remaining cases, let

$$\Psi_i : \tilde{G} \to \mathrm{GL}(V)$$

be a complex representation affording the character $\tilde{\zeta}_{i,n}$.

Lemma 3.2. *Assume $n \geq 3$ is* **odd** *and q is odd.*

(i) $\Psi_0(\mathrm{GU}_n(q)) \cong \mathrm{GU}_n(q)/C_{(q+1)/2}$ *is contained in* $\mathrm{Sp}(V)$ *and contains* $\Psi_0(\mathrm{SU}_n(q)) \cong \mathrm{PSU}_n(q)$ *with index* $2d$*, where* $d := \gcd(n, q+1)$.

(ii) *If* $1 \leq i \leq q$*, then* $\mathrm{Ker}(\Psi_i)$ *is a central subgroup of order* $\gcd(i+(q+1)/2, q+1)$*, and* $\mathrm{Ker}(\Psi_i|_{\mathrm{SU}_n(q)})$ *is a central subgroup of order* $\gcd(i+(q+1)/2, n, q+1)$.

(iii) $\mathrm{PGU}_n(q) \cong \Psi_{(q+1)/2}(\mathrm{GU}_n(q)) \leq \mathrm{SO}(V)$ *contains* $\Psi_{(q+1)/2}(\mathrm{SU}_n(q)) \cong \mathrm{PSU}_n(q)$ *with index* d.

(iv) *If* $1 \leq i \leq q$ *and* $i \neq (q+1)/2$*, then* $\Psi_i(\mathrm{GU}_n(q)) \cap \mathrm{SL}(V)$ *contains* $\Psi_i(\mathrm{SU}_n(q))$ *with index* $\gcd(i+(q+1)/2, n, q+1)$.

(v) *Suppose* $H \leq \mathrm{GU}_n(q)$*. Then* $\Psi_i(H) \leq \mathrm{SL}(V)$ *for all* $0 \leq i \leq q$ *if and only if* $H \leq \mathrm{SU}_n(q)$.

Proof. According to [17, §4], one can label Ψ_i in such a way that

$$\Psi_i(z) = -\boldsymbol{\rho}^i \cdot 1_V = \boldsymbol{\rho}^{i+(q+1)/2} \cdot 1_V \tag{3.2.1}$$

for the generator $z = \rho \cdot 1_W$ of $\mathbf{Z}(\tilde{G}) \cong C_{q+1}$. Note that we need to add the minus-sign, because $\tilde{\zeta}_n$ is obtained from ζ_n by the quadratic twist $\tilde{\chi}_2$. In particular, $\mathrm{Ker}(\Psi_0) \cap \mathbf{Z}(\tilde{G}) = \langle z^2 \rangle$, and (i) follows.

Now we can assume $1 \leq i \leq q$. By (3.2.1), $z^j \in \mathrm{Ker}(\Psi_i)$ if and only if j is divisible by $(q+1)/\gcd(i+(q+1)/2, q+1)$. Furthermore, $z^{j(q+1)/d} \in \mathrm{Ker}(\Psi_i|_{\mathrm{SU}_n(q)})$ if and only if j is divisible by $d/\gcd(i,d) = d/\gcd(i+(q+1)/2, n, q+1)$ for $d = \gcd(n, q+1)$, equivalently, if $j(q+1)/d$ is divisible by $(q+1)/\gcd(i+(q+1)/2, n, q+1)$. Hence (ii) follows.

Consider the element $g := \mathrm{diag}(\rho, 1, 1, \ldots, 1) \in \tilde{G}$; note that $\tilde{G} = \langle G, g \rangle$ and $\tilde{\chi}_2(g) = -1$. Then (3.1.2) implies that

$$\tilde{\zeta}_{i,n}(g^k) = (-1)^k \left(-\frac{q^{n-1} - (-1)^{n-1}}{q+1} + (-1)^{n-1} \boldsymbol{\rho}^{ik} \right)$$

when $1 \leq k \leq q$. It follows that $\Psi_i(g)$ has eigenvalues $-\boldsymbol{\rho}^j$, $1 \leq j \leq q$, with multiplicity $(q^{n-1}-1)/(q+1)$ if $j \neq i$ and $1 + (q^{n-1}-1)/(q+1)$ if $j = i$, and so

$$\det(\Psi_i(g)) = -\boldsymbol{\rho}^i = \boldsymbol{\rho}^{i+(q+1)/2}.$$

In particular, $\Psi_i(g^j) \in \mathrm{SL}(V)$ if and only if j is divisible by $(q+1)/\gcd(i+(q+1)/2, q+1)$. Since $\mathrm{SU}_n(q)$ is perfect, (iii), (iv), and the "if" direction of (v) follow.

For the "only if" direction of (v), assume that $\Psi_1(H) \leq \mathrm{SL}(V)$, and consider any $h \in H$. If $\det(h) = \rho^j$ for $0 \leq j \leq q$, then $hg^{-j} \in \mathrm{SU}_n(q)$ and so $\Psi_{(q+3)/2}(hg^{-j}) \in \mathrm{SL}(V)$ by the previous statement. It follows that

$$1 = \det(\Psi_{(q+3)/2}(h)) = \det(\Psi_{(q+3)/2}(hg^{-j}))\det(\Psi_{(q+3)/2}(g^j))$$
$$= \det(\Psi_{(q+3)/2}(g^j)) = \boldsymbol{\rho}^j,$$

whence $j = 0$ and $\det(h) = 1$, as stated. $\square$

We will now show that, when $n \geq 3$ is odd and q is odd, the splitting (3.1.1) of a big Weil character Weil_i of $\mathrm{Sp}_{2n}(q)$ on its restriction to $\mathrm{SU}_n(q)$ into a sum of $q+1$ irreducible constituents of prescribed degrees characterizes $\mathrm{SU}_n(q)$ uniquely (up to conjugacy).

Recall [19] that if $a \geq 2$ and $n \geq 2$ are any integers with $(a, n) \neq (2, 6)$, $(2^k - 1, 2)$, then $a^n - 1$ has a *primitive prime divisor*, that is, a prime divisor ℓ that does not divide $\prod_{i=1}^{n-1}(a^i - 1)$; write $\ell = \mathrm{ppd}(a, n)$ in this case. Furthermore, if in addition $a, n \geq 3$ and $(a, n) \neq (3, 4)$, $(3, 6)$, $(5, 6)$, then $a^n - 1$ admits a *large primitive prime divisor*, i.e. a primitive prime divisor ℓ where either $\ell > n + 1$ (whence $\ell \geq 2n + 1$), or $\ell^2|(a^n - 1)$, see [4].

We will need the following recognition theorem [12, Theorem 4.6], which was obtained relying on [7].

Theorem 3.3. *Let $q = p^f$ be a power of an odd prime p and let $d \geq 2$. If $d = 2$, suppose that $p^{df} - 1$ admits a primitive prime divisor $\ell \geq 5$ with $(p^{df} - 1)_\ell \geq 7$. If $d \geq 3$, suppose in addition that $(p, df) \neq (3, 4)$, $(3, 6)$, $(5, 6)$, so that $p^{df} - 1$ admits a large primitive prime divisor ℓ. In either case, we choose such an ℓ to maximize the ℓ-part of $p^{df} - 1$. Let $W = \mathbb{F}_q^d$ and let G be a subgroup of $\mathrm{GL}(W) \cong \mathrm{GL}_d(q)$ of order divisible by the ℓ-part $Q := (q^d - 1)_\ell$ of $q^d - 1$. Also, let $L := \mathbf{O}^{\ell'}(G)$ be the smallest among normal subgroups of G of index coprime to ℓ. Then either L is a cyclic ℓ-group of order Q, or there is a divisor $j < d$ of d such that one of the following statements holds.*

- (i) *$L = \mathrm{SL}(W_j) \cong \mathrm{SL}_{d/j}(q^j)$, $d/j \geq 3$, and W_j is W viewed as a d/j-dimensional vector space over $\mathbb{F}_{q^j}$.*
- (ii) *$2j|d$, W_j is W viewed as a d/j-dimensional vector space over $\mathbb{F}_{q^j}$ endowed with a non-degenerate symplectic form, and $L = \mathrm{Sp}(W_j) \cong \mathrm{Sp}_{d/j}(q^j)$.*
- (iii) *$2|jf$, $2 \nmid d/j$, W_j is W viewed as a d/j-dimensional vector space over $\mathbb{F}_{q^j}$ endowed with a non-degenerate Hermitian form, and $L = \mathrm{SU}(W_j) \cong \mathrm{SU}_{d/j}(q^{j/2})$.*
- (iv) *$2j|d$, $d/j \geq 4$, W_j is W viewed as a d/j-dimensional vector space over $\mathbb{F}_{q^j}$ endowed with a non-degenerate quadratic form of type $-$, and $L = \Omega(W_j) \cong \Omega^-_{d/j}(q^j)$.*
- (v) *$(p, df, L/\mathbf{Z}(L)) = (3, 18, \mathrm{PSL}_2(37))$, $(17, 6, \mathrm{PSL}_2(13))$.*

The main result of this section is the following theorem:

Theorem 3.4. *Let $q = p^a$ be a power of an odd prime p and let $n \geq 3$ be an odd integer. Let $W = \mathbb{F}_q^{2n}$ be a non-degenerate symplectic space, and $H := \mathrm{Sp}(W) \cong \mathrm{Sp}_{2n}(q)$, and let Φ be a complex Weil representation Weil_i of H of degree q^n for some $i = 1, 2$ as in [12, §2]. Suppose that $G \leq H$ is a subgroup such that $\Phi|_G = \oplus_{j=0}^q$ is a sum of $q + 1$*

irreducible summands, Φ_0 of degree $(q^n - q)/(q+1)$ and Φ_j of degree $(q^n+1)/(q+1)$ for $1 \leq j \leq q$. Then W can be viewed as an n-dimensional vector space over $\mathbb{F}_{q^2}$ endowed with a G-invariant non-degenerate Hermitian form such that

$$\mathrm{SU}_n(q) \cong \mathrm{SU}(W) \lhd G \leq \mathrm{GU}(W) \cong \mathrm{GU}_n(q).$$

Proof. (a) First we assume that $(n, q) \neq (3, 3)$ and $(3, 5)$; in particular, so that $p^{2na} - 1$ admits a large primitive prime divisor ℓ, in which case we choose such an ℓ to maximize the ℓ-part of $p^{2na} - 1$. Note the assumptions imply that $|G|$ is divisible by both $(q^n - q)/(q+1)$ and $(q^n + 1)/(q + 1)$. In particular, $G < \mathrm{GL}(W)$ has order divisible by

$$qQ := q(p^{2na} - 1)_\ell. \tag{3.4.1}$$

Let $L := \mathbf{O}^{\ell'}(G)$ and $d(L)$ denote the smallest degree of nontrivial complex irreducible characters of L. Note that

$$d(L) \leq (q^n + 1)/(q + 1) \leq (q^n + 1)/4. \tag{3.4.2}$$

(Otherwise $L \leq \mathrm{Ker}(\Phi_1)$, whence Φ_1 could be viewed as an irreducible representation of G/L and so would have been of ℓ'-degree.) Furthermore, if L is cyclic of order Q, then by Ito's theorem (6.15) of [8], the degree of any irreducible character of G divides $|G/L|$, an integer coprime to ℓ, and so again G cannot be irreducible on Φ_1. Now we can apply Theorem 3.3 to arrive at one of the following cases.

(i) $L \cong \mathrm{SL}_{2n/j}(q^j)$ for some divisor $1 \leq j \leq n$ of $2n$. In this case, if $j \leq 2n/3$ then by [16, Theorem 3.1] we have

$$d(L) > q^{j(2n/j-1)} = q^{2n-j} > q^n,$$

contradicting (3.4.2). If $j = n$, then $q^j = q^n \geq 27$ and so

$$d(L) \geq (q^n - 1)/2 > (q^n + 1)/4,$$

again contradicting (3.4.2).

(ii) $L \cong \mathrm{Sp}_{2n/j}(q^j)$ for some divisor $1 \leq j < n/2$ of n. Then by [16, Theorem 1.1] we have

$$d(L) \geq (q^n - 1)/2 > (q^n + 1)/4,$$

contradicting (3.4.2).

(iii) There is some even divisor $j = 2k$ of $2n$ with $k|n$ and $2 \nmid n/k > 1$, such that W can be viewed as a $2n/j$-dimensional vector space over $\mathbb{F}_{q^j}$ endowed with a non-degenerate Hermitian form and $L = \mathrm{SU}(W) \cong \mathrm{SU}_{n/k}(q^k)$. Suppose first that $k > 1$, and let ψ be an irreducible constituent of the L-character afforded by Φ_0, so that $\psi(1) < (q^n+1)/4$. By [16, Theorem 4.1],

$$\psi(1) \in \left\{1, \frac{q^n+1}{q^k+1}, \frac{q^n-q^k}{q^k+1}\right\}.$$

The proof of (3.4.2) rules out the possibility $\psi(1) = 1$. Next,

$$\psi(1) | \dim \Phi_0 = (q^n - q)/(q+1)$$

by Clifford's theorem, implying $\psi(1) \neq (q^n - q^k)/(q^k + 1)$ as $k > 1$. The remaining possibility $\psi(1) = (q^n+1)/(q^k+1)$ is also ruled out since $\ell \nmid \dim \Phi_0$. We have shown that $k = 1$, i.e. $L = \mathrm{SU}(W) \cong \mathrm{SU}_n(q)$. This implies that

$$L \lhd G \leq \mathbf{N}_{\mathrm{Sp}(W)}(L) = \mathrm{GU}(W) \rtimes \langle\sigma\rangle \cong \mathrm{GU}_n(q) \rtimes C_2.$$

Here, σ is an involutive automorphism of $\mathrm{GU}(W)$ that acts as inversion on

$$\langle z\rangle = \mathbf{Z}(\mathrm{GU}(W)) \cong C_{q+1}. \tag{3.4.3}$$

Recall the decomposition

$$\Phi|_{\mathrm{GU}(W)} = \oplus_{i=0}^{q}\Psi_i, \tag{3.4.4}$$

with Ψ_0 of degree $(q^n - q)/(q+1)$ and Ψ_i of degree $(q^n+1)/(q+1)$ for $1 \leq i \leq q$, see the discussion preceding Lemma 3.2. In fact, one can find a primitive $(q+1)$th root of unity $\xi \in \mathbb{C}^\times$ such that $\Psi_i(z)$ is the multiplication by ξ^i. In particular, σ fuses Ψ_1 and Ψ_q. The assumption on $\Phi|_G$ now implies that $G \leq \mathrm{GU}(W)$, as stated.

(iv) $L \cong \Omega^-_{2n/j}(q^j)$ for some divisor $1 \leq j < n/2$ of the odd integer n. If $j \leq n/5$, then by [16, Theorem 1.1] we have

$$d(L) > q^n + 1,$$

contradicting (3.4.2). If $j = n/3$, then L is a quasisimple quotient of $\mathrm{PSU}_4(q^{n/3})$ with $q^{n/3} > 5$, and so by [16, Theorem 1.1] we have

$$d(L) = \frac{q^{4n/3}-1}{q^{n/3}+1} > q^n/2,$$

again contradicting (3.4.2).

(v) $(p, na, L/\mathbf{Z}(L)) = (3, 9, \mathrm{PSL}_2(37))$. Note that the smallest dimension of a nontrivial irreducible representation of L over $\overline{\mathbb{F}}_3$ is 18 (see e.g. [16, Table I]), so $(q, n) = (3, 9)$ and $L = \mathrm{SL}_2(37)$ acts absolutely irreducibly on $W = \mathbb{F}_3^{18}$. This in turn implies that

$$\mathbf{C}_{\mathrm{Sp}(W)}(L) = \mathbf{Z}(L) = C_2,$$

and so $L \lhd G \leq \mathbf{N}_{\mathrm{Sp}(W)}(L) \leq L \cdot C_2$. But in this case, G cannot have an irreducible complex representation of degree

$$\dim \Phi_1 = (q^n + 1)/(q + 1) = (3^9 + 1)/4.$$

(vi) $(p, na, L/\mathbf{Z}(L)) = (17, 6, \mathrm{PSL}_2(13))$. In this case $(q, n) = (17, 3)$ and $L = \mathrm{SL}_2(13)$ acts absolutely irreducibly on $W = \mathbb{F}_{17}^6$. As in (v), this implies that

$$\mathbf{C}_{\mathrm{Sp}(W)}(L) = \mathbf{Z}(L) = C_2,$$

and $L \lhd G \leq \mathbf{N}_{\mathrm{Sp}(W)}(L) \leq L \cdot C_2$, whence G cannot have an irreducible complex representation of degree

$$\dim \Phi_1 = (q^n + 1)/(q + 1) = (17^3 + 1)/18.$$

(b) It remains to consider the two cases $(n, q) = (3, 3)$ and $(3, 5)$. Let M be a maximal subgroup of $\mathrm{Sp}(W)$ that contains G. Then condition (3.4.1) also holds for $|M|$; furthermore, the maximal degree of complex irreducible characters of M must be at least $(q^n + 1)/(q + 1) = 7$, respectively 21, since $\Phi_1 \in \mathrm{Irr}(G)$. First suppose that $q = 5$. Then, according to Tables 8.27 and 8.28 of [3], one of the following possibilities occurs.

- $M = 2J_2$. In this case, since $|G|$ is divisible by $3 \cdot 5 \cdot 7$, see (3.4.1), we see by inspecting maximal subgroups of J_2 [2] that $G = M$. But then G does not admit any complex irreducible representation of degree $\dim \Phi_0 = 20$.
- $M = \mathrm{SL}_2(125) \rtimes C_3$. In this case, since $|G \cap [M, M]|$ is divisible by $5 \cdot 7$, see (3.4.1), we see by inspecting maximal subgroups of $\mathrm{PSL}_2(125)$ [3, Table 8.1] that $G \rhd \mathrm{SL}_2(125)$. But then $d(G) \geq 62$ (see e.g. [16, Table I]), violating (3.4.2).
- $M = \mathrm{GU}_3(5) \rtimes C_2$. If $G \geq N := \mathrm{SU}_3(5)$, then we can argue as in (iii) above. Suppose $G \not\geq N$. Since $L := G \cap N \lhd G$ has order divisible by $5 \cdot 7$, see (3.4.1), we see by inspecting maximal subgroups of $\mathrm{PSL}_3(5)$ and Alt_7 [2] that $L = 3\mathrm{Alt}_7$, and $\mathbf{Z}(L) = \langle z^2 \rangle$ with $\langle z \rangle = \mathbf{Z}(\mathrm{GU}_3(5))$ as defined in (3.4.3). Using the decomposition (3.4.4), we may assume that $\Phi_i = (\Psi_i)|_G$ for $0 \leq i \leq q$. As mentioned in (iii), the subgroup C_2 fuses Ψ_1 with Ψ_5, hence Φ_1 with Φ_5. Thus $G \leq \mathrm{GU}_3(5)$, and so $|G/L|$ and $|\mathbf{N}_{\mathrm{GU}_3(5)}(L)/L|$ both divide 6. Note that $\mathbf{N}_{\mathrm{GU}_3(5)}(L)$ contains the central involution of $\mathrm{GU}_3(5)$ which lies outside of $\mathrm{SU}_3(5)$. It follows that G induces a subgroup X of outer automorphisms of L of order dividing 3, whence $X = 1$ as $|\mathrm{Out}(\mathrm{Alt}_7)| = 2$ [2]. Now let $g \in L$ be of order 7. Then $\Phi_0(g) = \Psi_0(g)$ has trace -1. On the other hand, as G induces only inner automorphisms on L, we see that $(\Phi_0)|_L$ must be a direct sum of two copies of a single irreducible complex representation Φ' (of dimension 10) of L and we arrive at the contradiction that $\Phi'(g)$ has trace $-1/2$.

(c) Finally, we consider the case $q = 3$. Inspecting the list of maximal subgroups of $\mathrm{PSp}_6(3)$ in [2], we arrive at the following possibilities for M. By (3.4.1), G contains an element $g \in G$ of order 7. According to [2], we may assume that $\Phi_0 \oplus \Phi_2 = \Lambda|_G$, where Λ is an irreducible Weil representation of degree 13 of $\mathrm{Sp}_6(3)$ and contains the central involution t of $\mathrm{Sp}_6(3)$ in its kernel, and that $\Lambda(g)$ has trace -1.

- $M = \mathrm{SL}_2(13)$. In this case, since $|G|$ is divisible by $3 \cdot 7$, see (3.4.1), we see by inspecting maximal subgroups of $\mathrm{PSL}_2(13)$ [2] that $G = M$. Note that t is the central involution of G. Now the conditions that $t \in \mathrm{Ker}(\Lambda)$ and $\Lambda(g)$ has trace -1 imply by [2] that $\Lambda|_G$ is irreducible, a contradiction.
- $M = \mathrm{SL}_2(27) \cdot 3$. In this case, since $|G|$ is divisible by 7, we see by inspecting maximal subgroups of $\mathrm{PSL}_2(27)$ [2] that either $G \geq [M, M] = \mathrm{SL}_2(27)$ or $G \cap [M, M]$ is contained in a dihedral group D_{28}. It is easy to see that in the former case $d(G) \geq 13$ contradicting (3.4.2), and in the latter case G does not admit any complex irreducible representation of dimension $\dim \Phi_1 = 7$.
- $M = \mathrm{GU}_3(3) \rtimes C_2$. If $G \geq N := \mathrm{SU}_3(3)$, then we can argue as in (iii) above. Suppose $G \not\geq N$. Since $L := G \cap N \lhd G$ has order divisible by $3 \cdot 7$, see (3.4.1), we see by inspecting maximal subgroups of $\mathrm{SU}_3(3)$ and $\mathrm{PSL}_2(7)$ [2] that either L is of order 21 or $L = \mathrm{PSL}_2(7)$. The former case is ruled out since $(\Phi_1)|_L$ is irreducible of dimension 7. In the latter case, fix an involution $s \in L$. We may assume that

$$(\Phi_i)|_L = (\Psi_i)|_L$$

for the representations Ψ_i defined in (3.4.4), and furthermore Ψ_2 is self-dual of dimension 7. Using [2] we see that $\Psi_1(s)$ has trace 3 and $\Psi_1(g)$ has trace 0, whence $(\Phi_1)|_L = (\Psi_1)|_L$ is the sum of two irreducible representations of dimensions 1 and 6, contradicting the irreducibility of Φ_1 on $G \rhd L$. $\square$

In the next statement, we consider a non-degenerate symplectic space $W = \mathbb{F}_p^{2N}$, a (reducible) big Weil representation of degree p^N of $G = \mathrm{Sp}(W) \cong \mathrm{Sp}_{2N}(p)$ with character ω as in [12]; in particular,

$$|\omega(g)| = |\mathbf{C}_W(g)|^{1/2} \tag{3.4.5}$$

for any $g \in G$. Let $N = AB$ and $B = bj$ for some positive integers A, B, b, j. We may then assume that W is obtained from the symplectic space $W_1 := \mathbb{F}_{p^B}^{2A}$ (with a Witt basis $(e_1, \ldots, e_A, f_1, \ldots, f_A)$) by base change from $\mathbb{F}_{p^B}$ to $\mathbb{F}_p$. Using this basis we can consider the transformation

$$\sigma : \sum_{i=1}^{A}(x_i e_i + y_i f_i) \mapsto \sum_{i=1}^{A}(x_i^r e_i + y_i^r f_i)$$

induced by the Galois automorphism $x \mapsto x^r$ for $r := p^j$. Then, as in [12, §4] we can consider the standard subgroup

$$H = \mathrm{Sp}(2A, p^B) \rtimes C_b$$

of G, where $C_b = \langle \sigma \rangle$.

Theorem 3.5. *Each value $|\omega(x)|^2$, $x \in H$, is a power of $r = p^j$. Furthermore, there is some $h \in H$ such that $|\omega(h)|^2 = r$.*

Proof. Note that H embeds in $\mathrm{Sp}(2Ab, p^j)$, and so the first statement follows by applying (3.4.5) to a big Weil representation of $\mathrm{Sp}(2Ab, p^j)$. To define h, consider the $\mathbb{F}_r$-linear map

$$f : \mathbb{F}_{p^B} \to \mathbb{F}_{p^B},\ x \mapsto x - x^r.$$

Viewed as a vector space over $\mathbb{F}_r$, $\mathrm{Ker}(f)$ has dimension 1. Hence f cannot be surjective, and so we can find

$$\alpha \in \mathbb{F}_{p^B} \smallsetminus \mathrm{Im}(f).$$

Let J denote the Jordan block of size $A \times A$ with eigenvalue α^{-1}, and let $g \in H$ have the following matrix

$$\begin{pmatrix} {}^t(\alpha J)^{-1} & \alpha^2 J \\ 0 & \alpha J \end{pmatrix}$$

in the chosen basis $(e_1, \dots, e_A, f_1, \dots, f_A)$ of W_1. We will show that $h = g\sigma$ satisfies $|\omega(h)|^2 = r$. According to (3.4.5), it suffices to show that h fixes exactly r vectors in W_1. To this end, suppose that $w = \sum_{i=1}^{A}(x_i e_i + y_i f_i)$ is fixed by h, where $x_i, y_i \in \mathbb{F}_{p^B}$. Comparing the coefficient for f_A we have

$$y_A^r = y_A$$

implying $y_A \in \mathbb{F}_r$. Next, comparing the coefficient for f_{A-1} we see that

$$y_{A-1}^r + \alpha y_A^r = y_{A-1},$$

and so $\alpha y_A = f(y_{A-1})$. By the choice of α, $y_A = 0$, whence $y_{A-1} \in \mathbb{F}_r$. Continuing in the same fashion, we conclude that

$$y_1 \in \mathbb{F}_r,\ y_2 = y_3 = \ldots = y_A = 0.$$

Thus we have shown that $v := \sum_{i=1}^{A} y_i f_i = y_1 f_1$. Letting $u := w - v = \sum_{i=1}^{A} x_i e_i$, we have

$$^t(\alpha J)^{-1}\sigma(u) + \alpha^2 J\sigma(v) = u,$$

i.e.

$$\sigma(u) + {}^t(\alpha J)\alpha^2 J\sigma(v) = {}^t(\alpha J)(u).$$

Comparing the coefficient for e_1, we get

$$x_1^r + \alpha y_1 = x_1,$$

and so $\alpha y_1 = f(x_1)$. Again by the choice of α, we must have that $y_1 = 0$ and $x_1 \in \mathbb{F}_r$. Next, comparing the coefficient for e_2, we get

$$x_2^r = \alpha x_1 + x_2,$$

and so $-\alpha x_1 = f(x_2)$. By the choice of α, we must have that $x_1 = 0$ and $x_2 \in \mathbb{F}_r$. Continuing in the same fashion, we conclude that

$$x_A \in \mathbb{F}_r,\ x_1 = x_2 = \ldots = x_{A-1} = 0.$$

Thus $w = x_A e_A$ with $x_A \in \mathbb{F}_r$, as desired. □

Lemma 3.6. *Let $q = p^a \geq 3$ be a prime power and let A, B, b, c be positive integers, and let $H = \mathrm{Sp}_{2A}(p^B) \rtimes C_b$ as above. Then the following statements hold.*

(i) *If $c \geq 3$, then $\mathrm{SU}_{Ac}(q)$ cannot embed in H.*
(ii) *Assume in addition that $(p, A, B) \neq (3, 1, 1)$. Then the only quotient groups of H are H, $H/\mathbf{Z}(H) = \mathrm{PSp}_{2A}(p^B) \rtimes C_b$, and quotients of C_b.*

Proof. (i) Assume the contrary. Since $c, q \geq 3$, $\mathrm{SU}_{Ac}(q)$ is perfect, and so it embeds in $\mathrm{Sp}_{2A}(p^B) < \mathrm{Sp}_{2A}(\overline{\mathbb{F}}_p)$. In particular, $\mathrm{SU}_{Ac}(q)$ has a nontrivial absolutely irreducible representation in characteristic p of dimension $\leq 2A \leq Ac - 1$. But this contradicts [13, Proposition 5.4.11].

(ii) The assumption on (p, A, B) ensures that $L := [H, H] = \mathrm{Sp}_{2A}(p^B)$ is quasisimple, with $S = L/\mathbf{Z}(H) \cong \mathrm{PSp}_{2A}(p^B)$ being simple. Furthermore, $H/\mathbf{Z}(H)$ acts faithfully on S.

Suppose that $N \lhd H$. If $N \geq L$, then H/N is a quotient of $H/L \cong C_b$. In the remaining case, we have that $N \cap L$ is a proper normal subgroup of L, and so contained in $\mathbf{Z}(H)$. In particular, $[N, L] \leq N \cap L$ centralizes L, i.e. $[[N, L], L] = 1$. Since $L = [L, L]$, the Three Subgroups Lemma implies that $[N, L] = 1$, whence

$$N \leq \mathbf{C}_H(L) \leq \mathbf{C}_H(S) = \mathbf{Z}(H).$$

Thus either $N = 1$ or $N = \mathbf{Z}(H)$. □

4. Local systems for $\mathrm{SU}_n(q)$ and $\mathrm{Sp}_{2n}(q)$ with n odd

In this section, we fix an odd prime p, and a prime $\ell \neq p$, so that we can avail ourselves of $\overline{\mathbb{Q}_\ell}$-adic cohomology. We also fix a nontrivial additive character ψ of $\mathbb{F}_p$. We denote by χ_2 the quadratic character of $\mathbb{F}_p^\times$. Given a power $q = p^a$ of p, and a power q^n of q, we define

$$\boldsymbol{A} := \boldsymbol{A}_{\mathbb{F}_p,q^n} := -\sum_{x\in\mathbb{F}_p^\times} \psi((-1)^{(q^n-1)/2}2x)\chi_2(x). \tag{4.0.1}$$

For $k/\mathbb{F}_p$ a finite extension, we define

$$\boldsymbol{A}_k := \boldsymbol{A}^{\deg(k/\mathbb{F}_p)}.$$

We denote by ψ_k the additive character of k given by

$$\psi_k := \psi \circ \mathrm{Trace}_{k/\mathbb{F}_p}.$$

In [12, §3], we introduced, for each integer $n \geq 2$ and each power $q = p^a$ of the odd prime p, the 2-parameter local system

$$\mathcal{W}_{2\text{-param}}(\psi, n, q)$$

on $\mathbb{A}^2/\mathbb{F}_p$ whose trace function at a point $(s,t) \in \mathbb{A}^2(k)$, k a finite extension of $\mathbb{F}_p$, is the sum

$$(-1/\boldsymbol{A}_k)\sum_{x\in k}\psi_k(x^{q^n+1} + sx^{q+1} + tx^2).$$

Here the normalizing factor $\boldsymbol{A}_k$ is the one built from $\boldsymbol{A}_{\mathbb{F}_p,q^n}$ as defined in the previous paragraph.

We proved there [12, Theorems 3.1, 6.8] that when both n and $a := \log_p(q)$ are prime to p, the geometric monodromy group G_{geom} of $\mathcal{W}_{2\text{-param}}(\psi, n, q)$ was $\mathrm{Sp}_{2n}(q)$ in one of its big Weil representations (of degree q^n), and that after extension of scalars from $\mathbb{A}^2/\mathbb{F}_p$ to $\mathbb{A}^2/\mathbb{F}_q$, its arithmetic monodromy group G_{arith} coincided with G_{geom}.

Without these "prime to p" hypotheses, we have the following result.

Theorem 4.1. *For $n \geq 2$ and $q = p^a$ a power of the odd prime p, we have the following results.*

(i) *There exists a factorization $na = AB$ and a factorization $B = bj$ such that the geometric monodromy group $G_{geom,2\text{-param}}$ of $\mathcal{W}_{2\text{-param}}(\psi, n, q)$ is $\mathrm{Sp}_{2A}(p^B) \rtimes C_b$ in one of its big Weil representations.*

(ii) *Moreover, p^j is a power of q, say $p^j = q^r$ (so that $j = ar, B = arb$), and hence we have inclusions of groups*

$$\mathrm{Sp}_{2A}(p^B) \rtimes C_b = \mathrm{Sp}_{2A}(q^{rb}) \rtimes C_b \hookrightarrow \mathrm{Sp}_{2Ab}(q^r) \hookrightarrow \mathrm{Sp}_{2Abr}(q) = \mathrm{Sp}_{2n}(q).$$

Proof. To prove (i), we argue as follows. From [12, Theorems 4.1, 4.2, and the proof of Proposition 6.6], we see that there exist factorizations $na = AB, B = bj$ and $na = CD, D = dk$ such that $G_{geom,\text{2-param}}$ is a subgroup of the product group

$$(\mathrm{Sp}_{2A}(p^B) \rtimes C_b) \times (\mathrm{PSp}_{2C}(p^D) \rtimes C_d)$$

which maps onto each factor.

We apply Goursat's lemma. Note that $AB = na \geq 2$, so by Lemma 3.6(ii), the only quotient groups of $\mathrm{Sp}_{2A}(p^B) \rtimes C_b$ are

$$\mathrm{Sp}_{2A}(p^B) \rtimes C_b, \mathrm{PSp}_{2A}(p^B) \rtimes C_b, \text{ and quotients of } C_b.$$

Their commutator subgroups are

$$\mathrm{Sp}_{2A}(p^B), \mathrm{PSp}_{2A}(p^B), \{1\}$$

respectively. Similarly, the only quotient groups of $\mathrm{PSp}_{2C}(p^D) \rtimes C_d$ are

$$\mathrm{PSp}_{2C}(p^D) \rtimes C_d, \text{ and quotients of } C_d,$$

and their commutator subgroups are

$$\mathrm{PSp}_{2C}(p^D), \{1\}$$

respectively.

We first rule out the case when $G_{geom,\text{2-param}}$ is the pullback by the quotient maps of the graph of an isomorphism between a quotient of C_b with a quotient of C_d. In this case, $G_{geom,\text{2-param}}$ would contain the product group $\mathrm{Sp}_{2A}(p^B) \times \mathrm{PSp}_{2C}(p^D)$. This group contains elements of trace zero in the representation at hand, whereas every element of the arithmetic monodromy group $G_{arith,\text{2-param}}$, and a fortiori every element of $G_{geom,\text{2-param}}$ has nonzero trace, cf. [12, Proposition 6.6] and its proof.

The only remaining possibility is that $G_{geom,\text{2-param}}$ is the graph of an isomorphism between $\mathrm{PSp}_{2A}(p^B) \rtimes C_b$ and $\mathrm{PSp}_{2C}(p^D) \rtimes C_d$. Such an isomorphism induces an isomorphism of commutator subgroups. Hence $(A, B) = (C, D)$. Comparing cardinalities, we then infer that $b = d$. Thus $G_{geom,\text{2-param}}$ is as asserted.

To prove (ii), we use Theorem 3.5, according to which $p^j = p^{B/b}$ is the lowest value attained as the square absolute value of the trace of an element of $\mathrm{Sp}_{2A}(p^B) \rtimes C_b$ in either big Weil representation. On the other hand, from [12, Theorem 5.5], the group

$G_{arith,2\text{-param}}$ is also finite. The quotient $G_{arith,2\text{-param}}/G_{geom,2\text{-param}}$ is then a finite quotient of $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$. Hence over some $\mathbb{F}_Q/\mathbb{F}_q$, we have $G_{geom,2\text{-param}} = G_{arith,2\text{-param}}$. From [12, Lemma 5.2], exploiting an idea of van der Geer and van der Flugt, we see that for any finite extension $k_0/\mathbb{F}_Q$, all square absolute values of traces are powers of q, and that for any point $(s,t) \in \mathbb{A}^2(k_0)$, there is a finite extension k_1/k_0 for which the same point, now viewed in $\mathbb{A}^2(k_1)$ has trace of square absolute value q^{2n}. In particular, the least square absolute value attained is some strictly positive power $q^r, r \geq 1$ of q. $\square$

We now introduce a new local system $\mathcal{W}(\psi,n,q)$ when $n \geq 3$ is **odd**, which we get by setting $t = 0$ in $\mathcal{W}_{2\text{-param}}(\psi,n,q)$. Thus the trace function of $\mathcal{W}(\psi,n,q)$ at a point $s \in \mathbb{A}^1(k)$, $k/\mathbb{F}_p$ a finite extension, is

$$(-1/\boldsymbol{A}_k)\sum_{x\in k}\psi_k(x^{q^n+1}+sx^{q+1}).$$

On $\mathbb{A}^1/\mathbb{F}_{q^2}$, we can break up this local system as the direct sum of $q+1$ local systems, by making use of the $q+1$ multiplicative characters, including the trivial one, of order dividing $q+1$. We have

$$\mathcal{W}(\psi,n,q) = \bigoplus_{\chi \text{ with } \chi^{q+1}=\mathbb{1}} \mathcal{G}(\psi,n,q,\chi).$$

The trace function of $\mathcal{G}(\psi,n,q,\chi)$ at a point $s \in \mathbb{A}^1(k)$, $k/\mathbb{F}_{q^2}$ a finite extension, is

$$(-1/\boldsymbol{A}_k)\sum_{x\in k}\psi_k(x^{\frac{q^n+1}{q+1}}+sx)\chi_k(x).$$

Here we write χ_k for $\chi\circ\mathrm{Norm}_{k/\mathbb{F}_{q^2}}$, and adopt the usual convention that for χ nontrivial, we have $\chi_k(0)=0$, but $\mathbb{1}(0)=1$.

These $\mathcal{G}(\psi,n,q,\chi)$ are pairwise non-isomorphic, geometrically irreducible local systems on $\mathbb{A}^1/\mathbb{F}_{q^2}$ (thanks to their descriptions as Fourier Transforms, cf. [11, Section 2]). The ranks of these local systems are

$$\mathrm{rank}(\mathcal{G}(\psi,n,q,\mathbb{1})) = \frac{q^n+1}{q+1}-1,$$
$$\mathrm{rank}(\mathcal{G}(\psi,n,q,\chi)) = \frac{q^n+1}{q+1}, \chi \neq \mathbb{1}.$$

Recall that for any n, and q any power of the odd prime p, there are inclusions

$$\mathrm{SU}_n(q) \lhd \mathrm{GU}_n(q) \hookrightarrow \mathrm{Sp}_{2n}(q),$$

Theorem 4.2. *For $n \geq 3$ odd, and $q = p^a$ a power of the odd prime p, the geometric monodromy group $G_{geom,\mathcal{W}}$ for $\mathcal{W}(\psi,n,q)$ is $\mathrm{SU}_n(q)$ in its big Weil representation (of degree q^n).*

Proof. Because $\mathcal{W}(\psi, n, q)$ is the pullback (by $(s,t) \mapsto (s,0)$) of the local system $\mathcal{W}_{2\text{-param}}(\psi, n, q)$, its $G_{geom,\mathcal{W}}$ is a subgroup of $G_{geom,2\text{-param}}$. By Theorem 4.1, we have

$$G_{geom,2\text{-param}} \hookrightarrow \mathrm{Sp}_{2n}(q).$$

Thus $G_{geom,\mathcal{W}}$ is a subgroup of $\mathrm{Sp}_{2n}(q)$ under which a big Weil representation of $\mathrm{Sp}_{2n}(q)$ breaks up into $q+1$ pieces, one of rank $\frac{q^n-q}{q+1}$ and q of rank $\frac{q^n+1}{q+1}$. By Theorem 3.4, we have inclusions

$$\mathrm{SU}_n(q) \leq G_{geom,\mathcal{W}} \leq \mathrm{GU}_n(q).$$

The group $\mathrm{GU}_n(q)$ has a quotient, via the determinant, of order $q+1$, which is prime to p. Because $G_{geom,\mathcal{W}}$ is the monodromy group of a local system on $\mathbb{A}^1/\overline{\mathbb{F}_p}$, it has no nontrivial prime to p quotients. Thus we have $G_{geom,\mathcal{W}} = \mathrm{SU}_n(q)$. $\square$

Theorem 4.3. *For $n \geq 3$ odd and q an odd prime power, the geometric monodromy group $G_{geom,2\text{-param}}$ of $\mathcal{W}_{2\text{-param}}(\psi, n, q)$ is $\mathrm{Sp}_{2n}(q)$ in one of its big Weil representations (of degree q^n). Moreover, after extension of scalars to $\mathbb{A}^2/\mathbb{F}_q$, we have $G_{geom,2\text{-param}} = G_{arith,2\text{-param}}$.*

Proof. Recall the inclusion

$$\mathrm{SU}_n(q) = G_{geom,\mathcal{W}} \leq G_{geom,2\text{-param}} = \mathrm{Sp}_{2A}(p^B) \rtimes C_b$$

and the relation $n = Abr$ of Theorem 4.1. By Lemma 3.6(i), $br \leq 2$, but $2 \nmid n$, hence $br = 1$ and $(A, p^B, b) = (n, q, 1)$, yielding the first assertion.

Since $G_{geom,2\text{-param}} = \mathrm{Sp}_{2n}(q) = \mathrm{Sp}_{2n}(p^a)$, $G_{arith,2\text{-param}}$ is contained in $\mathrm{Sp}_{2n}(p^a) \rtimes C_a$, cf. [12, proof of Lemma 6.7]. Thus the quotient $G_{arith,2\text{-param}}/G_{geom,2\text{-param}}$ has order dividing a, so after extension of scalars from $\mathbb{A}^2/\mathbb{F}_p$ to $\mathbb{A}^2/\mathbb{F}_{p^a} = \mathbb{A}^2/\mathbb{F}_q$ we have $G_{geom,2\text{-param}} = G_{arith,2\text{-param}}$. $\square$

Theorem 4.4. *For $n \geq 3$ odd and q a power of the odd prime p, the geometric monodromy group of the local system $\mathcal{G}(\psi, n, q, \mathbb{1})$ is $\mathrm{PSU}_n(q)$, the image of $\mathrm{SU}_n(q)$ in its unique irreducible representation of dimension $\frac{q^n-q}{q+1}$, with character $\zeta_{0,n}$. The geometric monodromy group of $\mathcal{G}(\psi, n, q, \chi_2)$ (where χ_2 is the quadratic character) is $\mathrm{PSU}_n(q)$, the image of $\mathrm{SU}_n(q)$ in its unique orthogonal representation of dimension $\frac{q^n+1}{q+1}$, with character $\zeta_{(q+1)/2,n}$. For the remaining $q-1$ local systems $\mathcal{G}(\psi, n, q, \chi)$ with χ^2 nontrivial, $\chi^{q+1} = \mathbb{1}$, their geometric monodromy groups are the images of $\mathrm{SU}_n(q)$ in its $q-1$ non-selfdual irreducible representations of dimension $\frac{q^n+1}{q+1}$.*

Proof. Because $G_{geom,\mathcal{W}}$ is $\mathrm{SU}_n(q)$, the geometric monodromy groups in question are the images of $\mathrm{SU}_n(q)$ in various of its irreducible representations. Recall the fact [16, Theorem 4.1] that $\mathrm{SU}_n(q)$ has, up to equivalence, one irreducible representation of dimension $\frac{q^n-q}{q+1}$

(with character $\zeta_{0,n}$) and q irreducible representations of dimension $\frac{q^n+1}{q+1}$ (with character $\zeta_{j,n}$, $1 \leq j \leq q$), with exactly one of the q latter representations being self-dual (and necessarily orthogonal, as it has odd dimension). Using this fact and looking at the dimensions, we get the asserted matching. $\square$

5. Arithmetic monodromy groups of local systems for $\mathrm{SU}_n(q)$ with n odd

Theorem 5.1. *Let $n \geq 3$ be odd and q be a power of the odd prime p. After extension of scalars to $\mathbb{A}^1/\mathbb{F}_{q^4}$, the arithmetic monodromy group $G_{arith,\mathcal{W}}$ is equal to $G_{geom,\mathcal{W}} = \mathrm{SU}_n(q)$. Furthermore, the arithmetic monodromy group $G_{arith,\chi}$ of each of the $q+1$ local systems $\mathcal{G}(\psi,n,q,\chi)$ is equal to its geometric monodromy group $G_{geom,\chi}$, as described in Theorem 4.4.*

Proof. For $k/\mathbb{F}_{q^2}$, let H_k denote the arithmetic monodromy group $G_{arith,\mathcal{W}}$ of the local system $\mathcal{W}(\psi,n,q)$ after extension of scalars to $\mathbb{A}^1/k$. By Theorem 4.3, $H_k \leq \mathrm{Sp}_{2n}(q)$, and by Theorem 4.4, $H_k \rhd \mathrm{SU}_n(q)$. As in the proof of Theorem 4.2, H_k is a subgroup of $\mathrm{Sp}_{2n}(q)$ under which a big Weil representation of $\mathrm{Sp}_{2n}(q)$ breaks up into $q+1$ pieces Ψ_i, $0 \leq i \leq q$, with Ψ_0 of rank $\frac{q^n-q}{q+1}$ and $\Psi_1,\dots,\Psi_q$ of rank $\frac{q^n+1}{q+1}$. By Theorem 3.4, we have $H_k \leq \mathrm{GU}_n(q)$.

Now we pay particular attention to the situation over $\mathbb{F}_{q^4}$. The normalizing factor $\boldsymbol{A} := \boldsymbol{A}_{\mathbb{F}_p,q^n}$ used for $\mathcal{W}$ is minus a choice of quadratic Gauss sum over $\mathbb{F}_p$, so its square is either p, if p is 1 mod 4, or it is $-p$. Taken over $\mathbb{F}_{q^4}$, the normalizing factor $\boldsymbol{A}_{\mathbb{F}_{q^4}}$ is thus q^2. On the other hand, according to [11, Lemma 8.3], this same normalizing factor q^2 insures that each of the $q+1$ local systems $\mathcal{G}(\psi,n,q,\chi)$ on $\mathbb{A}^1/\mathbb{F}_{q^4}$ has its $G_{arith,\chi}$ contained in $\mathrm{SL}_{\mathrm{rank}\,\mathcal{G}(\psi,n,q,\chi)}(\mathbb{C})$. Applying Lemma 3.2(v) to $H_{\mathbb{F}_{q^4}}$, we conclude that $H_{\mathbb{F}_{q^4}} = \mathrm{SU}_n(q)$.

Once we have $G_{arith,\mathcal{W}} = \mathrm{SU}_n(q)$, it follows that each $G_{arith,\chi}$ is the image of $\mathrm{SU}_n(q)$. $\square$

Theorem 5.2. *Let $n \geq 3$ be odd and q be a power of the odd prime p. Denote by $G_{arith,\mathcal{W}/\mathbb{F}_{q^2}}$ the arithmetic monodromy group of the local system $\mathcal{W}(\psi,n,q)$ after extension of scalars to $\mathbb{A}^1/F_{q^2}$. We have*

$$\begin{aligned} G_{arith,\mathcal{W}/\mathbb{F}_{q^2}} &= \mathrm{SU}_n^{\pm}(q) := \{X \in \mathrm{GU}(W) \mid \det(X) = \pm 1\} \\ &= \mathrm{SU}(W) \times \langle -1_W \rangle \cong \mathrm{SU}_n(q) \times C_2. \end{aligned}$$

Furthermore, the arithmetic monodromy groups $G_{arith,\chi}$ of the local system $\mathcal{G}(\psi,n,q,\chi)$ with $\chi^{q+1} = \mathbb{1}$ are described as follows.

(a) *If $q \equiv 3 \pmod 4$, $G_{arith,\mathbb{1}}$ is $\mathrm{PSU}_n(q)$, the image of $\mathrm{SU}_n(q)$ in its unique irreducible representation of dimension $\frac{q^n-q}{q+1}$, with character $\zeta_{0,n}$. If $q \equiv 1 \pmod 4$, $G_{arith,\mathbb{1}}$ is $\mathrm{PSU}_n(q) \times C_2$, the image of $\mathrm{SU}_n(q) \times C_2$ in its irreducible representation of dimension*

$\frac{q^n-q}{q+1}$, with character $\zeta_{0,n} \otimes \nu$, where ν is the unique nontrivial irreducible character of C_2.

(b) *G_{arith,χ_2} (where χ_2 is the quadratic character) is the image of $\mathrm{SU}_n(q)$ in its unique orthogonal representation of dimension $\frac{q^n+1}{q+1}$, with character $\zeta_{(q+1)/2,n}$.*

(c) *For the remaining $q-1$ characters χ with χ^2 nontrivial, $\chi^{q+1} = \mathbb{1}$, the groups $G_{arith,\chi}$ are the images of $\mathrm{SU}_n^{\pm}(q) \times C_2$ in its $q-1$ non-selfdual irreducible representations of dimension $\frac{q^n+1}{q+1}$, obtained by restricting down Ψ_i, $1 \le i \le q$ and $i \ne (q+1)/2$, from $\mathrm{GU}_n(q)$.*

Proof. The key point here is that the normalizing factor for $\mathcal{W}$ over $\mathbb{F}_{q^2}$, being a power of a quadratic Gauss sum over the prime field, is q when $q \equiv 1 (\mathrm{mod}\, 4)$ and $-q$ when $q \equiv 3 (\mathrm{mod}\, 4)$. But according to [11, Lemma 8.3], the normalizing factors for the various $\mathcal{G}(\psi, n, q, \chi)$ on $\mathbb{A}^1/\mathbb{F}_{q^2}$ which force their $G_{arith,\chi}$ to lie in SL are not all the same: some are q and some are $-q$. [The exact recipe is that for χ of order m dividing $q+1$, one should use $-(-1)^{(q+1)/m}q$ as the normalizing factor.]

Since $\mathrm{SU}_n(q)$ is perfect, this implies that $G_{arith,\mathcal{W}/\mathbb{F}_{q^2}}$ contains $\mathrm{SU}_n(q)$ strictly. On the other hand, $\mathrm{SU}_n(q)$ is $G_{arith,\mathcal{W}/\mathbb{F}_{q^4}}$, as proven above, hence $\mathrm{SU}_n(q)$ has index at most 2 in $G_{arith,\mathcal{W}/\mathbb{F}_{q^2}} \le \mathrm{GU}_n(q)$. As n is odd, we also observe that $\det(z^{(q+1)/2}) = -1$, where z is the generator of $\mathbf{Z}(\mathrm{GU}_n(q))$ introduced in the proof of Lemma 3.2; in particular, $z^{(q+1)/2} = -1_W$. Hence

$$\begin{aligned} G_{arith,\mathcal{W}/\mathbb{F}_{q^2}} &= \mathrm{SU}_n^{\pm}(q) = \{X \in \mathrm{GU}_n(q) \mid \det(X) = \pm 1\} \\ &= \mathrm{SU}_n(q) \times \langle z^{(q+1)/2} \rangle \cong \mathrm{SU}_n(q) \times C_2, \end{aligned}$$

and $G_{arith,\chi}$ is the image of $\mathrm{SU}_n^{\pm}(q)$ under some Ψ_i. If $\chi = \chi_2$, we know that Ψ_i is of dimension $(q^n+1)/(q+1)$ and self-dual, whence $i = (q+1)/2$ and $\Psi_i(z^{(q+1)/2}) = 1_V$ by (3.2.1), yielding (b). Suppose $\chi = \mathbb{1}$. Then $i = 0$ by dimension comparison, and $\Psi_0(z^{(q+1)/2}) = (-1)^{(q+1)/2} \cdot 1_V$ by (3.2.1), leading to (a). For the remaining $q = 1$ characters χ with $\chi^2 \ne \mathbb{1}$, we arrive at (c). □

With this information in hand, we can prove Conjecture 9.2 of [11].

Theorem 5.3. *Let $n \ge 3$ be odd and q be a power of the odd prime p. For each multiplicative character χ of $\mathbb{F}_{q^2}^{\times}$ of order denoted m_χ dividing $q+1$, define*

$$\boldsymbol{B}_\chi := -(-1)^{(q+1)/m_\chi} q.$$

Denote by $\mathcal{H}_\chi$ the local system on $\mathbb{A}^1/\mathbb{F}_{q^2}$ whose trace function at a point $s \in K$, $K/\mathbb{F}_{q^2}$ a finite extension, is

$$s \mapsto \left(-1/(\boldsymbol{B}_\chi)^{\deg(K/\mathbb{F}_{q^2})}\right) \sum_{x \in K} \psi_K(x^{(q^n+1)/(q+1)} + sx)\chi(x).$$

[Thus $\mathcal{H}_\chi$ is the constant field twist of $\mathcal{G}(\psi, n, q, \chi)$ by the unique choice of sign ± 1 for which $G_{arith,\mathcal{H}_\chi} < \mathrm{SL}_{\mathrm{rank}\mathcal{H}_\chi}(\mathbb{C})$, cf. [11, Lemma 8.3].] *Then* $G_{arith,\mathcal{H}_\chi} = G_{geom,\mathcal{H}_\chi}$ *is the image of* $\mathrm{SU}_n(q)$ *in the given representation.*

Proof. Pick a faithful character $\Lambda : \mu_{q+1}(\mathbb{F}_{q^2}^\times) \cong \mu_{q+1}(\mathbb{C}^\times)$. The indexing of the small Weil representations Ψ_i of $\mathrm{GU}_n(q)$ is by the powers of Λ. For each power Λ^i of Λ, the multiplicative character χ_i of $\mathbb{F}_{q^2}^\times$ given by

$$\chi_i : x \mapsto \Lambda^i(x^{q-1})$$

has order dividing $q+1$, and we get all the $q+1$ such characters this way. In view of the previous result, what we must show is that the scalar $-1_W \in \mathrm{SU}_n^\pm(q)$ acts trivially on each $\mathcal{H}_\chi$. We know this element acts trivially after quadratic extension of the ground field from $\mathbb{F}_{q^2}$ to $\mathbb{F}_{q^4}$, so it must be attained by a Frobenius in a odd degree extension of $\mathbb{F}_{q^2}$. In the representation $\mathcal{G}(\psi, n, q, \chi)$, we have

$$\Psi_i(-1_W) = \epsilon_q(-1)^i \cdot 1_{V_i},$$

where $\Psi_i : \mathrm{SU}_n^\pm(q) \to \mathrm{GL}(V_i)$, and

$$\epsilon_q := (-1)^{q+1)/2},$$

the sign ϵ_q being 1 or -1 depending on whether -1 is a square or not in the group $\mu_{q+1}(\mathbb{F}_{q^2})$, cf. (3.2.1).

Thus $\epsilon_q = 1$ if $q \equiv 3 (\mathrm{mod}\, 4)$, and $\epsilon_q = -1$ if $q \equiv 1 (\mathrm{mod}\, 4)$. Now the clearing factor used for $\mathcal{W}$, and hence also for $\mathcal{G}(\psi, n, q, \chi)$, was $-\epsilon_q q$, whereas the clearing factor for $\mathcal{H}_\chi$ is $-(-1)^{(q+1)/m_\chi} q$. So the change of clearing factor for $\mathcal{H}_\chi$ is $\epsilon_q(-1)^{(q+1)/m_\chi}$.

Consider the case $\chi = \mathbb{1}$, i.e. $m_\chi = 1$. By dimension comparison, we see that the representation on $\mathcal{G}(\psi, n, q, \mathbb{1})$ is Ψ_0, i.e. $i = 0$. Hence the action of -1_W in the representation $\mathcal{H}_\mathbb{1}$ is

$$(\epsilon_q)^2(-1)^{q+1} \cdot 1_{V_0} = 1_{V_0}.$$

Now assume that $\chi \neq \mathbb{1}$. Then $i \neq 0$, and $\dim V_i = (q^n+1)/(q+1)$ is odd. As we mentioned above, the action of -1_W on $\mathcal{H}_\chi$ has determinant 1. As the central involution -1_W of $\mathrm{SU}_n^\pm(q)$ acts as $\gamma \cdot 1_{V_i}$ for some $\gamma = \pm 1$, the oddness of $\dim V_i$ implies that $\gamma = 1$.

Thus in either case, -1_W acts trivially on $\mathcal{H}_\chi$, as stated. $\square$

6. Introduction to the n even case

The key insight in the n odd case was to start with the 2-parameter local system

$$\mathcal{W}_{2\text{-param}}(\psi, n, q)$$

on $\mathbb{A}^2/\mathbb{F}_p$ whose trace function at a point $(s,t) \in \mathbb{A}^2(k)$, k a finite extension of $\mathbb{F}_p$, was the sum

$$(-1/\boldsymbol{A}_k)\sum_{x\in k}\psi_k(x^{q^n+1}+sx^{q+1}+tx^2),$$

and then study the one-parameter local system obtained by setting $t=0$.

In the n even case, it is precisely the "same" one parameter local system, the one obtained by setting $t=0$ in $\mathcal{W}_{\text{2-param}}(\psi,n,q)$, that is the key object of study. Because n is even, the gcd of $q+1$ and q^n+1 is just 2, so this one parameter system only breaks up into two visible pieces. Each of these two pieces itself turns out to be a suitable Kummer pullback to $\mathbb{A}^1$ of a particular hypergeometric sheaf on $\mathbb{G}_m$. It is this fact, and the group-theoretic analysis it makes possible, that leads to our results in this n even case.

The main results about monodromy groups in the n even case are Theorems 10.3, 10.4, 10.6, and 10.7.

7. A special class of hypergeometric sheaves

We fix an odd prime p, a prime $\ell \neq p$, and two integers $\mathsf{A} > \mathsf{B} > 0$ with $\gcd(\mathsf{A},\mathsf{B}) = 1$ and AB prime to p. We also fix a nontrivial additive character $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}_\ell}^\times$. For $K/\mathbb{F}_p$ a finite extension, we denote by ψ_K the additive character of K given by $x \mapsto \psi(\mathrm{Trace}_{K/\mathbb{F}_p}(x))$.

We denote by $\mathcal{H}(\psi,\mathsf{A}_{ntriv},\mathsf{B}_{ntriv})$ the hypergeometric sheaf

$$\begin{aligned}&\mathcal{H}(\psi,\mathsf{A}_{ntriv},\mathsf{B}_{ntriv}) :=\\ &\mathcal{H}yp(\psi,\ \text{all nontrivial characters of order dividing }\mathsf{A},\\ &\qquad\qquad \text{all nontrivial characters of order dividing }\mathsf{B}).\end{aligned}$$

Lemma 7.1. *Up to a constant field twist, $\mathcal{H}(\psi,\mathsf{A}_{ntriv},\mathsf{B}_{ntriv})$ is the lisse sheaf on $\mathbb{G}_m/\mathbb{F}_p$ whose trace function at $u \in \mathbb{G}_m(K)$, K a finite extension of $\mathbb{F}_p$, is*

$$u \mapsto -\sum_{x,y\in K\ \ with\ y^{\mathsf{B}}=x^{\mathsf{A}}/u}\psi_K(\mathsf{A}x-\mathsf{B}y).$$

Proof. By definition, $\mathcal{H}(\psi,\mathsf{A}_{ntriv},\mathsf{B}_{ntriv})$ is the multiplicative ! convolution of

$$\mathcal{K}l(\psi,\mathsf{A}_{ntriv}) = \mathcal{H}(\psi,\mathsf{A}_{ntriv},1_{ntriv})$$

with the pullback by multiplicative inversion of

$$\mathcal{K}l(\overline{\psi},\mathsf{B}_{ntriv}) = \mathcal{H}(\overline{\psi},\mathsf{B}_{ntriv},1_{ntriv}).$$

As explained in [10, Lemma 1.2], up to a constant field twist, $\mathcal{K}l(\psi, \mathsf{A}_{ntriv})$ has a descent to $\mathbb{G}_m/\mathbb{F}_p$ whose trace function is given at $s \in \mathbb{G}_m(K)$, K a finite extension of $\mathbb{F}_p$ by

$$s \mapsto -\sum_{x \in K} \psi_K(-x^{\mathsf{A}}/s + \mathsf{A}x).$$

If $\mathsf{B} = 1$, there is nothing more to prove. Suppose now $\mathsf{B} > 1$.

Then the pullback by multiplicative inversion of $\mathcal{K}l(\overline{\psi}, \mathsf{B}_{ntriv})$ has, up to a constant field twist, a descent to $\mathbb{G}_m/\mathbb{F}_p$ whose trace function is given at $t \in \mathbb{G}_m(K)$, K a finite extension of $\mathbb{F}_p$ by

$$t \mapsto -\sum_{y \in K} \psi_K(ty^{\mathsf{B}} - \mathsf{B}y).$$

Their multiplicative convolution then has trace function at $u \in \mathbb{G}_m(K)$ given by

$$-\sum_{\substack{s,\, t \,\in\, K^\times \\ st \,=\, u}} \sum_{x \in K} \psi_K(-x^{\mathsf{A}}/s + \mathsf{A}x) \sum_{y \in K} \psi_K(ty^{\mathsf{B}} - \mathsf{B}y) =$$

(solving for $1/s = t/u$)

$$= -\sum_{x,y \in K} \psi_K(\mathsf{A}x - \mathsf{B}y) \sum_{t \in K^\times} \psi_K(t(y^{\mathsf{B}} - x^{\mathsf{A}}/u)) =$$

(the inner sum may as well be over all $t \in K$, since for $t = 0$ the sum $\sum_{x,y \in K} \psi_K(\mathsf{A}x - \mathsf{B}y)$ vanishes)

$$= -(\#K) \sum_{x,y \in K \text{ with } y^{\mathsf{B}} = x^{\mathsf{A}}/u} \psi_K(\mathsf{A}x - \mathsf{B}y),$$

as asserted. □

Corollary 7.2. *The pullback $[\mathsf{A}]^\star \mathcal{H}(\psi, \mathsf{A}_{ntriv}, \mathsf{B}_{ntriv})$ of $\mathcal{H}(\psi, \mathsf{A}_{ntriv}, \mathsf{B}_{ntriv})$ by $x \mapsto x^{\mathsf{A}}$ has, up to a constant field twist, a descent to (the restriction to $\mathbb{G}_m/\mathbb{F}_p$ of) the lisse sheaf*

$$\mathcal{G}(\mathsf{A}, \mathsf{B})$$

on $\mathbb{A}^1/\mathbb{F}_p$ whose trace function at $t \in K$ is given by

$$t \mapsto -\sum_{z \in K} \psi_K(-\mathsf{B}z^{\mathsf{A}} + t\mathsf{A}z^{\mathsf{B}}).$$

Proof. After pullback, write $u = t^{\mathsf{A}}$. Then the summation range $y^{\mathsf{B}} = x^{\mathsf{A}}/u$ becomes $y^{\mathsf{B}} = (x/t)^{\mathsf{A}}$. As A, B are relatively prime, $y^{\mathsf{B}} = (x/t)^{\mathsf{A}}$ means precisely that $y = z^{\mathsf{A}}$, $x/t = z^{\mathsf{B}}$ for a unique $z \in K$. □

Lemma 7.3. *The lisse sheaf* $\mathcal{G}(\mathsf{A},\mathsf{B})$ *is geometrically isomorphic to a multiplicative translate of the lisse sheaf*

$$\mathcal{G}_0(\mathsf{A},\mathsf{B})$$

on $\mathbb{A}^1/\mathbb{F}_p$ *whose trace function at* $t \in K$ *is given by*

$$t \mapsto -\sum_{z \in K} \psi_K(z^{\mathsf{A}} + tz^{\mathsf{B}}).$$

Proof. Geometrically, take the Ath root of $-\mathsf{B}$, say $\beta^{\mathsf{A}} = -\mathsf{B}$, and make the substitution $z \mapsto z/\beta$. The trace sum becomes

$$-\sum_{z \in K} \psi_K(z^{\mathsf{A}} + (t\mathsf{A}/\beta^{\mathsf{B}})z^{\mathsf{B}}). \quad \square$$

Lemma 7.4. *The lisse sheaves* $\mathcal{G}(\mathsf{A},\mathsf{B})$ *and* $\mathcal{G}_0(\mathsf{A},\mathsf{B})$ *on* $\mathbb{A}^1/\mathbb{F}_p$ *are geometrically irreducible.*

Proof. Since multiplicative translation does not affect geometric irreducibility, it suffices to treat $\mathcal{G}_0(\mathsf{A},\mathsf{B})$. Its trace function is

$$-\sum_{z \in K} \psi_K(z^{\mathsf{A}} + tz^{\mathsf{B}}) = -\sum_{u \in K} \psi_K(tu) \sum_{z \in K, z^{\mathsf{B}} = u} \psi_K(z^{\mathsf{A}}),$$

which is to say that $\mathcal{G}_0(\mathsf{A},\mathsf{B})$ is the Fourier transform FT_ψ of $[\mathsf{B}]_\star \mathcal{L}_{\psi(z^{\mathsf{A}})}$. This FT is geometrically irreducible, because the input $[\mathsf{B}]_\star \mathcal{L}_{\psi(z^{\mathsf{A}})}$ is geometrically irreducible, indeed $I(\infty)$-irreducible, because at ∞ it is totally wild with all of its B slopes equal to A/B, a fraction with exact denominator B, cf. [9, 1.14, 1.14.1].

Here is another proof of this result. It is equivalent to prove that $[\mathsf{A}]^\star \mathcal{H}(\psi, \mathsf{A}_{ntriv}, \mathsf{B}_{ntriv})$ is geometrically irreducible. By Frobenius reciprocity, we have

$$\begin{aligned} &\langle [\mathsf{A}]^\star \mathcal{H}(\psi, \mathsf{A}_{ntriv}, \mathsf{B}_{ntriv}), [\mathsf{A}]^\star \mathcal{H}(\psi, \mathsf{A}_{ntriv}, \mathsf{B}_{ntriv}) \rangle = \\ &= \langle \mathcal{H}(\psi, \mathsf{A}_{ntriv}, \mathsf{B}_{ntriv}), [\mathsf{A}]_\star [\mathsf{A}]^\star \mathcal{H}(\psi, \mathsf{A}_{ntriv}, \mathsf{B}_{ntriv}) \rangle. \end{aligned}$$

But

$$[\mathsf{A}]_\star [\mathsf{A}]^\star \mathcal{H}(\psi, \mathsf{A}_{ntriv}, \mathsf{B}_{ntriv}) \cong \bigoplus_{\chi; \chi^{\mathsf{A}} = \mathbb{1}} \mathcal{H}(\psi, \mathsf{A}_{ntriv}, \mathsf{B}_{ntriv}) \otimes \mathcal{L}_\chi.$$

Of these summands, only the $\chi = \mathbb{1}$ summand is isomorphic to $\mathcal{H}(\psi, \mathsf{A}_{ntriv}, \mathsf{B}_{ntriv})$, all the others have the wrong "downstairs" characters (precisely because B is relatively prime to A). $\square$

Lemma 7.5. *The wild part of the $I(\infty)$-representation of $\mathcal{G}(\mathsf{A},\mathsf{B})$ (or of $\mathcal{G}_0(\mathsf{A},\mathsf{B})$) is $I(\infty)$-irreducible, of dimension $\mathsf{A}-\mathsf{B}$, with all slopes $\frac{\mathsf{A}}{\mathsf{A}-\mathsf{B}}$.*

Proof. This wild part is the pullback by $[\mathsf{A}]$ of the wild part of $\mathcal{H}(\psi, \mathsf{A}_{ntriv}, \mathsf{B}_{ntriv})$, which has rank $\mathsf{A}-\mathsf{B}$ and all slopes $\frac{1}{\mathsf{A}-\mathsf{B}}$. Because $\gcd(\mathsf{A}, \mathsf{A}-\mathsf{B}) = 1$, its $[\mathsf{A}]$ pullback, which has dimension $\mathsf{A}-\mathsf{B}$, with all slopes $\frac{\mathsf{A}}{\mathsf{A}-\mathsf{B}}$, is itself $I(\infty)$-irreducible. $\square$

8. A second special class of hypergeometric sheaves

In this section, we continue with $p, \psi, \mathsf{A}, \mathsf{B}$ as in the previous section; $\mathsf{A} > \mathsf{B} > 0$ are integers with $\gcd(\mathsf{A}, \mathsf{B}) = 1$ and AB prime to p, but now assume in addition that A is odd. We denote by

$$\mathcal{H}(\mathsf{A}_{all}, \mathsf{B}_{\star}\chi_2)$$

the hypergeometric sheaf

$$\mathcal{H}yp(\psi,\ \text{all } \chi \text{ with } \chi^{\mathsf{A}} = \mathbb{1},\ \text{all } \rho \text{ with } \rho^{\mathsf{B}} = \chi_2).$$

Lemma 8.1. *Up to a constant field twist, $\mathcal{H}(\mathsf{A}_{all}, \mathsf{B}_{\star}\chi_2)$ is the lisse sheaf on $\mathbb{G}_m/\mathbb{F}_p$ whose trace function at $u \in \mathbb{G}_m(K)$, K a finite extension of $\mathbb{F}_p$, is*

$$u \mapsto -\sum_{x,y \in K, x^{\mathsf{A}} = uy^{\mathsf{B}}} \psi_K(\mathsf{A}x - \mathsf{B}y)\chi_2(y).$$

Proof. By definition, $\mathcal{H}(\psi, \mathsf{A}_{all}, \mathsf{B}_{\star}\chi_2)$ is the multiplicative ! convolution of

$$\mathcal{K}l(\psi, \mathsf{A}_{all})$$

with the pullback by multiplicative inversion of

$$\mathcal{K}l(\overline{\psi}, \mathsf{B}_{\star}\chi_2).$$

We have geometric isomorphisms

$$\mathcal{K}l(\psi, \mathsf{A}_{all}) \cong [\mathsf{A}]_{\star}\mathcal{L}_{\psi(\mathsf{A}x)},$$
$$\mathcal{K}l(\overline{\psi}, \mathsf{B}_{\star}\chi_2) \cong [\mathsf{B}]_{\star}(\mathcal{L}_{\psi(-\mathsf{B}x)} \otimes \mathcal{L}_{\chi_2(x)}).$$

The multiplicative convolution of $[\mathsf{A}]_{\star}\mathcal{L}_{\psi(\mathsf{A}x)}$ with the pullback by multiplicative inversion of $[\mathsf{B}]_{\star}(\mathcal{L}_{\psi(-\mathsf{B}x)} \otimes \mathcal{L}_{\chi_2(x)})$ thus has trace function at $u \in \mathbb{G}_m(K)$ given by

$$-\sum_{s,t \in K^\times, st=u} \ \sum_{x \in K, x^{\mathsf{A}}=s} \psi_K(\mathsf{A}x) \sum_{y \in K, y^{\mathsf{B}}=1/t} \psi_K(-\mathsf{B}y)\chi_2(y) =$$
$$= -\sum_{x,y \in K, x^{\mathsf{A}}=uy^{\mathsf{B}}} \psi_K(\mathsf{A}x - \mathsf{B}y)\chi_2(y).$$

[We do not need to specify that x, y are nonzero, since $\chi_2(y)$ vanishes unless $y \neq 0$, and once $y \neq 0$, the equation $x^{\mathsf{A}} = uy^{\mathsf{B}}$ forces $x \neq 0$ as well.] $\square$

Exactly as in the previous section, we get the following results.

Corollary 8.2. *The pullback* $[\mathsf{A}]^{\star}\mathcal{H}(\mathsf{A}_{all}, \mathsf{B}_{\star}\chi_2)$ *of* $\mathcal{H}(\mathsf{A}_{all}, \mathsf{B}_{\star}\chi_2)$ *by* $x \mapsto x^{\mathsf{A}}$ *has, up to a constant field twist, a descent to (the restriction to* $\mathbb{G}_m/\mathbb{F}_p$ *of) the lisse sheaf*

$$\mathcal{G}(\mathsf{A}_{all}, \mathsf{B}_{\star}\chi_2)$$

on $\mathbb{A}^1/\mathbb{F}_p$ *whose trace function at* $t \in K$ *is given by*

$$t \mapsto -\sum_{z \in K} \psi_K(-\mathsf{B}z^{\mathsf{A}} + t\mathsf{A}z^{\mathsf{B}})\chi_2(z).$$

Lemma 8.3. *The lisse sheaf* $\mathcal{G}(\mathsf{A}_{all}, \mathsf{B}_{\star}\chi_2)$ *is geometrically isomorphic to a multiplicative translate of the lisse sheaf*

$$\mathcal{G}_0(\mathsf{A}_{all}, \mathsf{B}_{\star}\chi_2)$$

on $\mathbb{A}^1/\mathbb{F}_p$ *whose trace function at* $t \in K$ *is given by*

$$t \mapsto -\sum_{z \in K} \psi_K(z^{\mathsf{A}} + tz^{\mathsf{B}})\chi_2(z).$$

Lemma 8.4. *The lisse sheaves* $\mathcal{G}(\mathsf{A}_{all}, \mathsf{B}_{\star}\chi_2)$ *and* $\mathcal{G}_0(\mathsf{A}_{all}, \mathsf{B}_{\star}\chi_2)$ *on* $\mathbb{A}^1/\mathbb{F}_p$ *are geometrically irreducible.*

Lemma 8.5. *The wild part of the* $I(\infty)$*-representation of* $\mathcal{G}(\mathsf{A}_{all}, \mathsf{B}_{\star}\chi_2)$ *(or of* $\mathcal{G}_0(\mathsf{A}_{all}, \mathsf{B}_{\star}\chi_2)$*) is* $I(\infty)$*-irreducible, of dimension* $\mathsf{A} - \mathsf{B}$*, with all slopes* $\frac{\mathsf{A}}{\mathsf{A}-\mathsf{B}}$.

9. Local systems for $\mathrm{Sp}_{4n}(q)$

In this section, with the odd prime p and its ψ fixed, we denote by $\alpha := \alpha_{\mathbb{F}_p}$ the negative of the Gauss sum

$$\alpha := \boldsymbol{A}_{\mathbb{F}_p, q^{2n}} = -\sum_{x \in \mathbb{F}_p^\times} \psi(2x)\chi_2(x),$$

cf. (4.0.1). For $K/\mathbb{F}_p$ a finite extension, we define

$$\alpha_K := -\sum_{x \in K^\times} \psi_K(2x)\chi_{2,K}(x).$$

One knows (Hasse-Davenport relation) that

$$\alpha_K = (\alpha_{\mathbb{F}_p})^{\deg(K/\mathbb{F}_p)}.$$

We fix also an even integer $2n \geq 2$ and

$$q := \text{a power of } p,\ \mathsf{A} := \frac{q^{2n}+1}{2},\ \mathsf{B} := \frac{q+1}{2}.$$

We now work with the two local systems on $\mathbb{A}^1/\mathbb{F}_p$,

$$\mathcal{G}_{\text{even}}(\psi, 2n, q) := \mathcal{G}_0(\mathsf{A}, \mathsf{B}) \otimes \alpha^{-\deg},\quad \mathcal{G}_{\text{odd}}(\psi, 2n, q) := \mathcal{G}_0(\mathsf{A}_{all}, \mathsf{B}_{\star}\chi_2) \otimes \alpha^{-\deg},$$

and their direct sum

$$\mathcal{W}(\psi, 2n, q) := \mathcal{G}_{\text{even}}(\psi, 2n, q) \oplus \mathcal{G}_{\text{odd}}(\psi, 2n, q),$$

whose trace function at $s \in \mathbb{A}^1(K)$, $K/\mathbb{F}_p$ a finite extension, is given by

$$s \mapsto (-1/\alpha_K) \sum_{x \in K} \psi_K(x^{q^{2n}+1} + sx^{q+1}).$$

These local systems are the pullbacks to the line $t = 0$ on the local systems of the same name in [12, §3] on $\mathbb{A}^2/\mathbb{F}_p$ with coordinates (s, t). To avoid confusion, we will denote by

$$\mathcal{G}_{\text{even,2-param}}(\psi, 2n, q),\ \mathcal{G}_{\text{odd,2-param}}(\psi, 2n, q),\ \mathcal{W}_{\text{2-param}}(\psi, 2n, q)$$

the two-parameter local systems. Thus the trace function of $\mathcal{W}_{\text{2-param}}(\psi, 2n, q)$ at a point $(s, t) \in \mathbb{A}^2(K)$, $K/\mathbb{F}_p$ a finite extension, is given by

$$(s, t) \mapsto (-1/\alpha_K) \sum_{x \in K} \psi_K(x^{q^{2n}+1} + sx^{q+1} + tx^2).$$

This pullback relation gives us the following inclusions.

Lemma 9.1. *We have the following inclusions.*

(i) *For the local systems $\mathcal{G}_{\text{even}}(\psi, 2n, q)$ and $\mathcal{G}_{\text{even,2-param}}(\psi, 2n, q)$, their geometric and arithmetic monodromy groups satisfy the inclusions*

$$G_{geom,\text{even}} \leq G_{geom,\text{even,2-param}},\ G_{arith,\text{even}} \leq G_{arith,\text{even,2-param}}.$$

(ii) *For the local systems $\mathcal{G}_{\text{odd}}(\psi, 2n, q)$ and $\mathcal{G}_{\text{odd,2-param}}(\psi, 2n, q)$, their geometric and arithmetic monodromy groups satisfy the inclusions*

$$G_{geom,\text{odd}} \leq G_{geom,\text{odd,2-param}},\ G_{arith,\text{odd}} \leq G_{arith,\text{odd,2-param}}.$$

(iii) *For the local systems* $\mathcal{W}(\psi, 2n, q)$ *and* $\mathcal{W}_{2\text{-param}}(\psi, 2n, q)$*, their geometric and arithmetic monodromy groups satisfy the inclusions*

$$G_{geom,\mathrm{sum}} \leq G_{geom,\mathrm{sum,2\text{-}param}}, \;\; G_{arith,\mathrm{sum}} \leq G_{arith,\mathrm{sum,2\text{-}param}}.$$

Lemma 9.2. *For* $\epsilon_p := (-1)^{(p-1)/2}$*, the local systems* $\mathcal{G}_{\mathrm{even}}(\psi, 2n, q)$*, and* $\mathcal{G}_{\mathrm{odd}}(\psi, 2n, q)$ *have all their Frobenius traces in the quadratic field* $\mathbb{Q}(\sqrt{\epsilon_p p})$*.*

Proof. This is proved in [12, Lemma 6.1] for the two parameter versions. □

Lemma 9.3. *Suppose* $q \equiv 1 (\mathrm{mod}\, 4)$*. Then we have the following results about our local systems on* $\mathbb{A}^1/\mathbb{F}_p$*.*

(i) *Let* $k/\mathbb{F}_p$ *be a finite extension in which* -1 *is a square. After pullback to* $\mathbb{A}^1/k$*, the arithmetic monodromy group* $G_{arith,\mathrm{even}}$ *for* $\mathcal{G}_{\mathrm{even}}(\psi, 2n, q)$ *lies in* $\mathrm{Sp}_{(q^{2n}-1)/2}(\mathbb{C})$*.*
(ii) *The arithmetic monodromy group* $G_{arith,\mathrm{odd}}$ *for* $\mathcal{G}_{\mathrm{odd}}(\psi, 2n, q)$ *lies in* $\mathrm{SO}_{(q^{2n}+1)/2}(\mathbb{C})$*.*

Proof. This is proved in [12, Lemma 6.2] for the two parameter versions. □

Lemma 9.4. *We have the following results about our local systems on* $\mathbb{A}^1/\mathbb{F}_p$*.*

(i) *The arithmetic monodromy group* $G_{arith,\mathrm{even}}$ *for* $\mathcal{G}_{\mathrm{even}}(\psi, 2n, q)$ *lies in* $\mathrm{SL}_{(q^{2n}-1)/2}(\mathbb{C})$*.*
(ii) *The arithmetic monodromy group* $G_{arith,\mathrm{odd}}$ *for* $\mathcal{G}_{\mathrm{odd}}(\psi, 2n, q)$ *lies in* $\mathrm{SL}_{(q^{2n}+1)/2}(\mathbb{C})$*.*
(iii) *The arithmetic monodromy group* $G_{arith,\mathrm{sum}}$ *for* $\mathcal{W}(\psi, 2n, q)$ *lies in* $\mathrm{SL}_{q^{2n}}(\mathbb{C})$*.*

Proof. This is proved in [12, Lemma 6.3] for the two parameter versions. □

Corollary 9.5. *The geometric and arithmetic monodromy groups of the local systems* $\mathcal{G}_{\mathrm{even}}(\psi, 2n, q)$ *on* $\mathbb{A}^1$ *and* $\mathcal{G}_{\mathrm{even,2\text{-}param}}(\psi, 2n, q)$ *on* $\mathbb{A}^2/\mathbb{F}_p$ *are irreducible subgroups of* $\mathrm{SL}_{(q^{2n}-1)/2}(\mathbb{C})$*. The geometric and arithmetic monodromy groups of the local systems* $\mathcal{G}_{\mathrm{odd}}(\psi, 2n, q)$ *on* $\mathbb{A}^1$ *and* $\mathcal{G}_{\mathrm{odd,2\text{-}param}}(\psi, 2n, q)$ *on* $\mathbb{A}^2/\mathbb{F}_p$ *are irreducible subgroups of* $\mathrm{SL}_{(q^{2n}+1)/2}(\mathbb{C})$*.*

Proof. Because we have the inclusion $G_{geom} < G_{arith}$, it suffices to prove the irreducibility for the geometric monodromy groups. For the local systems on $\mathbb{A}^1$, this was proven in Lemmas 7.4 and 8.4. Because these local systems on $\mathbb{A}^1/\mathbb{F}_p$ are pullbacks, by $t \mapsto 0$, of the local systems on $\mathbb{A}^2$, these latter local systems on $\mathbb{A}^2$ are a fortiori geometrically irreducible. □

From van der Geer-van der Vlugt [18], we get

Theorem 9.6. *The groups G_{geom} and G_{arith} for $\mathcal{W}(\psi, 2n, q)$ on $\mathbb{A}^1/\mathbb{F}_p$ are finite, as are the groups G_{geom} and G_{arith} for each of its direct summands $\mathcal{G}_{\mathrm{odd}}(\psi, 2n, q)$ and $\mathcal{G}_{\mathrm{even}}(\psi, 2n, q)$.*

Proof. This is proved in [12, Theorem 5.5] for the two parameter versions. □

Lemma 9.7. *The order of G_{geom} for $\mathcal{G}_{\mathrm{even}}(\psi, 2n, q)$ is divisible by both $(q^{2n} - 1)/2$ and $(q^{2n} - q)/2$. The order of G_{geom} for $\mathcal{G}_{\mathrm{odd}}(\psi, 2n, q)$ is divisible by both $(q^{2n} + 1)/2$ and $(q^{2n} - q)/2$.*

Proof. The divisibilities are instances of the fact that for a finite group, the degree of an irreducible representation divides the order of the group, applied first to G_{geom} and its given representation, and second to the image in G_{geom} of $I(\infty)$ acting on the wild part of the $I(\infty)$ representation. □

Lemma 9.8. *The image of the wild inertia group $P(\infty)$ in the geometric monodromy group G_{geom} of each of $\mathcal{G}_{\mathrm{even}}(\psi, 2n, q)$ and $\mathcal{G}_{\mathrm{odd}}(\psi, 2n, q)$ is a p-group, whose action on the wild part of the given representation is the direct sum of $(q^{2n-1} - 1)/2$ pairwise inequivalent irreducible representations of dimension q.*

Proof. In each case, the wild part of the $I(\infty)$-representation is irreducible (by Lemma 7.5 and Lemma 8.5), of dimension $(q^{2n} - q)/2$. So the assertion results from [9, 1.14 (3) and 1.14.1]. □

Corollary 9.9. *The geometric monodromy group $G_{\mathrm{geom,sum}}$ of the local system $\mathcal{W}(\psi, 2n, q)$ contains a p-group that admits a representation which is the direct sum of $(q^{2n-1} - 1)/2$ pairwise inequivalent irreducible representations of dimension q.*

Proof. We will show that the image $P(\infty)_{\mathrm{sum}}$ of $P(\infty)$ in $G_{\mathrm{geom,sum}}$ is such a group. The group $G_{\mathrm{geom,sum}}$ is a subgroup of the product $G_{\mathrm{geom,even}} \times G_{\mathrm{geom,odd}}$ which maps onto each factor. Viewing all these groups as quotients of $\pi_1(\mathbb{A}^1_{\overline{\mathbb{F}_p}})$, we see that $P(\infty)_{\mathrm{sum}}$ maps onto the image of $P(\infty)$ in, say, the first factor $G_{\mathrm{geom,even}}$. Via this quotient, we see from Lemma 9.8 that $P(\infty)_{\mathrm{sum}}$ admits a representation which is the direct sum of $(q^{2n-1} - 1)/2$ pairwise inequivalent irreducible representations of dimension q. □

Corollary 9.10. *Each of the arithmetic and geometric monodromy groups for each of the six local systems*

$$\mathcal{G}_{\mathrm{odd}}(\psi, 2n, q),\ \mathcal{G}_{\mathrm{even}}(\psi, 2n, q),\ \mathcal{W}(\psi, 2n, q),$$
$$\mathcal{G}_{\mathrm{odd,2\text{-}param}}(\psi, 2n, q),\ \mathcal{G}_{\mathrm{even,2\text{-}param}}(\psi, 2n, q),\ \mathcal{W}_{\mathrm{2\text{-}param}}(\psi, 2n, q)$$

contain a p-group that admits an irreducible representation of dimension q.

Proof. By Lemma 9.8 and Corollary 9.9, the assertion holds for the one-parameter local systems. The assertion for the two-parameter local systems results from the one-parameter case and the inclusions of Lemma 9.1. □

Corollary 9.11. *The geometric monodromy group $G_{geom,\mathrm{even,2\text{-}param}}$ for $\mathcal{G}_{\mathrm{even,2\text{-}param}}(\psi, 2n, q)$ has order divisible by both $(q^{2n}-1)/2$ and $(q^{2n}-q)/2$. The geometric monodromy group $G_{geom,\mathrm{odd,2\text{-}param}}$ for $\mathcal{G}_{\mathrm{odd,2\text{-}param}}(\psi, 2n, q)$ has order divisible by both $(q^{2n}+1)/2$ and $(q^{2n}-q)/2$.*

Proof. Immediate from Lemma 9.7 and the inclusions of Lemma 9.1. □

Theorem 9.12. *Write $q = p^a$. Then we have the following results.*

(i) *The arithmetic monodromy group $G_{arith,\mathrm{even}}$ for $\mathcal{G}_{\mathrm{even}}(\psi, 2n, q)$ lies in $\mathrm{Sp}_{4an}(p)$, the latter group viewed inside $\mathrm{SL}_{(q^{2n}-1)/2}(\mathbb{C})$ by one of its even Weil representations.*
(ii) *The arithmetic monodromy group $G_{arith,\mathrm{odd}}$ for $\mathcal{G}_{\mathrm{odd}}(\psi, 2n, q)$ lies in $\mathrm{PSp}_{4an}(p)$, the latter group viewed inside $\mathrm{SL}_{(q^{2n}+1)/2}(\mathbb{C})$ by one of its odd Weil representations.*

Proof. In the two parameter versions, the named groups contain $\mathrm{SL}_2(p^{2an})$ (respectively $\mathrm{PSL}_2(p^{2an})$), so the asserted inclusions for them result from [12, Theorem 4.1]. Our local systems are pullbacks of these by $t \mapsto 0$. □

Theorem 9.13. *Suppose that $(q, 2n) \neq (3, 2)$. Then each of the arithmetic $G_{arith,\mathrm{odd}}$ and geometric $G_{geom,\mathrm{odd}}$ monodromy groups for $\mathcal{G}_{\mathrm{odd}}(\psi, 2n, q)$ is (separately) of the form $\mathrm{PSp}_{2A}(p^B) \rtimes C_b$ for some factorization $2an = AB$ and some divisor b of B.*

Proof. This is part (ii) of [12, Theorem 4.7]. □

Theorem 9.14. *Write $q = p^a$. We have the following results.*

(i) *For the local system $\mathcal{G}_{\mathrm{even,2\text{-}param}}(\psi, 2n, q)$, each of its geometric $G_{geom,\mathrm{even,2\text{-}param}}$ and arithmetic $G_{arith,\mathrm{even,2\text{-}param}}$ monodromy groups is (separately) of the form $\mathrm{Sp}_{2A}(p^B) \rtimes C_b$ for some factorization $2an = AB$ and some p-power divisor b of B.*
(ii) *For the local system $\mathcal{G}_{\mathrm{odd,2\text{-}param}}(\psi, 2n, q)$, each of its geometric $G_{geom,\mathrm{odd,2\text{-}param}}$ and arithmetic $G_{arith,\mathrm{odd,2\text{-}param}}$ monodromy groups is (separately) of the form $\mathrm{PSp}_{2A}(p^B) \rtimes C_b$ for some factorization $2an = AB$ and some p-power divisor b of B.*

Proof. This is proved inside the proof of [12, Corollary 6.5]. □

Theorem 9.15. *Write $q = p^a$. For the local system $\mathcal{W}_{\mathrm{2\text{-}param}}(\psi, 2n, q)$, we have the following results.*

(i) *Its geometric monodromy group* $G_{geom,\mathrm{sum},\text{2-param}}$ *is isomorphic to the diagonal image of* $\mathrm{Sp}_{2A}(p^B) \rtimes C_b$ *in* $\mathrm{Sp}_{4an}(p) \times \mathrm{PSp}_{4an}(p)$ *for some factorization* $2an = AB$ *and some* p*-power divisor* b *of* B*.*

(ii) *Its arithmetic monodromy group* $G_{arith,\mathrm{sum},\text{2-param}}$ *is isomorphic to the diagonal image of* $\mathrm{Sp}_{2A}(p^B) \rtimes C_b$ *in* $\mathrm{Sp}_{4an}(p) \times \mathrm{PSp}_{4an}(p)$ *for some factorization* $2an = AB$ *and some* p*-power divisor* b *of* B*.*

Proof. Let us begin with the geometric group. From Theorem 9.14, we get that $G_{geom,\mathrm{sum},\text{2-param}}$ is a subgroup of a product group

$$\big(\mathrm{Sp}_{2A}(p^B) \rtimes C_b\big) \times \big(\mathrm{PSp}_{2C}(p^D) \rtimes C_d\big),$$

for some factorizations $2an = AB$, $2an = CD$, with b some p-power divisor of B and d some p-power divisor of D, which maps onto each factor. By Goursat's lemma, $G_{geom,\mathrm{sum},\text{2-param}}$ is the inverse image of the graph of an isomorphism of some quotient of the first factor with some quotient of the second factor. The only quotients of the first factor are itself, $\mathrm{PSp}_{2A}(p^B) \rtimes C_b$ and the quotients of C_b. The only quotients of the second factor are itself and quotients of C_d.

There are no isomorphisms between any $\mathrm{Sp}_{2A}(p^B) \rtimes C_b$ and any $\mathrm{PSp}_{2C}(p^D) \rtimes C_d$, because their derived groups, namely $\mathrm{Sp}_{2A}(p^B)$ and $\mathrm{PSp}_{2C}(p^D)$, are not isomorphic.

There is an isomorphism between $\mathrm{PSp}_{2A}(p^B) \rtimes C_b$ and $\mathrm{PSp}_{2C}(p^D) \rtimes C_d$ precisely when $(A, B) = (C, D)$ and $b = d$.

The are no isomorphisms of any nonabelian quotient of one factor with an abelian quotient of the other.

The only remaining possibilities are isomorphisms between quotients of C_b with quotients of C_d. But in this case, the group $G_{geom,\mathrm{sum},\text{2-param}}$ would contain the entire product

$$\mathrm{Sp}_{2A}(p^B) \;\times\; \mathrm{PSp}_{2C}(p^D),$$

and this is ruled out by the trace zero argument of [12, Proposition 6.6].

Repeat the same argument for the arithmetic group $G_{arith,\mathrm{sum},\text{2-param}}$. $\square$

Lemma 9.16. *At the point* $s = -1 \in \mathbb{A}^1(\mathbb{F}_q)$*, we have*

$$|\mathrm{Trace}(Frob_{-1,\mathbb{F}_q}|\mathcal{W}(\psi, 2n, q))|^2 = q.$$

Moreover, for any finite extension $K/\mathbb{F}_q$*, and any* $s \in \mathbb{A}^1(K)$*, we have*

$$|\mathrm{Trace}(Frob_{s,K}|\mathcal{W}(\psi, 2n, q))|^2 \in \{1, q, q^2, \ldots, q^{4n}\}.$$

Proof. From [12, §5], with t set to 0, we see that, for $K/\mathbb{F}_p$ a finite extension, and $s \in \mathbb{A}^1(K)$, this square absolute value

$$|\mathrm{Trace}(Frob_{s,K}|\mathcal{W}(\psi, 2n, q))|^2$$

is the number of zeroes in K of the polynomial

$$x^{q^{4n}} + s^{q^{2n}} x^{q^{2n+1}} + s^{q^{2n-1}} x^{q^{2n-1}} + x.$$

When K is a finite extension of $\mathbb{F}_q$, the set of its zeroes in K is an $\mathbb{F}_q$ vector space (under addition and scalar multiplication by $\mathbb{F}_q$) of dimension $\leq 4n$. With $s = -1$, this becomes the polynomial

$$x^{q^{4n}} - x^{q^{2n+1}} - x^{q^{2n-1}} + x.$$

Every $x \in \mathbb{F}_q$ is a zero of this polynomial. $\square$

In fact, we have the following result.

Lemma 9.17. *Let $K \subset \mathbb{F}_q$ be a subfield. At the point $s = -1 \in \mathbb{A}^1(K)$, we have*

$$|\mathrm{Trace}(Frob_{-1,K}|\mathcal{W}(\psi, 2n, q))|^2 = \#K.$$

In particular,

$$|\mathrm{Trace}(Frob_{-1,\mathbb{F}_p}|\mathcal{W}(\psi, 2n, q))|^2 = p.$$

Proof. As noted at the beginning of this section, the local system $\mathcal{W}(\psi, 2n, q)$ on $\mathbb{A}^1/\mathbb{F}_p$ has trace function at $s \in \mathbb{A}^1(K)$, $K/\mathbb{F}_p$ a finite extension, given by

$$s \mapsto (-1/\alpha_K) \sum_{x \in K} \psi_K(x^{q^{2n}+1} + s x^{q+1}).$$

Taking $s = -1$, we get

$$\mathrm{Trace}(Frob_{-1,K}|\mathcal{W}(\psi, 2n, q)) = (-1/\alpha_K) \sum_{x \in K} \psi_K(x^{q^{2n}+1} - x^{q+1}).$$

When K is a subfield of $\mathbb{F}_q$, for each $x \in K$ we have

$$x^{q^{2n}+1} = x^{q+1} = x^2,$$

so that the sum

$$\sum_{x \in K} \psi_K(x^{q^{2n}+1} - x^{q+1}) = \sum_{x \in K} \psi_K(0) = \#K.$$

Thus for K a subfield of $\mathbb{F}_q$,

$$\mathrm{Trace}(Frob_{-1,K}|\mathcal{W}(\psi,2n,q)) = (-1/\alpha_K)\#K,$$

whose square absolute value is indeed $\#K$. □

Corollary 9.18. *For the* 2*-parameter local system* $\mathcal{W}_{2\text{-param}}(\psi,2n,q)$*, we have the following results.*

(i) *At the point* $(s,t)=(-1,0)\in\mathbb{A}^2(\mathbb{F}_q)$*, we have*

$$|\mathrm{Trace}(Frob_{(-1,0),\mathbb{F}_q}|\mathcal{W}_{2\text{-param}}(\psi,2n,q))|^2 = q.$$

Moreover, for any finite extension $K/\mathbb{F}_q$*, and any* $(s,t)\in\mathbb{A}^2(K)$*, we have*

$$|\mathrm{Trace}(Frob_{(s,t),K}|\mathcal{W}_{2\text{-param}}(\psi,2n,q))|^2 \in \{1,q,q^2,\ldots,q^{4n}\}.$$

(ii) *Let* $K\subset\mathbb{F}_q$ *be a subfield. At the point* $(s,t)=(-1,0)\in\mathbb{A}^2(K)$*, we have*

$$|\mathrm{Trace}(Frob_{(-1,0),K}|\mathcal{W}_{2\text{-param}}(\psi,2n,q))|^2 = \#K.$$

In particular,

$$|\mathrm{Trace}(Frob_{(-1,0),\mathbb{F}_p}|\mathcal{W}_{2\text{-param}}(\psi,2n,q))|^2 = p.$$

Proof. The statements about the point $(-1,0)$ are the statements about the point $s=-1$ in Lemmas 9.16 and 9.17. The second assertion of (i) is the fact [12, §5] that the square absolute value in question is the number of zeroes in K of the polynomial

$$x^{q^{4n}} + s^{q^{2n}}x^{q^{2n+1}} + 2t^{q^{2n}}x^{q^{2n}} + s^{q^{2n-1}}x^{q^{2n-1}} + x. \quad \square$$

10. Identifications of monodromy groups with $\mathrm{Sp}_{4n}(q)$

Recall, see [19], that if $a>2$ and $m\geq 3$, then a^m-1 admits a *primitive prime divisor* $\mathrm{ppd}(a,m)$, that is, a prime divisor that does not divide $\prod_{i=1}^{m-1}(a^i-1)$.

Theorem 10.1. *Let* $A,B,a,n,b\geq 1$ *be some integers with* $b|B$ *and* $AB=2an$*. Suppose that* $H\cong\mathrm{Sp}_{2A}(p^B)\rtimes C_b\leq\mathrm{Sp}_{4an}(p)$ *as in §3 and that* H *satisfies the following conditions:*

(i) *If* $n\geq 2$ *then* $|H|$ *is divisible by a primitive prime divisor* $\ell_2=\mathrm{ppd}(p,(2n-1)a)$*.*
(ii) *If* $n=1$*, then a* p*-subgroup of* H *is acting irreducibly on a complex space of dimension* $q:=p^a$*.*

(iii) *If ω denotes one of the two big Weil characters (of degree p^{2an}) of $\mathrm{Sp}_{4an}(p)$, then $|\omega(h)|^2$ is a power of q for any $h \in H$.*

Then $(A, B, b) = (2n, a, 1)$, that is, $H \cong \mathrm{Sp}_{4n}(p^a)$.

Proof. First we note by Theorem 3.5 that $p^{B/b} = |\omega(g)|^2$ for some $g \in H$. Hence condition (iii) implies that

$$B = bas \tag{10.1.1}$$

for some integer $s \geq 1$.

(a) Consider the case $n \geq 2$. Note that $\ell_2 \geq (2n-1)a + 1$ by the choice of ℓ_2. On the other hand, any odd prime divisor of b divides $AB = 2an$ and so is at most $an < (2n-1)a$. Hence $\ell_2 \nmid b$, whence ℓ_2 divides $|\mathrm{Sp}_{2A}(p^B)|$. Thus there is some $1 \leq j \leq A$ such that ℓ_2 divides $p^{2Bj} - 1$, whence

$$(2n-1)a \text{ divides } 2Bj \tag{10.1.2}$$

again by the choice of ℓ_2.

Suppose $1 \leq j \leq A/2$. Then $2Bj \leq AB = 2an$. As $2(2n-1)a > 2an$, (10.1.2) implies that $2Bj = (2n-1)a = AB - a$, and so $a = B(A - 2j)$ is divisible by B.

Suppose $A/2 < j \leq A$. Then

$$(2n-1)a < 2an = AB < 2Bj \leq 2AB = 4an < 3(2n-1)a.$$

Now (10.1.2) implies that $2Bj = 2(2n-1)a = 2AB - 2a$, and so $a = B(A - j)$ is again divisible by B. Thus we have shown that $B|a$ in either case. Now using (10.1.1) we conclude that $b = s = 1$, $B = a$, and $A = 2n$ as stated.

(b) Now assume that $n = 1$. Then (10.1.1) implies that $Abs = 2$. If furthermore $A = 2$, then again $b = s = 1$ and we are done. So assume that $A = 1$, i.e. $H \cong \mathrm{Sp}_2(p^{2a}) \rtimes C_b$, with $bs = 2$. In this case, Sylow p-subgroups of H are abelian, contradicting (ii). $\square$

Lemma 10.2. *Let $q = p^a$, and let $\tilde{G}$ be such that $G := \mathrm{Sp}_{4n}(q) \lhd \tilde{G} \leq \mathrm{Sp}_{4an}(p)$ and $|\omega(g)|^2 = p$ for some $g \in \tilde{G}$, where ω is one of the big Weil representation of degree p^{2an} of $\mathrm{Sp}_{4an}(p)$. Then $\tilde{G} = \mathbf{N}_{\mathrm{Sp}_{4an}(p)}(G) = \mathrm{Sp}_{4n}(q) \rtimes C_a$.*

Proof. Note that $\mathbf{N}_{\mathrm{Sp}_{4an}(p)}(G) = \langle G, \sigma \rangle$, where σ is the automorphism of G induced by the map $x \mapsto x^p$, of order a. It follows that $\tilde{G} = \langle G, \sigma^j \rangle$ for some $j|a$. By Theorem 3.5, $|\omega(h)|^2$ is always a power of p^j for any $h \in \tilde{G}$. Hence we conclude that $j = 1$. $\square$

Theorem 10.3. *Let $q = p^a$. For the local system $\mathcal{W}_{2\text{-param}}(\psi, 2n, q)$, we have the following results.*

(i) *The geometric monodromy group* $G_{geom,\text{sum},\text{2-param}}$ *is isomorphic to* $\mathrm{Sp}_{4n}(q)$.
(ii) *For any finite extension* $K/\mathbb{F}_q$, *the arithmetic monodromy group* $G_{arith,\text{sum},\text{2-param}}$ *is isomorphic to* $\mathrm{Sp}_{4n}(q)$.
(iii) *If* $K = \mathbb{F}_p$, *then the arithmetic monodromy group* $G_{arith,\text{sum},\text{2-param}}$ *is isomorphic to* $\mathrm{Sp}_{4n}(q) \rtimes C_a$.

Proof. (a) First we consider $H = G_{geom,\text{sum},\text{2-param}}$. By Theorem 9.15, H has the shape specified in Theorem 10.1. Note that H projects onto (in fact, isomorphic to) $G_{geom,\text{even},\text{2-param}}$. Therefore, by Corollaries 9.10 and 9.11, H satisfies conditions (i) and (ii) of Theorem 10.1. Condition 10.1(iii) is fulfilled by Corollary 9.18. Hence, we conclude by Theorem 10.1 that $G_{geom,\text{sum},\text{2-param}} \cong \mathrm{Sp}_{4n}(q)$.

(b) If K is any finite extension of $\mathbb{F}_q$, then the same arguments as in (a), but applied to $G_{arith,\text{sum},\text{2-param}}$, show that $G_{arith,\text{sum},\text{2-param}} \cong \mathrm{Sp}_{4n}(q)$.

Finally, let $K = \mathbb{F}_p$ and $\tilde{H} = G_{arith,\text{sum},\text{2-param}}$. By Theorem 9.15 and by (i) we know that

$$\mathrm{Sp}_{4n}(q) \cong G_{geom,\text{sum},\text{2-param}} \lhd \tilde{H} \leq \mathrm{Sp}_{4an}(p).$$

Applying Corollaries 9.18 and Lemma 10.2, we conclude that $\tilde{H} \cong \mathrm{Sp}_{4n}(q) \rtimes C_a$. $\square$

Theorem 10.4. *Let* $q = p^a$. *For the local system* $\mathcal{G}_{\text{even},\text{2-param}}(\psi, 2n, q)$, *we have the following results.*

(i) *The geometric monodromy group* $G_{geom,\text{even},\text{2-param}}$ *is isomorphic to* $\mathrm{Sp}_{4n}(q)$.
(ii) *For any finite extension* $K/\mathbb{F}_q$, *the arithmetic monodromy group* $G_{arith,\text{even},\text{2-param}}$ *is isomorphic to* $\mathrm{Sp}_{4n}(q)$.
(iii) *If* $K = \mathbb{F}_p$, *then the arithmetic monodromy group* $G_{arith,\text{even},\text{2-param}}$ *is isomorphic to* $\mathrm{Sp}_{4n}(q) \rtimes C_a$.

For the local system $\mathcal{G}_{\text{odd},\text{2-param}}(\psi, 2n, q)$, *we have the following results.*

(i) *The geometric monodromy group* $G_{geom,\text{odd},\text{2-param}}$ *is isomorphic to* $\mathrm{PSp}_{4n}(q)$.
(ii) *For any finite extension* $K/\mathbb{F}_q$, *the arithmetic monodromy group* $G_{arith,\text{odd},\text{2-param}}$ *is isomorphic to* $\mathrm{PSp}_{4n}(q)$.
(iii) *If* $K = \mathbb{F}_p$, *then the arithmetic monodromy group* $G_{arith,\text{odd},\text{2-param}}$ *is isomorphic to* $\mathrm{PSp}_{4n}(q) \rtimes C_a$.

Proof. Note that each of the arithmetic and geometric monodromy groups of each of the two local systems $\mathcal{G}_{\text{even},\text{2-param}}(\psi, 2n, q)$ and $\mathcal{G}_{\text{odd},\text{2-param}}(\psi, 2n, q)$ is a quotient of the corresponding group for the local system $\mathcal{W}_{\text{2-param}}(\psi, 2n, q)$. Also, observe that $\mathrm{Sp}_{4n}(q)$ acts faithfully on the even-dimensional Weil representations of degree $(q^{2n} - 1)/2$, and acts with kernel C_2 on the odd-dimensional Weil representations of degree $(q^{2n} + 1)/2$.

Now using Corollary 7.4 and Theorem 10.3(i), we conclude that $G_{geom,\text{even},2\text{-param}} \cong \mathrm{Sp}_{4n}(q)$ and $G_{geom,\text{odd},2\text{-param}} \cong \mathrm{PSp}_{4n}(q)$.

The same arguments, but now using Theorem 10.3(ii) show that $G_{arith,\text{even},2\text{-param}} \cong \mathrm{Sp}_{4n}(q)$ and $G_{arith,\text{odd},2\text{-param}} \cong \mathrm{PSp}_{4n}(q)$ for any finite extension $K/\mathbb{F}_q$.

In the case $K = \mathbb{F}_p$, we use in addition the fact that the arithmetic group contains the geometric group as a normal subgroup and Theorem 10.3(iii) to see that $G_{arith,\text{even},2\text{-param}} \cong \mathrm{Sp}_{4n}(q) \rtimes C_a$ and $G_{arith,\text{odd},2\text{-param}} \cong \mathrm{PSp}_{4n}(q) \rtimes C_a$. □

Theorem 10.5. *Suppose that $q = p^a$ as before and that a subgroup G of $H = \mathrm{Sp}_{4n}(q)$ satisfies the following conditions:*

- (i) *G is irreducible on a Weil module V of dimension $(q^{2n}+1)/2$ of H.*
- (ii) *If $n \geq 2$ then $|G|$ is divisible by a primitive prime divisor $\ell_2 = \mathrm{ppd}(p,(2n-1)a)$.*
- (iii) *If $n = 1$, then a p-subgroup of G is acting irreducibly on a complex space of dimension p^a.*

Then $G = H = \mathrm{Sp}_{4n}(q)$.

Proof. (a) First we consider the case $(n,q) = (1,3)$. Then (i) implies that 5 divides $|G|$. Furthermore, Sylow 3-subgroups of G are non-abelian by (iii), whence 3^3 divides $|G|$. Since no maximal subgroup of $H = \mathrm{Sp}_4(3)$ can have order divisible by $3^3 \cdot 5$, see [2], we conclude that $G = H$.

From now on, we may assume that $(n,q) \neq (1,3)$. Hence, $p^{4an}-1$ admits a *large primitive prime divisor* $\ell = \mathrm{ppd}(p,4an)$ by [4], and we choose such an ℓ to maximize the ℓ-part Q of $p^{4an}-1 = q^{4n}-1$. By (i), $|G|$ is divisible by Q, and we can apply [12, Theorem 4.6] (with $d = 4n$ and $f = a$) to G. Let $L := \mathbf{O}^{\ell'}(G)$. Note that $L \not\cong C_Q$, as otherwise by Ito's theorem [8, (6.15)] any irreducible complex character of G has degree coprime to ℓ, violating (i). In what follows we will consider the possibilities for L as listed in [12, Theorem 4.6]. We also denote by $d_{\mathbb{C}}(L)$ the smallest degree > 1 of complex irreducible representations of L, and freely use lower bounds for $d_{\mathbb{C}}(L)$ as listed in [16].

(b) $L \cong \mathrm{SL}_{4n/j}(q^j)$ for some $j|4n$ with $4n/j \geq 3$. Then

$$d_{\mathbb{C}}(L) \geq q^{j(4n/j-1)} = q^{4n-j} > (q^{2n}+1)/2 = \dim(V).$$

It follows that the quasisimple group L acts trivially on V. But in this case G cannot be irreducible on V as G/L is an ℓ'-group.

(c) $L \cong \mathrm{SU}_{4n/j}(q^j)$ for some $j|4n$ with $4n/j \geq 3$ being odd; in particular, $4|j$ and $n \geq 3$. Recall that $L \lhd G \leq \mathrm{GL}_{4n}(q)$. Now part (e) of the proof of [12, Theorem 4.7] (with $N = 2an \geq 6$) shows that no such subgroup G can be irreducible on V.

(d) $L \cong \Omega^-_{4n/j}(q^j)$ with $j|2n$ and $j \leq n$. If, moreover, $j \leq n/2$, then

$$d_{\mathbb{C}}(L) \geq q^{j(4n/j-3)} = q^{4n-3j} > q^{2n} > \dim(V),$$

whence L acts trivially on V and we arrive at a contradiction as in (b). If $j = 2n/3$ (and so $3|n$), then L is a cover of $\mathrm{PSU}_4(q^{2n/3})$, and so

$$d_{\mathbb{C}}(L) = \frac{q^{8n/3} - 1}{q^{2n/3} + 1} > \frac{q^{2n} + 1}{2} = \dim(V),$$

and we again arrive at a contradiction. In the remaining case we have $j = n$, $L \cong \mathrm{PSL}_2(q^{2n})$, and $d_{\mathbb{C}}(L) = (q^{2n} + 1)/2$. This possibility cannot however occur, since $L \leq H = \mathrm{Sp}_{4n}(q)$ has a faithful representation of degree $(q^{2n} - 1)/2$.

(e) $L = \mathrm{Sp}(W_j) \cong \mathrm{Sp}_{4n/j}(q^j)$ for some $j|2n$ (and the natural module $W_j = \mathbb{F}_{q^j}^{4n/j}$ for L is obtained from the natural module $\mathbb{F}_q^{4n}$ of H by base change). Arguing as in part (d) of the proof of [12, Theorem 4.7], we see that $G = \langle L, \sigma \rangle$, where σ is a field automorphism of L order say $b|j$. If furthermore $j = 1$, then we obtain $G = L = \mathrm{Sp}_{4n}(q)$, as stated.

Assume furthermore that $n \geq 2$. Note any odd prime divisor of b is $\leq n < (2n-1)a < \ell_2$, hence ℓ_2 divides $|L| = |\mathrm{Sp}_{4n/j}(q^j)|$ by (ii). It follows that ℓ_2 divides $q^{2ij} - 1$ for some integer $1 \leq i \leq 2n/j$, whence $2n - 1$ divides $2ij$. This is possible only when $ij = 2n - 1$ as $n \geq 2$. But $j|2n$, so we conclude $j = 1$, as desired.

Finally, we consider the case $n = 1$, but $j > 1$. Then $\mathrm{Sp}_2(q^2) = L \lhd G \leq \mathrm{Sp}_2(q^2) \rtimes C_2$. In particular, the Sylow p-subgroups of G are abelian, contradicting (iii). $\square$

Theorem 10.6. *Let $q = p^a$. For the local system $\mathcal{W}(\psi, 2n, q)$, we have the following results.*

- (i) *The geometric monodromy group $G_{geom,\mathrm{sum}}$ is isomorphic to $\mathrm{Sp}_{4n}(q)$.*
- (ii) *For any finite extension $K/\mathbb{F}_q$, the arithmetic monodromy group $G_{arith,\mathrm{sum}}$ is isomorphic to $\mathrm{Sp}_{4n}(q)$.*
- (iii) *If $K = \mathbb{F}_p$, then the arithmetic monodromy group $G_{arith,\mathrm{sum}}$ is isomorphic to $\mathrm{Sp}_{4n}(q) \rtimes C_a$.*

Proof. (a) First we consider $G = G_{geom,\mathrm{sum}}$. By Theorem 10.3(i) and Lemma 9.1,

$$G \leq H := G_{geom,\mathrm{sum},\text{2-param}} \cong \mathrm{Sp}_{4n}(q).$$

Next, by Corollary 7.4 we have that G acts irreducibly on a Weil module of dimension $(q^{2n} + 1)/2$ of H and thus fulfills condition 10.5(i). Furthermore, Lemma 9.7 and Corollary 9.10 show that G satisfies conditions (ii) and (iii) of Theorem 10.5. Hence, applying Theorem 10.5 to G, we obtain that $G_{geom,\mathrm{sum}} = H \cong \mathrm{Sp}_{4n}(q)$.

(b) If K is any finite extension of $\mathbb{F}_q$, then the same arguments as in (a), but applied to $G_{arith,\mathrm{sum}}$, show that $G_{arith,\mathrm{sum}} \cong \mathrm{Sp}_{4n}(q)$.

Finally, let $K = \mathbb{F}_p$ and $\tilde{G} = G_{arith,\mathrm{sum}}$. By Theorem 9.15, Lemma 9.15 and by (i) we know that

$$\mathrm{Sp}_{4n}(q) \cong G_{geom,\mathrm{sum}} \lhd \tilde{G} \leq G_{arith,\mathrm{sum},\text{2-param}} \leq \mathrm{Sp}_{4an}(p).$$

Applying Lemmas 9.17 and 10.2, we conclude that $\tilde{G} \cong \mathrm{Sp}_{4n}(q) \rtimes C_a$. $\square$

Theorem 10.7. *Let $q = p^a$. For the local system $\mathcal{G}_{\mathrm{even}}(\psi, 2n, q)$, we have the following results.*

(i) *The geometric monodromy group $G_{geom,\mathrm{even}}$ is isomorphic to $\mathrm{Sp}_{4n}(q)$.*
(ii) *For any finite extension $K/\mathbb{F}_q$, the arithmetic monodromy group $G_{arith,\mathrm{even}}$ is isomorphic to $\mathrm{Sp}_{4n}(q)$.*
(iii) *If $K = \mathbb{F}_p$, then the arithmetic monodromy group $G_{arith,\mathrm{even}}$ is isomorphic to $\mathrm{Sp}_{4n}(q) \rtimes C_a$.*

For the local system $\mathcal{G}_{\mathrm{odd}}(\psi, 2n, q)$, we have the following results.

(i) *The geometric monodromy group $G_{geom,\mathrm{odd}}$ is isomorphic to $\mathrm{PSp}_{4n}(q)$.*
(ii) *For any finite extension $K/\mathbb{F}_q$, the arithmetic monodromy group $G_{arith,\mathrm{odd}}$ is isomorphic to $\mathrm{PSp}_{4n}(q)$.*
(iii) *If $K = \mathbb{F}_p$, then the arithmetic monodromy group $G_{arith,\mathrm{odd}}$ is isomorphic to $\mathrm{PSp}_{4n}(q) \rtimes C_a$.*

Proof. Argue similarly to the proof of Theorem 10.4, but using Theorem 10.6 instead of Theorem 10.3. $\square$

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