IMRN International Mathematics Research Notices
1999, No. 16

Estimates for "Singular" Exponential Sums

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Introduction

Fix a finite field k and a nontrivial additive character

$$\psi\colon\ k \to \mathbb{C}^\times.$$

In [D1, Thm. 8.4], Deligne proved the following beautiful estimate for exponential sums in $n \geq 1$ variables. Given a polynomial $f(x_1, \ldots, x_n)$ in n variables over k of some degree $d \geq 1$, write it as

$$f = F_d + F_{d-1} + \cdots F_0$$

with F_i homogeneous of degree i. Suppose that the following two conditions are satisfied.

(1) The degree d is prime to $p := \mathrm{char}(k)$.

(2) The locus $F_d = 0$ is a *nonsingular* hypersurface in $\mathbb{P}^{n-1}$ (for $n = 1$, this second condition always holds).

Then we have the estimate

$$\left|\sum_{x \in k^n} \psi(f(x))\right| \leq (d-1)^n (\#k)^{n/2}.$$

Now let us consider a slightly more general situation. We begin with a projective, *nonsingular*, geometrically connected variety X/k, of some dimension $n \geq 1$, given with a projective embedding. We give ourselves a linear form on X,

$$L \in H^0(X, \mathcal{O}(1)),$$

an integer $d \geq 1$, and a form of degree d on X,

$$H \in H^0(X, \mathcal{O}(d)).$$

Received 25 February 1999.
Communicated by Peter Sarnak.

Suppose that the following three conditions are satisfied.

(0) The locus $L = 0$ in X, denoted $X \cap L$, is *nonsingular* of codimension 1 in X.

(1) The degree is prime to p.

(2) The locus $L = H = 0$ in X, denoted $X \cap L \cap H$, is *nonsingular* of codimension 1 in $X \cap L$.

The ratio $f := H/L^d$ makes sense as a function on the smooth affine variety $V := X[1/L] := X - X \cap L$ of dimension n,

$$f := H/L^d \colon V \to \mathbb{A}^1.$$

Under these hypotheses, we have the estimate (see [L], [K, 5.1.1])

$$\left|\sum_{x \in V(k)} \psi(f(x))\right| \leq \mathrm{Const}(X, \mathcal{O}(1), d)(\#k)^{n/2},$$

where the constant is topological, given in terms of the total Chern class $c(X)$ of X and the class L of $\mathcal{O}(1)$ by the explicit formula

$$\mathrm{Const}(X, \mathcal{O}(1), d) = \left|\int_X c(X)/((1+L)(1+dL))\right|.$$

Let us recall how this second result includes the first one as a special case. Take X to be $\mathbb{P}^n$ with homogeneous coordinates $X_0, \ldots, X_n$, take L to be X_0, and take H to be the homogenization of f,

$$H := F_d + (X_0)F_{d-1} + (X_0)^2 F_{d-2} + \cdots = (X_0)^d f(X_1/X_0, \ldots, X_n/X_0).$$

We recover Deligne's result above: V is $\mathbb{A}^n$, and f is f. To see that the constant works out correctly (to be $(d-1)^n$), recall that the total Chern class of $\mathbb{P}^n$ is $(1+L)^{n+1}$, so the constant is

$$\begin{aligned}
\left|\int_{\mathbb{P}^n} (1+L)^n/(1+dL)\right| &= \left|\text{coef. of } L^n \text{ in } (1+L)^n/(1+dL)\right| \\
&= \left|\text{coef. of } L^n \text{ in } \left(\sum_a \mathrm{Binom}(n, a)L^a\right)\left(\sum_b (-d)^b L^b\right)\right| \\
&= \left|\sum_a \mathrm{Binom}(n, a)(-d)^{n-a}\right| = |(1-d)^n| = (d-1)^n.
\end{aligned}$$

In this paper, we drop, as much as we can, all the hypotheses of nonsingularity made in the results discussed above. It is for this reason that we speak of "singular" exponential sums in the title. Of course, our estimates suffer, but by surprisingly little. It turns out that, under quite general conditions, only the dimension δ of the singular locus of $X \cap L \cap H$ costs us anything, if we are willing to have results valid for all p sufficiently large. (The question of effectively estimating just how large p need be is itself quite interesting and far from being solved. It is closely related to questions of independence of ℓ. We discuss it below.) If we want results that are valid in a fixed characteristic p, we need to assume further that d is prime to p, and we need sometimes to take into account the dimension of the singular locus of $X \cap L$.

One cautionary remark is perhaps in order: our result may reveal essentially nothing about sums that are in fact quite nice. We illustrate with two examples.

(1) Suppose we wish to sum $\psi(f)$, f a polynomial of degree $d > 1$ in n variables, over $\mathbb{A}^n$. If the subscheme of $\mathbb{P}^{n-1}$ defined by the highest-degree part F_d of f is highly singular, then our result is quite weak. For instance, if F_d is a d-th power, then $\delta = n - 2$, and we get $2n - 1$ as an upper bound for the weights in large characteristic. Yet we know from [KL, 5.5.1] that if we add to f a general linear term, i.e., replace $f(x_1, \dots, x_n)$ by $f + \sum_i a_i x_i$ with the a_i sufficiently general, then despite the fact that we have not changed F_d, the sum becomes pure of weight n.

(2) The $(n+1)$-variable Kloosterman sum

$$\sum_{x_1 x_2 \dots x_{n+1} = 1} \psi\left(\sum_i x_i\right)$$

is known to be pure of weight n (see [D2, 7.5]). Yet a naive treatment would be to view the domain of summation as the open set $X[1/L]$, where $L := X_0$ is invertible in the singular projective hypersurface X of equation $(X_0)^{n+1} = \prod_{i=1}^{n+1} X_i$, and to view the summand as $\psi(H/L)$ for $H := \sum_{i=1}^{n+1} X_i$. In this case, $X \cap L \cap H$ is the reducible hypersurface of equation $\left(\prod_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} X_i\right) = 0$ in $\mathbb{P}^{n-1}$. For $n \geq 2$, its singular locus is of codimension 1; i.e., $\delta = n - 3$. Thus we get the upper bound $n + \delta + 1 = 2n - 2$ for the weights, quite far from the correct upper bound, which is n, as soon as n is large.

Despite these caveats, there are some situations where we get significant improvements over what was previously known. Some of these are given in the final section.

Notation, assumptions, and statements of the main results

We fix a perfect field k, a (large) integer N, an integer $r \geq 1$, and a list of r strictly positive integers $D_1, \dots, D_r$. We work in the projective space $\mathbb{P}^N$ over k. Inside $\mathbb{P}^N$, we give ourselves a closed subscheme X, which is definable, scheme-theoretically, in $\mathbb{P}^N$ by a set of r homogeneous equations of degrees $D_1, \dots, D_r$. We call the data $(N, r, D_1, \dots, D_r)$ a *numerical type* for X. (Of course, a given X in $\mathbb{P}^N$ admits many different numerical types.)

We assume throughout that at least one of the conditions (H1) or (H1′) holds.

(H1) $X \otimes_k \bar{k}$ is irreducible and integral, of dimension $n \geq 1$.

(H1′) X is Cohen-Macaulay and equidimensional, of dimension $n \geq 1$.

We give ourselves a linear form on X,

$$L \in H^0(X, \mathcal{O}(1)),$$

an integer $d \geq 1$, and a form of degree d on X,

$$H \in H^0(X, \mathcal{O}(d)).$$

We assume throughout that the following condition holds:

(H2) The scheme-theoretic intersection $X \cap L \cap H$ has dimension $n-2$.

We denote by $\mathrm{Sing}(X \cap L \cap H)$ the singular locus of $X \cap L \cap H$. Because we are over a perfect field, this is the set of points of $X \cap L \cap H$ whose local ring is not regular. We denote by δ the dimension of this singular locus:

$$\delta := \text{dimension of } \mathrm{Sing}(X \cap L \cap H).$$

We adopt the convention that the empty scheme has dimension -1.

By (H2), the scheme-theoretic intersection $X \cap L$ has dimension $n-1$. We denote by ε the dimension of its singular locus:

$$\varepsilon := \text{dimension of } \mathrm{Sing}(X \cap L).$$

Lemma 3. We have an a priori inequality

$$\varepsilon \leq \delta + 1. \qquad \square$$

Proof. Zariski locally on $X \cap L \cap H$, $X \cap L \cap H$ is defined in $X \cap L$ by one equation, say, h. So any closed point x of $X \cap L \cap H$ that is regular in $X \cap L \cap H$ is regular in $X \cap L$. (Just lift back the $n-2$ parameters defining x in $X \cap L \cap H$ and tack on the defining equation h to get $n-1$ parameters defining x in $X \cap L$.) In other words, we have

$$\mathrm{Reg}(X \cap L \cap H) \subset H \cap \mathrm{Reg}(X \cap L).$$

Taking complements in $X \cap L \cap H$, we have

$$H \cap \mathrm{Sing}(X \cap L) \subset \mathrm{Sing}(X \cap L \cap H).$$

Because (a high enough power of) H is (the restriction to X of) a hypersurface in the ambient $\mathbb{P}^N$, for every closed subscheme Z of X, we have (see [H, 7.2])

$$\dim(H \cap Z) \geq \dim(Z) - 1.$$

Thus we get

$$\delta \geq \dim(H \cap \mathrm{Sing}(X \cap L)) \geq \dim(\mathrm{Sing}(X \cap L)) - 1 = \varepsilon - 1. \qquad \blacksquare$$

We put

$$V := X[1/L], f := H/L^d\colon V \to \mathbb{A}^1.$$

For a nontrivial $\mathbb{C}$-valued additive character ψ of k, we are interested in bounding the sum $\sum_{x \in V(k)} \psi(f(x))$.

Theorem 4. Given a numerical type $(N, r, D_1, \dots, D_r)$ and an integer $d \geq 1$, denote by C the explicit Bombieri constant

$$C := C(N, r, D_1, \dots, D_r, d) := (4\,\mathrm{Sup}(1 + D_1, \dots, 1 + D_r, d) + 5)^{N+r}.$$

For any finite field k in which the integer d is invertible, any nontrivial $\mathbb{C}$-valued additive character ψ of k, and any data (X in $\mathbb{P}^N, L, H$) over k as above, with X of numerical type $(N, r, D_1, \dots, D_r)$ and H of degree d, which satisfies (H1) (or (H1)′) and (H2), we have the following estimates.

(1) If $\varepsilon \leq \delta$, then

$$\left|\sum_{x \in V(k)} \psi(f(x))\right| \leq C \times (\#k)^{(n+1+\delta)/2}.$$

(2) If $\varepsilon = \delta + 1$, then

$$\left|\sum_{x \in V(k)} \psi(f(x))\right| \leq C \times (\#k)^{(n+2+\delta)/2}. \qquad \square$$

Remark on the constant $C(N, r, D_1, \dots, D_r, d)$. Bombieri [B, Theorems 1 and 2] showed that if we consider any exponential sum of any polynomial in N variables of degree at most d over any closed subscheme of $\mathbb{A}^N$ defined by r equations of degrees at most $D_1, \dots, D_r$, then the total degree (i.e., total number of zeroes and poles in $\mathbb{C}$) of the corresponding L-function is bounded by

$$C := C(N, r, D_1, \dots, D_r D, d) := (4\,\mathrm{Sup}(1 + D_1, \dots, 1 + D_r, d) + 5)^{N+r}.$$

If we think of L as the homogeneous coordinate X_0 on $\mathbb{P}^N$, then $X[1/L]$ is a closed subscheme of $\mathbb{A}^N$ defined by r equations of degrees at most $D_1, \dots, D_r$, and the function inside the ψ is a polynomial of degree at most d. So if we are able, using ℓ-adic cohomology, to show that the (at most C in number, after cancellation) Frobenius eigenvalues that occur all have weight $\leq n + 1 + \delta$ in case (1), and weight $\leq n + 1 + 2$ in case (2), then we get the asserted estimate.

Theorem 5. Given a numerical type $(N, r, D_1, \dots, D_r)$ and an integer $d \geq 1$, there exists a constant

$$B := B(N, r, D_1, \dots, D_r, d)$$

such that the following result holds. Take any finite field k of characteristic $p > B$, any nontrivial $\mathbb{C}$-valued additive character ψ of k, any data (X in $\mathbb{P}^N, L, H$) over k as above, with X of numerical type $(N, r, D_1, \ldots, D_r)$ and H of degree d, which satisfies (H1) (or (H1)′) and (H2), and any nonzero form G of degree $d-1$ on X,

$$G \in H^0(X, \mathcal{O}(d-1)).$$

Consider the ratio H/LG as a function on $X[1/LG]$; i.e., put

$$V := X[1/LG], f := H/LG\colon\ V \to \mathbb{A}^1.$$

Then for C_1 the effective constant,

$$\begin{aligned} C_1 &:= C(N+1, r+1, D_1, \ldots, D_r, d, d+1) \\ &:= (4\,\mathrm{Sup}(1+D_1, \ldots, 1+D_r, d+1, d+1)+5)^{N+r+2}, \end{aligned}$$

we have the estimate

$$\left|\sum_{x \in V(k)} \psi(f(x))\right| \le C_1 \times (\#k)^{(n+1+\delta)/2}. \qquad \square$$

Remarks. How do Theorems 4 and 5 compare? A major deficiency of Theorem 5 is that the constant B is, for the moment, not effective, and hence we have no idea just how large p need be. Its major advantage over Theorem 4 is that for large p, it gives the “good” upper bound $n+1+\delta$ for the weight, independent of the value of ε. What about the constant C_1? It arises when we view $V := X[1/LG]$ as the open set of $X[1/L]$ where the polynomial $g := G/L^{d-1}$ is invertible. On $X[1/L]$ we have the polynomial $h := H/L^d$. We introduce a new variable x_{n+1}, i.e., pass to $X[1/L] \times \mathbb{A}^1$. In our exponential sum, we are summing, inside the ψ, the polynomial $x_{n+1}h$ of degree at most $d+1$ over the locus $gx_{n+1} = 1$ in $X[1/L] \times \mathbb{A}^1$. The locus of summation is defined in $\mathbb{A}^{N+1}$ by the previous r equations of degrees at most $D_1, \ldots, D_r$ in the first N variables, and by one more equation $gx_{n+1} = 1$ of degree at most d. So to prove the theorem, we may use ℓ-adic cohomology and show that all Frobenius eigenvalues that occur have weight at most $n+\delta+1$.

Proof of Theorem 5: The method of pencils

In this section, we fix a numerical type $(N, r, D_1, \ldots, D_r)$ and an integer $d \ge 1$.

Lemma 6. Fix a prime number ℓ. There exists a constant

$$\mathrm{Betti}(\ell, N, r, D_1, \ldots, D_r)$$

with the following property. For any algebraically closed field K of characteristic not ℓ, any closed subscheme X of $\mathbb{P}^N$ over K that has numerical type $(N, r, D_1, \ldots, D_r)$, and any integer $i \geq 0$, we have

$$\dim_{\mathbb{Q}_\ell} H^i(X, \mathbb{Q}_\ell) \leq \mathrm{Betti}(\ell, N, r, D_1, \ldots, D_r). \qquad \square$$

Proof. Given a field k and a closed subscheme X of $\mathbb{P}^N$ over k that has numerical type $(N, r, D_1, \ldots, D_r)$, let us define an *equation list of type* $(N, r, D_1, \ldots, D_r)$ for X to be the auxiliary choice of homogeneous coordinates $X_0, \ldots, X_N$ for $\mathbb{P}^N$ together with a choice of r homogeneous forms F_{D_i} of degree D_i in these variables that define X. There is an obvious universal family

$$\pi\colon\ \mathcal{X} \to S$$

of data (X in $\mathbb{P}^N$ of numerical type $(N, r, D_1, \ldots, D_r)$, together with an equation list of numerical type $(N, r, D_1, \ldots, D_r)$). Namely, one takes for S the product, over Z, of the r projective spaces whose points are the homogeneous forms in $n+1$ variables of degree D_i for $i = 1$ to r. Inside $\mathbb{P}^N \times S$, one takes $\mathcal{X}$ to be the closed subscheme (incidence variety) consisting of those points

$$\{(x \in \mathbb{P}^N, F_{D_i} \text{ for } i = 1 \text{ to } r) \text{ at which } F_{D_i}(x) = 0 \text{ for } i = 1 \text{ to } r\},$$

with the map $\pi\colon \mathcal{X} \to S$ induced by the projection pr_2 of $\mathbb{P}^N \times S$ onto S. This map is proper. If we invert a prime number ℓ, then we get the proper map

$$\pi[1/\ell]\colon\ \mathcal{X}[1/\ell] \to S[1/\ell].$$

By [SGA 4, Exposé XIV, Thm. 1.1], the sheaves $R^i(\pi[1/\ell])_*\mathbb{Q}_\ell$ on $S[1/\ell]$ are each constructible (i.e., lisse on each piece of a stratification of $S[1/\ell]$ by finitely many connected regular locally closed subschemes $W_{j,\ell}$ of $S[1/\ell]$), and they vanish for i outside the range $[0, 2N]$. Moreover, by proper base change (see [SGA 4, Exposé XII, Thm 5.1]), the stalk of $R^i(\pi[1/\ell])_*\mathbb{Q}_\ell$ at a geometric point s of $S[1/\ell]$ is the cohomology group $H^i(\mathcal{X}_s, \mathbb{Q}_\ell)$ of the corresponding geometric fibre. As S varies over all geometric points of $S[1/\ell]$, the fibre $\mathcal{X}_s$ runs over, with many repetitions, all possible X in $\mathbb{P}^N$ over algebraically closed fields of characteristic not ℓ that are of numerical type $(N, r, D_1, \ldots, D_r)$. So we have only to take

$$\mathrm{Betti}(\ell, N, r, D_1, \ldots, D_r) := \mathrm{Sup}_i \mathrm{Sup}_j \text{rank of } R^i(\pi[1/\ell])_*\mathbb{Q}_\ell | W_{j,\ell}. \qquad \blacksquare$$

Remark on the constants B and Betti. It is an old result of Milnor (see [M]) that for X in

$\mathbb{P}^N$ over $\mathbb{C}$ of numerical type $N, D_1, \ldots, D_r$, if we define

$$D_{\sup} := \mathrm{Sup}_i(D_i),$$

then we have, for every ℓ, the inequality

$$\sum_i \dim_{\mathbb{Q}_\ell} H^i(X, \mathbb{Q}_\ell) \leq ND_{\sup}(2D_{\sup} - 1)^{2N+1}.$$

(In fact, Milnor's inequality holds with coefficients in any field (see [M]).)

We expect that this same Milnor inequality [M] holds in arbitrary characteristic not ℓ. If it does, then we can take $\mathrm{Betti}(\ell, N, r, D_1, \ldots, D_r)$ to be $ND_{\sup}(2D_{\sup} - 1)^{2N+1}$ for every ℓ, and so we can take the constant $B(N, r+1, D_1, \ldots, D_r, d)$ to be

$$1 + \mathrm{Betti}(\ell, N, r+1, D_1, \ldots, D_r, d),$$

as we see below (see the remark following the proof of Theorem 15).

Lemma 7. Let K be the fraction field of a Henselian discrete valuation ring R, whose residue field k is perfect of characteristic $p > 0$, K^{sep} a separable closure of K, D the decomposition group $\mathrm{Gal}(K^{sep}/K)$, $I \subset D$ the inertia group, and $P \subset I$ the wild inertia group (see [S]). Let ℓ be a prime not p, and suppose V is a finite-dimensional $\mathbb{Q}_\ell$ or $\mathbb{F}_\ell$-vector space on which I acts continuously, by $\rho\colon I \to GL(V)$. Then we have the following:

(1) If $p > \ell^{\dim(V)}$, then ρ is tame; i.e., P acts trivially.

(2) If $p - 1 > \dim(V)$ and if $\ell \bmod p$ is a generator of the multiplicative group $(\mathbb{F}_p)^\times$, or more generally, if $\dim(V) <$ (the order of $\ell \bmod p$), then ρ is tame. □

Proof. In the $\mathbb{Q}_\ell$ case, pick an I-stable $\mathbb{Z}_\ell$ lattice $\mathbf{V}$ in V, so that the representation lands in $GL(\mathbf{V})$. The kernel of the reduction mod ℓ map from $GL(\mathbf{V})$ to $GL(\mathbf{V}/\ell\mathbf{V})$ is pro-ℓ, being $1 + \ell\,\mathrm{End}(\mathbf{V})$, so ρ is tame on V if and only if $\rho \bmod \ell$ is tame on $\mathbf{V}/\ell\mathbf{V}$. So it suffices to treat the $\mathbb{F}_\ell$-case. We show that under either of the hypotheses (1) or (2), the group $GL(V) \cong GL(\dim(V), \mathbb{F}_\ell)$ has order prime to p. Indeed, this group has order

$$\prod_{i=0}^{\dim(V)-1} (\ell^{\dim(V)} - \ell^i) = (\text{a power of } \ell) \times \prod_{i=1}^{\dim(V)} (\ell^i - 1).$$

Under either hypothesis (1) or (2), no factor $(\ell^i - 1)$ is divisible by p. ■

Corollary 8. Let k be a perfect field of characteristic $p > 0$, let C/k be a smooth, geometrically connected curve over k, let $\bar{C}/k$ be its complete nonsingular model, let ℓ be a prime number not p, and let $\mathcal{F}$ be a constructible $\mathbb{Q}_\ell$-sheaf on C of generic rank r; i.e., on any dense open set U in C on which $\mathcal{F}$ is lisse, it is lisse of rank r. Suppose that either $p > \ell^r$, or that the multiplicative order of $\ell \bmod p$ is greater than r. Then $\mathcal{F}$ is everywhere tamely ramified on $\bar{C}$; i.e., for every closed point $x \in \bar{C}$, the action of the inertia group $I(x)$ on the geometric generic fibre $\mathcal{F}_{\bar{\eta}}$ is tame. $\square$

We now return to the given numerical type $(N, r, D_1, \ldots, D_r)$ and integer $d \geq 1$. Let k be a perfect field of characteristic $p > 0$, and (X in $\mathbb{P}^N$, L, H) over k with X of numerical type $(N, r, D_1, \ldots, D_r)$, which satisfies (H1) (or (H1)′) and (H2). Given any nonzero $G \in H^0(X, \mathcal{O}(d-1))$, we wish to consider the one-parameter family $H = \lambda LG$ of hypersurface sections of X. More precisely, inside $X \times \mathbb{A}^1$ consider the incidence variety

$$\tilde{X} := \{\text{points } (x, \lambda), \text{ where } H(x) = \lambda(LG)(x)\}$$

and the projection onto the second factor

$$\tilde{f}\colon\ \tilde{X} \to \mathbb{A}^1.$$

The fibre over a point λ is precisely the hypersurface section $H = \lambda LG$ of X.

Lemma 9. Under the hpotheses given in the paragraph above, the morphism

$$\tilde{f}\colon\ \tilde{X} \to \mathbb{A}^1$$

is flat. $\square$

Proof. We reduce immediately to the case where k is algebraically closed (the hypothesis (H1)′ is stable under extension of the ground field—see [EGA, IV, 6.7.1 and 7.3.8]). Because the base $\mathbb{A}^1$ is reduced and the morphism is projective, it suffices to check that all the fibres have the same Hilbert polynomial (see [H, III, Thm. 9.9] for the case of an integral base, which is good enough here, but the given proof works over a reduced base as well, using [Mu, Lemma 1, p. 51] and [EGA, III, 2.2.3 and 7.9.14]).

If (H1) and (H2) hold, we argue as follows. Let us temporarily admit that for each $\lambda \in \mathbb{A}^1$, the form $H - \lambda LG$ is nonzero in $H^0(X, \mathcal{O}(d))$. Then for each fibre $\tilde{X}_\lambda$, we have a short exact sequence on X,

$$0 \to \mathcal{O}_X(-d) \xrightarrow{\times(H-\lambda LG)} \mathcal{O}_X \to \mathcal{O}_{\tilde{X}_\lambda} \to 0.$$

(The named map is injective because $H - \lambda LG$ is nonzero and X is integral.) Twisting by $\mathcal{O}_X(k)$ for every k, we get an equality of Hilbert polynomials

$$\chi(\tilde{X}, \mathcal{O}_{\tilde{X}_\lambda}(k)) = \chi(X, \mathcal{O}_X(k)) - \chi(X, \mathcal{O}_X(k-d)),$$

which shows that $\chi(\tilde{X}, \mathcal{O}_{\tilde{X}_\lambda}(k))$ is independent of $\lambda \in \mathbb{A}^1$.

To see that $H - \lambda LG$ is nonzero, it suffices to see that its vanishing defines a subscheme of X (namely, $\tilde{X}_\lambda$) of dimension $n - 1$. But the intersection of this subscheme with the hyperplane $L = 0$ is the subscheme $X \cap L \cap H$, which by (H2) has dimension $n-2$, and hence $\tilde{X}_\lambda$ has dimension $n-1$.

If (H1)$'$ and (H2) hold, then $(H - \lambda LG, L)$ is a regular sequence for the graded ring $\oplus_{m \geq 0} \Gamma(X, \mathcal{O}_X(m))$. In particular, the sheaf sequence above remains exact, and we argue as above. ■

Lemma 10. Suppose that k is finite, and ψ is a nontrivial $\mathbb{C}$-valued additive character of k. With the notations

$$V := X[1/LG], \qquad f := H/LG\colon\ V \to \mathbb{A}^1,$$

the sum

$$\sum_{x \in V(k)} \psi(f(x))$$

may be expressed in terms of the fibration $\tilde{f}\colon\ \tilde{X} \to \mathbb{A}^1$ by the identity

$$\sum_{x \in V(k)} \psi(f(x)) = \sum_{\lambda \in \mathbb{A}^1(k)} \psi(\lambda)\#(\tilde{X}_\lambda(k)). \qquad \square$$

Proof. We may rewrite the sum as

$$\begin{aligned} &\sum_{\lambda \in \mathbb{A}^1(k)} \psi(\lambda)\#\{x \in V(k) \text{ with } f(x) = \lambda\} \\ &\qquad = \sum_{\lambda \in \mathbb{A}^1(k)} \psi(\lambda)\#\{x \in (X[1/LG])(k) \text{ with } H(x)/(LG)(x) = \lambda\} \\ &\qquad = \sum_{\lambda \in \mathbb{A}^1(k)} \psi(\lambda)\#\{x \in \tilde{X}_\lambda(k) \text{ with } (LG)(x) \text{ nonzero}\}. \end{aligned}$$

The difference of this from $\sum_{\lambda \in \mathbb{A}^1(k)} \psi(\lambda)\#(\tilde{X}_\lambda(k))$ is

$$\sum_{\lambda \in \mathbb{A}^1(k)} \psi(\lambda)\#\{x \in \tilde{X}_\lambda(k) \text{ with } (LG)(x) = 0\}.$$

Now $\tilde{X}_\lambda$ has equation $H = \lambda LG$ in X, so $LG = 0$ in $\tilde{X}_\lambda$ defines the subscheme $X \cap H \cap (LG)$, independent of $\lambda \in \mathbb{A}^1$. Thus our difference is

$$\#\{x \in X(k) \text{ with } H(x) = (LG)(x) = 0\} \times \sum_{\lambda \in \mathbb{A}^1(k)} \psi(\lambda),$$

and this last sum vanishes, because ψ is nontrivial. ■

We now express the sum cohomologically.

Lemma 11. Suppose that k is finite. For any prime ℓ invertible in k, we have the cohomological formula

$$\sum_{\lambda\in\mathbb{A}^1(k)}\psi(\lambda)\#(\tilde{X}_\lambda(k)) = \sum_{a,b}(-1)^{a+b}\mathrm{Trace}\left(\mathrm{Frob}_k, H^a_c(\mathbb{A}^1\otimes_k \bar{k}, \mathcal{L}_\psi\otimes R^b\tilde{f}_*\mathbb{Q}_\ell)\right).$$

(In this formula, $\mathcal{L}_\psi$ makes sense as a $\bar{\mathbb{Q}}_\ell$ sheaf, the tensor product in $\mathcal{L}_\psi\otimes R^b\tilde{f}_*\mathbb{Q}_\ell$ is over $\mathbb{Q}_\ell$, the resulting sheaf and its cohomology group are viewed as $\bar{\mathbb{Q}}_\ell$-objects, and the trace is taken in that sense.) □

Proof. For each $\lambda\in\mathbb{A}^1(k)$, the Lefschetz trace formula on $\tilde{X}_\lambda$ with constant $\mathbb{Q}_\ell$ coefficients gives

$$\#(\tilde{X}_\lambda(k)) = \sum_b(-1)^b\mathrm{Trace}(\mathrm{Frob}_{k,\lambda}R^b\tilde{f}_*\mathbb{Q}_\ell).$$

Thus,

$$\sum_{\lambda\in\mathbb{A}^1(k)}\psi(\lambda)\#(\tilde{X}_\lambda(k)) = \sum_b(-1)^b\sum_{\lambda\in\mathbb{A}^1(k)}\mathrm{Trace}(\mathrm{Frob}_{k,\lambda}, \mathcal{L}_\psi\otimes R^b\tilde{f}_*\mathbb{Q}_\ell).$$

For each b, applying the Lefschetz trace formula on $\mathbb{A}^1$ with $\mathcal{L}_\psi\otimes R^b\tilde{f}_*\mathbb{Q}_\ell$ coefficients gives

$$\begin{aligned}&\sum_{\lambda\in\mathbb{A}^1(k)}\mathrm{Trace}(\mathrm{Frob}_{k,\lambda}, \mathcal{L}_\psi\otimes R^b\tilde{f}_*\mathbb{Q}_\ell)\\ &\qquad=\sum_a(-1)^a\mathrm{Trace}(\mathrm{Frob}_k, H^a_c(\mathbb{A}^1\otimes_k\bar{k}, \mathcal{L}_\psi\otimes R^b\tilde{f}_*\mathbb{Q}_\ell)).\end{aligned}$$ ■

Key Lemma 12. For any algebraically closed overfield E of k, and any point $\lambda\in\mathbb{A}^1(E)=E$, the $(n-1)$-dimensional fibre $\tilde{X}_\lambda := X\cap(H=\lambda LG)$ has the dimension of its singular locus bounded by $1+\delta$:

$$\dim\mathrm{Sing}(\tilde{X}_\lambda)\le 1+\delta.$$ □

Proof. The scheme-theoretic intersection of $\tilde{X}_\lambda$ with the hyperplane $L=0$ is $X\cap L\cap H$, as noted above in the proof of Lemma 9. So any regular point of $X\cap L\cap H$ is regular on $\tilde{X}_\lambda$, and hence

$$L\cap\mathrm{Sing}(\tilde{X}_\lambda)\subset\mathrm{Sing}(X\cap L\cap H).$$

Thus a hyperplane section of $\mathrm{Sing}(\tilde{X}_\lambda)$ has dimension $\le\delta$, and hence (compare to the proof of Lemma 3) $\mathrm{Sing}(\tilde{X}_\lambda)$ has dimension at most $1+\delta$. ■

Theorem 13. Let k be a perfect field of characteristic $p > 0$, and $(X \text{ in } \mathbb{P}^N, L, H)$ over k with X of numerical type $(N, r, D_1, \dots, D_r)$, which satisfies (H1) (or (H1)′) and (H2). Given a nonzero $G \in H^0(X, \mathcal{O}(d-1))$, denote by

$$\tilde{f}\colon \tilde{X} \to \mathbb{A}^1$$

the one-parameter family $H = \lambda LG$ of hypersurface sections of X. For any prime number ℓ invertible in k, we have the following results concerning the sheaves $R^i\tilde{f}_*\mathbb{Q}_\ell$ on $\mathbb{A}^1$.

(1) For $i > n + \delta + 1$, the sheaf $R^i\tilde{f}_*\mathbb{Q}_\ell$ is lisse on $\mathbb{A}^1$.

(2) For $i = n + \delta + 1$, denote by $j\colon U \to \mathbb{A}^1$ the inclusion of a dense open set on which the sheaf $R^i\tilde{f}_*\mathbb{Q}_\ell$ is lisse. Then we have a short exact sequence of sheaves on $\mathbb{A}^1$,
$0 \to (\text{punctual, support in } \mathbb{A}^1 - U) \to R^i\tilde{f}_*\mathbb{Q}_\ell \to j_*j^*R^i\tilde{f}_*\mathbb{Q}_\ell \to 0$,
and the sheaf $j_*j^*R^i\tilde{f}_*\mathbb{Q}_\ell$ is lisse on $\mathbb{A}^1$. □

Proof. We have seen above that $\tilde{f}$ is a proper flat map to $\mathbb{A}^1$ whose fibres have dimension $n-1$ and singular loci of dimension at most $1+\delta$. The result is now immediate from [SGA 7 I, Exposé I, Cor. 4.3], which tells us that for each $\bar{k}$-valued point $s \in \mathbb{A}^1$, the $I(s)$-equivariant specialization map

$$(R^i\tilde{f}_*\mathbb{Q}_\ell)_s \to (R^i\tilde{f}_*\mathbb{Q}_\ell)_{\bar{\eta}}$$

is an isomorphism for $i > n + \delta + 1$ and is surjective for $i = n + \delta + 1$. Thus $R^i\tilde{f}_*\mathbb{Q}_\ell$ is lisse on $\mathbb{A}^1$ for $i > n + \delta + 1$. For $i = n + \delta + 1$, the $I(s)$-equivariance of the surjective specialization map shows that $I(s)$ acts trivially on $(R^i\tilde{f}_*\mathbb{Q}_\ell)_{\bar{\eta}}$, which means precisely that $j_*j^*R^i\tilde{f}_*\mathbb{Q}_\ell$ is lisse on $\mathbb{A}^1$. (In [SGA 7 I, Exposé I, Cor. 4.3], the "n" is the fibre dimension, our "$n-1$," and the "d" there is an upper bound for the dimension of the singular locus of the special fibre, our "$1+\delta$.") ■

Corollary 14. With hypotheses and notation as in Theorem 13 above, we have the following.

(1) Suppose that for each $i \geq n + \delta$, the sheaf $R^i\tilde{f}_*\mathbb{Q}_\ell$ on $\mathbb{A}^1$ is tame at ∞. Then the cohomology groups
$E_2^{a,b} := H_c^a(\mathbb{A}^1 \otimes_k \bar{k}, \mathcal{L}_\psi \otimes R^b\tilde{f}_*\mathbb{Q}_\ell)$
vanish for $a + b \geq n + \delta + 2$.

(2) Suppose that for each $i \geq n + \delta + 1$, the sheaf $R^i\tilde{f}_*\mathbb{Q}_\ell$ on $\mathbb{A}^1$ is tame at ∞. Then the cohomology groups
$E_2^{a,b} := H_c^a(\mathbb{A}^1 \otimes_k \bar{k}, \mathcal{L}_\psi \otimes R^b\tilde{f}_*\mathbb{Q}_\ell)$
vanish for $a+b \geq n+\delta+2$ with the possible exception of $(a = 2,\ b = n+\delta)$.
In particular, $E_2^{a,b} = 0$ for $a + b \geq n + \delta + 3$. □

Proof. For reasons of cohomological dimension, the terms $E^2_{a,b}$ vanish unless a lies in $[0,2]$ and b lies in $[0,2n-2]$. What must be shown is as follows.

(1) For $b \geq n+\delta+2$, $E_2^{a,b}=0$ for all a.

(2) For $b = n+\delta+1$, $E_2^{a,b}=0$ for $a=1,2$.

(3) If $R^{n+\delta}\tilde{f}_*\mathbb{Q}_\ell$ is tame at ∞, then for $b=n+\delta$, $E_2^{2,n+\delta}=0$.

To prove (1), recall that for $b \geq n+\delta+2$, the sheaf $R^b\tilde{f}_*\mathbb{Q}_\ell$ is lisse on $\mathbb{A}^1$. Since it is also tame at ∞, it is geometrically constant on $\mathbb{A}^1$, say, with constant value V^b. Then

$$E_2^{a,b} := H^a_c(\mathbb{A}^1 \otimes_k \bar{k}, \mathcal{L}_\psi \otimes R^b\tilde{f}_*\mathbb{Q}_\ell) = H^a_c(\mathbb{A}^1 \otimes_k \bar{k}, \mathcal{L}_\psi) \otimes V^b.$$

But $H^a_c(\mathbb{A}^1 \otimes_k \bar{k}, \mathcal{L}_\psi) = 0$ for all a.

To prove (2), use the short exact sequence

$$0 \to (\text{punctual}) \to R^{n+\delta+1}\tilde{f}_*\mathbb{Q}_\ell \to j_*j^*R^{n+\delta+1}\tilde{f}_*\mathbb{Q}_\ell \to 0,$$

in which the last term $j_*j^*R^{n+\delta+1}\tilde{f}_*\mathbb{Q}_\ell$ is lisse on $\mathbb{A}^1$ and tame at ∞ and, hence, geometrically constant, say, with constant value $V^{n+\delta+1}$.

Consider the long exact cohomology sequence. We have

$$H^a_c(\mathbb{A}^1 \otimes_k \bar{k}, \text{punctual}) = 0 \quad \text{for } a \geq 1,$$

so we get, for $a \geq 1$, an isomorphism

$$E_2^{a,b} \cong H^a_c(\mathbb{A}^1 \otimes_k \bar{k}, \mathcal{L}_\psi \otimes j_*j^*R^b\tilde{f}_*\mathbb{Q}_\ell) = H^a_c(\mathbb{A}^1 \otimes_k \bar{k}, \mathcal{L}_\psi) \otimes V^{n+\delta+1} = 0.$$

To prove (3), that $H^2_c(\mathbb{A}^1 \otimes_k \bar{k}, \mathcal{L}_\psi \otimes R^{n+\delta}\tilde{f}_*\mathbb{Q}_\ell)$ vanishes, simply note that the sheaf $\mathcal{F} := R^{n+\delta}\tilde{f}_*\mathbb{Q}_\ell$ is tame at ∞, so the sheaf $\mathcal{L}_\psi \otimes \mathcal{F}$ is totally wild at ∞. Now apply [K, Lemme Clef, (1), p. 131] to get the vanishing of $H^2_c(A^1 \otimes_k \bar{k}, \mathcal{L}_\psi \otimes \mathcal{F})$. ■

End of the proof of Theorem 5

Given a numerical type $(N, r, D_1, \dots, D_r)$ and an integer $d \geq 1$, recall that for each prime ℓ, we have the constant

$$\mathrm{Betti}(\ell, N, r+1, D_1, \dots, D_r, d);$$

this is an upper bound for the ℓ-adic Betti numbers of any closed subscheme of $\mathbb{P}^N$ over an algebraically closed field of characteristic not ℓ, which is definable by $r+1$ forms of degrees $D_1, \dots, D_r, d$.

Theorem 15 (= Theorem 5 bis). Given a numerical type $(N, r, D_1, \ldots, D_r)$ and an integer $d \geq 1$, take for B the constant

$$B := 2^{\mathrm{Betti}(\ell=2,N,r+1,D_1,\ldots,D_r,d)}.$$

Let k be a finite field k of characteristic $p > B$, and let ψ be any nontrivial $\mathbb{C}$-valued additive character of k. Suppose we are given data $(X \text{ in } \mathbb{P}^N, L, H)$ over k with X of numerical type $(N, r, D_1, \ldots, D_r)$ and H of degree d, which satisfy (H1) (or (H1)$'$) and (H2):

(H1) $X \otimes_k \bar{k}$ is irreducible and integral, of dimension $n \geq 1$;

(H1)$'$ X is Cohen-MacCaulay and equidimensional, of dimension $n \geq 1$;

(H2) the scheme-theoretic intersection $X \cap L \cap H$ has dimension $n - 2$.

Denote by δ the dimension of the singular locus of $X \cap L \cap H$, with the convention that the empty scheme has dimension -1.

For any nonzero form G of degree $d - 1$ on X,

$$G \in H^0(X, \mathcal{O}(d-1)),$$

consider the ratio H/LG as a function on $X[1/LG]$; i.e., put

$$V := X[1/LG], \quad f := H/LG\colon\ V \to \mathbb{A}^1.$$

Then for C_1 the effective constant

$$C_1 := C(N+1, r, D_1, \ldots, D_r, d, d+1),$$

we have the estimate

$$\left| \sum_{x \in V(k)} \psi(f(x)) \right| \leq C \times (\#k)^{(n+1+\delta)/2}. \qquad \square$$

Proof. Since p is odd, we may use 2-adic cohomology. Consider

$$\tilde{f}\colon\ \tilde{X} \to \mathbb{A}^1,$$

the one-parameter family $H = \lambda LG$ of hypersurface sections of X. The fibres of $\tilde{f}$ are all of numerical type $(N, r+1, D_1, \ldots, d_r, d)$, so their Betti numbers are all uniformly bounded by $\mathrm{Betti}(\ell, N, r+1, D_1, \ldots, D_r, d)$. Applying this to the geometric generic fibre of $\tilde{f}$, we find

$$\dim_{\mathbb{Q}_2}((R^i\tilde{f}_*\mathbb{Q}_2)_{\bar{\eta}}) \leq \mathrm{Betti}(\ell, N, r+1, D_1, \ldots, D_r, d).$$

Viewing $(R^i\tilde{f}_*\mathbb{Q}_2)_{\bar{\eta}}$ as a representation of $I(\infty)$, the inertia group at ∞, it follows from Lemma 7 and the hypothesis $p > B$ that this representation is tame for every i. In other words, all the sheaves $R^i\tilde{f}_*\mathbb{Q}_2$ on $\mathbb{A}^1$ are tame at ∞. By Corollary 14, we have

$$H^a_c(\mathbb{A}^1 \otimes_k \bar{k}, \mathcal{L}_\psi \otimes R^b\tilde{f}_*\mathbb{Q}_2) \text{ vanishes for } a + b \geq n + \delta + 2.$$

So by Lemmas 10 and 11, we have the following cohomological expression for $\sum_{x \in V(k)} \psi(f(x))$:

$$\sum_{x \in V(k)} \psi(f(x)) = \sum_{\lambda \in \mathbb{A}^1(k)} \psi(\lambda) \#(\tilde{X}_\lambda(k))$$
$$= \sum_{a+b \le n+\delta+1} (-1)^{a+b} \mathrm{Trace}(\mathrm{Frob}_k, H^a_c(\mathbb{A}^1 \otimes_k \bar{k}, \mathcal{L}_\psi \otimes R^b \tilde{f}_* \mathbb{Q}_2)).$$

By the main theorem of [D3, 3.3.3 and 3.3.4], we know that $H^a_c(\mathbb{A}^1 \otimes_k \bar{k}, \mathcal{L}_\psi \otimes R^b \tilde{f}_* \mathbb{Q}_2)$ is mixed of weight less than $a + b$. Thus all the eigenvalues that enter have weight at most $n + 1 + \delta$. ∎

Remark. It is conjectured that for any scheme X of finite type over an algebraically closed field k, any prime number ℓ invertible in k, and any integer $i \ge 0$, the dimension of $H^i_c(X, \mathbb{Q}_\ell)$ is independent of the choice of ℓ. If this is true, then $\mathrm{Betti}(\ell, N, r+1, D_1, \ldots, D_r, d)$ is independent of ℓ. In that case, we would have Theorem 5 with the choice $B := 1 + \mathrm{Betti}(\ell, N, r+1, D_1, \ldots, D_r, d)$. To see this, choose a prime ℓ which, mod p, has multiplicative order $p - 1$ (this choice is possible, thanks to Dirichlet), and use ℓ-adic cohomology. For $p > B$, we have

$$p - 1 > \dim_{\mathbb{Q}_\ell}((R^i \tilde{f}_* \mathbb{Q}_\ell)_{\bar{\eta}}) \text{ for every } i.$$

We conclude from Lemma 7 that $R^i \tilde{f}_* \mathbb{Q}_\ell$ is tame at ∞ for all i, and the rest of the proof is unchanged.

A cautionary example. Here is an example to show that the requirement in Theorem 15 (= Theorem 5 bis) that the characteristic p of k be sufficiently large cannot be dropped, even when $d = 1$. Fix an integer $n \ge 2$. In $\mathbb{P}^{n+1}$ over k with homogeneous coordinates $Y, Z, X_1, \ldots, X_n$, take for X the hypersurface of equation

$$Y^p - YZ^{p-1} = X_1 Z^{p-1},$$

p being the characteristic of k.

Take for L the linear form Z, and for H the linear form X_1. Then $X \cap L$ is the everywhere singular hypersurface $Y^p = 0$ in the $\mathbb{P}^n$ defined by $Z = 0$, and $X \cap L \cap H$ is the entirely singular hypersurface $Y^p = 0$ in the $\mathbb{P}^{n-1}$ defined by $Z = X_1 = 0$. Thus we have $\delta = n - 2$. Thus $n + 1 + \delta$ is $2n - 1$.

Take for ψ a nontrivial additive character of k of the form $\psi_1 \circ \mathrm{Trace}_{k/\mathbb{F}_p}$, with ψ_1 a nontrivial additive character of $\mathbb{F}_p$; i.e., ψ is nontrivial and $\psi(a^p - a) = 1$ for every $a \in k$. For

$$V := X[1/L], \quad f := H/L\colon\ V \to \mathbb{A}^1,$$

the sum

$$\sum_{x \in V} \psi(f(x))$$

is pure of weight $2n$. Indeed, V is the variety $y^p - y = x_1 \in \mathbb{A}^{n+1}$ with coordinates $y, x_1, \ldots, x_n$, f is the function x_1, and our sum is

$$\sum_{x \in V(k)} \psi(x_1).$$

If we solve for x_1 in terms of y, then V becomes the affine space $\mathbb{A}^n$ with coordinates $y, x_2, \ldots, x_n$, and the sum becomes

$$\sum_{y, x_2, \ldots, x_n \in k} \psi(y^p - y) = (\#k)^n.$$

This same example is also a case in which the estimate of Theorem 4(2) (the case "$\varepsilon = \delta + 1$") is sharp.

Another cautionary remark. Consider the special case of Theorem 15 in which X is $\mathbb{P}^n$, with homogeneous coordinates $X_0, X_1, \ldots, X_n$, L is X_0, and $d \geq 2$. Then we are looking at

$$\sum_{x \in (\mathbb{A}^n[1/g])(k)} \psi(h(x)/g(x)),$$

where h is a polynomial of degree d, and where g is an arbitrary nonzero polynomial of degree $\leq d - 1$. The theorem asserts that if the highest-degree term H_d of h defines a smooth hypersurface in $\mathbb{P}^{n-1}$, then in any sufficiently large characteristic, this sum is bounded by $C \times (\#k)^{n/2}$.

It is natural to ask what one can say about sums of (ψ evaluated at) more general rational functions. What can we say about

$$\sum_{x \in (\mathbb{A}^n[1/g])(k)} \psi(h(x)/g(x)),$$

if we keep h as above, of degree d with $H_d = 0$ smooth in $\mathbb{P}^{n-1}$, but now allow g to be a nonzero polynomial of degree $\geq d$?

If g has degree d, this reduces to a question we can treat by the same techniques. We can view h/g as H/G with H and G homogeneous of degree d, and we are looking at the sum

$$\sum_{x\in(\mathbb{P}^n[1/LG])(k)} \psi((H/G)(x)),$$

which is the difference $\mathrm{Sum}_1 - \mathrm{Sum}_2$ of the two sums

$$\sum_{x\in(\mathbb{P}^n[1/G])(k)} \psi((H/G)(x)) - \sum_{x\in((\mathbb{P}^n\cap L)[1/G])(k)} \psi((H/G)(x)).$$

If we take the d-fold Segre embeddings of P^n and of $\mathbb{P}^n \cap L$, respectively, into giant projective spaces, then H and G become linear forms, and we can apply Theorem 5 to these. Thus, for example, suppose that $\mathbb{P}^n \cap G \cap H$ is smooth of dimension $n-2$, and suppose that $\mathbb{P}^n \cap G \cap H \cap L$ is of dimension $n-3$ with at worst isolated singularities. Then from Theorem 4 we get an estimate $O((\#k)^{n/2})$ for both Sum_1 and Sum_2 above, in all large enough characteristics.

What can we say if $\mathrm{degree}(g) > \mathrm{degree}(h)$? For instance, what happens if we look at

$$\sum_{x\in(\mathbb{A}^n[1/h])(k)} \psi(1/h(x)),$$

where h is a polynomial of degree d, which is as nice as you please? To fix ideas, suppose that h is H/L^d for $L = X_0$, and that both $\mathbb{P}^n \cap H$ and $\mathbb{P}^n \cap H \cap L$ are smooth, of dimensions $n-1$ and $n-2$, respectively. In this case, for $n \geq 3$, we claim that we have

$$\sum_{x\in(\mathbb{A}^n[1/h])(k)} \psi(1/h(x)) = (\#k)^{n-1} + O(\#k)^{n-2}.$$

Thus the sum has weight $2n-2$, far worse than the “n” one might naively expect. To see this, recall from [K, 5.1.2] that the complete sum has a good estimate:

$$\sum_{x\in(\mathbb{P}^n[1/H])(k)} \psi((L^d/H)(x)) = O((\#k)^{n/2}).$$

But our affine sum differs from this “complete” sum by

$$\begin{aligned}\sum_{x\in(\mathbb{P}^n\cap L)[1/H])(k)} \psi((L^d/H)(x)) &= \#((\mathbb{P}^n \cap L)[1/H])(k)\\ &= (\#k)^{n-1} + O((\#k)^{n-2}).\end{aligned}$$

Proof of Theorem 4

Let us recall the statement of Theorem 4.

Theorem 16 (= Theorem 4 bis). Given a numerical type $(N, r, D_1, \ldots, D_r)$ and an integer $d \geq 1$, denote by C the explicit Bombieri constant

$$C := C(N, r, D_1, \ldots, D_r, d) := (4\,\mathrm{Sup}(1 + D_1, \ldots, 1 + D_r, d) + 5)^{N+r}.$$

Let k be a finite field in which d is invertible, and let ψ be any nontrivial $\mathbb{C}$-valued additive character of k. Suppose we are given data (X in $\mathbb{P}^N$, L, H), which satisfy (H1) (or (H1)$'$) and (H2):

(H1) $X \otimes_k \bar{k}$ is irreducible and integral, of dimension $n \geq 1$;

(H1)$'$ X is Cohen-Macaulay and equidimensional, of dimension $n \geq 1$;

(H2) the scheme-theoretic intersection $X \cap L \cap H$ has dimension $n - 2$.

Denote by δ the dimension of the singular locus of $X \cap L \cap H$, and denote by ε the dimension of the singular locus of $X \cap L$, with the convention that the empty scheme has dimension -1.

Then we have the following estimates.

(1) If $\varepsilon \leq \delta$, then

$$\left|\sum_{x \in V(k)} \psi(f(x))\right| \leq C \times (\#k)^{(n+1+\delta)/2}.$$

(2) If $\varepsilon = \delta + 1$, then

$$\left|\sum_{x \in V(k)} \psi(f(x))\right| \leq C \times (\#k)^{(n+2+\delta)/2}. \qquad \square$$

Proof. We pick any prime ℓ invertible in k, and we work with ℓ-adic cohomology. We claim that it suffices to show that the following statements (A) and (B) hold.

(A) If $\varepsilon \leq \delta$, then $R^b\tilde{f}_*\mathbb{Q}_\ell$ is tame at ∞ for $b \geq n + \delta$.

(B) If $\varepsilon = \delta + 1$, then $R^b\tilde{f}_*\mathbb{Q}_\ell$ is tame at ∞ for $b \geq n + \delta + 1$.

Indeed, if (A) holds, then Corollary 14(1) gives

$$H^a_c(\mathbb{A}^1 \otimes_k \bar{k}, \mathcal{L}_\psi \otimes R^b\tilde{f}_*\mathbb{Q}_\ell) = 0 \text{ for } a + b \geq n + \delta + 2,$$

and we conclude by invoking [D3] as in the proof of Theorem 5. If (B) holds, then Corollary 14(2) gives

$$H^a_c(\mathbb{A}^1 \otimes_k \bar{k}, \mathcal{L}_\psi \otimes R^b\tilde{f}_*\mathbb{Q}_\ell) = 0 \text{ for } a + b \geq n + \delta + 3,$$

and we conclude as above.

We first show that (A) and (B) hold in the case $d = 1$. In this case, we consider the one-parameter family hyperplane sections $\mu H - \lambda L = 0$ of X, parameterized by (λ, μ) in $\mathbb{P}^1$. Let us denote by $\mathcal{X}$ in $X \times \mathbb{P}^1$ the incidence variety

$$\{(x, (\lambda, \mu)) \text{ with } \mu H(x) = \lambda L(x)\},$$

and denote by π: $\mathcal{X} \to \mathbb{P}^1$ the projection. (Thus over $\mathbb{A}^1 \subset \mathbb{P}^1$, we have the map $\tilde{f}$: $\tilde{X} \to \mathbb{A}^1$, to which we have added $X \cap L$ as the fibre over $\infty := (1,0)$). The map π is proper and flat, and its fibres have dimension $n-1$. Moreover, all fibres have a singular locus of dimension at most $1+\delta$. So exactly as in Theorem 13, we have:

(1) for $i > n+\delta+1$, the sheaf $R^i\pi_*\mathbb{Q}_\ell$ is lisse on $\mathbb{P}^1$;

(2) for $i = n+\delta+1$, denote by j: $U \to \mathbb{P}^1$ the inclusion of a dense open set on which the sheaf $R^i\pi_*\mathbb{Q}_\ell$ is lisse. Then we have a short exact sequence of sheaves on $\mathbb{P}^1$,
$0 \to (\text{punctual, support in } \mathbb{P}^1 - U) \to R^i\pi_*\mathbb{Q}_\ell \to j_*j^*R^i\pi_*\mathbb{Q}_\ell \to 0$,
and the sheaf $j_*j^*R^i\tilde{f}_*\mathbb{Q}_\ell$ is lisse on $\mathbb{P}^1$.

Restricting to $\mathbb{A}^1$, we find that $R^b\tilde{f}_*\mathbb{Q}_\ell$ is tame at ∞ for $b \geq n+\delta+1$, which is (B).

If in addition $\varepsilon \leq \delta$, then the ∞-fibre of π, whose dimension is $n-1$, has singular locus of dimension $\leq \delta$. By [SGA 7 I, Exposé I, Cor. 4.3], the specialization map

$$(R^{n+\delta}\pi_*\mathbb{Q}_\ell)_\infty \to (R^{n+\delta}\pi_*\mathbb{Q}_\ell)_{\bar{\eta}} = (R^{n+\delta}\tilde{f}_*\mathbb{Q}_\ell)_{\bar{\eta}}$$

is surjective. Thus, the inertia group $I(\infty)$ acts trivially, and hence tamely, on $(R^{n+\delta}\tilde{f}_*\mathbb{Q}_\ell)_{\bar{\eta}}$, which proves (A).

We now show that (A) and (B) hold when $d \geq 1$ is prime to p. For this, we introduce the variety $Y := X[H^{1/d}]$ obtained from X by adjoining the d-th root of H. More precisely, in $\mathbb{P}^{N+1}$ with coordinates $(Z, X_0, \ldots, X_N)$, we consider the locus defined by the r forms F_i in the X-variables $(X_0, \ldots, X_N)$, which define X in $\mathbb{P}^N$, together with the equation

$$Z^d = H(X_0, \ldots, X_N).$$

On Y, both L and Z make sense as linear forms, and we consider the one parameter family of hyperplane sections $\mu Z - \lambda L = 0$ of Y, parameterized by $(\lambda, \mu) \in \mathbb{P}^1$. We denote by $\mathcal{Y}$ in $Y \times \mathbb{P}^1$ the corresponding incidence variety, and we denote by

$$\pi\colon\ \mathcal{Y} \to \mathbb{P}^1$$

the projection. The fibre $\pi^{-1}(\infty)$ over the point $(\lambda, \mu) = (1,0)$ at ∞ is

$$Y \cap (L = 0) \cong (X \cap L)[H^{1/d}].$$

The intersection of this ∞-fibre with the hyperplane $Z = 0$ is $X \cap L \cap H$. In particular, the singular locus of the ∞-fibre has dimension at most $1+\delta$.

Lemma 17. If $\varepsilon := \dim\mathrm{Sing}(X\cap L) \leq \delta := \dim\mathrm{Sing}(X\cap L\cap H)$, then $\dim\mathrm{Sing}((X\cap L)[H^{1/d}]) \leq \delta$. $\square$

Proof. Over the open set of $X \cap L$, where H is invertible, the scheme $(X \cap L)[H^{1/d}]$ is finite étale (because d is prime to p), so $\mathrm{Sing}((X \cap L)[H^{1/d}])[1/H]$ is finite étale over

$\mathrm{Sing}(X \cap L)[1/H]$. If x is a point of $X \cap L \cap H$ that is regular, then x is a regular point in $X \cap L$, H is transverse to $X \cap L$ at x, and $(X \cap L)[H^{1/d}]$ is regular at the unique point lying over x. Thus the singularities of $(X \cap L)[H^{1/d}]$ lie over, by a finite flat map, singularities of either $X \cap L \cap H$ or of $(X \cap L)[1/H]$, and both of these singular loci have dimension at most δ. ∎

Lemma 18. The morphism $\pi\colon \mathcal{Y} \to \mathbb{P}^1$ is flat. □

Proof. The fibre over a finite point $(\lambda, 1)$ is the hyperplane section $Z = \lambda L$ of Y. Now Y is defined over X by the equation $Z^d = H$, so its intersection with $Z = \lambda L$ is the hypersurface section $H = \lambda^d L^d$ of X. Exactly as in the proof of Lemma 9, we see that all the fibres over finite points $(\lambda, 1)$ have the same Hilbert polynomial.

It remains to see that the ∞-fibre $(X \cap L)[H^{1/d}]$ has this same Hilbert polynomial. For this, we argue as follows. Think of X as $\mathrm{Proj}(R_\bullet)$, with $R_\bullet$ the graded ring $\oplus_{m \geq 0} H^0(X, \mathcal{O}_X(m))$ in which $L \in R_1$ and $H \in R_d$ are nonzero elements. If (H1) and (H2) hold, $R_\bullet$ is a graded integral domain. If (H1)′ and (H2) hold, both (L, H) and (H, L) are regular sequences in $R_\bullet$. Under either set of hypotheses, the maps $f \mapsto fL$ and $f \mapsto fH$ of $R_\bullet$ to itself are both injective.

We may obtain $Y = X[H^{1/d}]$ as follows. Form the graded ring $R_\bullet[Z]$, with Z an indeterminate of degree 1. Then we obtain Y as $\mathrm{Proj}(R_\bullet[Z]/(Z^d - H))$, and the ∞-fibre $\pi^{-1}(\infty) = (X \cap L)[H^{1/d}]$ is $\mathrm{Proj}(R_\bullet[Z]/(Z^d - H, L)) = \mathrm{Proj}((R_\bullet/(L))[Z]/Z^d - H))$. In this description, $(R_\bullet/(L))[Z]/(Z^d - H)$ is free of rank d as a graded module over $R_\bullet/(L)$, with basis $1, Z, \ldots, Z^{d-1}$. Therefore, for each integer i, the k-dimension of its graded piece of degree i is given by

$$\begin{aligned}\dim((R_\bullet/(L))[Z]/(Z^d - H))_i &= \sum_{j=0}^{d-1} \dim(R_\bullet/(L))_{i-j} = \sum_{j=0}^{d-1} \{\dim R_{i-j} - \dim R_{i-j-1}\} \\ &= \dim R_i - \dim R_{i-d} = \dim(R_\bullet/(H))_i.\end{aligned}$$

As this holds for all i, it holds for all large i, and hence the ∞-fibre $(X \cap L)[H^{1/d}]$ has the same Hilbert polynomial as the 0-fibre $X \cap H$. ∎

Lemma 19. The sheaves $R^i\pi_*\mathbb{Q}_\ell$ on $\mathbb{P}^1$ are unramified at ∞ (meaning that $I(\infty)$ acts trivially on $(R^i\pi_*\mathbb{Q}_\ell)_{\bar{\eta}}$) for $i \geq n + \delta + 1$. In particular, these sheaves for $i \geq n + \delta + 1$ are all tame at ∞. If $\varepsilon \leq \delta$, then the sheaf $R^{n+\delta}\pi_*\mathbb{Q}_\ell$ is also unramified, and hence tame, at ∞. □

Proof. The map π is proper and flat, with fibres of dimension $n - 1$. The ∞-fibre has a singular locus of dimension at most $1 + \delta$, because its intersection with the hyperplane $Z = 0$ is $X \cap L \cap H$. By Lemma 17, if $\varepsilon \leq \delta$, then the ∞-fibre has a singular locus of dimension less than or equal to δ. The result now follows from [SGA 7 I, Exposé I, Cor. 4.3] (see the proof of Theorem 13). ■

Lemma 20. The morphism $\tilde{f}\colon \tilde{X} \to \mathbb{A}^1$, with fibre over λ the hypersurface section $X \cap (H - \lambda L^d = 0)$ of X, is related to the morphism $\pi\colon \mathcal{Y} \to \mathbb{P}^1$, with fibre over (λ, μ), the hyperplane section $\mu Z = \lambda L$ of $Y := X[H^{1/d}]$, as follows. Denote by

$$[d]\colon \mathbb{A}^1 \to \mathbb{A}^1$$

the d-th power map. The restriction π_{aff} of π to $\pi^{-1}(\mathbb{A}^1)$ sits in a Cartesian diagram

$$\begin{tikzcd} \pi^{-1}(\mathbb{A}^1) \arrow[r] \arrow[d, "\pi_{\mathrm{aff}}"] & \tilde{X} \arrow[d, "\tilde{f}"] \\ \mathbb{A}^1 \arrow[r, "{[d]}"'] & \mathbb{A}^1 . \end{tikzcd}$$ □

Proof. In terms of the graded ring $R_\bullet$, whose Proj is X, the scheme $\tilde{X}$ is

$$\mathrm{Proj}_{k[\lambda]}(R_\bullet[\lambda]/(H - \lambda L^d)).$$

The fibre product of $\tilde{X}$ with $\mathbb{A}^1$ over $\mathbb{A}^1$ by $(\tilde{f}, [d])$ is

$$\mathrm{Proj}_{k[\lambda]}(R_\bullet[\lambda]/(H - \lambda^d L^d)).$$

But the scheme $\pi^{-1}(\mathbb{A}^1)$ is

$$\mathrm{Proj}_{k[\lambda]}(R_\bullet[\lambda, Z]/(Z^d - H, Z - \lambda L)) = \mathrm{Proj}_{k[\lambda]}(R_\bullet[\lambda]/((\lambda L)^d - H)).$$ ■

Corollary 21. The sheaves $R^i\tilde{f}_*\mathbb{Q}_\ell$ on $\mathbb{A}^1$ attached to $\tilde{f}\colon \tilde{X} \to \mathbb{A}^1$ with fibres the hypersurface sections $X \cap (H - \lambda L^d = 0)$ and the restrictions to $\mathbb{A}^1$ of the sheaves $R^i\pi_*\mathbb{Q}_\ell$ on $\mathbb{P}^1$ are related as follows. For every i, we have

$$R^i\pi_*\mathbb{Q}_\ell \mid \mathbb{A}^1 \cong [d]^* R^i\tilde{f}_*\mathbb{Q}_\ell.$$ □

Proof. The result follows immediately from Lemma 20 by proper base change (see [SGA 7 I, Exposé XII, Thm. 5.1]). ■

Corollary 22. The sheaves $R^i\tilde{f}_*\mathbb{Q}_\ell$ on $\mathbb{A}^1$ are tame at ∞ for $i \geq n + \delta + 1$. If $\varepsilon \leq \delta$, then the sheaf $R^{n+\delta}\tilde{f}_*\mathbb{Q}_\ell$ on $\mathbb{A}^1$ is also tame at ∞. □

Proof. Since d is prime to p, the sheaf $R^i\tilde{f}_*\mathbb{Q}_\ell$ on $\mathbb{A}^1$ is tame at ∞ if and only if its $[d]$ pullback $[d]^*R^i\tilde{f}_*\mathbb{Q}_\ell \cong R^i\pi_*\mathbb{Q}_\ell \mid \mathbb{A}^1$ is tame at ∞. So the result is immediate from Lemma 19. ■

Thus we have proven that when d is prime to p, we have the assertions (A) and (B) given at the beginning of the proof of Theorem 16 (= Theorem 4 bis). ■

Application to complete intersections

Let k be a field, and let X in $\mathbb{P}^{n+r}$ over k be a smooth complete intersection of dimension $n \geq 1$, of multidegree $(D_1, D_2, \dots, D_r)$. So (H1) holds. Thus X has numerical type $(n + r, D_1, D_2, \dots, D_r)$. We suppose that all D_i are at least 2, i.e., that X is not contained in a hyperplane. A remarkable result of Zak and Fulton-Lazarsfeld [F-L, Remark 7.5] asserts that for any nonzero linear form $L \in H^0(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}}(1)) \cong H^0(X, \mathcal{O}_X(1))$, the intersection $X \cap L$ is either smooth or has only isolated singularities. Now let L and H be nonzero linear forms on X, and suppose that $X \cap L$ is smooth. Then $X \cap L$ is itself a smooth complete intersection of dimension $n-1$ in the $\mathbb{P}^{n+r-1}$ of equation $L = 0$, of the same multidegree $(D_1, D_2, \dots, D_r)$. So $X \cap L \cap H$ is either smooth or has at worst isolated singularities. Thus from Theorem 4 we get the following estimates.

Theorem 23. Let k be a finite field, and let ψ be a nontrivial $\mathbb{C}$-valued additive character of k. Let X in $\mathbb{P}^{n+r}$ over k be a smooth complete intersection of dimension n, of multidegree $(D_1, D_2, \dots, D_r)$, all $D_i \geq 2$. Let L and H be nonzero linear forms in $H^0(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}}(1)) \cong H^0(X, \mathcal{O}_X(1))$, and suppose that $X \cap L$ is smooth. Put

$$V := X[1/L], \quad f := H/L\colon V \to \mathbb{A}^1,$$
$$C := C(n, r, D_1, \dots, D_r, 1) := (4\,\mathrm{Sup}(1 + D_1, \dots, 1 + D_r, 1) + 5)^{n+r}.$$

Then we have the following estimates:

$$\left| \sum_{x \in X[1/L](k)} \psi(f(x)) \right| \leq C \times (\#k)^{n/2} \text{ if } X \cap L \cap H \text{ is smooth,}$$

and

$$\left| \sum_{x \in X[1/L](k)} \psi(f(x)) \right| \leq C \times (\#k)^{(n+1)/2} \text{ if } X \cap L \cap H \text{ is not smooth.} \qquad \square$$

Proof. This is an instance of Theorem 4 with $\varepsilon = -1$ and δ either -1 or 0. ■

Remark. In the case where $X \cap L \cap H$ is smooth, one can give a slightly better constant of a topological nature (see [K, 5.1.1 and its proof] for details). Denote by $c(X)$ the total Chern class of X. Explicitly, we have

$$c(X) = (1 + L)^{n+r+1} / \prod_{i=1}^{r} (1 + D_i L).$$

Then, when $X \cap L \cap H$ is smooth, we can take C to be

$$\left| \int_X c(X)/(1 + L)^2 \right| = \left(\prod_{i=1}^{r} D_i \right) \times \left| \text{coef. of } L^n \text{ in } \left(1 + L\right)^{n+r-1} / \prod_{i=1}^{r} (1 + D_i L) \right|.$$

This expression has total degree $n + r$ in the D_i's, and so it is of the same order of magnitude as $C(n, r, D_1, \ldots, D_r, 1)$, at least if all the D_i are near a common large D.

In a similar vein, we might begin with X in $\mathbb{P}^{n+r}$ over k a complete intersection of dimension n, of multidegree $(D_1, D_2, \ldots, D_r)$, whose singular locus has some dimension $\gamma \geq 0$. (Recall that a complete intersection in $\mathbb{P}^{n+r}$ is Cohen-Macaulay and equidimensional, so (H1)′ holds.) We suppose that all D_i are at least 2, i.e., that X is not contained in a hyperplane. It follows from the result of Zak and Fulton-Lazarsfeld [F-L, Remark 7.5] that for any nonzero linear form $L \in H^0(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}}(1)) \cong H^0(X, \mathcal{O}_X(1))$, the intersection $X \cap L$ has a singular locus of dimension $\leq \gamma + 1$. To see this, take $\gamma + 1$ general linear forms $L_1, \ldots, L_{\gamma+1}$. The intersection

$$X \cap L_1 \cap \ldots \cap L_{\gamma+1}$$

is then a smooth complete intersection of dimension $n-\gamma-1$ and multidegree $(D_1, D_2, \ldots, D_r)$. By the result of Zak and Fulton and Lazarsfeld [F-L, Remark 7.5] applied to this smooth complete intersection, the further intersection

$$X \cap L \cap L_1 \cap \ldots \cap L_{\gamma+1}$$

has at worst isolated singularities. But any point of

$$L_1 \cap \ldots \cap L_{\gamma+1} \cap \mathrm{Sing}(X \cap L)$$

is singular on $X \cap L \cap L_1 \cap \ldots \cap L_{\gamma+1}$ (see the proof of Lemma 3), and hence $L_1 \cap \ldots \cap L_{\gamma+1} \cap \mathrm{Sing}(X \cap L)$ is finite or empty, which in turn implies that $\mathrm{Sing}(X \cap L)$ has dimension at most $\gamma + 1$. For a general L, $X \cap L$ has its singular locus of dimension $\gamma - 1$. Now let L and H be nonzero linear forms on X, and suppose that $X \cap L$ has $\varepsilon = \gamma - 1$. As $X \cap L$ is itself a complete intersection of dimension $n - 1$ in the $\mathbb{P}^{n+r-1}$ of equation $L = 0$, of the same multidegree $(D_1, D_2, \ldots, D_r)$, we conclude that $X \cap L \cap H$ has $\delta \leq \gamma$. Of course, for a general H, we have $\delta = \gamma - 2$.

Applying Theorem 4, we find the following result.

Theorem 24. Let k be a finite field, and let ψ be a nontrivial $\mathbb{C}$-valued additive character of k. Let X in $\mathbb{P}^{n+r}$ over k be a complete intersection of dimension n, of multidegree $(D_1, D_2, \ldots, D_r)$, all $D_i \geq 2$, whose singular locus has dimension γ. Let L and H be nonzero linear forms in $H^0(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}}(1)) \cong H^0(X, \mathcal{O}_X(1))$, and suppose that $X \cap L$ has singular locus of dimension $\varepsilon = \gamma - 1$. Put

$$V := X[1/L], \quad f := H/L\colon\ V \to \mathbb{A}^1,$$
$$C := C(n, r, D_1, \ldots, D_r, 1) := (4\,\mathrm{Sup}(1 + D_1, \ldots, 1 + D_r, 1) + 5)^{n+r}.$$

Then we have the following estimates:

$$\left|\sum_{x\in X[1/L](k)} \psi(f(x))\right| \le C \times (\#k)^{(n+\gamma)/2} \text{ if } \delta \le \gamma - 1$$

and

$$\left|\sum_{x\in X[1/L](k)} \psi(f(x))\right| \le C \times (\#k)^{(n+\gamma+1)/2} \text{ if not (i.e., if } \delta = \gamma). \qquad \square$$

Proof. This is an instance of Theorem 4 with $\varepsilon = \gamma - 1$ and δ either $\gamma - 2$ or $\gamma - 1$ or γ. ■

If we take arbitrary nonzero L and H, then $X \cap L$ has $\varepsilon \le \gamma + 1$ and $X \cap L \cap H$ has $\delta \le \gamma + 2$. So at worst we have the following estimate.

Theorem 25. Let k be a finite field, and let ψ be a nontrivial $\mathbb{C}$-valued additive character of k. Let $X \in \mathbb{P}^{n+r}$ over k be a complete intersection of dimension n, of multidegree $(D_1, D_2, \ldots, D_r)$, all $D_i \ge 2$, whose singular locus has dimension γ. Let L and H be *any* nonzero linear forms in $H^0(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}}(1)) \cong H^0(X, \mathcal{O}_X(1))$. Put

$$V := X[1/L], \quad f := H/L\colon V \to \mathbb{A}^1,$$
$$C := C(n, r, D_1, \ldots, D_r, 1) := (4\,\mathrm{Sup}(1 + D_1, \ldots, 1 + D_r, 1) + 5)^{n+r}.$$

Then

$$\left|\sum_{x\in X[1/L](k)} \psi(f(x))\right| \le C \times (\#k)^{(n+\gamma+3)/2}.$$

$\square$

Proof. This is an instance of Theorem 4 with $\varepsilon \le \gamma + 1$ and $\delta \le \gamma + 2$. ■

Acknowledgments

I learned from Luo the idea that results of this type, where the main player is the dimension δ of the singular locus of $X \cap L \cap H$, should be true for exponential sums. Especially, he thought that Theorem 24 should be true. It is a pleasure to thank him.

References

[B] E. Bombieri, *On exponential sums in finite fields II*, Invent. Math. **47** (1978), 29–39.

[D1] P. Deligne, *La conjecture de Weil I*, Inst. Hautes Études Sci. Publ. Math. **43** (1974), 273–307.

[D2] ———, "Application de la formule des traces aux sommes trigonométriques" dans *Cohomologie Étale (SGA 4 1/2)*, Lecture Notes in Math. **569**, Springer, New York, 1977, 168–232.

[D3] ———, *La conjecture de Weil II*, Inst. Hautes Études Sci. Publ. Math. **52** (1980), 137–252.

[EGA] J. Dieudonné and A. Grothendieck, *Eléments de Géométrie Algébrique*, Inst. Hautes Études Sci. Publ. Math. **4** (1960); **8** (1961); **11** (1961); **17** (1963); **20** (1964); **24** (1965); **28** (1966); **32** (1967).

[F-L] W. Fulton and R. Lazarsfeld, "Connectivity and its applications in algebraic geometry" in *Algebraic Geometry (Chicago, Ill., 1980)*, Lecture Notes in Math. **862**, Springer, Berlin, 1981, 26–92.

[H] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. **52**, Springer Verlag, New York, 1977.

[K] N. Katz, *Sommes exponentielles, rédigé par G. Laumon*, Astérisque **79**, Soc. Math. France, Paris, 1980.

[KL] N. Katz and G. Laumon, *Transformation de Fourier et majoration des sommes exponentielles*, Inst. Hautes Études Sci. Publ. Math. **62** (1986), 361–418.

[L] G. Laumon, "Majoration de sommes trigonométriques (d'apres P. Deligne et N. Katz)" dans *Caracéristique d'Euler-Poincare (Seminaire de l'ENS 1978/9)*, Astérisque **82-83**, Soc. Math. France, Paris, 1981, 221–258.

[M] J. Milnor, *On the Betti numbers of real varieties*, Proc. Amer. Math. Soc. **15** (1964), 275–280.

[Mu] D. Mumford, *Abelian Varieties*, Oxford Univ. Press, London, 1970.

[S] J.-P. Serre, *Corps Locaux*, 2d ed., Hermann, Paris, 1968.

[SGA 7 I] A. Grothendieck, M. Raynaud, P. Deligne, and D. Rim, *Groupes de Monodromie en Géometrie Algébrique*, *I*, Séminaire de Géométrie Algebrique du Bois-Marie (SGA 7 1), Lecture Notes in Math. **288**, Springer, Berlin, 1972.

[SGA 4] M. Artin, A. Grothendieck, and J.-L. Verdier, *Théorie des Topos et Cohomologie Étale des Schémas*, *Vol. 3*, Séminaire de Géométrie Algebrique du Bois-Marie (SGA 4), Lecture Notes in Math. **305**, Springer, Berlin, 1973.

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