Hooley parameters for families of exponential sums over finite fields

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1. Introduction In this note, we introduce the notion of Hooley parameters for certain families of exponential sums over finite fields, namely those obtained by Fourier Transform from suitable inputs. These parameters constitute a generalization of the notion of "A-number" found in [Ka-PES] and [Fou-Ka]. We explain the diophantine significance of these Hooley parameters, and we prove some basic "independence of p" results about them.

2. Shape parameters

(2.1) We fix an integer $n \geq 1$ and a prime number ℓ invertible in k. We work in $\mathbb{A}^n$ over k. Suppose that K in $D_b{}^c(\mathbb{A}^n, \overline{\mathbb{Q}}_\ell)$ is a perverse sheaf on $\mathbb{A}^n$ which is geometrically irreducible. Then there is a geometrically irreducible reduced closed subscheme $i : X \subset \mathbb{A}^n$, a dense open set $j: U \subset X$, which is both affine and smooth over k, and a geometrically irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on U, such that K is the extension by zero $i_\star$ from X of the middle extension $j_{!\star}$ to X of the perverse sheaf $\mathcal{F}[\dim U]$ on U:

(2.1.1) $$K \cong i_\star j_{!\star}(\mathcal{F}[\dim U]).$$

Intrinsically, X is the support of K, i.e., X is the smallest closed set outside of which K vanishes, and U in X is any dense open set which is both affine and lisse over k, on which K has lisse cohomology sheaves.

(2.2) The pair of integers

(2.2.1) $$(\dim X, \operatorname{rank}\mathcal{F})$$

in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$ we will call the **shape parameters** of the geometrically irreducible perverse sheaf K on $\mathbb{A}^n$.

(1.3) Here is a way to "calculate" the shape parameters of a geometrically irreducible perverse sheaf K on $\mathbb{A}^n$, in terms of stratifications. Suppose we are given a smooth stratification $\mathcal{Y}$ of $\mathbb{A}^n$, i.e., a set-theoretic decomposition of $\mathbb{A}^n$ into finitely many reduced, locally closed subschemes Y_i, with each Y_i smooth over k, and equidimensional of some dimension $d_i \geq 0$. Suppose further that the perverse sheaf K is "adapted to the stratification $\mathcal{Y}$", in the sense that for each strat Y_i, and for each ordinary cohomology sheaf $\mathcal{H}^j(K)$ of K, the restriction

$$\mathcal{H}^j(K) \mid Y_i$$

is a lisse sheaf on Y_i. We do not assume the Y_i to be connected, so the rank of $\mathcal{H}^j(K) \mid Y_i$ is a locally constant function $y \mapsto r_{i,j}(y)$ on Y_i. By the perversity of K, and the fact that it is supported on X, we know that

(2.3.1) $$\mathcal{H}^{-j}(K) = 0 \text{ unless } 0 \leq j \leq \dim X,$$

(2.3.2) $$\mathcal{H}^{-j}(K) \mid Y_i \neq 0 \text{ implies } j \geq d_i := \dim Y_i.$$

Look among all those Y_i such that $\mathcal{H}^j(K) \mid Y_i$ is nonzero for some j. Among these Y_i, there is a unique one of largest dimension, namely the unique strat which contains the generic point of X. Call this the "essential" strat, Y_{ess}. We have

(2.3.3) $$\dim X = \dim Y_{ess}.$$

The restriction $\mathcal{H}^{-\dim X}(K) \mid Y_{ess}$ is a lisse sheaf on Y_{ess}, which vanishes outside the connected component of Y_{ess} containing the generic point of X, and whose rank on that connected component is $\operatorname{rank}\mathcal{F}$. [Moreover, $\mathcal{H}^{-j}(K) \mid Y_{ess} = 0$ for $j \neq \dim X$.] From the fact that K is a middle extension from U, we know that for $0 \leq j < \dim X$, we have

(2.3.4) $$\dim \operatorname{Supp}(\mathcal{H}^{-j}(K)) \leq j-1.$$

Thus we have

(2.3.5) $\mathcal{H}^{-j}(K) \mid Y_i \neq 0$ and $j < \dim X$ implies $j-1 \geq d_i := \dim Y_i$.

(2.4) From this description, we see that another characterization of the shape parameters is this. On each strat Y_i, the Euler characteristic

$$y \mapsto \chi(y) := \Sigma_j (-1)^j \mathrm{rank} \mathcal{H}^j(K)_y$$

is a locally constant $\mathbb{Z}$-valued function. Among all those strats on which this function does not vanish identically, there is a unique one of largest dimension, namely Y_{ess}, whose dimension is r. We recover A as the the maximum value on Y_{ess} of the function

$$y \mapsto (-1)^r \chi(y),$$

or equivalently as the maximum value on Y_{ess} of the function

$$y \mapsto |\chi(y)|.$$

The set of points of Y_{ess} where this function attains its maximum a single connected component of Y_{ess} , outside of which the function vanishes.

(2.5) We now introduce weights into the story.

Lemma 2.6 Suppose in addition that our geometrically irreducible perverse sheaf K on $\mathbb{A}^n$ is pure of weight zero. Then

1) the lisse sheaf $\mathcal{H}^{-\dim X}(K) \mid Y_{ess}$ is pure of weight $-\dim X$, it has rank $\mathrm{rank}\mathcal{F}$ on one connected component of Y_{ess}, and it vanishes on all other connected components of Y_{ess}.

2) for $j \neq \dim X$, $\mathcal{H}^{-j}(K) \mid Y_{ess} = 0$.

3) For any $Y_i \neq Y_{ess}$, and any cohomology sheaf $\mathcal{H}^{-j}(K)$, we have

the lisse sheaf $\mathcal{H}^{-j}(K) \mid Y_i$ is mixed of weight $\leq -1 - d_i$.

proof Assertion 1) holds because a middle extension of the shape $i_\star j_{!\star}(\mathcal{F}[\dim U])$, with $\mathcal{F}$ lisse on U, is pure of weight zero if and only if $\mathcal{F}$ is pure of weight $-r = -\dim U$. Assertion 2) has already been noted above. For assertion 3), we argue as follows. If $\mathcal{H}^{-j}(K) \mid Y_i$ is nonzero and $Y_i \neq Y_{ess}$, then $d_i < \dim X$, and so $j-1 \geq d_i$, i.e., $-j \leq -1 - d_i$. Since $\mathcal{H}^{-j}(K)$ is mixed of weight $\leq -j$, the assertion follows. QED

Theorem 2.7 Let N be a perverse sheaf on $\mathbb{A}^n$ over the finite field k. Suppose N is mixed of weight ≤ 0, and that its weight zero quotient, $K := gr_W{}^0(N)$ is geometrically irreducible. Denote by

$$(r, A) := \text{the shape parameters for } K := gr_W{}^0(N).$$

Pick a smooth stratification $\mathcal{Y}$ on $\mathbb{A}^n$ to which both N and K are adapted (or equivalently to which the object $K \oplus N$ is adapted). Then the strat Y_{ess} for $K := gr_W{}^0(N)$ is the unique strat Y_i for which it is **not** true that

for every j, $\mathcal{H}^{-j}(N) \mid Y_i$ is mixed of weight $\le -1 - d_i$.

Once we have identified Y_{ess}, then we recover r as $\dim Y_{ess}$, and we recover A as the maximum, on any connected component of Y_{ess}, of the rank of $gr_W{}^{-r}(\mathcal{H}^{-r}(N)|Y_{ess})$, or, what is the same, as the rank of $gr_W{}^{-r}(\oplus_j \mathcal{H}^{-j}(N)|Y_{ess})$.

proof We have a tautological surjection of perverse sheaves

$$N \twoheadrightarrow K := gr_W{}^0(N),$$

so a short exact sequence of perverse sheaves

$$0 \to Ker \to N \to K \to 0,$$

with Ker mixed of weight ≤ -1. From the long cohomology sequence of ordinary cohomology sheaves

$$\ldots \to \mathcal{H}^{j-1}(N) \to \mathcal{H}^{j-1}(K) \to \mathcal{H}^j(Ker) \to \mathcal{H}^j(N) \to \mathcal{H}^j(K) \to .$$

we see that Ker is also adapted to the smooth stratification $\mathcal{Y}$. Since Ker is mixed of weight ≤ -1 and is perverse, if

$$\mathcal{H}^{-j}(Ker) \mid Y_i \neq 0$$

then $j \ge d_i$, i.e., $-j-1 \le -1 - d_i$. Because Ker is mixed of weight ≤ -1, $\mathcal{H}^{-j}(Ker)$ is mixed of weight $\le -1 - j$, so for every i and every j we have

$$\mathcal{H}^{-j}(Ker) \mid Y_i \text{ is mixed of weight } \le -1 - d_i.$$

Now rewrite the same cohomology sequence as

$$\to \mathcal{H}^{j-1}(K) \to \mathcal{H}^j(Ker) \to \mathcal{H}^j(N) \to \mathcal{H}^j(K) \to \mathcal{H}^{j+1}(Ker) \to$$

Recall that

the lisse sheaf $\mathcal{H}^{-r}(K) \mid Y_{ess}$ is pure of weight $-r$

and of rank A on one connected component of Y_{ess}, and vanishes on all other connected components of Y_{ess}, and for $j \neq \dim X$, $\mathcal{H}^{-j}(K) | Y_{ess} = 0$.

For any $Y_i \neq Y_{ess}$, and any cohomology sheaf $\mathcal{H}^{-j}(K)$, we have

the lisse sheaf $\mathcal{H}^{-j}(K) | Y_i$ is mixed of weight $\leq -1 - d_i$.

On any strat Y_i, we have, for every j,

$$\mathcal{H}^{-j}(Ker) | Y_i \text{ is mixed of weight } \leq -1 - d_i.$$

So if $Y_i \neq Y_{ess}$, we see from the long cohomology sequence that for every j,

the lisse sheaf $\mathcal{H}^{-j}(N) | Y_i$ is mixed of weight $\leq -1 - d_i$.

What happens on Y_{ess}? By the perversity of N, we know that if $\mathcal{H}^{-j}(N) | Y_{ess}$ is nonzero, then $r \geq j$. As N is mixed of weight ≤ 0, the only possibly nonzero $\mathcal{H}^{-j}(N) | Y_{ess}$ are all mixed of weight $\leq -j \leq -r$, and of these only $\mathcal{H}^{-r}(N) | Y_{ess}$ fails to be mixed of weight $\leq -1 -r$. As all the nonzero $\mathcal{H}^{-j}(Ker) | Y_{ess}$ are mixed of weight $\leq 1 - r$, the cohomology sequence gives a four term exact sequence of lisse sheaves on Y_{ess},

$$0 \to (\text{mixed of weight } \leq -1-r) \to \mathcal{H}^{-r}(N) | Y_{ess}$$
$$\to \mathcal{H}^{-r}(K) | Y_{ess} \to (\text{mixed of weight } \leq -1 -r) \to 0.$$

Passing to $gr_W{}^{-r}$, we get

$$gr_W{}^{-r}(\mathcal{H}^{-r}(N) | Y_{ess}) \cong gr_W{}^{-r}(\mathcal{H}^{-r}(K) | Y_{ess}).$$

Since the nonzero $\mathcal{H}^{-j}(N) | Y_{ess}$ with $j \neq r$ are mixed of weight $\leq -1 -r$, we also have

$$gr_W{}^{-r}(\oplus_j \mathcal{H}^{-j}(N) | Y_{ess}) \cong gr_W{}^{-r}(\mathcal{H}^{-r}(N) | Y_{ess}).$$

As already noted in our discussion of the pure and geometrically irreducible case, we recover A as the maximum rank of

$$\mathcal{H}^{-r}(K) | Y_{ess} = gr_W{}^{-r}(\mathcal{H}^{-r}(K) | Y_{ess}. \quad \text{QED}$$

Definition 2.8 Given a perverse sheaf N on $\mathbb{A}^n$ which is mixed of weight ≤ 0, and whose weight zero quotient, $K := gr_W{}^0(N)$ is assumed to be geometrically irreducible, we define the shape parameters (r, A) of N to be the shape parameters of its weight zero quotient, $K := gr_W{}^0(N)$.

3. Hooley parameters

(3.1) We continue to work in $\mathbb{A}^n$ over k. We fix a nontrivial $\overline{\mathbb{Q}}_\ell{}^\times$-valued additive character ψ of k, and a choice of Sqrt(q) in $\overline{\mathbb{Q}}_\ell$. This allows us to define Tate twists by half-integers. Using them, we next define the Fourier Transform

(3.1.1) $$FT_\psi : D_b{}^c(\mathbb{A}^n, \overline{\mathbb{Q}}_\ell) \to D_b{}^c(\mathbb{A}^n, \overline{\mathbb{Q}}_\ell).$$

In terms of the two projections

(3.1.2) $$pr_1 \text{ and } pr_2 : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n$$

and the standard inner product

(3.1.3) $$\langle .,. \rangle : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^1,$$
$$\langle (x_i)_i, (y_i)_i \rangle := \Sigma_i\, x_i y_i,$$

we define FT_ψ as follows:

(3.1.4) $$FT_\psi(K) := R(pr_2)_!(\mathcal{L}_{\psi(\langle .,. \rangle)} \otimes pr_1{}^*(K)[n](n/2)).$$

(3.2) One knows that FT_ψ preserves perversity (and more generally commutes with the formation of perverse cohomology sheaves), and is an exact autoequivalence on the abelian category of perverse sheaves. In particular, it preserves the property of being geometrically irreducbible. It is stable on the full subcategory of mixed perverse sheaves, and on that full subcategory it preserves the filtration by the weight, and so in particular preserves the property of being pure of given weight.

(3.3) Now suppose we are given once again a perverse sheaf N on $\mathbb{A}^n$ which is mixed of weight ≤ 0, whose weight zero quotient, $K := gr_W{}^0(N)$ is assumed to be geometrically irreducible. Then $FT_\psi(N)$ is a perverse sheaf on $\mathbb{A}^n$ which is mixed of weight ≤ 0, whose weight zero quotient, $gr_W{}^0(FT_\psi(N)) = FT_\psi(gr_W{}^0(N)) = FT_\psi(K)$, is geometrically irreducible. Thus it makes sense to speak of the shape parameters of $FT_\psi(N)$, which are by definition the shape parameters of $FT_\psi(K)$.

Definition 3.3.1 The Hooley parameters (r, A) of N are the shape parameters of $FT_\psi(N)$.

(3.4) In view of the above results on shape parameters, the following result on Hooley parameters is a tautology.

Theorem 3.5 Let N be a perverse sheaf on $\mathbb{A}^n$ over the finite field k. Suppose N is mixed of weight ≤ 0, and that its weight zero quotient, $K := gr_W{}^0(N)$ is geometrically irreducible. Denote by

$$(r, A)$$

the Hooley parameters of N. Pick a smooth stratification $\mathcal{Y}$ on $\mathbb{A}^n$ to which both $FT_\psi(N)$ and $FT_\psi(K)$ are adapted (or equivalently to which the object $FT_\psi(N) \oplus FT_\psi(K)$ is adapted). Then we have the following results.

1) The strat Y_{ess} for $FT_\psi(K)$ is the unique strat Y_i for which it is **not** true that

for every j, $\mathcal{H}^{-j}(FT_\psi(N)) \mid Y_i$ is mixed of weight $\le -1 - d_i$.

Once we have identified Y_{ess}, then we recover r as $\dim Y_{ess}$, and we recover A as the maximum, over all connected components of Y_{ess}, of the rank of $gr_W{}^{-r}(\mathcal{H}^{-r}(FT_\psi(N))|Y_{ess})$, or, what is the same, as the rank of $gr_W{}^{-r}(\oplus_j \mathcal{H}^{-j}(FT_\psi(N))|Y_{ess})$.

2) On each strat Y_i, the Euler characteristic

$$y \mapsto \chi(y) := \Sigma_j (-1)^j \mathrm{rank}\mathcal{H}^j(FT_\psi(K))_y$$

is a locally constant $\mathbb{Z}$-valued function. Among all those strats on which this function does not vanish identically, there is a unique one of largest dimension, namely Y_{ess}, whose dimension is r. We recover A as the the maximum value on Y_{ess} of the function

$$y \mapsto (-1)^r \chi(y),$$

or equivalently as the maximum value on Y_{ess} of the function

$$y \mapsto |\chi(y)|.$$

4. Relation to A-number

(4.1) When N as above has Hooley parameters (r, A) with r = n, then the second parameter A ≥ 1 is what is called the A-number of N in [Ka-PES, 4.6] and in [Ka-Fou, 4.0]. Hooley has emphasized (in private communications) the importance of understanding the situation when that A-number vanishes, or what is the same, when

$r < n$.

(4.2) The simplest example of such a situation is to take N to be the (shifted and Tate-twisted) constant sheaf $\overline{\mathbb{Q}}_\ell[n](n/2)$. Then $FT_\psi(N)$ is δ_0, the constant sheaf at the origin, extended by zero. So this N has Hooley parameters (0, 1).

(4.3) We can make more examples by taking products with this situation. Indeed, by the compatibility of Fourier Transform with external direct products on products of affine spaces,

(4.3.1) $$FT_\psi(N\boxtimes M) = FT_\psi(N)\boxtimes FT_\psi(M),$$

we see that if N on $\mathbb{A}^n$ has Hooley parameters (r, A), and if M on $\mathbb{A}^m$ has Hooley parameters (s, B), then $N\boxtimes M$ on $\mathbb{A}^{n+m}$ has Hooley parameters (r + s, AB). In particular, if we take M to be $\overline{\mathbb{Q}}_\ell[m](m/2)$, then N on $\mathbb{A}^n$ and $N\boxtimes M$ on $\mathbb{A}^{n+m}$ have the same Hooley parameters (r, A), no matter how large m. In section 9, we give less trivial examples of objects N on $\mathbb{A}^{2m}$ which both have Hooley parameters (2m-1, 1).

5. Calculation of $\mathrm{gr}_W{}^0(N)$

(5.1) We continue to work in $\mathbb{A}^n$ over a finite field k, with ℓ a chosen prime number which is invertible in k. In the interesting applications of the general theory we have developed so far, one is given the perverse object N which is mixed of weight ≤ 0, and one somehow knows that its weight zero quotient $K := \mathrm{gr}_W{}^0(N)$ is geometrically irreducible. For instance, one might know this by applying [Ka-MMP, 1.8.3, part 3)]. But in general N itself need not be K, and one is not given K explicitly. To what extent can we describe K "explicitly" in terms of N? In the discussion below, we will address, to some extent, this question.

(5.2) We first give a quite general lemma.

Lemma 5.3 Let X/k be an affine scheme of finite type. Let N be a perverse sheaf on X, mixed of weight ≤ 0. Let f be a nonzero function on X.Suppose that $N \mid X[1/f]$ is geometrically irreducible on X[1/f], and pure of weight 0. Suppose that $N[-1\} \mid \mathrm{Var}(f)$ is semiperverse on Var(f). Denote by

$$j : X[1/f] \to X,$$
$$i : \mathrm{Var}(f) \to X$$

the inclusions. Then $gr_W{}^0(N)$ is geometrically irreducible on X, and we have an isomorphism

$$gr_W{}^0(N) \cong j_{!*}j^*N$$

of perverse sheaves on X.

proof Since j^*N is geometrically irreducible and pure of weight 0 on $X[1/f]$, its middle extension $j_{!*}j^*N$ is geometrically irreducible and pure of weight 0 on X. So it suffices to prove the second statement.

Step 1. We first show that $i_*i^*N[-1]$ is perverse, and that N and $j_!j^*N$ have the same $gr_W{}^0$.

We have a distinguished triangle

$$\ldots \to i_*i^*N[-1] \to j_!j^*N \to N \to \ldots,$$

in which both N and $j_!j^*N$ are perverse. By hypothesis, $i_*i^*N[-1]$ is semiperverse. Since N is mixed of weight ≤ 0, $N[-1]$ is mixed of weight ≤ -1, and so $i_*i^*N[-1]$ is mixed of weight ≤ -1. Consider the long exact cohomology sequence of perverse cohomology sheaves

$$\ldots\ldots \to {}^p\mathcal{H}^{j-1}(N) \to {}^p\mathcal{H}^j(i_*i^*N[-1]) \to {}^p\mathcal{H}^j(j_!j^*N) \to {}^p\mathcal{H}^j(N) \to \ldots$$

Because N and $j_!j^*N$ are both perverse, their ${}^p\mathcal{H}^j$ vanish for $j \neq 0$. So the only possibly nonvanishing ${}^p\mathcal{H}^j(i_*i^*N[-1])$ have $j=0$ and $j=1$. But as $i_*i^*N[-1]$ is semiperverse, its only possibly nonvanishing ${}^p\mathcal{H}^j(i_*i^*N[-1])$ have $j \le 0$. Hence ${}^p\mathcal{H}^j(i_*i^*N[-1])$ vanishes for $j \neq 0$, so $i_*i^*N[-1]$ is perverse. So the distinguished triangle is a short exact sequence of perverse sheaves

$$.0 \to i_*i^*N[-1] \to j_!j^*N \to N \to 0,$$

with kernel mixed of weight ≤ -1. Since $gr_W{}^0$ is an exact functor, we find the asserted isomorphism

$$gr_W{}^0(j_!j^*N) \cong gr_W{}^0(N).$$

Step 2. We next show that $j_!j^*N$ and $j_{!*}j^*N$ have the same $gr_W{}^0$. Then we are done, because by Gabber's purity theorem, $j_{!*}j^*N$ is pure of weight 0. In the perverse category, $j_{!*}j^*N$ is a quotient of $j_!j^*N$ (being the image of $j_!j^*N$ in Rj_*j^*N). So we have a short exact sequence of perverse sheaves

$$.0 \to i_*i^*j_{!*}j^*N[-1] \to j_!j^*N \to j_{!*}j^*N \to 0,$$

Since $j_{!*}j^*N$ is pure of weight 0, it is mixed of weight ≤ 0, hence $i_*i^*j_{!*}j^*N[-1]$ is mixed of weight ≤ -1. Applying the exact functor $gr_W{}^0$ gives the asserted isomorphism

$$gr_W{}^0(j_!j^*N) \cong gr_W{}^0(j_{!*}j^*N). \qquad \text{QED}$$

(5.4) The utility of Lemma 5.3 is enhanced by the following elementary lemma.

Lemma 5.5 Suppose that X/k is reduced and irreducible of dimension $d \ge 0$, and that N is a perverse sheav N on X of the form $\mathcal{G}[d]$ for some constructible sheaf $\mathcal{G}$ on X. Then for any nonzero function f on X, $N[-1] \mid Var(f)$ is semiperverse on Var(f).
proof Indeed, the only nonvanishing ordinary cohomology sheaf of $N[-1] \mid Var(f) = \mathcal{G}[d-1] \mid Var(f)$ is $\mathcal{H}^{1-d}$, whose support lies in Var(f), which has dimension at most d-1. QED

(5.6) We also have the following well known lemma, which we include for the reader's convenience.

Lemma 5.7 Suppose now that X/k is a local complete intersection which is equidimensional of dimension $d \ge 0$. Then for any lisse sheaf $\mathcal{F}$ on X, the object $N := \mathcal{F}[d]$ is perverse on X.
proof For $N_0 := \overline{\mathbb{Q}}_\ell[d]$, this is [Ka-PES II, 2.1]. In general, it is obvious that $N := \mathcal{F}[d]$ is semiperverse. Its dual DN is $\mathcal{F}^\vee \otimes DN_0$, which is semiperverse, because DN_0 is semiperverse (being perverse). QED

(5.8) Putting together the previous three lemmas, we get the following theorem.

Theorem 5.9 Let X/k be a closed subscheme of $\mathbb{A}^n$.which is geometrically reduced and irreducible, and is a local complete intersection over k of dimension $d \geq 0$. Let $V \subset X$ be a dense affine open set, inclusion denoted $j_V : V \to X$. Let f be a nonzero function on X such that $X[1/f] \subset V$ and such that $X[1/f]$ is smooth over k. Denote by $j : X[1/f] \to X$ the inclusion. Let $\mathcal{F}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on V which is mixed of weight $\leq -d$. Suppose that $\mathcal{F} \mid X[1/f]$ is a geometrically irreducible lisse sheaf on $X[1/f]$ which is pure of weight $-d$. Then we have the following results.

1) The object $N := (j_V)_!\mathcal{F}[d]$ on X is a perverse sheaf on X, mixed of weight ≤ 0.

2) The object $j_{!*}(N \mid X[1/f])$ on X is a geometrically irreducible perverse sheaf on X which is pure of weight 0.

3) We have an isomorphism of perverse sheaves on X,

$$\mathrm{gr}_W{}^0(N) \cong j_{!*}(N \mid X[1/f]).$$

6. Behavior of Hooley parameters in arithmetic families

(6.1) We continue with our previously chosen integer n. We work on $\mathbb{A}^n$ over an affine parameter scheme $S = \mathrm{Spec}(R)$ of finite type over $\mathbb{Z}[1/\ell]$, i.e., R is finitely generated as a $\mathbb{Z}[1/\ell]$-algebra. We suppose given a stratification $\mathcal{Y}$ of $\mathbb{A}^n{}_S$, i.e., a set-theoretic partition of $\mathbb{A}^n{}_S$ as a disjoint union of finitely many locally closed reduced subschemes Y_i in $\mathbb{A}^n{}_S$. We also suppose given a function h on $\mathbb{A}^n{}_S$, i.e., an S-morphism $h: \mathbb{A}^n{}_S \to \mathbb{A}^1{}_S$, i.e., h is an n-variable polynomial over R.

Basic Stratification Theorem 6.2 Given S and $\mathcal{Y}$ as above, there exist

an integer $M \geq 1$,

a real constant C,

a stratification $\mathfrak{W}$ of $\mathbb{A}^n{}_{S[1/M]}$ (i.e., a set-theoretic partition of $\mathbb{A}^n{}_{S[1/M]}$ as a disjoint union of finitely many locally closed subschemes W_i). and

an additive map

$\{\mathbb{Z}$-valued functions on $\mathbb{A}^n{}_{S[1/M]}$ adapted to $\mathcal{Y}[1/M]\}$

$\to \{\mathbb{Z}$-valued functions on $\mathbb{A}^n{}_{S[1/M]}$ adapted to $\mathfrak{W}\}$,

$$\rho \mapsto \rho^\vee,$$
with the following properties.

1) For any prime number ℓ, for any etale map $S_1 \to S[1/M\ell]$, any object M in $D_b{}^c(\mathbb{A}^n{}_{S_1}, \overline{\mathbb{Q}}_\ell)$ which is adapted to the stratification $\mathcal{Y}_{S_1}$ of $\mathbb{A}^n{}_{S_1}$, for any finite field k, for any k valued point s_1 of $S_1(k)$, for any direct factor L of M_{s_1} in $D_b{}^c(\mathbb{A}^n{}_{s_1}, \overline{\mathbb{Q}}_\ell)$, for any nontrivial additive $\overline{\mathbb{Q}}_\ell{}^\times$-valued character ψ of k, and for any choice of Sqrt(#k) in $\overline{\mathbb{Q}}_\ell$, the object $FT_\psi(L\otimes\mathcal{L}_{\psi(h)})$ is adapted to $\mathcal{W}_{s_1}$, and

$$\| FT_\psi(L\otimes\mathcal{L}_{\psi(h)})\| \le C\|L\|.$$

[For any object L in $D_b{}^c(\mathbb{A}^n{}_{s_1}, \overline{\mathbb{Q}}_\ell)$, we write $\|L\|$ for

$$\|L\| := \mathrm{Sup}_{x \text{ in } \mathbb{A}^n(\overline{k})} \Sigma_i \text{ rank } \mathcal{H}^i(L)_x.]$$

2) Moreover, if L is χ-adapted to $\mathcal{Y}_{s_1}$, then $FT_\psi(L\otimes\mathcal{L}_{\psi(h)})$ is χ-adapted to $\mathcal{W}_{s_1}$, and their χ-functions χ_L and $\chi_{FT_\psi(L)}$ are related by

$$\chi_{FT_\psi(L\otimes\mathcal{L}_{\psi(h)})} = (\chi_L)^\vee.$$

proof When h = 0, this is [Ka-Lau, 4.1], itself a consequence of [Ka-Lau, 3.1.2 and 4.3.2]. The general case results from combining the same ingredients, but now using the recipe of [Fou-Ka, 2.1 and 3.1], where this theorem is proved (but not stated!) in the special case when S is Spec($\mathbb{Z}$). QED

(6.3) In most applications, we want to work over the integers of a number field F. So suppose that S = Spec($\mathcal{O}_F$). Then every dense open set of S contains S[1/M] for some integer M ≥ 1. This allows us to apply [Ka-PES, 1.4.4], which in general requires replacing S by a dense open set in S[1/M_1] for some integer M_1 ≥ 1.

Corollary 6.4 to Theorem 6.2 When in addition S = Spec($\mathcal{O}_F$), we may in addition choose the integer M and the stratification $\mathcal{W}$ in such a way that $\mathcal{W}$ is a smooth stratification of $\mathbb{A}^n{}_{S[1/M]}$, i.e., each W_i is smooth and surjective over S[1/M] with all fibres equidimensional of some common dimension d_i.

proof This is an instance of [Ka-PES, 1.4.4], which in general requires replacing $S[1/M]$ by a dense open set of $S[1/MM_1]$ for some integer $M_1 \geq 1$, but any such set contains $S[1/M_2]$ for some integer $M_2 \geq 1$ QED

Uniformity Theorem 6.5 Hypotheses and notations as in Theorem 6.2, suppose in addition that $S = \mathrm{Spec}(\mathcal{O}_F)$ and that we are given an integer $M_2 \geq 1$, and objects N and K in $D_b{}^c(\mathbb{A}^n{}_{S[1/\ell MM_2]}, \overline{\mathbb{Q}}_\ell)$, which satisfy all the following conditions.

1) The objects N and K are each adapted (:= their ordinary cohomology sheaves are lisse on each strat) and also χ-adapted (:= their χ-functions are constant on each strat, a supplementary condition which is automatic if all strats are connected) to the stratification $\mathcal{Y}$.

2) The objects N is fibrewise perverse and fibrewise mixed of weight ≤ 0: for every finite field k, and for every point s in $S[1/\ell MM_2](k)$, the object N_s on $\mathbb{A}^n{}_k$ is perverse, and mixed of weight ≤ 0.

3) The object K is fibrewise perverse, fibrewise pure of weight 0, and fibrewise geometrically irreducible: for every finite field k, and for every point s in $S[1/\ell MM_2](k)$, the object K_s on $\mathbb{A}^n{}_k$ is perverse, pure of weight 0, and geometrically irreducible.

4) The object K is fibrewise the weight 0 quotient of N: for every finite field k, and for every point s in $S[1/\ell MM_2](k)$, there exists an isomorphism $K_s \cong \mathrm{gr}_W{}^0(N_s)$ of perverse sheaves on $\mathbb{A}^n{}_k$.

Then we have the following two conclusions.

1) There exists a strat W_{ess} of $\mathfrak{W}$ with the following property. For every finite field k, and every point s in $S[1/\ell MM_2](k)$, denote by $W_{s,\, ess}$ the strat of $\mathfrak{W}_s$ which is "essential" for $FT_\psi(K_s \otimes \mathcal{L}_{\psi(h)})$. Then $W_{s,ess} = (W_{ess})_s$ for every finite field k, and every point s in $S[1/\ell MM_2](k)$.

2) There exists a pair of integers (r, A) with the following property: For every finite field k, and every point s in $S[1/\ell MM_2](k)$, denote by

(r_s, A_s) the Hooley parameters of the object $N_s\otimes\mathcal{L}_{\psi(h)}$ on $\mathbb{A}^n{}_k$. We have

$$(r_s, A_s) = (r, A)$$

for every finite field k, and every point s in $S[1/\ell MM_2](k)$.

proof Since on $\mathbb{A}^n{}_k$, $\mathcal{L}_{\psi(h)}$ is a lisse sheaf of rank one which is pure of weight zero, $N_s\otimes\mathcal{L}_{\psi(h)}$ on $\mathbb{A}^n{}_k$ is mixed of weight ≤ 0, $K_s\otimes\mathcal{L}_{\psi(h)}$ is geometrically irreducible and pure of weight zero, we have

$$K_s\otimes\mathcal{L}_{\psi(h)} \cong \mathrm{gr}_W{}^0(N_s\otimes\mathcal{L}_{\psi(h)}),$$

and both $K_s\otimes\mathcal{L}_{\psi(h)}$ and $N_s\otimes\mathcal{L}_{\psi(h)}$ are adapted and χ-adapted to $\mathcal{Y}_s$. The χ-function of $K_s\otimes\mathcal{L}_{\psi(h)}$ is equal to that of K, hence is independent of s. The χ-function of $FT_\psi(K_s\otimes\mathcal{L}_{\psi(h)})$ is, by the uniformity theorem, determined entirely by the χ-function of $K_s\otimes\mathcal{L}_{\psi(h)}$. Hence the χ-function of $FT_\psi(K_s\otimes\mathcal{L}_{\psi(h)})$ is independent of s. Assertion 1) now results from the description [ref previous result] of $W_{s,ess}$ as the unique strat of highest dimension among all the strats on which the χ-function of $FT_\psi(K_s\otimes\mathcal{L}_{\psi(h)})$ does not identically vanish. Then r is the relative dimension of W_{ess} over $S[1/M]$, and A is the absolute value of the χ-function on this strat. QED

Corollary 6.7 For every finite field k, and every point s in $S[1/\ell MM_2](k)$, the strat $W_{s,ess}$ is geometrically connected over k.

proof Given k and s, It suffices to show that $W_{s,ess} = (W_{ess})_s$ is connected, since for any finite extension field E/k, we can view s as an E-valued point, call it s_E, of $S[1/\ell MM_2](E)$, and we have

$$(W_{ess})_s\otimes_k E = (W_{ess})_{s_E}.$$

That $W_{s,ess}$ is connected results from the fact that the χ-function of $FT_\psi(K_s\otimes\mathcal{L}_{\psi(h)})$, which has the constant value $A > 0$ on this strat, is nonzero precisely on one connected component of this strat, cf.2.4. QED

Corollary 6.8 Fix a strat W_i of $\mathfrak{W}$, of relative dimension d_i over Let k be a finite field, s a point in $S[1/\ell MM_2](k)$, and y a k-valued point of $\mathbb{A}^n{}_s$ which lies in the strat $(W_i)_s$. Then we have the following results.

1) If $W_i \neq W_{ess}$, then each of the cohomology groups

$$H_c{}^j(\mathbb{A}^n{}_s \otimes_k \bar{k}, N[n] \otimes \mathcal{L}_{\psi(h(x) + \langle y,x\rangle)})(n/2)$$

is mixed of weight $\leq -1 - d_i$, and the sum of their ranks is bounded by $C||N||$. The L-function of $\mathbb{A}^n{}_s$ with coefficients in

$$N[n] \otimes \mathcal{L}_{\psi(h(x) + \langle y,x\rangle)}(n/2)$$

has total degree at most $C||N||$, and all its reciprocal zeroes and poles have weight $\leq -1 - d_i$.

2) 1) If $W_i = W_{ess}$, each of the cohomology groups

$$H_c{}^j(\mathbb{A}^n{}_s \otimes_k \bar{k}, N[n] \otimes \mathcal{L}_{\psi(h(x) + \langle y,x\rangle)})(n/2),\ j \neq -r$$

is mixed of weight $\leq -1-r$. The group

$$H_c{}^{-r}(\mathbb{A}^n{}_s \otimes_k \bar{k}, N \otimes \mathcal{L}_{\psi(h(x) + \langle y,x\rangle)})(n/2)$$

is mixed of weight $\leq -r$, and its weight $-r$ quotient has rank A. The sum of the ranks of all these $H_c{}^j$ is bounded by $C||N||$. The L-function of $\mathbb{A}^n{}_s$ with coefficients in

$$N[n] \otimes \mathcal{L}_{\psi(h(x) + \langle y,x\rangle)}(n/2)$$

has total degree at most $C||N||$. If r is odd [resp even] it has precisely A reciprocal zeroes [resp. A reciprocal poles] of weight $-r$, and all its other reciprocal zeroes and poles have weight $\leq -1 - r$.

proof The assertion about weights of cohomology groups is just a rewsriting of Theorem 2.7 on each fibre, given the fundamental fact that for each k and s, the strat $(W_s)_{ess}$ is the fibre over S of the strat W_{ess}. The assertion about sum of ranks is just the final assertion of the Basic Stratification Theorem 6.2. The assertion about L-functions is just the translation of these results on weights and ranks through the Lefschetz trace formula [Gro-FL]. QED

Remark 6.9 Theorem 2.7 gives a characterization, for each finite field and each point s in $S[1/\ell MM_2](k)$, of the strat $W_{s,ess}$ among all the strats of $\mathfrak{W}_s$, purely in terms of the weights which occur at various points in $FT_\psi(N_s \otimes \mathcal{L}_{\psi(h)})$. It is not clear a priori from this characterization alone that $W_{s,ess}$ is always the s-fibre of a single strat W_{ess}.

7. Calculation of $gr_W{}^0(N)$ in arithmetic families

(7.1) In this section, we explain how to create input for the Uniformity Theorem 6.5 above. We work in $\mathbb{A}^n$ over an affine parameter scheme $S = \mathrm{Spec}(R)$ of finite type over $\mathbb{Z}[1/\ell]$. We assume further that S is reduced and irreducible, i.e., we assume that R is an integral domain. We suppose given a closed subscheme $X \subset \mathbb{A}^n{}_S$, an integer $d \geq 0$, and a polynomial function f on X, about which we make the following two assumptions 7.1.1 and 7.1.2.

(7.1.1) Every geometric fibres of X/S is reduced and irreducible, and is a local complete intersection of dimension d.

(7.1.2) $X[1/f]/S$ is smooth and surjective.

Lemma 7.2 In the above situation 7.1, there exists an integer $M \geq 1$, a dense open set $U \subset S[1/M]$, and a finite sequence of open sets in X_U,

$$X_U = V_0 \supset V_1 \supset V_2 \supset \ldots V_b = X_U[1/f]$$

such that the following hold:

1) For each i = 1 to b, the complement $Y_i := (V_{i-1} - V_i)^{red}$ is smooth and surjective over U, with fibres equidimensional of some common dimension d_i.

2) For any i = 1 to b, denote by

$$j_i : V_i \to V_{i-1}$$

the inclusion. For any object M_i in $D_b{}^c(V_i, \overline{\mathbb{Q}}_\ell)$ which is adapted to the stratification $\mathcal{Y}_i := \{V_b, Y_b, Y_{b-1}, \ldots, Y_i\}$ of V_i , the object $R(j_i)_\star M_i$ on V_{i-1} is adapted to the stratification $\mathcal{Y}_i := \{V_b, Y_b, Y_{b-1}, \ldots, Y_{i-1}\}$ of V_{i-1}, and its formation is compatible with arbitrary change of base on U.

proof This is proven (but not stated!) in [Ka-PES, 1.8.3]. QED

Lemma 7.3 In the above situation 7.1, suppose that we are given a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on X[1/f]. Form the object $M_r := \mathcal{F}[d]|X_U[1/f]$ on V_r, which is fibrewise perverse over U. For i = b to 1, consider the objects M_{i-1} on V_{i-1} defined successively as

$$M_{i-1} = \tau^{Y_i}{}_{< -d_i} R(j_i)_\star M_i.$$

For each i = 0 to b-1, the object M_i on V_i is adapted to the stratification $\mathcal{Y}_i$, and its formation commutes with arbitrary change of base on U. It is fibrewise perverse over U, and fibre by fibre it is the middle extension of the perverse sheaf M_b.

proof Immediate from the previous lemma and the explicit description [BBD,.2.1.11] of middle extension. QED

Lemma 7.4 In Lemma 7.3, suppose that the lisse sheaf $\mathcal{F}$ on $X_U[1/f]$ is pure of weight -d, and is geometrically irreducible on each geometric fibre of $X_U[1/f]$ over U. Then the object M_0 on X_U is, fibre by fibre, a geometrically irreducible perverse sheaf which is pure of weight zero.

Lemma 7.5 In Lemma 7.4, suppose we are given in addition an affine open set $W \subset X$ with $X[1/f] \subset W \subset X$, and a lisse sheaf $\mathcal{F}$ on W, which is mixed of weight $\le$ -d, whose restriction to $X_U[1/f]$ is pure of weight -d and geometrically irreducible on each geometric fibre of $X_U[1/f]$ over U. Denote by $j_W : W \to X$ the inclusion, and form the object

$$N := (j_W)_! \mathcal{F}[d]$$

on X. Then N is fibrewise perverse on X_U/U, and fibre by fibre its $\mathrm{gr}_W{}^0$ is M_0. The objects N and M_0 on X_U are adapted to any stratification $\mathcal{X}_U$ of X_U which refines both the stratifications

$$\mathcal{Y}_0 := \{Y_0, Y_1,..., Y_r, X_U[1/f]\}$$

and

$$\{X_U - V_U, V_U\}$$

of X_U. If we take a common refinement $\mathcal{X}_U$ which has connected strats, then both N and M_0 are both adapted and χ-adapted to $\mathcal{X}_U$.

proof That N is fibrewise perverse is just [ref earlier lemma], applied fibre by fibre. That fibre by fibre its $gr_W{}^0$ is M_0 is just [ref earlier thm], applied fibre by fibre. The other assertions are obvious, and asserted for ease of later reference QED

8. An instance of the general theory: character sums

(8.1) Let F be a number field, $\nu \geq 1$ an integer, and suppose that F contains the ν'th roots of unity. For each character

$$\chi : \mu_\nu(F) \to \overline{\mathbb{Q}}_\ell{}^\times,$$

we have the Kummer sheaf $\mathcal{L}_\chi$ on $\mathbb{G}_m/\mathcal{O}_F[1/\nu\ell]$.

(8.2) We work in $\mathbb{A}^n$ over $S = \mathrm{Spec}(\mathcal{O}_F[1/N_0\nu\ell])$, for some chosen integer $N_0 \geq 1$. We suppose given the following data:

(8.2.1) a closed subscheme $X \subset \mathbb{A}^n{}_S$,

(8.2.2) an integer $d \geq 0$.

(8.2.3) polynomial functions f and h on X,

(8.2.4) a finite list of pairs (g_i, χ_i) with g_i a polynomial function on X, and χ_i a character of $\mu_\nu(F)$.

(8.3) We suppose this data satisfies the following conditions 8.3.1-3.

(8.3.1) Every fibre of X/S is geometrically reduced and irreducible and is a local complete intersection of dimension d.

(8.3.2) X[1/f]/S is smooth and surjective.

(8.3.3) We have inclusions $X[1/f] \subset X[1/(\prod_j g_j)] \subset X$

Theorem 8.4 Given the above data

$$(S, X, d, f, h, \text{the } (g_j, \chi_j)),$$

there exists an integer $M \geq 1$, a smooth stratification $\mathfrak{W} = \{W_i\}_i$ of $\mathbb{A}^n{}_{S[1/M]}$, with $W_i/S[1/M]$ smooth and surjective with fibres equidimensional of dimension d_i, a real constant C, a pair of integers (r, A) with $r \geq 0$ and $A \geq 1$, and a distinguished strat W_{ess} of $\mathfrak{W}$, with the following properties.

1) The fibres of $W_{ess}/S[1/M]$ are geometrically connected and smooth of dimension r.

2) For any finite field k, any nontrivial additive character ψ of k,

any point s in S[1/M](k), and any point y in $(W_{ess})_s(k)$, the cohomology groups

$$H_c^i(X_s[1/g]\otimes\bar{k}, (\otimes_j \mathcal{L}_{\chi_j(g_j(x))})\otimes\mathcal{L}_{\psi(h(x) + \langle y,x\rangle)}), i \neq d+n-r,$$

is mixed of weight ≤ d+n-r-1, while the cohomology group

$$H_c^{d+n-r}(X_s[1/g]\otimes\bar{k}, (\otimes_j \mathcal{L}_{\chi_j(g_j(x))})\otimes\mathcal{L}_{\psi(h(x) + \langle y,x\rangle)})$$

is mixed of weight ≤ d+n-r, and has precisely A eigenvalues of weight d+n-r.

3) For any strat $W_i \neq W_{ess}$, any finite field k, any nontrivial additive character ψ of k, any point s in S[1/M](k), and any point y in $(W_i)_s(k)$, the cohomology groups

$$H_c^i(X_s[1/g]\otimes\bar{k}, (\otimes_j \mathcal{L}_{\chi_j(g_j(x))})\otimes\mathcal{L}_{\psi(h(x) + \langle y,x\rangle)})$$

all have weight ≤ $d+n-d_i-1$.

4) In both situations 3) and 4), the sum of the ranks of the H_c^i is bounded by C.

Corollary 8.5 Hypotheses and notations as in Theorem 8.4, we have the following estimates for exponential sums.

1) For any finite field k, any nontrivial additive character ψ of k, any point s in S[1/M](k), and any point y in $(W_{ess})_s(k)$, we have the estimate

$$|\Sigma_{x \text{ in } X_s[1/g](k)} (\Pi_j\ \chi_j(g_j(x)))\psi(h(x) + \langle y,x\rangle)|$$
$$\leq A(\#k)^{(d+n-r)/2} + (C-A)(\#k)^{(d+n-r-1)/2}$$

2) For any strat $W_i \neq W_{ess}$, any finite field k, any nontrivial additive character ψ of k, any point s in S[1/M](k), and any point y in $(W_i)_s(k)$, we have the estimate

$$|\Sigma_{x \text{ in } X_s[1/g](k)} (\Pi_j\ \chi_j(g_j(x)))\psi(h(x) + \langle y,x\rangle)|$$
$$\leq C(\#k)^{(d+n-d_i -1)/2}.$$

proof of Corollary 8.5 The corollary is immediate from the theorem, via the Lefschetz trace formula [Gro-FL]. QED

proof of Theorem 8.4 Put $g = \Pi_j\ g_j$. Begin with the lisse sheaf

$$\mathcal{F} = (\otimes_j \mathcal{L}_{\chi_j(g_j(x))})(d/2)$$

on the open set X[1/g], and restrict it to X[1/f]. Then apply Lemma 7.3 to the inclusion of X[1/f] into X. This produces an integer $M_1 \geq 1$, an open dense set U in $S[1/M_1]$, and a particularly nice finite sequence of open sets in X_U,

$$X_U = V_0 \supset V_1 \supset V_2 \supset \ldots V_k = X_U[1/f].$$

Since any dense open set U in $S[1/M_1]$ contains one of the form $S[1/M_2]$, we may and will assume that U is $S[1/M_2]$. Form the object $M_k := \mathcal{F}[d]|X_U[1/f] = \mathcal{F}[d]|X[1/M_2 f]$ on V_r. Then form the object M_0 on $X_U = X[1/M_2]$, which is adapted to the stratification $\mathcal{Y}$ of Lemma 7.3.

Denote by

$$j_g : X[1/gM_2] \to X[1/M_2]$$

the inclusion. Take for N the object $(j_g)_!\mathcal{F}[d]$ on X[1/M]. Pick a stratification $\mathcal{X}[1/M]$ of X[1/M] as in Lemma 7.5 with connected strats. Append to it the open set $\mathbb{A}^n{}_{S[1/M_2]} - X[1/M_2]$ to obtain a stratification, still denoted $\mathcal{Y}$, of $\mathbb{A}^n{}_{S[1/M_2]}$. Then both M_0 and N are adapted and χ-adapted to this stratification. Now apply the Basic Stratification Theorem 6.2 to this stratification, to produce an integer $M_3 \geq 1$ and a stratification $\mathcal{W}$ of $\mathbb{A}^n{}_{S[1/M_2 M_3]}$. We take the integer M to be the product $M_2 M_3$. Now apply the Uniformity Theorem 6.5 to this situation, with $\mathcal{W}$ and N as above, and with K takeni to be our M_0.

To get the statements about cohomology groups, recall (3.1.4) the Tate twists and the shifts in the definition of Fourier Transform. For a finite field k, a prime ℓ invertible in k, a nontrivial additive character ψ of k, a choice of Sqrt(#k), we have

$$FT_\psi(L) := R(pr_2)_!(\mathcal{L}_{\psi(\langle .,\, . \rangle)} \otimes pr_1{}^*(L)[n](n/2))$$

for any object L in $D_b{}^c(\mathbb{A}^n, \overline{\mathbb{Q}}_\ell)$.

For a finite field k, an nontrivial additive character ψ of k, and a k-valued point s in S[1/M], the perverse sheaf N_s induced by N on the fibre $\mathbb{A}^n{}_s$ over s is the object $\mathcal{L}_{\chi(g(x))}[d](d/2)$ $X_s[1/g]$ on $X_s[1/g]$, extended by zero to all of $\mathbb{A}^n{}_S$. So the stalk of $\mathcal{H}^j(FT_\psi(N_S \otimes \mathcal{L}_{\psi(h)}))$ at a k-valued point y of $\mathbb{A}^n{}_S$ is

$$\mathcal{H}^j(FT_\psi(N_S \otimes \mathcal{L}_{\psi(h)}))_y$$
$$= H^j(R\Gamma_c(X_S[1/g]\otimes \bar{k},$$
$$(\otimes_j \mathcal{L}_{\chi_j(g_j(x))})[d](d/2)\otimes \mathcal{L}_{\psi(h(x) + \langle y,x\rangle)}[n](n/2)))$$
$$= H_c{}^j(X_S[1/g]\otimes \bar{k},$$
$$(\otimes_j \mathcal{L}_{\chi_j(g_j(x))})[d](d/2)\otimes \mathcal{L}_{\psi(h(x) + \langle y,x\rangle)}[n](n/2))$$
$$= H_c{}^{j+d+n}(X_S[1/g]\otimes \bar{k},$$
$$(\otimes_j \mathcal{L}_{\chi_j(g_j(x))})\otimes \mathcal{L}_{\psi(h(x) + \langle y,x\rangle)})((n+d)/2).$$

So the assertion about cohomology groups is just Corollary 6.8 of the Uniformity Theorem 6.5. QED

9. Three examples

(9.1) **First example** Pick an integer $m \geq 2$. View $\mathbb{A}^{2m}$ as $\mathbb{A}^m \times \mathbb{A}^m$, with coordinates $(x_1, ..., x_m, y_1, ..., y_m)$. In $\mathbb{A}^{2m}$ over $\mathbb{Z}$, consider the (split quadric) hypersurface X of equation

$$\Sigma_i x_i y_i = 0.$$

Because $m \geq 2$, $X/\mathbb{Z}$ has all its fibres geometrically reduced and irreducible. Choose a prime ℓ, and take for N the extension by zero to $\mathbb{A}^{2m}[1/\ell]$ of the object $\bar{\mathbb{Q}}_{\ell,X}[2m-1]((2m-1)/2]$ on $X[1/\ell]$. We leave to the reader the elementary verification that in every characteristic $p \neq \ell$, the Hooley parameters of N are $(2m-1, 1)$.

Moreover, a stratification $\mathfrak{W}$ of the dual $\mathbb{A}^{2m} = \mathbb{A}^m \times \mathbb{A}^m$, with coordinates $(a_1, ..., a_m, b_1, ..., b_m)$ to which, in every characteristic $p \neq \ell$ both $FT_\psi(N)$ and $FT_\psi(gr_W{}^0(N))$ are adapted, is the following one.

W_0: the open set where $\Sigma_i a_i b_i$ is invertible,

W_1: the set where $\Sigma_i a_i b_i = 0$ but some a_i is invertible and some b_j is invertible,

W_2 : the set where all $a_i = 0$ but some b_j is invertible,

W_3, the set where all $b_j = 0$ but some a_i is invertible,

W_4 : the origin.

The "essential" strat W_{ess} is W_1.

(9.2) **Second example** Take $m \geq 1$. In $\mathbb{A}^{2m} = \mathbb{A}^m \times \mathbb{A}^m$ over $\mathbb{Z}$ with coordinates $(x_1, ..., x_m, y_1, ..., y_m)$, take for X the hypersurface of equation

$$\Pi_i x_i = \Pi_j y_j.$$

Again choose a prime ℓ, and take for N the extension by zero to $\mathbb{A}^{2m}[1/\ell]$ of the object $\overline{\mathbb{Q}}_{\ell,X}[2m-1]((2m-1)/2]$ on $X[1/\ell]$. In every characteristic $p \neq \ell$, the Hooley parameters of N are $(2m-1, 1)$. A stratification $\mathfrak{W}$ to which both $FT_\psi(N)$ and $FT_\psi(gr_W{}^0(N))$ are adapted is the following:

W_0: the open set where $(\Pi_i a_i)(\Pi_j b_j)[(\Pi_i a_i) - (-1)^m(\Pi_j b_j)]$ is invertible,

W_1: the set where $(\Pi_i a_i) = (-1)^m(\Pi_j b_j)$ and $(\Pi_i a_i)(\Pi_j b_j)$ is invertible,

and, for each pair of subsets (A, B) of $\{1, 2, ..., m\}$ which are not both empty, the set

$W_{A,B}$:= the points where the x_a for a in A and the y_b for b in B vanish, but all other x_i and y_j are invertible.

One can show, along the lines of [Ka-PESII, 15.1 and 15.4], that the "essential" strat W_{ess} is W_1.

(9.3) **Third example** Here is a slight generalization of the example above, where multiplicative characters enter. Pick an integer $\nu \geq 1$, and a number field F which contains the ν'th roots of unity. Pick an integer $m \geq 1$, and a list of m characters χ_j of $\mu_\nu(F)$. In $\mathbb{A}^{2m} = \mathbb{A}^m \times \mathbb{A}^m$ over $\mathcal{O}_F[1/\nu]$, with coordinates $(x_1, ..., x_m, y_1, ..., y_m)$, again take for X the hypersurface of equation

$$\Pi_i x_i = \Pi_j y_j.$$

Take $g := (\Pi_i x_i)(\Pi_j y_j)$. On the open set $X[1/g\ell]$ take the lisse $\overline{\mathbb{Q}}_\ell$-sheaf

$$\mathcal{F} := (\otimes_i \mathcal{L}_{\chi_i(x_i)}) \otimes (\otimes_j \mathcal{L}_{\overline{\chi}_j(y_j)}),$$

and take for N the extension by zero to $\mathbb{A}^{2m}[1/\ell]$ of the object

$\mathcal{F}[2m-1]((2m-1)/2)$ on $X[1/g]$. Once again, in every characteristic p prime to $\nu\ell$, the Hooley parameters of N are (2m-1, 1). Again $FT_\psi(N)$ and $FT_\psi(gr_W{}^0(N))$ are both adapted to the stratification $\mathfrak{W}$ of the example 9.2 above, and once again the "essential" strat W_{ess} is W_1.

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