1. Introduction

For an odd integer $n \geq 3$, and a prime power $q \geq 3$, the irreducible representations (over $\mathbb{C}$) of lowest degree after the trivial representation of the group $\text{SU}_n(q)$ are a symplectic representation of dimension $\frac{q^n+1}{q+1} - 1 = \frac{q^n-q}{q+1}$, and $q$ representations of dimension $\frac{q^n+1}{q+1}$. When $q$ is odd, exactly one of these $q$ representations is orthogonal, otherwise none is. The direct sum of these $q+1$ representations is called the big Weil representation of $\text{SU}_n(q)$.

In the paper [KT1], we wrote down $q+1$ rigid local systems on the affine line $\mathbb{A}^1/\mathbb{F}_p$ whose geometric monodromy groups we conjectured to be the images of $\text{SU}_n(q)$ in these $q+1$ representations. We were able to prove this only in the case when $n = 3$ and $\gcd(n, q+1) = 1$ (the condition that $\text{SU}_n(q) = \text{PSU}_n(q)$), where we made use of the results of Dick Gross [Gross]. In this paper, we use a completely different method, which starts\textsuperscript{1} with results of Gross, to prove these conjectures for any odd $n \geq 3$ and for any odd prime power $q$, see Theorem 3.4.

\textsuperscript{1}The results here use the results of [KT2], which in turn uses the results of [KT1] for $\text{SL}_2$, and those use [Gross] in an essential way.

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The method used here, which requires that \( q \) be odd, is based on a striking group-theoretic relation between the Weil representations of \( SU_n(q) \) and \( Sp_{2n}(q) \), and on the determination of those subgroups of \( Sp_{2n}(q) \) to which the Weil representation restricts “as though” it were the Weil representation of \( SU_n(q) \), cf. Theorem 2.3. We are able to apply this result to our local systems, in Section 3, by invoking results of [KT2], which was devoted to questions around \( Sp_{2n}(q) \). Furthermore, our Theorem 3.3 also improves the main results Theorems 1.1 and 4.8 of [KT2] in the case \( 2 \nmid n \), by removing the condition that \( p \nmid n \cdot \log_p(q) \) for the prime \( p | q \).

In the course of thinking about these questions, we stumbled upon a very striking representation-theoretic fact about the \( q \) irreducible representations of \( SU_n(q) \) (\( n \geq 3 \) odd, \( q \) odd) of dimension \( \frac{q^n+1}{q+1} \). For each of them, their \( n \)th moment (i.e. the dimension of the space of invariants in the \( n \)th tensor power of the representation in question) is one, cf. Theorem 4.11. For the irreducible representation of dimension \( \frac{q^n+1}{q+1} - 1 \), the \( n \)th moment vanishes. At present we do not have a conceptual explanation for this.

Given this result about \( n \)th moments for \( SU_n(q) \) when \( n \) is odd, it is natural to wonder about the situation for \( n \)th moments when \( n \) is even. [For \( n \) even and \( q \geq 3 \) a prime power, the irreducible representations (over \( \mathbb{C} \)) of lowest degree after the trivial representation of the group \( SU_n(q) \) are an orthogonal representation of dimension \( \frac{q^n-1}{q+1} + 1 = \frac{q^n-q}{q+1} \), and \( q \) representations of dimension \( \frac{q^n-1}{q+1} \)] Already for \( n = 4 \), the result is not so nice, cf. Theorem 5.1.

2. Unitary-type subgroups of finite symplectic groups

Let \( q = p^f \) be any prime power and \( n \geq 2 \). It is well known, see e.g. [TZ2, §4], that the function

\[
\zeta_{n,q} = \zeta_n : g \mapsto (-1)^n (-q)^{\dim_{F_q^2} \ker(g-1_W)}
\]

defines a complex character, called the (reducible) Weil character, of the general unitary group \( GU_n(q) = GU(W) \), where \( W = \mathbb{F}_{q^2}^n \) is a non-degenerate Hermitian space with Hermitian product \( \circ \). Note that the \( \mathbb{F}_q \)-bilinear form

\[
(u|v) = \text{Trace}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\theta u \circ u)
\]
on \( W \), for a fixed \( \theta \in \mathbb{F}_{q^2}^\times \) with \( \theta^{q-1} = -1 \), is non-degenerate symplectic. This leads to an embedding

\[
\tilde{G} := GU_n(q) \hookrightarrow Sp_{2n}(q).
\]

Moreover, if \( q \) is odd then the restriction of any of the two big Weil characters (of degree \( q^n \), and denoted \( \text{Weil}_{1,2} \) in [KT2]) of \( Sp_{2n}(q) \) to \( GU_n(q) \) is exactly the big Weil character \( \zeta_n \), cf. [TZ2, §4]. We will also denote by \( \zeta_n \) the restriction of this character to the special unitary group \( G := SU_n(q) \).
Fix a generator $\sigma$ of $\mathbb{F}_q^\times$ and set $\rho := \sigma^{q-1}$. We also fix a primitive $(q^2 - 1)^{\text{th}}$ root of unity $\sigma \in \mathbb{C}^\times$ and let $\rho = \sigma^{q-1}$. Then

\begin{equation}
(\text{Weil}_1)|_{\tilde{G}} = \zeta_n = \sum_{i=0}^{q} \tilde{\zeta}_{i,n}
\end{equation}

decomposes as the sum of $q+1$ characters of $\tilde{G}$, where

\begin{equation}
\tilde{\zeta}_{i,n}(g) = \frac{(-1)^n}{q+1} \sum_{l=0}^{q} \rho^{il}(-q)^{\dim \ker(g-\rho^l 1_W)},
\end{equation}

see [TZ2, Lemma 4.1]. In particular, $\tilde{\zeta}_{i,n}$ has degree $(q^n - (-1)^n)/(q+1)$ if $i > 0$ and $(q^n + (-1)^n q)/(q+1)$ if $i = 0$.

We will let $\zeta_{i,n}$ denote the restriction of $\tilde{\zeta}_{i,n}$ to $G = \text{SU}_n(q)$, for $0 \leq i \leq q$. If $n \geq 3$, then these $q+1$ characters are all irreducible and distinct. If $n = 2$, then $\zeta_{i,n}$ is irreducible, unless $q$ is odd and $i = (q+1)/2$, in which case it is a sum of two irreducible characters of degree $(q-1)/2$, see [TZ2, Lemma 4.7]. Formula (2.0.2) implies that Weil characters $\zeta_{i,n}$ enjoy the following branching rule while restricting to the natural subgroup $H := \text{Stab}_G(w) \cong \text{SU}_{n-1}(q)$ ($w \in W$ any anisotropic vector):

\begin{equation}
\zeta_{i,n}|_H = \sum_{j=0, j \neq i}^{q} \zeta_{j,n-1}.
\end{equation}

Furthermore, the complex conjugation fixes $\tilde{\zeta}_{0,n}$ and sends $\tilde{\zeta}_{j,n}$ to $\tilde{\zeta}_{q+1-j,n}$ when $1 \leq j \leq q$. As $n \geq 3$ is odd, it is also known that $\tilde{\zeta}_{0,n}$ is of symplectic type; let $\Psi_0 : \tilde{G} \to \text{Sp}(V)$ be a complex representation affording this character. If $2 \nmid q$, then $\tilde{\zeta}_{(q+1)/2,n}$ is of orthogonal type; let $\Psi_{(q+1)/2} : \tilde{G} \to \text{O}(V)$ be a complex representation affording this character. In the remaining cases, let $\Psi_i : \tilde{G} \to \text{GL}(V)$ be a complex representation affording the character $\tilde{\zeta}_{i,n}$.

**Lemma 2.1.** Assume $n \geq 3$ is odd and $(n,q) \neq (3,2)$.

(i) $\Psi_0(\text{GU}_n(q)) \cong \text{PGU}_n(q)$ is contained in $\text{Sp}(V)$ and contains $\Psi_0(\text{SU}_n(q))$ with index $d := \gcd(n, q+1)$.

(ii) If $1 \leq i \leq q$, then $\ker(\Psi_i)$ is a central subgroup of order $\gcd(i, q+1)$, and $\ker(\Psi_i|_{\text{SU}_n(q)})$ is a central subgroup of order $\gcd(i, n, q+1)$.

(iii) If $2 \nmid q$, then $\Psi_{(q+1)/2}(\text{GU}_n(q)) \cap \text{SO}(V)$ contains $\Psi_{(q+1)/2}(\text{SU}_n(q))$ with index $(q+1)/2$.

(iv) If $1 \leq i \leq q$ and $i \neq (q+1)/2$, then $\Psi_i(\text{GU}_n(q)) \cap \text{SL}(V)$ contains $\Psi_i(\text{SU}_n(q))$ with index $\gcd(i, q+1)$.

**Proof.** According to [TZ2, §4], one can label $\Psi_i$ in such a way that

$\Psi_i(z) = \rho^i \cdot 1_V$
for the generator $z = \rho \cdot 1_W$ of $Z(\tilde{G}) \cong C_{q+1}$. In particular, $z \in \ker(\Psi_0)$, and (i) follows.

Now we can assume $1 \leq i \leq q$. Then $z^j \in \ker(\Psi_i)$ if and only if $j$ is divisible by $(q+1)/\gcd(i, q+1)$. Furthermore, $z^{j(q+1)/d} \in \ker(\Psi_i|_{\text{SU}_n(q)})$ if and only if $j$ is divisible by $d/\gcd(i, d) = d/\gcd(i, n, q+1)$ for $d := \gcd(n, q+1)$, equivalently, if $j(q+1)/d$ is divisible by $(q+1)/\gcd(i, n, q+1)$. Hence (ii) follows.

Consider the element $g := \text{diag}(\rho, 1, \ldots, 1) \in \tilde{G}$; note that $\tilde{G} = \langle G, g \rangle$. Then (2.0.2) implies that

$$\tilde{\zeta}_{i,n}(g^k) = -\frac{q^{n-1} - (-1)^{n-1}}{q+1} + (-1)^{n-1} \rho^{|k|}$$

when $1 \leq k \leq q$. It follows that $\Psi_i(g)$ has eigenvalues $\rho^j$, $1 \leq j \leq q$, with multiplicity $(q^{n-1} - 1)/(q+1)$ if $k \neq i$ and $1 + (q^{n-1} - 1)/(q+1)$ if $k = i$, and so

$$\det(\Psi_i(g)) = \rho^i.$$ 

Since $\text{SU}_n(q)$ is perfect, (ii) and (iii) follow. □

We will now show that, when $n \geq 3$ is odd and $q$ is odd, the splitting (2.0.1) of a big Weil character $\text{Weil}_i$ of $\text{Sp}_{2n}(q)$ on its restriction to $\text{SU}_n(q)$ into a sum of $q+1$ irreducible constituents of prescribed degrees characterizes $\text{SU}_n(q)$ uniquely (up to conjugacy).

Recall [Zs] that if $a \geq 2$ and $n \geq 2$ are any integers with $(a, n) \neq (2, 6), (2^k - 1, 2)$, then $a^n - 1$ has a primitive prime divisor, that is, a prime divisor $\ell$ that does not divide $\prod_{i=1}^{n-1}(a^i - 1)$; write $\ell = \text{ppd}(a, n)$ in this case. Furthermore, if in addition $a, n \geq 3$ and $(a, n) \neq (3, 4), (3, 6), (5, 6)$, then $a^n - 1$ admits a large primitive prime divisor, i.e. a primitive prime divisor $\ell$ where either $\ell > m + 1$ (whence $\ell \geq 2m + 1$), or $\ell^2|(a^n - 1)$, see [F2].

We will need the following recognition theorem [KT2, Theorem 2.6], which was obtained relying on [GPS].

**Theorem 2.2.** Let $q = p^f$ be a power of an odd prime $p$ and let $d \geq 2$. If $d = 2$, suppose that $p^{df} - 1$ admits a primitive prime divisor $\ell > 5$. If $d \geq 3$, suppose in addition that $(p, df) \neq (3, 4), (3, 6), (5, 6)$, so that $p^{df} - 1$ admits a large primitive prime divisor $\ell$, in which case we choose such an $\ell$ to maximize the $\ell$-part of $p^{df} - 1$. Let $W = \mathbb{F}_q^d$ and let $G$ be a subgroup of $\text{GL}(W) \cong \text{GL}_d(q)$ of order divisible by the $\ell$-part $Q := (q^d - 1)\ell$ of $q^d - 1$. Then either $L := \text{O}^\ell(G)$ is a cyclic $\ell$-group of order $Q$, or there is a divisor $j < d$ of $d$ such that one of the following statements holds.

(i) $L = \text{SL}(W_j) \cong \text{SL}_{d/j}(q^j)$, $d/j \geq 3$, and $W_j$ is $W$ viewed as a $d/j$-dimensional vector space over $\mathbb{F}_q$.

(ii) $2j|d$, $W_j$ is $W$ viewed as a $d/j$-dimensional vector space over $\mathbb{F}_q$, endowed with a non-degenerate symplectic form, and $L = \text{Sp}(W_j) \cong \text{Sp}_{d/j}(q^j)$. 
(iii) $2 | j f$, $2 \nmid d/j$, $W_j$ is $W$ viewed as a $d/j$-dimensional vector space over $\mathbb{F}_{q^j}$ endowed with a non-degenerate Hermitian form, and $L = SU(W_j) \cong SU_{d/j}(q^{j/2})$.

(iv) $2 | d$, $d/j \geq 4$, $W_j$ is $W$ viewed as a $d/j$-dimensional vector space over $\mathbb{F}_{q^j}$ endowed with a non-degenerate quadratic form of type $-$, and $L = \Omega(W_j) \cong \Omega_{d/j}(q^j)$.

(v) $(p, df, L/\mathbb{Z}(L)) = (3, 18, PSL_2(37)), (17, 6, PSL_2(13))$.

The main result of this section is the following theorem:

**Theorem 2.3.** Let $q = p^f$ be a power of an odd prime $p$ and let $n \geq 3$ be an odd integer. Let $W = \mathbb{F}_{q^2}^{2n}$ be a non-degenerate symplectic space, and $H := \text{Sp}(W) \cong \text{Sp}_{2n}(q)$, and let $\Phi$ be a complex Weil representation $\text{Weil}$ of $H$ of degree $q^n$ as in [KT2, §1]. Suppose that $G \leq H$ is a subgroup such that $\Phi|_G = \oplus_{j=0}^q$ is a sum of $q+1$ irreducible summands, $\Phi_0$ of degree $(q^n - q)/(q+1)$ and $\Phi_j$ of degree $(q^n + 1)/(q+1)$ for $1 \leq j \leq q$. Then $W$ can be viewed as an $n$-dimensional vector space over $\mathbb{F}_{q^2}$ endowed with a $G$-invariant non-degenerate Hermitian form such that

$$SU_n(q) \cong SU(W) \triangleleft G \leq GU(W) \cong GU_n(q).$$

**Proof.** (a) First we assume that $(n, q) \neq (3, 3)$ and $(3, 5)$; in particular, so that $p^{2nf} - 1$ admits a large primitive prime divisor $\ell$, in which case we choose such an $\ell$ to maximize the $\ell$-part of $p^{2nf} - 1$. Note the assumptions imply that $|G|$ is divisible by both $(q^n - q)/(q+1)$ and $(q^n + 1)/(q+1)$. In particular, $G < \text{GL}(W)$ has order divisible by

$$qQ := q(p^{2nf} - 1)/\ell.$$  

Let $L := \mathcal{O}^d(G)$ and $d(L)$ denote the smallest degree of nontrivial complex irreducible characters of $L$. Note that

$$d(L) \leq (q^n + 1)/(q+1) \leq (q^n + 1)/4.$$  

(Otherwise $L \leq \text{Ker}(\Phi_1)$, whence $\Phi_1$ could be viewed as an irreducible representation of $G/L$ and so would have been of $\ell$-degree.) Furthermore, if $L$ is cyclic of order $Q$, then by Ito’s theorem, the degree of any irreducible character of $G$ divides $|G/L|$, an integer coprime to $\ell$, and so again $G$ cannot be irreducible on $\Phi_1$. Now we can apply Theorem 2.2 to arrive at one of the following cases.

(i) $L \cong \text{SL}_{2n/j}(q^j)$ for some divisor $1 \leq j \leq n$ of $2n$. In this case, if $j \leq 2n/3$ then by [TZ1, Theorem 3.1] we have

$$d(L) > q^{j(2n/j) - 1} = q^{2n - 2n/j} > q^n,$$

contradicting (2.3.2). If $j = n$, then $q^j = q^n \geq 27$ and so

$$d(L) \geq (q^n - 1)/2 > (q^n + 1)/4,$$

again contradicting (2.3.2).
(ii) \( L \cong \text{Sp}_{2n/j}(q^j) \) for some divisor \( 1 \leq j < n/2 \) of \( n \). Then by [TZ1, Theorem 1.1] we have
\[ d(L) > \frac{q^n}{2} > \frac{q^n + 1}{4}, \]
contradicting (2.3.2).

(iii) There is some even divisor \( j = 2k \) of \( 2n \) with \( k|n \) and \( 2 \nmid n/k > 1 \), such that \( W \) can be viewed as a \( 2n/j \)-dimensional vector space over \( \mathbb{F}_{q^j} \) endowed with a non-degenerate Hermitian form and \( L = \text{SU}(W) \cong \text{SU}_{n/k}(q^k) \). Suppose first that \( k > 1 \), and let \( \psi \) be an irreducible constituent of the \( L \)-character afforded by \( \Phi_0 \), so that \( \psi(1) < \frac{q^n + 1}{4} \). By [TZ1, Theorem 4.1],
\[ \psi(1) \in \left\{ 1, \frac{q^n + 1}{q^k + 1}, \frac{q^n - q^k}{q^k + 1} \right\}. \]
The proof of (2.3.2) rules out the possibility \( \psi(1) = 1 \). Next,
\[ \psi(1) | \dim \Phi_0 = \frac{q^n - q^k}{q^k + 1} \]
by Clifford’s theorem, implying \( \psi(1) \neq (q^n - q^k)/(q^k + 1) \). The remaining possibility \( \psi(1) = (q^n + 1)/(q^k + 1) \) is also ruled out since \( k \nmid \dim \Phi_0 \). We have shown that \( k = 1 \), i.e. \( L = \text{SU}(W) \cong \text{SU}_n(q) \). This implies that
\[ L \triangleleft G \leq N_{\text{Sp}(W)}(L) = \text{GU}(W) \rtimes \langle \sigma \rangle \cong \text{GU}_n(q) \rtimes \mathbb{C}_2. \]
Here, \( \sigma \) is an involutive automorphism of \( \text{GU}(W) \) that acts as inversion on (2.3.3)
\[ \langle z \rangle = \mathbb{Z}(\text{GU}(W)) \cong C_{q+1}. \]
Recall the decomposition
\[ \Phi|_{\text{GU}(W)} = \oplus_{i=0}^{q-1} \Psi_i, \]
with \( \Psi_0 \) of degree \( (q^n - q)/(q + 1) \) and \( \Psi_i \) of degree \( (q^n + 1)/(q + 1) \) for \( 1 \leq i \leq q \), see the discussion preceding Lemma 2.1. In fact, one can find a primitive \( (q + 1) \)-th root of unity \( \xi \in \mathbb{C}^\times \) such that \( \Psi_i(z) \) is the multiplication by \( \xi^i \). In particular, \( \sigma \) fuses \( \Psi_1 \) and \( \Psi_q \). The assumption on \( \Phi|_G \) now implies that \( G \leq \text{GU}(W) \), as stated.

(iv) \( L \cong \Omega^-_{2n/j}(q^j) \) for some divisor \( 1 \leq j < n/2 \) of the odd integer \( n \). If \( j < n/5 \), then by [TZ1, Theorem 1.1] we have
\[ d(L) > q^n + 1, \]
contradicting (2.3.2). If \( j = n/3 \), then \( L \) is a quasisimple quotient of \( \text{PSU}_4(q^{n/3}) \) with \( q^{n/3} > 5 \), and so by [TZ1, Theorem 1.1] we have
\[ d(L) = \frac{q^{4n/3} - 1}{q^{n/3} + 1} > \frac{q^n}{2}, \]
again contradicting (2.3.2).

(v) \((p,n,f,L/\mathbb{Z}(L)) = (3,9,\text{PSL}_2(37))\). Note that the smallest dimension of a nontrivial irreducible representation of \( L \) over \( \overline{\mathbb{F}}_3 \) is 18 (see e.g. [TZ1, Table I]), so
(g, n) = (3, 9) and L = SL_2(37) acts absolutely irreducibly on W = \mathbb{F}_3^{18}. This in turn implies that

\[ C_{Sp(W)}(L) = Z(L) = C_2, \]

and so \( L \triangleleft G \leq N_{Sp(W)}(L) \leq L \cdot C_2 \). But in this case, \( G \) cannot have an irreducible complex representation of degree

\[ \dim \Phi_1 = (q^n + 1)/(q + 1) = (3^9 + 1)/4. \]

(vi) \( (p, n, f, L/Z(L)) = (17, 6, PSL_2(13)) \). In this case \( (q, n) = (17, 3) \) and \( L = SL_2(13) \) acts absolutely irreducibly on \( W = \mathbb{F}_6^17 \). As in (v), this implies that

\[ C_{Sp(W)}(L) = Z(L) = C_2, \]

and \( L \triangleleft G \leq N_{Sp(W)}(L) \leq L \cdot C_2 \), whence \( G \) cannot have an irreducible complex representation of degree

\[ \dim \Phi_1 = (q^n + 1)/(q + 1) = (17^3 + 1)/18. \]

(b) It remains to consider the two cases \( (n, q) = (3, 3) \) and \( (3, 5) \). Let \( M \) be a maximal subgroup of \( Sp(W) \) that contains \( G \). Then condition (2.3.1) also holds for \( |G| \); furthermore, the maximal degree of complex irreducible characters of \( M \) must be at least \( (q^n + 1)/(q + 1) = 7 \), respectively 21, since \( \Phi_1 \in \text{Irr}(G) \). First suppose that \( q = 5 \). Then, according to Tables 8.27 and 8.28 of [BHR], one of the following possibilities occurs.

- \( M = 2J_2 \). In this case, since \( |G| \) is divisible by \( 3 \cdot 5 \cdot 7 \), see (2.3.1), we see by inspecting maximal subgroups of \( J_2 \) [Atlas] that \( G = M \). But then \( G \) does not admit any complex irreducible representation of degree \( \dim \Phi_0 = 20 \).

- \( M = SL_2(125) \rtimes C_3 \). In this case, since \( |G \cap [M, M]| \) is divisible by \( 5 \cdot 7 \), see (2.3.1), we see by inspecting maximal subgroups of \( PSL_2(125) \) [BHR, Table 8.1] that \( G > SL_2(125) \). But then \( d(G) \geq 62 \) (see e.g. [TZ1, Table I]), violating (2.3.2).

- \( M = GU_3(5) \rtimes C_2 \). If \( G \geq N := SU_3(5) \), then we can argue as in (iii) above. Suppose \( G \not\geq N \). Since \( L := G \cap N \triangleleft G \) has order divisible by \( 3 \cdot 7 \), see (2.3.1), we see by inspecting maximal subgroups of \( PSL_3(5) \) and \( Alt_7 \) [Atlas] that \( L = 3Alt_7 \), and \( Z(L) = \langle z^2 \rangle \) with \( \langle z \rangle = Z(GU_3(5)) \) as defined in (2.3.3). Using the decomposition (2.3.4), we may assume that \( \Phi_i = (\Psi_i)|_G \) for \( 0 \leq i \leq q \). As mentioned in (iii), the subgroup \( C_2 \) fuses \( \Psi_1 \) with \( \Psi_5 \), hence \( \Phi_1 \) with \( \Phi_5 \). Thus \( G \leq GU_3(5) \), and so \( |G/L| \) and \( |NGU_3(5)(L)/L| \) both divide 6. Note that \( NGU_3(5)(L) \) contains the central involution of \( GU_3(5) \) which lies outside of \( SU_3(5) \). It follows that \( G \) induces a subgroup \( X \) of outer automorphisms of \( L \) of order dividing 3, whence \( X = 1 \) as \( |Out(Alt_7)| = 2 \) [Atlas]. Now let \( g \in L \) be of order 7. Then \( \Phi_0(g) = \Psi_0(g) \) has trace \(-1\). On the other hand, as \( G \) induces only inner automorphisms on \( L \), we see that \( (\Phi_0)|_L \) must be a direct sum of two copies of a single irreducible complex representation \( \Phi' \) (of dimension 10) of \( L \) and we arrive at the contradiction that \( \Phi'(g) \) has trace \(-1/2\).
Finally, we consider the case \( q = 3 \). Inspecting the list of maximal subgroups of \( \text{PSp}_6(3) \) in [Atlas], we arrive at the following possibilities for \( M \). By (2.3.1), \( G \) contains an element \( g \in G \) of order 7. According to [Atlas], we may assume that \( \Phi_0 \oplus \Phi_2 = \Lambda \mid_G \), where \( \Lambda \) is an irreducible Weil representation of degree 13 of \( \text{Sp}_6(3) \) and contains the central involution \( t \) of \( \text{Sp}_6(3) \) in its kernel, and that \( \Lambda(g) \) has trace \(-1\).

- \( M = \text{SL}_2(13) \). In this case, since \( |G| \) is divisible by \( 3 \cdot 7 \), see (2.3.1), we see by inspecting maximal subgroups of \( \text{PSL}_2(13) \) [Atlas] that \( G = M \). Note that \( t \) is the central involution of \( G \). Now the conditions that \( t \in \text{Ker}(\Lambda) \) and \( \Lambda(g) \) has trace \(-1\) imply by [Atlas] that \( \Lambda \mid_G \) is irreducible, a contradiction.

- \( M = \text{SL}_2(27) \cdot 3 \). In this case, since \( |G| \) is divisible by 7, we see by inspecting maximal subgroups of \( \text{PSL}_2(27) \) [Atlas] that either \( G \geq [M,M] = \text{SL}_2(27) \) or \( G \cap [M,M] \) is contained in a dihedral group \( D_{28} \). It is easy to see that in the former case \( d(G) \geq 13 \) contradicting (2.3.2), and in the latter case \( G \) does not admit any complex irreducible representation of dimension \( \dim \Phi_1 = 7 \).

- \( M = \text{GU}_3(3) \ltimes C_2 \). If \( G \geq N := \text{SU}_3(3) \), then we can argue as in (iii) above. Suppose \( G \not\geq N \). Since \( L := G \cap N < G \) has order divisible by \( 3 \cdot 7 \), see (2.3.1), we see by inspecting maximal subgroups of \( \text{SU}_3(3) \) and \( \text{PSL}_2(7) \) [Atlas] that either \( L \) is of order 21 or \( L = \text{PSL}_2(7) \). The former case is ruled out since \( (\Phi_1) \mid_L \) is irreducible of dimension 7. In the latter case, fix an involution \( s \in L \). We may assume that

\[
(\Phi_i) \mid_L = (\Psi_i) \mid_L
\]

for the representations \( \Psi_i \) defined in (2.3.4), and furthermore \( \Psi_2 \) is self-dual of dimension 7. Using [Atlas] we see that \( \Psi_1(s) \) has trace 3 and \( \Psi_1(g) \) has trace 0, whence \( (\Phi_1) \mid_L = (\Psi_1) \mid_L \) is the sum of two irreducible representations of dimensions 1 and 6, contradicting the irreducibility of \( \Phi_1 \) on \( G \triangleright L \). \( \square \)

In the next statement, we consider a non-degenerate symplectic space \( W = \mathbb{F}_p^{2N} \), a (reducible) big Weil representation of degree \( q^N \) of \( G = \text{Sp}(W) \cong \text{Sp}_{2N}(p) \) with character \( \omega \) as in [KT2]; in particular,

\[
(2.3.5) \quad |\omega(g)| = |C_W(g)|^{1/2}
\]

for any \( g \in G \). Let \( N = AB \) and \( B = bj \) for some positive integers \( A, B, b, j \). We may then assume that \( W \) is obtained from the symplectic space \( W_1 := \mathbb{F}_p^{2A} \) (with a Witt basis \( (e_1, \ldots, e_A, f_1, \ldots, f_A) \)) by base change from \( \mathbb{F}_{p^A} \) to \( \mathbb{F}_p \). Using this basis we can consider the transformation

\[
\sigma : \sum_{i=1}^A (x_ie_i + y_if_i) \mapsto \sum_{i=1}^A (x_i^s e_i + y_i^r f_i)
\]
induced by the Galois automorphism $x \mapsto x^r$ for $r := p^j$. Then, as in [KT2, §2] we can consider the standard subgroup

$$H = \text{Sp}(2A, p^B) \rtimes C_b$$

of $G$, where $C_b = \langle \sigma \rangle$.

**Theorem 2.4.** Each value $|\omega(x)|^2$, $x \in H$, is a power of $r = p^j$. Furthermore, there is some $h \in H$ such that $|\omega(h)|^2 = r$.

**Proof.** Note that $H$ embeds in $\text{Sp}(2A_b, p^j)$, and so the first statement follows by applying (2.3.5) to a big Weil representation of $\text{Sp}(2A_b, p^j)$. To define $h$, consider the $F_r$-linear map

$$f : F_{p^B} \to F_{p^B}, \quad x \mapsto x - x^r.$$

Viewed as a vector space over $F_r$, $\ker(f)$ has dimension 1. Hence $f$ cannot be surjective, and so we can find

$$\alpha \in F_{p^B} \setminus \text{Im}(f).$$

Let $J$ denote the Jordan block of size $A \times A$ with eigenvalue $\alpha^{-1}$, and let $g \in H$ have the following matrix

$$
\begin{pmatrix}
(\alpha J)^{-1} & \alpha^2 J \\
0 & \alpha J
\end{pmatrix}
$$

in the chosen basis $(e_1, \ldots, e_A, f_1, \ldots, f_A)$ of $W_1$. We will show that $h = g \sigma$ satisfies $|\omega(h)|^2 = r$. According to (2.3.5), it suffices to show that $h$ fixes exactly $r$ vectors in $W_1$. To this end, suppose that $w = \sum_{i=1}^A (x_i e_i + y_i f_i)$ is fixed by $h$, where $x_i, y_i \in F_{p^B}$. Comparing the coefficient for $f_A$ we have

$$y_A^r = y_A$$

implying $y_A \in F_r$. Next, comparing the coefficient for $f_{A-1}$ we see that

$$y_A^{r-1} + \alpha y_A^r = y_{A-1},$$

and so $\alpha y_A = f(y_{A-1})$. Continuing in the same fashion, we conclude that

$$y_1 \in F_r, \quad y_2 = y_3 = \ldots = y_A.$$

Thus we have shown that $v := \sum_{i=1}^A y_i f_i = y_1 f_1$. Letting $u := w - v = \sum_{i=1}^A x_i e_i$, we have

$$t(\alpha J)^{-1} \sigma(u) + \alpha^2 J \sigma(v) = u,$$

i.e.

$$\sigma(u) + t(\alpha J) \alpha^2 J \sigma(v) = t(\alpha J)(u).$$

Comparing the coefficient for $e_1$, we get

$$x_1^r + \alpha y_1 = x_1.$$
and so $\alpha y_1 = f(x_1)$. Again by the choice of $\alpha$, we must have that $y_1 = 0$ and $x_1 \in \mathbb{F}_r$. Next, comparing the coefficient for $e_2$, we get

$$x_2^\alpha = \alpha x_1 + x_2,$$

and so $-\alpha x_1 = f(x_2)$. By the choice of $\alpha$, we must have that $x_1 = 0$ and $x_2 \in \mathbb{F}_r$. Continuing in the same fashion, we conclude that

$$x_A \in \mathbb{F}_r, \quad x_1 = x_2 = \ldots = x_{A-1}.$$

Thus $w = x_A e_A$ with $x_A \in \mathbb{F}_r$. \hfill $\square$

**Lemma 2.5.** Let $q = p^f \geq 3$ be a prime power and let $A, B, b, c$ be positive integers, and let $H = \text{Sp}_{2A}(p^B) \rtimes C_b$ as above. Then the following statements hold.

(i) If $c \geq 3$, then $\text{SU}_{Ac}(q)$ cannot embed in $H$.

(ii) Assume in addition that $(p, A, B) \neq (3, 1, 1)$. Then the only quotient groups of $H$ are $H$, $H/\mathbb{Z}(H) = \text{PSp}_{2A}(p^B) \rtimes C_b$, and quotients of $C_b$.

**Proof.** (i) Assume the contrary. Since $c, q \geq 3$, $\text{SU}_{Ac}(q)$ is perfect, and so it embeds in $\text{Sp}_{2A}(p^B) < \text{Sp}_{2A}(\mathbb{F}_p)$. In particular, $\text{SU}_{Ac}(q)$ has a nontrivial absolutely irreducible representation in characteristic $p$ of dimension $\leq 2A \leq Ac - 1$. But this contradicts [KLI, Proposition 5.4.11].

(ii) The assumption on $(p, A, B)$ ensures that $L := [H, H] = \text{Sp}_{2A}(p^B)$ is quasisimple, with $S = L/\mathbb{Z}(H) \cong \text{PSp}_{2A}(p^B)$ being simple. Furthermore, $H/\mathbb{Z}(H)$ acts faithfully on $S$.

Suppose that $N < H$. If $N \geq L$, then $H/N$ is a quotient of $H/L \cong C_b$. In the remaining case, we have that $N \cap L$ is a proper normal subgroup of $L$, and so contained in $\mathbb{Z}(H)$. In particular, $[N, L] \leq N \cap L$ centralizes $L$, i.e. $[[N, L], L] = 1$. Since $L = [L, L]$, the Three Subgroups Lemma implies that $[N, L] = 1$, whence

$$N \leq C_H(L) \leq C_H(S) = \mathbb{Z}(H).$$

Thus either $N = 1$ or $N = \mathbb{Z}(H)$. \hfill $\square$

3. Local systems and Weil representations

In this section, we fix an odd prime $p$, and a prime $\ell \neq p$, so that we can avail ourselves of $\overline{\mathbb{Q}}_\ell$-adic cohomology. We also fix a nontrivial additive character $\psi$ of $\mathbb{F}_p$. We denote by $\chi_2$ the quadratic character of $\mathbb{F}_p^\times$, and we define

$$A := A_{\mathbb{F}_p} := - \sum_{x \in \mathbb{F}_p^\times} \psi(-2x)\chi_2(x).$$

For $k/\mathbb{F}_p$ a finite extension, we define

$$A_k := A^{\text{deg}(k/\mathbb{F}_p)}.$$
We denote by $\psi_k$ the additive character of $k$ given by
$$\psi_k := \psi \circ \text{Trace}_{k/F_p}.$$ 

In [KT2, Section 1], we introduced, for each integer $n \geq 2$ and each power $q = p^a$ of the odd prime $p$, the local system
$$\mathcal{W}(\psi, n, q)$$
on $A^2/F_p$ whose trace function at a point $(s, t) \in A^2(k)$, $k$ a finite extension of $F_p$, is
$$(-1/A_k) \sum_{x \in k} \psi_k(x^{q^{n+1}} + sx^{q^n+1} + tx^2).$$

We proved there [KT2, Theorem 1.1, 4.8] that when both $n$ and $a := \log_p(q)$ are prime to $p$, the geometric monodromy group $G_{\text{geom}}$ of $\mathcal{W}(\psi, n, q)$ was $\text{Sp}_{2n}(q)$ in one of its big Weil representations (of degree $q^n$), and that after extension of scalars from $A^2/F_p$ to $A^2/F_q$, its arithmetic monodromy group $G_{\text{arith}}$ coincided with $G_{\text{geom}}$.

Without these "prime to $p$" hypotheses, we have the following result.

**Theorem 3.1.** For $n \geq 2$ and $q = p^a$ a power of the odd prime $p$, we have the following results.

(i) There exists a factorization $na = AB$ and a factorization $B = bj$ such that the group $G_{\text{geom}}$ of $\mathcal{W}(\psi, n, q)$ is $\text{Sp}_{2A}(p^B) \times C_b$ in one of its big Weil representations.

(ii) Moreover, $p^j$ is a power of $q$, say $p^j = q^r$ (so that $j = ar, B = arb$), and hence we have inclusions of groups
$$\text{Sp}_{2A}(p^B) \times C_b \subseteq \text{Sp}_{2A}(q^{rb}) \times C_b \subseteq \text{Sp}_{2Ab}(q^r) \subseteq \text{Sp}_{2Atr}(q) = \text{Sp}_{2n}(q).$$

**Proof.** To prove (i), we argue as follows. From [KT2, Theorems 2.1, 2.2, and the argument of Proposition 4.6], we see that there exist factorizations $na = AB, B = bj$ and $na = CD, D = dk$ such that $G_{\text{geom}}$ is a subgroup of the product group
$$(\text{Sp}_{2A}(p^B) \times C_b) \times (\text{PSp}_{2C}(p^D) \times C_d)$$
which maps onto each factor.

We apply Goursat’s lemma. Note that $AB = na \geq 2$, so by Lemma 2.5(ii), the only quotient groups of $\text{Sp}_{2A}(p^B) \times C_b$ are
$$\text{Sp}_{2A}(p^B) \times C_b, \text{PSP}_{2A}(p^B) \times C_b,$$
and quotients of $C_b$.

Their commutator subgroups are
$$\text{Sp}_{2A}(p^B), \text{PSP}_{2A}(p^B), \{1\}$$
respectively. Similarly, the only quotient groups of $\text{PSP}_{2C}(p^D) \times C_d$ are
$$\text{PSP}_{2C}(p^D) \times C_d, \text{ and quotients of } C_d,$$
and their commutator subgroups are
$$\text{PSP}_{2C}(p^D), \{1\}.$$
respectively.

We first rule out the case when $G_{\text{geom}}$ is the graph of an isomorphism between a quotient of $C_b$ with a quotient of $C_d$. In this case, $G_{\text{geom}}$ would contain the product group $\text{Sp}_{2A}(p^B) \times \text{PSp}_{2C}(p^D)$. This group contains elements of trace zero in the representation at hand, whereas every element of $G_{\text{arith}}$, and a fortiori every element of $G_{\text{geom}}$ has nonzero trace, cf. [KT2, Proposition 4.6] and its proof.

The only remaining possibility is that $G_{\text{geom}}$ is the graph of an isomorphism between $\text{PSp}_{2A}(p^B) \rtimes C_b$ and $\text{PSp}_{2C}(p^D) \rtimes C_d$. Such an isomorphism induces an isomorphism of commutator subgroups. Hence $(A, B) = (C, D)$. Comparing cardinalities, we then infer that $b = d$. Thus $G_{\text{geom}}$ is as asserted.

To prove (ii), we use Theorem 2.4, according to which $p^j = p^{B/b}$ is the lowest value attained as the square absolute value of the trace of an element of $\text{Sp}_{2A}(p^B) \rtimes C_b$ in either big Weil representation. On the other hand, from [KT2, Theorem 3.5], the group $G_{\text{arith}}$ is also finite. The quotient $G_{\text{arith}}/G_{\text{geom}}$ is then a finite quotient of $\text{Gal}(\overline{F_p}/F_p)$. Hence over some $\overline{F}_q/\overline{F}_q$, we have $G_{\text{geom}} = G_{\text{arith}}$. From [KT2, Lemma 3.2], exploiting an idea of van der Geer and van der Flugt, we see that for any finite extension $k_0/\overline{F}_q$, all square absolute values of traces are powers of $q$, and that for any point $(s, t) \in \mathbb{A}^2(k_0)$, there is a finite extension $k_1/k_0$ for which the same point, now viewed in $\mathbb{A}^2(k_1)$ has trace of square absolute value $q^{2n}$. In particular, the least square absolute value attained is some strictly positive power $q^r, r \geq 1$ of $q$. □

We now introduce a new local system $\mathbb{W}(\psi, n, q)$ when $n \geq 3$ is odd, which we get by setting $t = 0$ in $\mathbb{W}(\psi, n, q)$. Thus the trace function of $\mathbb{W}(\psi, n, q)$ at a point $s \in \mathbb{A}^1(k), k/\overline{F}_q$ a finite extension, is

$$(-1/A_k) \sum_{x \in k} \psi_k(x^{q^n+1} + sx^{q^1}).$$

On $\mathbb{A}^1/\overline{F}_q$, we can break up this local system as the direct sum of $q + 1$ local systems, by making use of the $q + 1$ multiplicative characters, including the trivial one, of order dividing $q + 1$. We have

$$\mathbb{W}(\psi, n, q) = \bigoplus_{\chi \text{ with } \chi^{q+1} = 1} G(\psi, n, q, \chi).$$

The trace function of $G(\psi, n, q, \chi)$ at a point $s \in \mathbb{A}^1(k), k/\overline{F}_q$ a finite extension, is

$$(-1/A_k) \sum_{x \in k} \psi_k(x^{q^n+1} + sx)\chi_k(x).$$

Here we write $\chi_k$ for $\chi \circ \text{Norm}_{k/\overline{F}_q}$, and adopt the usual convention that for $\chi$ nontrivial, we have $\chi_k(0) = 0$, but $\mathbb{1}(0) = 1$.

These $G(\psi, n, q, \chi)$ are pairwise non-isomorphic, geometrically irreducible local systems on $\mathbb{A}^1/\overline{F}_q$ (thanks to their descriptions as Fourier Transforms, cf. [KT1, Section
The ranks of these local systems are
\[
\text{rank}(G(\psi, n, q, 1)) = \frac{q^n + 1}{q + 1} - 1,
\]
\[
\text{rank}(G(\psi, n, q, \chi)) = \frac{q^n + 1}{q + 1}, \chi \neq 1.
\]

Recall that for any \( n \), and \( q \) any power of the odd prime \( p \), there are inclusions
\[
\text{SU}_n(q) \subset \text{GU}_n(q) \hookrightarrow \text{Sp}_{2n}(q).
\]

**Theorem 3.2.** For \( n \geq 3 \) odd, and \( q = p^a \) a power of the odd prime \( p \), the group \( G_{\text{geom}} \) for \( \mathcal{W}(\psi, n, q) \) is \( \text{SU}_n(q) \) in its big Weil representation (of degree \( q^n \)).

**Proof.** Because \( \mathcal{W}(\psi, n, q) \) is the pullback (by \( (s, t) \mapsto (s, 0) \)) of the local system \( \mathcal{W}(\psi, n, q) \), its \( G_{\text{geom, } \mathcal{W}} \) is a subgroup of \( G_{\text{geom, } \mathcal{W}} \). By Theorem 3.1, we have
\[
G_{\text{geom, } \mathcal{W}} \hookrightarrow \text{Sp}_{2n}(q).
\]
Thus \( G_{\text{geom, } \mathcal{W}} \) is a subgroup of \( \text{Sp}_{2n}(q) \) under which a big Weil representation of \( \text{Sp}_{2n}(q) \) breaks up into \( q + 1 \) pieces, one of rank \( \frac{q^n - 2}{q + 1} \) and \( q \) of rank \( \frac{q^n + 1}{q + 1} \). By Theorem 2.3, we have inclusions
\[
\text{SU}_n(q) \leq G_{\text{geom, } \mathcal{W}} \leq \text{GU}_n(q).
\]
The group \( \text{GU}_n(q) \) has a quotient, via the determinant, of order \( q + 1 \), which is prime to \( p \). Because \( G_{\text{geom, } \mathcal{W}} \) is the monodromy group of a local system on \( \mathbb{A}^1/\overline{\mathbb{F}}_p \), it has no nontrivial prime to \( p \) quotients. Thus we have \( G_{\text{geom, } \mathcal{W}} = \text{SU}_n(q) \).

**Theorem 3.3.** For \( n \geq 3 \) odd and \( q \) an odd prime power, the geometric monodromy group \( G_{\text{geom, } \mathcal{W}} \) of \( \mathcal{W}(\psi, n, q) \) is \( \text{Sp}_{2n}(q) \) in one of its big Weil representations \( \mathcal{W}_{1,2} \) (of degree \( q^n \)). Moreover, after extension of scalars to \( \mathbb{A}^2/\overline{\mathbb{F}}_q \), we have \( G_{\text{geom}} = G_{\text{arith}} \).

**Proof.** Recall the inclusion
\[
\text{SU}_n(q) = G_{\text{geom, } \mathcal{W}} \leq G_{\text{geom, } \mathcal{W}} = \text{Sp}_{2n}(p^a) \rtimes C_b
\]
and the relation \( n = \text{Abr} \) of Theorem 3.1. By Lemma 2.5(i), \( br \leq 2 \), but \( 2 \nmid n \), hence \( ar = 1 \) and \( (A, p^B, b) = (n, q, 1) \), yielding the first assertion.

Once \( G_{\text{geom, } \mathcal{W}} = \text{Sp}_{2n}(q) = \text{Sp}_{2n}(p^a) \), \( G_{\text{arith, } \mathcal{W}} \) is contained in \( \text{Sp}_{2n}(p^a) \rtimes C_a \), cf. [KT2, proof of Lemma 4.7]. Thus the quotient \( G_{\text{arith, } \mathcal{W}} / G_{\text{geom, } \mathcal{W}} \) has order dividing \( a \), so after extension of scalars to \( \mathbb{A}^2/\overline{\mathbb{F}}_p \) to \( \mathbb{A}^2/\overline{\mathbb{F}}_{p^2} = \mathbb{A}^2/\overline{\mathbb{F}}_q \) we have \( G_{\text{geom}} = G_{\text{arith}} \).

**Theorem 3.4.** For \( n \geq 3 \) odd and \( q \) a power of the odd prime \( p \), the geometric monodromy group of the local system \( \mathcal{G}(\psi, n, q, 1) \) is \( \text{PSU}_n(q) \), the image of \( \text{SU}_n(q) \) in its unique irreducible representation of dimension \( \frac{q^n - q}{q + 1} \), with character \( \zeta_{0,n} \). The geometric monodromy group of \( \mathcal{G}(\psi, n, q, \chi_2) \) (where \( \chi_2 \) is the quadratic character) is the image of \( \text{SU}_n(q) \) in its unique orthogonal representation of dimension \( \frac{q^n + 1}{q + 1} \), with character \( \zeta_{(q+1)/2,n} \). For the remaining \( q - 1 \) local systems \( \mathcal{G}(\psi, n, q, \chi) \) with \( \chi^2 \)
nontrivial, \( \chi^{q+1} = 1 \), their geometric monodromy groups are the images of \( SU_n(q) \) in its \( q - 1 \) non-selfdual irreducible representations of dimension \( \frac{q^n - q}{q+1} \).

**Proof.** Because \( G_{\text{geom}, \mathcal{W}} = SU_n(q) \), the geometric monodromy groups in question are quotients of \( SU_n(q) \) in various of its irreducible representations. Recall the fact [TZ1, Theorem 4.1] that \( SU_n(q) \) has, up to equivalence, one irreducible representation of dimension \( \frac{q^n - q}{q+1} \) (with character \( \zeta_{0,n} \)) and \( q \) irreducible representations of dimension \( \frac{q^n + q}{q+1} \) (with character \( \zeta_{j,n} \), \( 1 \leq j \leq q \)), with exactly one of the \( q \) latter representations being self-dual (and necessarily orthogonal, as it has odd dimension). Using this fact and looking at the dimensions, we get the asserted matching. \( \square \)

**Corollary 3.5.** After extension of scalars to \( \mathbb{A}^1/\mathbb{F}_{q^2(q+1)} \), we have

\[
G_{\text{geom}, \mathcal{W}} = G_{\text{arith}, \mathcal{W}}
\]

for \( \mathcal{W}(\psi, n, q) \). The same is true for each of the \( q + 1 \) local systems \( G(\psi, n, q, \chi) \).

**Proof.** After extension of scalars to \( \mathbb{A}^1/\mathbb{F}_q \), we have \( G_{\text{arith}, \mathcal{W}} = Sp_{2n}(q) \), and hence

\[
G_{\text{arith}, \mathcal{W}} \leq Sp_{2n}(q).
\]

By Theorem 2.3, which we may apply after further extension of scalars to \( \mathbb{A}^1/\mathbb{F}_{q^2} \), we have

\[
SU_n(q) \leq G_{\text{arith}, \mathcal{W}} \leq GU_n(q).
\]

As we have \( G_{\text{geom}, \mathcal{W}} = SU_n(q) \), we see that the quotient \( G_{\text{arith}, \mathcal{W}}/G_{\text{geom}, \mathcal{W}} \) has order dividing \( q + 1 \). Thus after extension of scalars to \( \mathbb{A}^1/\mathbb{F}_{q^2(q+1)} \), we have \( G_{\text{geom}, \mathcal{W}} = G_{\text{arith}, \mathcal{W}} \). Each of the irreducible constituents then has \( G_{\text{geom}} = G_{\text{arith}} \) as well. \( \square \)

**Remark 3.6.** Theorem 3.3 is an improvement, in the \( n \) odd case, of Theorem 1.1 of [KT2, Theorem 1.1, 4.8], which required that both \( n \) and \( a := \log_p(q) \) be prime to \( p \). Theorem 3.4 verifies the \( G_{\text{geom}} \) conjectures of [KT1, Conjecture 9.2] in the case that \( q \) is odd. Corollary 3.5 establishes a weak version of the \( G_{\text{arith}} \) conjectures of [KT1, Conjecture 9.2], again in the case when \( q \) is odd. We should also point out that the normalizing factor \( A_k \) used here to define the local systems \( G(\psi, n, q, \chi) \) here can differ by a sign from the normalizing factors \( \beta \) used to define these local systems in [KT1, Lemma 8.3]. Over \( \mathbb{F}_{q^2} \), each normalizing factor is either \( q \) or \( -q \), so over extensions of \( \mathbb{F}_{q^4} \) there is no conflict. But we cannot hope to have the conjectural equality of \( G_{\text{geom}} \) with \( G_{\text{arith}} \) over \( \mathbb{F}_{q^2} \) for both \( G(\psi, n, q, \chi) \) as normalized here and for \( G(\psi, n, q, \chi) \) as normalized in [KT1, Lemma 8.3] in any situation where the normalizing factors do in fact differ by a sign.

The virtue of the normalizing factors \( \beta \) is that with them, when we work over \( \mathbb{F}_{q^2} \), the group \( G_{\text{arith}} \) for the renormalized \( G(\psi, n, q, \chi) \) lands in \( Sp_{\frac{q^n - q}{q+1}, \mathbb{Q}_l} \) for \( \chi = 1 \), it lands in \( SO_{\frac{q^n + q}{q+1}, \mathbb{Q}_l} \) for \( \chi = \chi_2 \) the quadratic character, and it lands in \( SL_{\frac{q^{n+1}}{q+1}, \mathbb{Q}_l} \) for the \( \chi \) with \( \chi^2 \neq 1 \). So with the exception of the \( \chi = 1 \) case, where a sign change of
normalizing factor won’t alter landing in $\text{Sp}(\frac{q^n-q}{q+1}, \overline{\mathbb{Q}})$, any sign change of normalizing factor in the other cases will destroy landing in SL (simply because $\frac{q^n+1}{q+1}$ is odd).

In the case of the quadratic character $\chi_2$, there is no sign change: the $\beta$ over $\mathbb{F}_{q^2}$ is equal to $A_{\mathbb{F}_{q^2}}$. Indeed, that $\beta$ is, cf. [KT1, Lemma 8.3 (3)]

$$\beta := (-1)^{(q+1)/2}q = (-1)^{(q-1)/2}q = (A_{\mathbb{F}_{q^2}}^2)_{\text{deg}(\mathbb{F}_{q^2}/\mathbb{F}_q)} = A_{\mathbb{F}_{q^2}}.$$  

[The normalizing factor $\beta$ for the renormalized $G(\psi, n, q, \chi)$ is $(-1)^{(q+1)/m}q$ for $m$ the order of $\chi$. This will be equal to $A_{\mathbb{F}_{q^2}}$ precisely when $(q + 1)/2$ and $(q + 1)/m$ have the same parity.]

For $G(\psi, n, q, 1)$, we have

$$G_{\text{geom}} = \Psi_0(\text{SU}_n(q)), \quad G_{\text{geom}} \leq G_{\text{arith}} \leq \Psi_0(\text{GU}_n(q)).$$

So we see from Lemma 2.1(i) that it suffices to extend scalars from $\mathbb{F}_{q^2}$ to $\mathbb{F}_{q^2 \text{gcd}(n, q+1)}$ (instead of to $\mathbb{F}_{q^2(q+1)}$) to achieve $G_{\text{geom}} = G_{\text{arith}}$ for $G(\psi, n, q, 1)$.

For $G(\psi, n, q, 2)$, we have

$$G_{\text{geom}} = \Psi_{\text{geom}}(\text{SU}_n(q)), \quad G_{\text{geom}} \leq G_{\text{arith}} \leq \Psi_{\text{geom}}(\text{GU}_n(q)) \cap \text{SO}(\text{deg}(\text{SU}_n(q)))_{\mathbb{Q}_\ell}.$$

So we see from Lemma 2.1(iii) that for $G(\psi, n, q, \chi_2)$, it suffices to extend scalars from $\mathbb{F}_{q^2}$ to $\mathbb{F}_{q^2}$ (instead of to $\mathbb{F}_{q^2(q+1)}$) to achieve $G_{\text{geom}} = G_{\text{arith}}$. Both these statements are far from the conjectured equality $G_{\text{geom}} = G_{\text{arith}}$ over $\mathbb{F}_{q^2}$ (except, of course, in the special case when $\text{gcd}(n, q + 1) = 1$).

4. Moments of Weil representations of odd-dimensional unitary groups

In this section, we will consider special unitary groups $G := \text{SU}_n(q) = \text{SU}(W)$ where $q$ is any prime power. The main result is Theorem 4.11 showing that when $n \geq 3$ is odd, the Weil representations of $G$ have $n$th moment 1 or 0.

First we assume that $n = 2k+1 \geq 5$ is odd, and fix a basis $(e_1, \ldots, e_k, f_1, \ldots, f_k, w)$ of the Hermitian space $W = \mathbb{F}_{q^2}$, in which the Hermitian form $\circ$ takes values

\[(4.0.1) \quad e_i \circ e_j = f_i \circ f_j = e_i \circ w = f_i \circ w = 0, \quad e_i \circ f_j = \delta_{i,j}, \quad w \circ w = 1.\]

We also fix the notation

$$P_1 := \text{Stab}_G(\langle e_1 \rangle) = Q_1 L_1, \quad P_k := \text{Stab}_G(\langle e_1, \ldots, e_k \rangle) = Q_k L_k,$$

where $Q_1 = \text{O}_p(P_1)$, $Q_k = \text{O}_p(P_k)$, $L_k \cong \text{GL}_k(q^2)$. The action of any $X \in L_k = \text{GL}_k(q^2)$ in the indicated basis of $W$ is given by $\text{diag}(X, X^{-1}, \det(X)^q, \det(X)^{-1})$, see [ST, §5.1].

As shown in [GMST, Lemmas 12.5, 12.6], the Levi subgroup $L$ has a unique orbit $\mathcal{O}$ on $\text{Irr}(\mathbb{Z}(Q_k)) \setminus \{1_{\mathbb{Z}(Q_k)}\}$ of smallest length $(q^{2k} - 1)/(q + 1)$, which then occurs in the restriction of any Weil character $\zeta_{i,n}$. Moreover, any $\lambda \in \mathcal{O}$ can only lie under an irreducible character of degree $q$ of $Q_k$. In particular, this shows that
Lemma 4.1. Suppose \( n = 2k + 1 \geq 5 \). Then \( \zeta_{i,n} \) is irreducible over \( P_k \). If \( 1 \leq i \leq q \), then \( \zeta_{i,n}|_{P_k} = \nu_i + \theta_i \), where \( \theta_i \in \text{Irr}(P_k) \) affords the orbit \( O \), and \( \nu_i \) is a linear character of \( P_k \) trivial at \( \mathbb{Z}(Q_k) \).

Lemma 4.2. In the notation of Lemma 4.1, assume that \( 1 \leq i \leq q \). Then \( \text{Ker}(\nu_i) \geq Q_k \), and if \( X \in L_k \) has determinant \( \sigma^t \) as an element in \( \text{GL}_k(q^2) \) with \( t \in \mathbb{Z} \), then \( \nu_i(X) = \sigma^{(q-1)t} \).

Proof. As noted in Lemma 4.1, \( \nu_i \) is trivial at \( \mathbb{Z}(Q_k) \), and it is \( P_k \)-invariant. But \( L_k \) acts transitively on the \( q^{2k} - 1 \) nontrivial linear characters of \( Q_k/\mathbb{Z}(Q_k) \), so \( \text{Ker}(\nu_i) \geq Q_k \). Next, \([L_k,L_k] \cong \text{SL}_k(q^2)\) is perfect, so \( \nu_i \) is trivial at \([L_k,L_k]\). Thus there is some \( 0 \leq s \leq q^2 - 2 \) such that \( \nu_i(X) = \sigma^{ts} \) for the listed \( X \in L_k \). To find \( s \), it suffices to evaluate \( \nu_i(X) \) for some \( X_0 \) that generates \( L_k \) modulo \([L_k,L_k]\). Let \( \gamma \) be a generator of \( \mathbb{F}_q^\times \) such that \( \gamma^{(q^{2k}-1)/(q^2-2)} = \sigma \), and choose \( X_0 \in L_k \) conjugate to

\[
\text{diag}(\gamma, \gamma^{q^2}, \ldots, \gamma^{q^{2k-2}})
\]

over \( \mathbb{F}_q \), so that \( \text{det}(X_0) = \sigma \). Since no eigenvalue of \( X_0 \) belongs to \( \mathbb{F}_q \), \( X_0 \) cannot fix any \( \lambda \in O \), see formula (20) of [ST]), and so \( \theta_i(X_0) = 0 \) and \( \nu_i(X_0) = \zeta_{i,n}(X_0) \). The absence of eigenvalues in \( \mathbb{F}_q \) and the equality \( \text{det}(X_0)^{q-1} = \rho \) imply by (2.0.2) that \( \zeta_{i,n}(X_0) = \rho^i = \sigma^{(q-1)i} \), i.e. \( s = (q-1)i \) as stated. \( \square \)

Proposition 4.3. Suppose \( n = 2k + 1 \geq 5 \). Then \((\zeta_n)^{n-1}\) contains \( \zeta_{i,n} \) with multiplicity one if \( i > 0 \), and zero if \( i = 0 \).

Proof. Note that \((\zeta_n)^2\) is just the permutation character of \( G \) acting on the point set of \( W \). Hence \((\zeta_n)^{n-1}\) is the permutation character of \( G \) acting on the set \( \Omega \) of ordered \( k \)-tuples \( \omega = (v_1, \ldots, v_k) \), \( v_i \in W \). Let \( \pi_\omega = \text{Ind}_{G_\omega}^G(1_{G_\omega}) \) denote the permutation character of \( G \) acting on the \( G \)-orbit of \( \omega = (v_1, \ldots, v_k) \), where \( G_\omega = \text{Stab}_G(\omega) \), and suppose that \( \zeta_{i,n} \) is an irreducible constituent of \( \pi_\omega \). Then

\[
(4.3.1) \quad 0 < [\pi_\omega, \zeta_{i,n}]_G = [1_{G_\omega}, \zeta_{i,n}|_{G_\omega}];
\]

in particular, \( 1_{G_\omega} \) is an irreducible constituent of \( \zeta_{i,n}|_{G_\omega} \).

(i) First we consider the case where \( X := \langle v_1, \ldots, v_k \rangle_{\mathbb{F}_q^2} \) is contained in a non-degenerate subspace \( Y \) of \( W \) of codimension \( \geq 2 \). Without loss we may assume that \( e_1, f_1 \in Y^\perp \). Then \( G_\omega \) contains a natural subgroup \( M := \text{SU}(\langle e_1, f_1 \rangle_{\mathbb{F}_q^2}) \cong \text{SU}_2(q) \) (that acts trivially on \( Y \)). The branching rule (2.0.3) then shows that \( \zeta_{i,n}|_M \) is a sum of Weil characters \( \zeta_{j,2} \) of \( M \). As mentioned above, an irreducible constituent \( \lambda \) of \( \zeta_{j,2} \) can have degree \( 1 \) only when \((q,j) = (2, \neq 0)\) or \((q,j) = (3, (q+1)/2)\). In the former case, one can check that \( \lambda \) is actually the sign character of \( M = \text{SU}_2(2) \cong \text{Sym}_3 \). In the latter case, \( \lambda(\zeta) \neq 1 \) for some element \( \zeta \) of \( M \cong \text{SU}_2(3) \) of order \( 3 \). Thus \( \lambda \) can never be equal to \( 1_M \), contradicting (4.3.1).

In particular, we have shown that \( X \) cannot be non-degenerate.
(ii) Suppose now that \(0 \neq X \cap X^\perp\) has dimension \(j \leq k - 1\). By Witt’s lemma, we may then assume that \(X = \langle e_1, \ldots, e_j, w_1, \ldots, w_{k-j}\rangle_{\mathbb{F}_{q^2}}\), where \(\langle w_1, \ldots, w_{k-j}\rangle_{\mathbb{F}_{q^2}}\) is a non-degenerate subspace of

\[
\langle e_{j+1}, \ldots, e_k, f_{j+1}, \ldots, f_k\rangle_{\mathbb{F}_{q^2}}.
\]

But then \(X\) is contained in the non-degenerate subspace

\[
Y := \langle e_1, \ldots, e_j, f_1, \ldots, f_j, w_1, \ldots, w_{k-j}\rangle_{\mathbb{F}_{q^2}}
\]

of codimension \(n - (k + j) \geq 2\), contradicting (i).

(iii) We have shown that \(\dim(X \cap X^\perp) = k\), i.e. \(X\) is totally singular of dimension \(k\). There is only one \(G\)-orbit of such \(\omega\), and we may assume that \(\omega = (e_1, \ldots, e_k)\). The description of \(P_k\) given in [ST, §5.1] shows that \(G_\omega = Q_k\). Now Lemmas 4.1, 4.2, and (4.3.1) show that \([\pi_\omega, \zeta_{i,n}]_G = 1 - \delta_{0,i}\), as stated. \(\square\)

Next we define the following linear characters \(\lambda_i\) of the parabolic subgroup \(P_1 := \text{Stab}_G(\langle e_1 \rangle_{\mathbb{F}_{q^2}})\) for \(1 \leq i \leq q\): if \(g \in P_1\) sends \(e_1\) to \(\sigma^t\) for \(0 \leq t \leq q^2 - 2\), then \(\lambda_i(g) = \sigma^{-(q-1)t}\), and set

\[\Lambda_i := \text{Ind}^{P_1}_{P_0}(\lambda_i)\]

**Proposition 4.4.** Suppose \(n = 2k + 1 \geq 5\), \((n, q) \neq (5, 2)\), and \(1 \leq i \leq q\). Then \(\Lambda_i\) enters the character \((\zeta_\alpha)^2\), and \([[(\zeta_{i,n})^2, \Lambda_i] \geq 1\).

**Proof.** (i) As discussed in [GMST, §11], \(P'_1 := \text{Stab}_G(e_1) = Q_1 \rtimes L'_1\), where \(L'_1 = \text{Stab}_G(e_1) \cap \text{Stab}_G(f_1) \cong \text{SU}_{n-2}(q)\). Note that \(\Lambda_i\) enters the character \(\text{Ind}^{P_1}_{P_0}(1_{P'_1})\), which in turn enters the character \((\zeta_\alpha)^2\). Furthermore, \(L_1\) acts transitively on the \(q - 1\) nontrivial linear characters of \(\text{Z}(Q_1)\) (which has order \(q\)), and for each such character \(\alpha\) there is a unique irreducible character of \(Q_1\) of degree \(q^{n-2}\), which then extends to a unique character \(M_\alpha\) of \(P'_1\). We fix some nontrivial \(\alpha \in \text{Irr}(\text{Z}(Q_1))\) and let \(K := \text{Stab}_{P_1}(\alpha) = P'_1 \cdot C_{q+1}\). By its uniqueness, \(M_\alpha\) extends to \(K\). Note that

\[\zeta_{i,n}(1) = (q^n + 1)/(q + 1) < 2q^{n-2}(q - 1) = 2(q - 1)M_\alpha(1)\]

It follows by Clifford’s theorem that

\[\zeta_{i,n}|_{P_1} = \beta_i + \text{Ind}^{P_1}_{K}(M_\alpha),\]

for some extension to \(K\) of \(M_\alpha\) which we also denote by \(M_\alpha\), and for some character \(\beta_i\) of \(P_1\) of degree \((q^{n-2} + 1)/(q + 1)\), with \(\text{Z}(Q_1) \leq \text{Ker}(\beta_i)\). Next, \(M_\alpha|_{L'_1} = \zeta_{n-2}\). Applying (2.0.3) to the standard subgroup \(L'_1\) and using (4.4.1), we get

\[\beta_i|_{L'_1} = \zeta_{i,n}|_{L'_1} - (q - 1)\zeta_{n-2} = \sum_{j \neq i, j' \neq j} \zeta_{n-2,j'} - (q - 1)\sum_{j'=0}^{q} \zeta_{n-2,j'} = \zeta_{n-2,i}.
\]

In particular, \(\beta_i \in \text{Irr}(P_1)\).
(ii) As usual, $\bar{\chi}$ denotes the complex conjugate of any character $\chi$. Note that $\text{Stab}_{P_1}(\bar{\alpha}) = K$. Hence, (4.4.1) implies that

\begin{equation}
\zeta_{i,n}|_{P_1} = \beta_i + \text{Ind}_{K}^{P_1}(M_{\alpha}).
\end{equation}

Observe that $M_{\alpha}$ affords the $\mathbb{Z}(q_1)$-character $q^{n-2}\bar{\alpha}$ and is irreducible over $P'_1$. By the aforementioned uniqueness, $M_{\alpha}$ agrees with $M_{\bar{\alpha}}$ on $P'_1$, where $M_{\bar{\alpha}}$ is the $K$-character of the $\bar{\alpha}$-isotypic component in $\zeta_{i,n}|_{P_1}$. As $K/P_1 \cong C_{q+1}$, these two characters differ from each other by a linear character of $K/P'_1$, which extends to a linear character $\delta$ of $P_1/P'_1 \cong C_{q^2-1}$. We have shown that

\begin{equation}
\text{Ind}_{K}^{P_1}(M_{\alpha}) = \text{Ind}_{K}^{P_1}(M_{\bar{\alpha}}) \cdot \delta.
\end{equation}

and

\begin{equation}
\zeta_{i,n}|_{P_1} = \beta_i + \text{Ind}_{K}^{P_1}(M_{\bar{\alpha}}),
\end{equation}

(iii) We aim to show that we one can take $\delta = \bar{\lambda}_i$ in (4.4.3). Let $\tau$ be an element of $\mathbb{F}_{q^{4k-2}}$ of order $q^{2k-1} + 1$ chosen such that $\tau^{(q^{2k-1}+1)/(q+1)} = \rho$. Then we can find an element $h \in K$ such that $h(e_1) = \rho e_1$ and $h$ is conjugate to

\[ \text{diag}(\rho, \rho, \tau^{-2}, \tau^{2q}, \tau^{-2q^2}, \ldots, \tau^{-2(q^{2k-2})}) \]

over $\mathbb{F}_{q^2}$. Since $k \geq 2$ and $(k, q) \neq (2, 2)$, by [Zs] there is a prime divisor $\ell$ of $q^{4k-2} - 1$ that does not divide $\prod_{j=1}^{4k-3}(q^j - 1)$. In particular, $\ell$ divides $(q^{2k-1} + 1)$, and moreover the $\ell$-part of $|P_1|$ is equal to the $\ell$-part of $\beta_i(1)$, whence $\beta_i$ is an irreducible character of $P_1$ of $\ell$-defect zero. On the other hand, for any $1 \leq t \leq q$, $\ell$ divides $|h^t|$, whence $\beta_i(t) = 0$, and so we obtain by using (2.0.2), (4.4.2), (4.4.4) that

\[ \text{Ind}_{K}^{P_1}(M_{\alpha})(h^t) = \zeta_{i,n}(h^t) = -(q-1)^{i} \rho^{it}, \]

\[ \text{Ind}_{K}^{P_1}(M_{\bar{\alpha}})(h^t) = \zeta_{i,n}(h^t) = -(q-1)^{i} \rho^{-it}. \]

It now follows from (4.4.3) that

\[ \delta(h^t) = \rho^{-2it} = \rho^{(q-1)it} = \bar{\lambda}_i(h^t), \]

whence $\delta(g) = \bar{\lambda}_i(g)$ for all $g \in K$, since the choice of $h$ ensures that $h$ generates $K$ modulo $P'_1$. Together with (4.4.3), we have shown that

\begin{equation}
(\text{Ind}_{K}^{P_1}(M_{\alpha}) \cdot \delta)(g) = (\text{Ind}_{K}^{P_1}(M_{\bar{\alpha}}) \cdot \bar{\lambda}_i)(g)
\end{equation}

for all $g \in K$. If $g \in P_1 \setminus K$ then $\text{Ind}_{K}^{P_1}(M_{\alpha})(g) = 0$ since $K \lhd P_1$, and so (4.4.5) holds for $g$ as well. Consequently,

\[ \text{Ind}_{K}^{P_1}(M_{\alpha}) = \text{Ind}_{K}^{P_1}(M_{\bar{\alpha}}). \]
This identity, together with (4.4.2) and (4.4.4), implies by Frobenius’ reciprocity that
\[
[(\zeta_{i,n})^2, \Lambda_i]_G = [\zeta_{i,n} \bar{\Lambda}_i, \bar{\zeta}_{i,n}]_G = [\zeta_{i,n} \cdot \text{Ind}_{P_1}^G(\bar{\Lambda}_i), \bar{\zeta}_{i,n}]_G
\]
\[
= [\text{Ind}_{P_1}^G(\zeta_{i,n} | P_1 \cdot \bar{\Lambda}_i), \bar{\zeta}_{i,n}]_G = [\zeta_{i,n} | P_1 \cdot \bar{\Lambda}_i, \bar{\zeta}_{i,n}]_{P_1}
\]
\[
\geq [\text{Ind}_{P_1}^G(M_\alpha) \cdot \bar{\Lambda}_i, \text{Ind}_{P_1}^G(\bar{\Lambda}_i)]_{P_1} = 1,
\]
as stated.

\[\square\]

**Proposition 4.5.** Suppose \( n = 2k + 1 \geq 5 \) and \( 0 < i \leq q \). Then \( [(\Lambda_i)^k, \bar{\zeta}_{i,n}] = 1 \).

**Proof.** Recall \( G \) acts transitively on the set \( \Xi \) of isotropic 1-spaces in \( W = \mathbb{F}^{n}_{q^2} \), with \( P_1 = \text{Stab}_G(\pi_1) \), where we set \( \pi_j := (e_j)_{\mathbb{F}^{n}_{q^2}} \) for \( 1 \leq j \leq k \). Hence the character \( \Lambda_i \) is afforded by a \( CG \)-module
\[
V = \text{Ind}_{P_1}^G(V_{\pi_1}) = \bigoplus_{gP_1 \in G/P_1} V_{g(\pi_1)},
\]
where \( V_{\pi_1} = \langle v_{\pi_1} \rangle_C \) is a one-dimensional \( P_1 \)-module with character \( \lambda_i \), and \( G \) permutes the summands via \( h(V_{g(\pi_1)}) = V_{h(g(\pi_1))} \). It follows that \( (\Lambda_i)^k \) is afforded by the \( G \)-module
\[
V \otimes^k = \langle v_\xi \ | \ \xi \in \Xi^k \rangle_C,
\]
where \( v_\xi = v_{\xi_1} \otimes v_{\xi_2} \otimes \ldots \otimes v_{\xi_k} \) for \( \xi = (\xi_1, \xi_2, \ldots, \xi_k) \).

Consider the \( G \)-orbit \( \Pi \) of the \( k \)-tuple \( \pi := (\pi_1, \pi_2, \ldots, \pi_k) \in \Xi^k \). Then the \( G \)-submodule
\[
V(\Pi) := \langle v_\xi \ | \ \xi \in \Pi \rangle_C
\]
of \( V \otimes^k \) affords the character \( \text{Ind}_{R}^G(\mu) \), where \( R := \cap_{j=1}^k \text{Stab}_G(\langle e_j \rangle_{\mathbb{F}^{n}_{q^2}}) \), and
\[
\mu(h) = \sigma^{-(q-1)i \sum_{j=1}^k t_j}
\]
if \( h(e_j) = \sigma^{t_j} \) for \( 0 \leq t_j \leq q^2 - 2 \) and \( 1 \leq j \leq k \).

Note that \( Q_k < R < P_k \) and \( Q_k \leq \text{Ker}(\mu) \). Furthermore, if \( h \in L_k \) belongs to \( R \) and \( h(e_j) = \sigma^{t_j} \), then \( \det(h) \) (as an element in \( \text{GL}_k(q^2) \)) is \( \sigma^{\sum_{j=1}^k t_j} \), and so
\[
\overline{\sigma}_i(h) = \sigma^{-(q-1)i \sum_{j=1}^k t_j} = \mu(h)
\]
for the character \( \nu_i \) considered in Lemma 4.2, i.e. \( \overline{\sigma}_i |_R = \mu \). By Lemma 4.1, we have therefore shown that
\[
0 < [\mu, \bar{\zeta}_{i,n}]_R = [\text{Ind}_{R}^G(\mu), \bar{\zeta}_{i,n}]_G \leq [(\Lambda_i)^k, \bar{\zeta}_{i,n}]_G.
\]

On the other hand, \( (\Lambda_i)^k \) enters the character \( (\zeta_n)^n \) by Proposition 4.4, whence the upper bound \( [(\Lambda_i)^k, \bar{\zeta}_{i,n}] \leq 1 \) follows from Proposition 4.3.

\[\square\]

Next we will study some see-saw dual pairs (cf. [Ku]) to determine various branching rules. Our consideration is based on the following well-known formula [LBST, Lemma 5.5]:
Lemma 4.6. Let $\omega$ be a character of the direct product $S \times G$ of finite groups $S$ and $G$. Then

$$\omega = \sum_{\alpha \in \text{Irr}(S)} D_\alpha \otimes \alpha,$$

where

$$D_\alpha : g \mapsto \frac{1}{|S|} \sum_{x \in S} \bar{\alpha(x)} \omega(xg)$$

is either zero, or a character of $G$.

We will work with a finite group $\Gamma$ that contains two dual pairs $S_1 \times G_1$ and $S_2 \times G_2$, where $G_1 \geq G_2$ and $S_2 \geq S_1$.

Lemma 4.7. Let $\omega$ be a character of $\Gamma$, and decompose

$$\omega|_{G_1 \times S_1} = \sum_{\alpha \in \text{Irr}(S_1)} D_\alpha \otimes \alpha, \quad \omega|_{G_2 \times S_2} = \sum_{\gamma \in \text{Irr}(G_2)} \gamma \otimes E_\gamma$$

as in Lemma 4.6. Then, for any $\alpha \in \text{Irr}(S_1)$ and any $\gamma \in \text{Irr}(G_2)$ we have that

$$[D_\alpha|_{G_2}, \gamma]|_{G_2} = [\alpha, E_\gamma]|_{S_1}$$

and hence

$$D_\alpha|_{G_2} = \sum_{\gamma \in \text{Irr}(G_2)} [E_\gamma|_{S_1}, \alpha]|_{S_1} \cdot \gamma.$$

Proof. Write $a_{\alpha, \gamma} := [D_\alpha|_{G_2}, \gamma]|_{G_2}$, so that

$$D_\alpha|_{G_2} = \sum_{\gamma \in \text{Irr}(G_2)} a_{\alpha, \gamma} \gamma.$$

Then

$$\omega|_{G_2 \times S_1} = \sum_{\alpha \in \text{Irr}(S_1), \gamma \in \text{Irr}(G_2)} a_{\alpha, \gamma} \gamma \otimes \alpha$$

$$= \sum_{\gamma \in \text{Irr}(G_2)} \gamma \otimes \sum_{\alpha \in \text{Irr}(S_1)} a_{\alpha, \gamma} \alpha.$$

Thus $E_\gamma|_{S_1} = \sum_{\alpha \in \text{Irr}(S_1)} a_{\alpha, \gamma} \alpha$, and the statements follow.

First we consider the dual pair

$$(4.7.1) \quad G_2 \times S_2$$

inside $\Gamma := \text{GU}_{2n}(q)$, where $S_2 = \text{GU}_2(q)$ and $G_2 = \text{SU}_n(q)$, and $\omega = \zeta_{2n} = \zeta_{2n,q}$. More precisely, we view $S_2$ as $\text{GU}(U)$, where $U = \langle v_1, v_2 \rangle q_2$ is endowed with the Hermitian form $\circ$, with an orthonormal basis $(v_1, v_2)$. Next, $G_2 = \text{SU}_n(q)$ is $\text{SU}(W)$,
where \( W = \mathbb{F}_{q^2}^n \) is endowed with the Hermitian form \( \circ \) defined in (4.0.1). Now we consider \( V = U \otimes_{\mathbb{F}_q^2} W \) with the Hermitian form \( \circ \) defined via

\[
(u \otimes w) \circ (u' \otimes w') = (u \circ u')(w \circ w')
\]

for \( u \in U \) and \( w \in W \). The action of \( G_2 \times S_2 \) on \( V \) induces a homomorphism \( G_2 \times S_2 \rightarrow \Gamma := \text{GU}(V) \).

Now \( V \) is the orthogonal sum \( V_1 \oplus V_2 \), where \( V_i := v_i \otimes W \). This gives us a subgroup \( G_1 := \text{SU}(V_1) \times \text{SU}(V_2) \cong \text{SU}_n(q) \times \text{SU}_n(q) \) of \( \Gamma \) that contains (the image of) \( G_2 \). In fact, \( G_2 \) embeds diagonally in \( G_1 \) via \( g \mapsto \text{diag}(g, g) \).

In the above basis \( (v_1, v_2) \) of \( U \) and for \( 0 \leq i, j \leq q \), we consider the character

\[
\lambda_{i,j} : \text{diag}(\rho^a, \rho^b) \mapsto \rho^{ia+jb}
\]

of \( S_1 \). Then, as explained in [TZ2, §4], \( \zeta_{i,n} \) corresponds to the \( \rho^i \)-eigenspace of the generator \( \rho \cdot 1_W \) of \( \mathbf{Z}(\text{GU}_n(q)) \), so that

\[
D_{\lambda_{i,j}} = \zeta_{i,n} \otimes \zeta_{j,n}
\]

for the dual pair \( G_1 \times S_1 \).

We use the notation of [E] for the irreducible characters of \( S_2 = \text{GU}_2(q) \) (with the parameter \( q + 1 \) in the superscripts of characters changed to 0). For instance

\[
\chi^{(t)}_{1}|_{S_1} = \lambda_{t,t}.
\]

The decomposition

\[
\omega|_{S_2 \times G_2} = \sum_{\alpha \in \text{Irr}(S_2)} \alpha \otimes C_{\alpha}
\]

was described in [LBST, Proposition 6.3]. In particular, the \( G_2 \)-characters

\[
C_{\alpha}^* := C_{\alpha} - k_{\alpha} \cdot 1_{G_2},
\]

where \( \alpha \in \text{Irr}(S_2) \), are irreducible and pairwise distinct, and \( k_{\alpha} \in \{0, 1\} \) is listed in Table I.

This implies

**Corollary 4.8.** For the decomposition

\[
\omega|_{G_2 \times S_2} = \sum_{\gamma \in \text{Irr}(G_2)} \gamma \otimes E_{\gamma},
\]
Among these two irreducible constituents, only \( (4.7.3) \) in the notation of Proposition 4.9.

We have that

\[
\begin{align*}
\chi_1^{(0)} & \quad 1 \quad (q^n - (-1)^n)(q^{n-1} + (-1)^n q^2)/(q + 1)(q^2 - 1) = 1 \\
\chi_1^{(t)}, t \neq 0 & \quad 1 \quad (q^n - (-1)^n)(q^{n-1} + (-1)^n)/(q + 1)(q^2 - 1) = 0 \\
\chi_q^{(0)} & \quad q \quad (q^n + (-1)^n q)(q^n - (-1)^n q^2)/(q + 1)(q^2 - 1) = 1 \\
\chi_q^{(t)}, t \neq 0 & \quad q \quad (q^n - (-1)^n)(q^n + (-1)^n q)/(q + 1)(q^2 - 1) = 0 \\
\chi_{q-1}^{(0,u)}, u \neq 0 & \quad q - 1 \quad (q^n - (-1)^n)(q^{n-1} - (-1)^n q)/(q + 1)^2 = 0 \\
\chi_{q-1}^{(2,u)}, t, u \neq 0 & \quad q - 1 \quad (q^n - (-1)^n)(q^{n-1} + (-1)^n)/(q + 1)^2 = 0 \\
\chi_{q+1}^{(t)} & \quad q + 1 \quad (q^n - (-1)^n)(q^{n-1} + (-1)^n)/q^2 - 1) = 0
\end{align*}
\]

we have that

\[
E_{\gamma} = \begin{cases} 
\alpha, & \gamma = C_\alpha^0 \text{ for some } \alpha \in \text{Irr}(S_2), \\
\chi_1^{(0)} + \chi_q^{(0)}, & \gamma = 1_{G_2}, \\
0, & \text{otherwise.}
\end{cases}
\]

**Proposition 4.9.** Suppose \( n = 2k + 1 \geq 5 \) and \( (n, q) \neq (5, 2) \). For \( 0 < i \leq q \), and in the notation of (4.7.3)–(4.7.4) we have

\[
\Lambda_i = C_{\chi_{1}^{(i)}} + C_{\chi_{q}^{(i)}}.
\]

Among these two irreducible constituents, only \( C_{\chi_{1}^{(i)}} \) enters \( (\zeta_{i,n})^2 \).

**Proof.** (i) First, an application of Mackey’s formula reveals that \( \Lambda_i \) is the sum of two distinct irreducible characters of \( G_2 = SU_n(q) \). Clearly, \( [\Lambda_i, 1_{G_2}] = 0 \). By Proposition 4.5, \( \Lambda_i \) enters \( (\zeta_n)^2 = \omega_{|G_2} \), so

\[
\Lambda_i = C_{\beta_1}^0 + C_{\beta_2}^0
\]

for some \( \beta_1 \neq \beta_2 \in \text{Irr}(S_2) \). Next,

\[
\Lambda_i(1) = (q^n - (-1)^n)(q^{n-1} + (-1)^n)/(q^2 - 1),
\]

so \( \beta_1, \beta_2 \neq \chi_{q+1}^{(t)} \), see Table I.

By Proposition 4.4, at least one of \( \gamma_j := C_{\beta_j}^0, j = 1, 2 \), is an irreducible constituent of

\[
(\zeta_{i,n})^2 = D_{\lambda_{i,i}}|G_2,
\]

see (4.7.2). As \( \gamma_j \neq 1_{G_2} \), by Lemma 4.6 and Corollary 4.8 we have

\[
[D_{\lambda_{i,i}}|G_2; \gamma_j]_{G_2} = [\lambda_{i,i}, E_{\gamma_j}|S_1]_{S_1} = [\lambda_{i,i}, \beta_j|S_1]_{S_1}.
\]

We have shown that \( C_{\beta_j}^0 \) is an irreducible constituent of \( (\zeta_{i,n})^2 \) precisely when \( \lambda_{i,i} \) is an irreducible constituent of \( \beta_j|S_1 \).
(ii) As in the proof of Proposition 4.4, let \( \tau \) be an element of \( \mathbb{F}_q^\times \) of order \( q^{2k-1} + 1 \) chosen such that \( \tau^{(q^{2k-1}+1)/(q+1)} = \rho \). Then we fix an element \( g \in L_1 \) such that \( g(e_1) = \sigma e_1, g(f_1) = \sigma^{-q} f_1 \), and \( g \) is conjugate to
\[
\text{diag}(\sigma, \sigma^{-q}, \tau, \tau^{-q}, \tau^q, \ldots, \tau^{-(q-2k-2)})
\]
on \( \mathbb{F}_q^2 \). By [Zs] there is a prime divisor \( \ell \) of \( q^{4k-2} - 1 \) that does not divide \( \prod_{j=1}^{4k-3} (q^j - 1) \). In particular, \( \ell \) divides \( |\tau| \). It follows that \( \sigma \) and \( \sigma^{-q} \) are the only eigenvalues of \( g \) that belong to \( \mathbb{F}_q^2 \).

Assume in addition that \( q > 2 \); in particular, \( \sigma \neq \sigma^{-q} \). Then, \( (e_1)_{\mathbb{F}_q^2} \) and \( (f_1)_{\mathbb{F}_q^2} \) are the only two \( g \)-invariant isotropic 1-spaces in \( W \), and so
\[
(4.9.1) \quad \Lambda_i(g) = 2\rho^{-i}.
\]

Next, for any \( x \in S_2 = \text{GU}_2(q) \), \( \omega(gx) = 1 \), unless \( x \) has, at least one, and therefore both, of \( \sigma^{-1} \) and \( \sigma^q \) as its eigenvalues. In this exceptional case, \( x \) belongs to class \( C_4^{(-1)} \) in the notation of [E], and \( \omega(gx) = q^2 \). It follows from Lemma 4.6 that
\[
C_\alpha^\sigma(g) = \begin{cases} 
\rho^{-t}, & \alpha = \chi_1^{(t)}, \ 0 < t \leq q, \\
2, & \alpha = \chi_1^{(0)}, \\
\rho^{-t}, & \alpha = \chi_q^{(t)}, \ 0 < t \leq q, \\
0, & \alpha = \chi_q^{(0)}, \\
0, & \alpha = \chi_{q^{-1}}^{(t,u)}, \ 0 \leq t, u \leq q.
\end{cases}
\]

Together with (4.9.1), this readily implies that \( \{\beta_1, \beta_2\} = \{\chi_1^{(i)}, \chi_q^{(i)}\} \). Note that \( \chi_1^{(i)}|_{S_1} = \lambda_{i,i} \), but \( \chi_q^{(i)}|_{S_1} \) does not contain \( \lambda_{i,i} \), so we are done.

(iii) Now we consider the case \( q = 2 \). As shown in (i), we may assume that \( \beta_1|_{S_1} \) contains \( \lambda_{i,i} \). It follows that \( \beta_1 \in \{\chi_1^{(i)}, \chi_{q^{-1}}^{(i)}\} \). However degree consideration using Table I rules out \( \chi_{q^{-1}}^{(i)} \) and shows that \( \beta_1 = \chi_1^{(i)} \). Again by degree consideration we now see that \( \beta_2 = \chi_q^{(i)} \) for some \( t \in \{1, 2\} \). Furthermore, \( g \) fixes exactly three isotropic 1-spaces in \( W \) (namely, the ones spanned by \( e_1, f_1 \), and \( e_1 + f_1 \)), so \( \Lambda_i(g) = 3\rho^{-i} \).

Arguing as in (ii), we see that
\[
C_\alpha^\sigma(g) = \begin{cases} 
\rho^{-t}, & \alpha = \chi_1^{(t)}, \ 0 < t \leq q, \\
2, & \alpha = \chi_1^{(0)}, \\
2\rho^{-t}, & \alpha = \chi_q^{(t)}, \ 0 < t \leq q, \\
0, & \alpha = \chi_q^{(0)}.
\end{cases}
\]

Hence \( \beta_2 = \chi_q^{(i)} \), and we are done since \( \chi_q^{(i)}|_{S_1} \) does not contain \( \lambda_{i,i} \).

We will now work with three new dual pairs. First, we consider the dual pair \( G_3 \times S_3 \) inside \( \Gamma := \text{GU}_{2kn}(q) \), where \( S_3 = \text{GU}_{2k}(q) \) and \( G_3 = \text{SU}_{3n}(q) \), and \( \omega = \zeta_{2nk} = \zeta_{2nk,q} \). More precisely, we view \( S_3 \) as \( \text{GU}(U) \), where \( U = \langle v_1, \ldots, v_{2k} \rangle_{\mathbb{F}_q^2} \) is endowed with
the Hermitian form \( \circ \), with an orthonormal basis \((v_1, \ldots, v_{2k})\). Next, \( G_3 = \text{SU}_n(q) \) is \( \text{SU}(W) \), where \( W = \mathbb{F}^n_{q^2} \) is endowed with the Hermitian form \( \circ \) defined in (4.0.1).

Now we consider \( V = U \otimes_{\mathbb{F}_{q^2}} W \) with the Hermitian form \( \cdot \) defined via

\[
(u \otimes w) \circ (u' \otimes w') = (u \circ u')(w \circ w')
\]

for \( u \in U \) and \( w \in W \). The action of \( G_3 \times S_3 \) on \( V \) induces a homomorphism \( G_3 \times S_3 \to \Gamma := \text{GU}(V) \).

Now \( V \) is the orthogonal sum \( \bigoplus_{i=1}^{2k} V_i \), where \( V_i := v_i \otimes W \). This gives us a subgroup \( G_1 := \text{SU}(V_1) \times \text{SU}(V_2) \times \ldots \times \text{SU}(V_{2k}) \cong \text{SU}_n(q)^{2k} \) of \( \Gamma \) that contains (the image of) \( G_3 \). In fact, \( G_3 \) embeds diagonally in \( G_1 \): \( g \mapsto \text{diag}(g, g, \ldots, g) \). Next,

\[
S_1 := \text{GU}((v_1)_{\mathbb{F}_{q^2}}) \times \text{GU}((v_2)_{\mathbb{F}_{q^2}}) \times \ldots \times \text{GU}((v_{2k})_{\mathbb{F}_{q^2}}) \cong \text{GU}_1(q)^{2k}
\]

is just the non-split diagonal torus of \( S_3 \). In the above basis \((v_1, v_2, \ldots, v_{2k})\) of \( U \) and for \( 1 \leq i \leq q \), we consider the character

\[
\mu_i : \text{diag}(\rho^{a_1}, \rho^{a_2}, \ldots, \rho^{a_{2k}}) \mapsto \rho^{i S_{j=1}^{2k} a_j}
\]

of \( S_1 \).

Next, for each \( 1 \leq j \leq k \) we embed one copy of \( \text{SU}(W) \) in

\[
\text{SU}((v_{2j-1}, v_{2j})_{\mathbb{F}_{q^2}} \otimes W)
\]

(by letting it act only on \( W \)). This gives an embedding of \( G_2 := \text{SU}_n(q)^k \) in \( G_1 \) via

\[
\text{diag}(g_1, g_2, \ldots, g_k) \mapsto \text{diag}(g_1, g_1, g_2, g_2, \ldots, g_k, g_k).
\]

At the same times, \( G_3 \) embeds diagonally in \( G_2 \) via \( g \mapsto \text{diag}(g, g, \ldots, g) \). The action of \( G_2 \) is centralized by

\[
S_2 := \text{GU}((v_1, v_2)_{\mathbb{F}_{q^2}}) \times \text{GU}((v_3, v_4)_{\mathbb{F}_{q^2}}) \times \ldots \times \text{GU}((v_{2k-1}, v_{2k})_{\mathbb{F}_{q^2}}) \cong \text{GU}_2(q)^k.
\]

Recall the characters \( C_\alpha \) of \( \text{SU}_n(q) \) introduced in (4.7.3).

**Proposition 4.10.** Suppose \( n = 2k + 1 \geq 5 \), \((n, q) \neq (5, 2)\), and \( 0 < i \leq q \). Then both \((C_{\chi^{(i)}})^k\) and \((\zeta_{i,n})^{n-1}\) contain \( \bar{\zeta}_{i,n} \).

**Proof.** (i) First we decompose

\[
\omega|_{G_3 \times S_3} = \sum_{\gamma \in \text{irr}(G_3)} \gamma \otimes E_\gamma
\]

for the dual pair \( G_3 \times S_3 \). By Proposition 4.3, \( \omega|_{G_3} = (\zeta_n)^{n-1} \) contains \( \bar{\zeta}_{i,n} \) with multiplicity one. It follows that the \( G_3 \)-character \( E_{\bar{\zeta}_{i,n}} \) has degree 1, so there is some \( 0 \leq m = m_i \leq q \) such that

\[
E_{\bar{\zeta}_{i,n}}(X) = \rho^{mt}
\]
whenever $X \in \text{GU}_{2k}(q)$ has determinant equal to $\rho^j$.

(ii) Next we decompose

$$\omega|_{S_2 \times G_2} = \sum_{\beta \in \text{Irr}(S_2)} \beta \otimes F_\beta$$

for the dual pair $S_2 \times G_2$. Note by (4.7.3) that if

$$\beta = \beta_1 \otimes \beta_2 \otimes \ldots \otimes \beta_k,$$

then

(4.10.1) $$F_\beta = C_{\beta_1} \otimes C_{\beta_2} \otimes \ldots \otimes C_{\beta_k}.$$ 

By Lemma 4.7,

$$[F_\beta|_{G_3}, \zeta_{i,n}]_{G_3} = [\beta, E_{\zeta_{i,n}}|_{S_2}]_{S_2}.$$ 

Since $E_{\zeta_{i,n}}$ has degree 1, we see that $\zeta_{i,n}$ is an irreducible constituent of $F_\beta|_{G_3}$ precisely when $\beta = E_{\zeta_{i,n}}|_{S_2}$, that is when

$$\beta(X_1, X_2, \ldots, X_k) = \rho^m \sum_{1 \leq j \leq k} t_j$$

whenever $X_j \in \text{GU}_{2}(q)$ has determinant equal to $\rho^j$ for $1 \leq j \leq k$. In the notation of [E] we then have

(4.10.2) $$\beta = \chi_1^{(m)}(\chi_1^{(m)} \otimes \ldots \otimes \chi_1^{(m)})^k.$$ 

(iii) Recall by Proposition 4.4 that $\Lambda_i$ enters $(\zeta_n)^2$. It follows that $\Lambda_i^\otimes k = \Lambda_i \otimes \Lambda_i \otimes \ldots \otimes \Lambda_i$ enters $\omega|_{G_2}$. Next, by Proposition 4.5, $\zeta_{i,n}$ is an irreducible constituent of $(\Lambda_i)^k = \Lambda_i^\otimes k|_{G_3}$. Furthermore, by Proposition 4.9, $\Lambda_i = C_{\chi_1^{(i)} \chi_1^{(i)}} + C_{\chi_1^{(i)}}$. Hence, using (4.10.1) we see that

$$\Lambda_i^\otimes k = \sum_{1 \leq j \leq k, \beta_j \in \{\chi_1^{(i)} \chi_1^{(i)}\}} C_{\beta_1} \otimes C_{\beta_2} \otimes \ldots \otimes C_{\beta_k}$$

$$= \sum_{1 \leq j \leq k, \beta_j \in \{\chi_1^{(i)} \chi_1^{(i)}\}} F_{\beta_1 \otimes \beta_2 \otimes \ldots \otimes \beta_k}.$$ 

Applying the result (4.10.2) of (ii), we conclude that $m = i$ and $\zeta_{i,n}$ is an irreducible constituent of

$$F_{\chi_1^{(m)} \otimes \chi_1^{(m)} \otimes \ldots \otimes \chi_1^{(m)}}|_{G_3} = (C_{\chi_1^{(i)}})^k.$$ 

(iv) The same argument as in (ii), but applied to the decomposition

$$\omega|_{S_1 \times G_1} = \sum_{\alpha \in \text{Irr}(S_1)} \alpha \otimes D_\alpha.$$
for the dual pair $S_1 \times G_1$ implies that $\zeta_{i,n}$ is an irreducible constituent of $D_{\alpha}|_{G_3}$ precisely when $\alpha = E_{\zeta_{i,n}}|_{S_1}$, that is when $\alpha = \mu_m$ as introduced in (4.9.2). As $m$ was shown to be equal to $i$ in (iii), we now have that $\zeta_{i,n}$ is an irreducible constituent of $D_{\alpha}|_{G_3} = D_{\mu_i}|_{G_3} = (\zeta_{i,n})^{n-1}$.

We can now prove the main result of this section:

**Theorem 4.11.** Let $q$ be a prime power and let $G = SU_n(q)$ with $n = 2k + 1 \geq 3$. Suppose in addition that $(n, q) \neq (3, 2)$. Then $(\zeta_{i,n})^n$ contains $1_G$ with multiplicity exactly one if $1 \leq i \leq q$ and zero if $i = 0$.

**Proof.** For $n = 3$, the statement was checked by A. Schaeffer Fry using the package Chevie. Likewise, the case $(n, q) = (5, 2)$ was checked using the package GAP. So we may assume that $n \geq 5$ and $(n, q) \neq (5, 2)$. Now for $i = 0$ the statement follows from Proposition 4.3. For $1 \leq i \leq q$ we have

$$[(\zeta_{i,n})^{n-1}, \zeta_{i,n}]_G = [(\zeta_{i,n})^n, 1_G]$$

is at most 1 by Proposition 4.3 and at least 1 by Proposition 4.10. □

Theorem 4.11 means that the Weil representation of $SU_n(q)$ affording the character $\zeta_{i,n}$ with $1 \leq i \leq q$ has a unique (up to scalar) polynomial invariant of degree $n$. It would be interesting to know what is the geometric significance of this polynomial invariant, and to find an explicit construction of it.

5. **Moments of Weil representations of $SU_4(q)$**

Theorem 4.11 naturally brings up the question: what are the $n$-moments of Weil representations of $SU_n(q)$ when $2|n$? Preliminary analysis indicates that the even-dimensional case does not behave as nicely as in the odd-dimensional case (particularly because real-valued characters usually have large even moments). We restrict ourselves to record the following result:

**Theorem 5.1.** Consider the irreducible Weil characters $\zeta_{i,n}$, $0 \leq i \leq q$, of $G := SU_n(q)$ as given in (2.0.2), and suppose $n = 4$. Then

$$[(\zeta_{i,4})^4, 1_G] = \begin{cases} 
q + 1, & i = 0, \\
q + 2, & 2 \nmid q, i = (q + 1)/2, \\
q - 1, & 4|q + 1), i = (q + 1)/4, 3(q + 1)/4, \\
1, & \text{otherwise}.
\end{cases}$$
Proof. (i) We will use the dual pairs \( G_1 \times S_1 = \text{SU}_2(q) \times \text{GU}_1(q)^2 \) and \( G_2 \times S_2 = \text{SU}_n(q) \times \text{GU}_2(q) \) as in (4.7.1). By [LBST, Proposition 6.3],

\[
\omega|_{G_2 \times S_2} = \sum_{\alpha \in \text{Irr}(S_2)} C_{\alpha} \otimes \alpha = \sum_{\gamma \in \text{Irr}(G_2)} \gamma \otimes E_{\gamma} = \sum_{\alpha \in \text{Irr}(S_2)} C_{\alpha} \otimes \alpha + 1_{G_2} \otimes (\chi_1^{(0)} + \chi_q^{(0)})
\]

where \( C_{\alpha}(1) \) are listed in Table I. The only new feature that arises in the case \( n = 4 \) is that, according to [LBST, Proposition 6.5],

(a) If \( \alpha \neq \beta \), then \( C_{\alpha} = C_{\beta} \) precisely when \( \{\alpha, \beta\} = \{\chi_1^{(t)}, \chi_1^{(q+1-t)}\} \) for some \( t \in \{1, 2, \ldots, q\} \setminus \{(q + 1)/2\} \); and

(b) All \( C_{\alpha} \) are irreducible, except when \( 2 \nmid q \) and \( \alpha = \chi_1^{(q+1)/2} \), in which case \( C_{\alpha} \) is a sum of two distinct irreducible characters (of degree \( (q^2 + 1)(q^2 - q + 1)/2 \)).

Hence, instead of Corollary 4.8 now we have

\[
(5.1.1) \quad E_{\gamma} = \begin{cases} 
\alpha, & \text{if } \gamma \text{ is an irreducible constituent of } C_{\alpha} \text{ for some } \alpha \in \text{Irr}(\text{GU}_2(q)), \\
\chi_1^{(0)} + \chi_q^{(0)}, & \text{if } \gamma = 1_{G_2}, \\
0, & \text{otherwise.}
\end{cases}
\]

On the other hand,

\[
\omega|_{G_1 \times S_1} = \sum_{\alpha \in \text{Irr}(S_1)} D_{\alpha} \otimes \alpha,
\]

where \( D_{\alpha} \) is given in (4.7.2) for \( \alpha = \lambda_{i,j} \in \text{Irr}(\text{GU}_1(q)^2) \). Applying Lemma 4.7 we then get

\[
(5.1.2) \quad (\zeta_{i,j})^2|_{\text{SU}_4(q)} = D_{\lambda_{i,j}}|_{G_2} = \sum_{\gamma \in \text{Irr}(G_2)} [E_{\gamma}|_{\text{GU}_1(q)^2}, \lambda_{i,j}|_{\text{GU}_1(q)^2} \cdot \gamma].
\]

Direct computations show for \( \alpha \in \text{Irr}(\text{GU}_2(q)) \) that

\[
(5.1.3) \quad [\alpha|_{\text{GU}_1(q)^2}, \lambda_{i,j}|_{\text{GU}_1(q)^2}]^2 = \begin{cases} 
\delta_{t,i}, & \alpha = \chi_1^{(t)}, \\
\delta_{t,2i}, & \alpha = \chi_q^{(t)}, \\
\delta_{t+u,2i}, & \alpha = \chi_{q-1}^{(t)}, \\
\delta_{t,i+(q+1)/2}, & \alpha = \chi_q^{(t)}, \quad 2 \nmid q, \\
0, & \alpha = \chi_q^{(t)}, \quad 2 | q,
\end{cases}
\]

and \( \delta_{i,j} \) is defined to be 1 if \( i \equiv j (\text{mod } q + 1) \) and 0 otherwise. Recall that in the notation for \( \alpha \in \text{Irr}(\text{GU}_2(q)) \), the superscripts are viewed as elements of \( \mathbb{Z}/(q + 1)\mathbb{Z} \) if \( \alpha(1) \leq q \), and as elements of \( \mathbb{Z}/(q^2 - 1)\mathbb{Z} \) if \( \alpha(1) = q + 1 \). Moreover, \( \chi_q^{-1} = \chi_{q-1}^{(t_u)} \) and \( \chi_q^{(t)} = \chi_{q+1}^{(t_q)} \).
(ii) Consider the case $2|q$. Then (5.1.1)–(5.1.3) imply that
\[
(\zeta_{0,4})^2 = 1_G + C^0_{\chi_1(0)} + \sum_{1 \leq i \leq q/2} C^0_{\chi_{q-1}(t,-t)} + \sum_{1 \leq s \leq (q-2)/2} C^0_{\chi_{q+1}(s(q+1))}.
\]
As $\zeta_{0,4}$ is real-valued, it follows that $[(\zeta_{0,4})^4, 1_G]_G = q + 1$.

Likewise, if $i \neq 0$, then the irreducible summands of $(\zeta_{i,4})^2$ are $C^0_{\chi_1(i)}$, $C^0_{\chi_{q-1}(t,2i-t)}$ with $t \neq i$, and $C^0_{\chi_q}$ with $s \equiv 2i (\text{mod } q + 1)$ (and $s \not\equiv 0 (\text{mod } q - 1)$); all with multiplicity one. It follows that the only common irreducible constituent of $(\zeta_{i,4})^2$ and $(\zeta_{0,4})^2 = (\zeta_{q+1-i,4})^2$ is $C^0_{\chi_1(i)} = C^0_{\chi_1(q+1-i)}$, cf. (a) above. Thus $[(\zeta_{i,4})^4, 1_G]_G = 1$. In fact, this argument also applies to the case where $2 \nmid q$ and $(q + 1) \nmid 4i$, where there is an extra irreducible summand $C^0_{\chi_q^{(q+1)/2}}$ (also with multiplicity 1) in $(\zeta_{i,4})^2$.

(iii) Assume now that $2 \mid q$. Then (5.1.1)–(5.1.3) imply that
\[
(\zeta_{0,4})^2 = 1_G + C^0_{\chi_1(0)} + \sum_{1 \leq i \leq q/4} C^0_{\chi_{q-1}(t,-t)} + C^0_{\chi_{q+1}(q+1)} + \sum_{1 \leq s \leq (q-2)/4} C^0_{\chi_{q+1}(s(q+1))},
\]
yielding $[(\zeta_{0,4})^4, 1_G]_G = q + 1$. Likewise,
\[
(\zeta_{q+1,4})^2 = 1_G + C^0_{\chi_1^{(q+1)}} + \sum_{1 \leq i \leq q/4} C^0_{\chi_{q-1}(t,-t)} + C^0_{\chi_q(0)} + \sum_{1 \leq s \leq (q-2)/4} C^0_{\chi_{q+1}(s(q+1))},
\]
Since $\zeta_{q+1,4}$ is real-valued and $C^0_{\chi_1^{(q+1)}}$ is the sum of two distinct irreducible summands,$[(\zeta_{q+1,4})^4, 1_G]_G = q + 2$.

Finally, the irreducible summands of $(\zeta_{q+1,4})^2$ are $C^0_{\chi_1^{(q+1)}}, C^0_{\chi_1^{(q+1)}}$, $C^0_{\chi_{q-1}(t,2(q+1)-t)}$ with $t \neq \pm(q + 1)/2$, and $C^0_{\chi_q^{(q+1)(q+1)/2}}$; all with multiplicity one. As mentioned in (a), $C^0_{\chi_1^{(q+1)}} = C^0_{\chi_1^{-(q+1)}}$. Thus all of these characters, except for the first one, are common irreducible summands between $(\zeta_{q+1,4})^2$ and $(\zeta_{q+1,4})^2 = (\zeta_{3(q+1),4})^2$. It follows that $[(\zeta_{q+1,4})^4, 1_G]_G = q - 1$.

\[\square\]

References


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