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A NOTE ON RANDOM MATRIX INTEGRALS, MOMENT IDENTITIES, AND CATALAN NUMBERS

NICHOLAS M. KATZ

Abstract. We relate Catalan numbers and Catalan determinants to random matrix integrals and to moments of spin representations of odd orthogonal groups.

§1. *Introduction.* In their paper "Random matrix theory and L-functions at $s = 1/2$", Keating and Snaith give explicit formulas [3, (10), p. 94] for the matrix integrals

$$\int_{USp(2n)} \det(1 - A)^s \, dA.$$

Here $USp(2n)$ is the compact symplectic group of size $2n$, dA is its Haar measure of total mass one, and $\det(1 - A)$ is computed for the standard representation of $A \in USp(2n)$ as a matrix of size $2n$. Because the group $USp(2n)$ contains the scalar matrix -1, and because Haar measure is translation invariant, we have

$$\int_{USp(2n)} \det(1 - A)^s \, dA = \int_{USp(2n)} \det(1 + A)^s \, dA.$$

Their formula, valid for $s \in \mathbb{C}$ with $\Re(s) > -3/2$, is

$$\int_{USp(2n)} \det(1 + A)^s \, dA = 2^{2ns} \prod_{j=1}^{n} \frac{\Gamma(n + j + 1)\Gamma(1/2 + s + j)}{\Gamma(1/2 + j)\Gamma(1 + s + n + j)}.$$

We were particularly interested in the case when s is an integer $r \geqslant -1$. Out of idle curiosity, we looked at what their formula gave for the case $n = 1$, when $USp(2)$ is the group $SU(2)$, and for integer values of $r \geqslant -1$. For $r = -1, 0, 1, \ldots, 9$, we found the sequence

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16\,796,$$

which is the start of the sequence of Catalan numbers C_n, indexed by integers $n \geqslant 0$:

$$C_n := \frac{1}{n+1}\binom{2n}{n}.$$

This made it seem likely that for every integer $r \geqslant -1$, we had the relation

$$\int_{SU(2)} \det(1 + A)^r \, dA = C_{r+1}.$$

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We will show the following.

THEOREM 1.1. *For every integer $r \geqslant -1$, we have the relation*

$$\int_{SU(2)} \det(1+A)^r \, dA = C_{r+1}.$$

The Catalan numbers are themselves matrix integrals over $SU(2)$. For integers $r \geqslant 0$, we have

$$C_r = \int_{SU(2)} \operatorname{Tr}(A)^{2r} \, dA, \qquad 0 = \int_{SU(2)} \operatorname{Tr}(A)^{2r+1} \, dA.$$

So, from Theorem 1.1, we have the identity, for integers $r \geqslant 0$,

$$\int_{SU(2)} \operatorname{Tr}(A)^{2r+2} \, dA = \int_{SU(2)} \det(1+A)^r \, dA.$$

For $SU(2)$, we have the identity

$$\det(1+A) = 2 + \operatorname{Tr}(A).$$

When we expand $(2+\operatorname{Tr}(A))^r$ by the binomial theorem, we find the following.

COROLLARY 1.2.

$$C_{r+1} = \sum_{0 \leqslant d \leqslant r/2} 2^{r-2d} \binom{r}{2d} C_d.$$

As we learned from Richard Stanley, this is Touchard's identity, cf. [9, p. 472], [6], and [8, solution 60 on p. 67].

We then looked at what the Keating–Snaith formula gave for the case $n = 2$, i.e. for the group $USp(4)$, and for integer values of $r \geqslant -1$. For $r = -1, 0, 1, \ldots, 7$, we found the sequence

$$1, 1, 3, 14, 84, 594, 4719, 40\,898, 379\,236.$$

Inspired by what had happened in the $SU(2)$ case, we computed (using Mathematica) the integrals

$$\int_{USp(4)} \operatorname{Tr}(A)^{2r+2} \, dA$$

for $r = -1, 0, 1, \ldots, 7$, and found this same sequence. This led us to suspect that we had the identity

$$\int_{USp(4)} \det(1+A)^r \, dA = \int_{USp(4)} \operatorname{Tr}(A)^{2r+2} \, dA$$

for every $r \geqslant 0$. We will show the following.

THEOREM 1.3. *For every integer $r \geqslant -1$, we have*

$$\int_{USp(4)} \det(1+A)^r \, dA = \int_{USp(4)} \mathrm{Tr}(A)^{2r+2} \, dA.$$

However, this identity failed for every $n \geqslant 3$. Already for $r = 1$, the Keating–Snaith formula gives

$$\int_{USp(2n)} \det(1+A) \, dA = n+1.$$

However, one knows that for every $n \geqslant 2$, one has

$$\int_{USp(2n)} \mathrm{Tr}(A)^4 \, dA = 3.$$

What is to be done?

It turns out that in order to understand the Keating–Snaith integrals

$$\int_{USp(2n)} \det(1+A)^r \, dA$$

along these lines, we must introduce the compact spin group $USpin(2n+1)$ (the universal covering of the group $SO(2n+1, \mathbb{R})$ for the sum of squares quadratic form) and its 2^n-dimensional spin representation.

The general result is this.

THEOREM 1.4. *For $n \geqslant 1$ and $r \geqslant 0$, we have the identity*

$$\int_{USp(2n)} \det(1+A)^r \, dA = \int_{USpin(2n+1)} \mathrm{Tr}(\mathrm{spin}(A))^{2r+2} \, dA.$$

This result includes the identities for $USp(2) = SU(2)$ and for $USp(4)$. Indeed, for $n = 1$ and 2, we have the accidents that $USpin(2n+1)$ is the group $USp(2n)$ and that the spin representation of $USpin(2n+1)$ is the standard representation of $USp(2n)$.

§2. *Proof of Theorem 1.4 via the Weyl integration formula.* For a group G, we denote by $G^\#$ its space of conjugacy classes. When G is a topological group, we topologize $G^\#$ so that continuous functions on $G^\#$ are precisely the continuous central (invariant by conjugation) functions on G. The function $\det(1+A)$ is a continuous central function on $USp(2n)$ with values in $\mathbb{R}_{\geqslant 0}$, and the function $\mathrm{Tr}(\mathrm{spin}(A))$ is a continuous central function on $USpin(2n+1)$ with values in $\mathbb{R}$.

An element $A \in USp(2n)$ has n pairs of eigenvalues $e^{\pm i\theta_j}$, $j = 1, \ldots, n$, with angles $\theta_j \in [0, \pi]$, and A is determined up to conjugacy by the unordered n-tuple of its angles θ_j. So, a continuous (respectively Borel measurable and $\mathbb{R}_{\geqslant 0}$-valued) central function $A \mapsto f(A)$ on $USp(2n)^\#$ is a continuous (respectively Borel measurable and $\mathbb{R}_{\geqslant 0}$-valued) function

$$f(\theta_1, \ldots, \theta_n)$$

on $[0, \pi]^n$ which is invariant under the symmetric group S_n. For such a function f, the Weyl integration formula, cf. [2, 5.0.4] or [11, (7.8B) on p. 218], asserts that

$$\int_{USp(2n)} f(A)\,dA = \int_{[0,\pi]^n} f(\theta_1, \dots, \theta_n)\mu_{USp(2n)}$$

for $\mu_{USp(2n)}$ the measure on $[0, \pi]^n$ given by

$$\mu_{USp(2n)} = (1/n!)\Bigg(\prod_{1\leqslant i<j\leqslant n} (2\cos(\theta_i) - 2\cos(\theta_j))^2\Bigg)\prod_{i=1}^{n}(2/\pi)\sin(\theta_i)^2\,d\theta_i.$$

An element $A \in SO(2n+1, \mathbb{R})$ has the eigenvalue 1, and in addition it has n pairs of eigenvalues $e^{\pm i\theta_j}$, $j = 1, \dots, n$, with angles $\theta_j \in [0, \pi]$, and A is determined up to conjugacy by the unordered n-tuple of its angles θ_j. So, a continuous (respectively Borel measurable and $\mathbb{R}_{\geqslant 0}$-valued) central function $A \mapsto f(A)$ on $SO(2n+1, \mathbb{R})^{\#}$ is a continuous (respectively Borel measurable and $\mathbb{R}_{\geqslant 0}$-valued) function

$$f(\theta_1, \dots, \theta_n)$$

on $[0, \pi]^n$ which is invariant under the symmetric group S_n. For such a function f, the Weyl integration formula, cf. [2, 5.0.5] or [11, (9.7) on p. 224], asserts that

$$\int_{SO(2n+1,\mathbb{R})} f(A)\,dA = \int_{[0,\pi]^n} f(\theta_1, \dots, \theta_n)\mu_{SO(2n+1,\mathbb{R})}$$

for $\mu_{SO(2n+1,\mathbb{R})}$ the measure on $[0, \pi]^n$ given by

$$\mu_{SO(2n+1,\mathbb{R})} = (1/n!)\Bigg(\prod_{1\leqslant i<j\leqslant n} (2\cos(\theta_i) - 2\cos(\theta_j))^2\Bigg) \times \prod_{i=1}^{n}(2/\pi)\sin(\theta_i/2)^2\,d\theta_i.$$

Notice the similarity between the formulas for the measures $\mu_{USp(2n)}$ and $\mu_{SO(2n+1,\mathbb{R})}$. The only difference is that each factor $\sin(\theta_i)^2$ in the first is replaced by $\sin(\theta_i/2)^2$ in the second.

The key lemma is this.

LEMMA 2.1. *The measures $\mu_{USp(2n)}$ and $\mu_{SO(2n+1,\mathbb{R})}$ on $[0, \pi]^n$ are related by the identity*

$$\mu_{USp(2n)} = \Bigg(\prod_{i=1}^{n}(2 + 2\cos(\theta_i))\Bigg)\mu_{SO(2n+1,\mathbb{R})}.$$

Proof. This is immediate from the trigonometric identity

$$(2 + 2\cos(\theta))\sin(\theta/2)^2 = \sin(\theta)^2,$$

whose verification is left to the reader. □

The factor $\prod_{i=1}^{n}(2+2\cos(\theta_i))$ has the following two interpretations.

LEMMA 2.2. *We have the following identities.*

(1) *For $A \in SO(2n+1,\mathbb{R})$ with eigenvalues 1 and n pairs of eigenvalues $e^{\pm i\theta_j}$, $j=1,\dots,n$, with angles $\theta_j \in [0,\pi]$,*

$$(1/2)\det(1+A) = \prod_{i=1}^{n}(2+2\cos(\theta_i)).$$

(2) *For $A \in USp(2n)$ with n pairs of eigenvalues $e^{\pm i\theta_j}$, $j=1,\dots,n$, with angles $\theta_j \in [0,\pi]$,*

$$\det(1+A) = \prod_{i=1}^{n}(2+2\cos(\theta_i)).$$

The spin representation of $USpin(2n+1)$ does not descend to $SO(2n+1,\mathbb{R})$, but its tensor square $\mathrm{spin}^{\otimes 2}$ does. In terms of the standard representation std_{2n+1} of $SO(2n+1,\mathbb{R})$ and the double covering projection map

$$p : USpin(2n+1) \to SO(2n+1,\mathbb{R}),$$

one knows [10, Lemma 6.6.2] that

$$\mathrm{spin}^{\otimes 2} = \left(\sum_{i=0}^{n} \Lambda^i(std_{2n+1})\right) \circ p.$$

Each representation $\Lambda^i(std_{2n+1})$ is self-dual, and hence isomorphic to $\Lambda^{2n+1-i}(std_{2n+1})$. So, we have

$$2\mathrm{spin}^{\otimes 2} = \left(\sum_{i=0}^{2n+1} \Lambda^i(std_{2n+1})\right) \circ p.$$

For $A \in SO(2n+1,\mathbb{R})$ (indeed, for $A \in GL(2n+1,\mathbb{C})$), we have the identity

$$\mathrm{Tr}\left(\sum_{i=0}^{2n+1} \Lambda^i(A)\right) = \det(1+A).$$

So, for any $B \in USpin(2n+1)$ lying over A, we have

$$\mathrm{Tr}(\mathrm{spin}(B))^2 = (1/2)\det(1+A).$$

For any continuous function g on $SO(2n+1,\mathbb{R})$, with pullback function $G := g \circ p$ on $USpin(2n+1)$, we have

$$\int_{USpin(2n+1)} G(A)\,dA = \int_{SO(2n+1,\mathbb{R})} g(A)\,dA,$$

simply because the direct image of Haar measure is Haar measure for any surjective homomorphism of compact groups; cf. [2, Lemma 1.3.1].

Putting this all together, we have, for $r \geqslant -1$,

$$\begin{aligned}\int_{USp(2n)} \det(1+A)^r\,dA &= \int_{[0,\pi]^n} \left(\prod_{i=1}^{n}(2+2\cos(\theta_i))\right)^r \mu_{USp(2n)} \\ &= \int_{[0,\pi]^n} \left(\prod_{i=1}^{n}(2+2\cos(\theta_i))\right)^{r+1} \mu_{SO(2n+1,\mathbb{R})} \\ &= \int_{SO(2n+1,\mathbb{R})} ((1/2)\det(1+A))^{r+1}\,dA \\ &= \int_{USpin(2n+1)} \mathrm{Tr}(\mathrm{spin}(A))^{2r+2}\,dA.\end{aligned}$$

This concludes the proof of Theorem 1.4.

Remark 2.3. The odd moments of the spin representation all vanish:

$$\int_{USpin(2n+1)} \mathrm{Tr}(\mathrm{spin}(A))^{2r+1}\,dA = 0$$

for all $r \geqslant 0$. Indeed, this integral is the multiplicity of spin in the representation $\mathrm{spin}^{\otimes 2r}$, a representation which descends to $SO(2n+1,\mathbb{R})$. Hence, all of its irreducible constituents also descend to $SO(2n+1,\mathbb{R})$. But none of these irreducible components can be isomorphic to spin, because the spin representation does not descend to $SO(2n+1,\mathbb{R})$.

§3. *A Catalan determinant interpretation*

THEOREM 3.1. *For integers $n \geqslant 1$ and $r \geqslant 0$, we have the (equivalent) identities*

$$\int_{USpin(2n+1)} \mathrm{Tr}(\mathrm{spin}(A))^{2r}\,dA = \underset{0\leqslant i,j\leqslant n-1}{\overset{}{\det_{n\times n}}} (C_{r+i+j}),$$

$$\int_{SO(2n+1,\mathbb{R})} 2^{-r}\det(1+A)^r\,dA = \underset{0\leqslant i,j\leqslant n-1}{\det_{n\times n}} (C_{r+i+j}),$$

$$\int_{USp(2n)} \det(1+A)^r\,dA = \underset{0\leqslant i,j\leqslant n-1}{\det_{n\times n}} (C_{r+1+i+j}).$$

Proof. This is simply a matter of comparing the Keating–Snaith formula for

$$\int_{USp(2n)} \det(1+A)^r\,dA$$

with the formula, cf. [5, Theorem 3] and [1, p. 21, line 6], of Gessel–Viennot for the Catalan determinant, and checking that the two formulas give the same answer. Each is a product of n terms. The individual terms do not quite match, but their ratios turn out to have product one. □

Remark 3.2. For $n = 1, 2, 3, 4, 5$, these sequences of $n \times n$ Catalan determinants indexed by r appear in the *On-line Encyclopedia of Integer Sequences* [7] as the sequences A000108, A005700, A006149, A006150, A006151, respectively. The interpretation of A000108, the sequence of Catalan numbers, as the even moments of the standard representation of $SU(2)$, is classical. The interpretation of A005700 as the sequence of even moments of the standard representation of $USp(4)$ occurs in [4, (11) on p. 131]. For higher n, the moment interpretation of these determinants seems to be new. Is it?

§4. *A question.* Our proof of Theorem 1.4,

$$\int_{USp(2n)} \det(1 + A)^r \, dA = \int_{USpin(2n+1)} \mathrm{Tr}(\mathrm{spin}(A))^{2r+2} \, dA,$$

via the Weyl integration formula, comes down to the trigonometric identity of Lemma 2.1. From the point of view of representation theory, the first integral is, for $r \geqslant 0$, the multiplicity of the trivial representation in the rth tensor power of the exterior algebra $\Lambda(std_{2n}) := \bigoplus_{k=0}^{2n} \Lambda^k(std_{2n})$ as a representation of $USp(2n)$. The second integral is the multiplicity of the trivial representation in the $(2r+2)$nd tensor power of the spin representation of $USpin(2n+1)$. Is there a representation-theoretic proof that they are equal?

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Nicholas M. Katz,
Department of Mathematics,
Princeton University, Fine Hall,
Washington Road, Princeton,
NJ 08544-1000, U.S.A.
E-mail: nmk@math.princeton.edu