Syntax and Semantics

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The year is 2002 and here we are at a symposium on Foundations and the Ontological Quest.

The first thing to say is how bleak the present situation is. In foundational studies of mathematics and physics we have been stuck for seventy years; despite numerous books, articles, and meetings, there has been no real progress.

Seventy years ago Gödel established his incompleteness theorem [1], destroying the foundational program of David Hilbert [2]. Just a year or two later, Gödel showed [3] that Brouwer's intuitionism [4], advanced as a correct constructive alternative to an allegedly incorrect classical mathematics, was actually an extension of classical mathematics rather than an alternative to it. In short order Gödel dashed the hopes of the formalists and the intuitionists, seemingly leaving no choice other than blind Platonic belief. I shall say more about this later.

Just over seventy years ago, Heisenberg [5] and Schrödinger [6] created quantum mechanics. There have been innumerable attempts to develop a rational understanding of this eminently successful physical theory, but not one of them has met with general acceptance. The foundations of cosmology seem to change radically every few years. It is a field in which, more than in any other science, the problem of separating observational fact from theoretical assumption is refractory.

I look forward to learning about foundational problems of cognitive science. I hope there is progress there. But in mathematics and physics there has been little or none in seventy long years.

When James Cook made his marvelous voyages of discovery in the 1700s, he found people speaking essentially the same language in the immense triangle from New Zealand to Easter Island to Hawaii [7]. What a contrast to Europe with its short distances and babel of different tongues!

The Polynesians were immensely skilled sailors, traveling hundreds and even thousands of kilometers by canoe on the open ocean [8]. This was their syntax.

When a canoe was built, the goddess Papa, wife of Wakea, was prayed to lift the tabu from a chosen tree, and the goddess Lea, wife of Moku-halii, was prayed to lift the tabu from the finished canoe:

> She initiated, she pointed the canoe;
> She started it, she guided it;
> She lifted the tabu from it,
> Lifted was the tabu from the canoe of Wakea.

This was their semantics [9].

Now my question is this: what is the evidentiary value of the Polynesian success at navigation as to the reality and efficacy of Papa, Wakea, Lea, and Moku-halii? Surely the only answer is: none whatsoever. The success of a syntax is no evidence at all for the validity of an associated semantics.

We admire the skill of the Polynesian navigators, and while we respect their belief system and find beauty in it, we do not share it—it is alien to us.

The pursuit of mathematics is part of our culture and it has met with many successes. Associated with mathematics is a belief system, its semantics. But the success of what mathematicians actually do is no evidence at all for the validity of the associated belief system.

Let me first discuss mathematics psychologically and then logically. I have been doing mathematics for 57 years. What is my experience? Do I discover or invent? Am I a James Cook, finding what was already there, or a Thomas Edison, bringing something new into being? (Cook did not invent Polynesia; Edison did not discover the light bulb.) Each mathematician will have a different answer to this question, for doing mathematics is personal and persons are different. But my answer is unequivocal: for me, the experience is one of invention. I start with an idea, a light bulb. I try various things to realize the idea—perhaps something familiar will work. Or perhaps I try carbon, tin, et cetera, until finally (if I am lucky) I hit on tungsten, and it works.[1] To say that it works, in mathematics, means that I persuade other mathematicians by a proof, for doing mathematics is social as well as personal. We are far more tightly constrained in our inventions than a musician or painter.

What is a proof in mathematics? More than anyone else, David Hilbert deserves the credit for answering this question. For several thousand years, Euclid's great book [10] was held to be the model for a

[1] After reading Oliver Sacks's *Uncle Tungsten*, which one of the organizers of this symposium gave me for Christmas, I see that I was ignorant of how the light bulb was invented.

proof. (Though the work of Archimedes is a better model.) But Euclid's work had serious logical flaws that were corrected by Hilbert [11]. And Hilbert's plane geometry is flawless.

Mathematics is expressed in terms of *formulas*, which are strings of symbols of various kinds put together according to certain rules. As to whether a string of symbols is a formula or not, there is no dispute: one simply checks the rules of formation. Certain formulas are chosen as *axioms*. Here there is great scope for imagination and inspiration from one semantics or another, to choose fruitful axioms. Certain *rules of inference* are specified, allowing one to deduce a formula as conclusion from one or two formulas as premise or premises. Then a *proof* is a string of formulas such that each one is either an axiom or follows from one or two preceding formulas by a rule of inference. As to whether or not a string of formulas is a proof there is no dispute: one simply checks the rules of formation. This is the syntax of mathematics.

Is that all there is to mathematics? Yes, and it is enough. Constructing proofs is a serious, deep, and beautiful vocation, one worthy the devotion of a lifetime. And as Galileo observed, the book of nature is written in mathematics, so doing mathematics can increase our understanding of nature: certainly of physics, and increasingly of the other sciences.

But some mathematicians feel that syntax is not enough: they want to add semantics. Now in defense of semantics it can be said that it is a useful source of inspiration and that it is essential in pedagogy—students who do well in calculus invariably have an understanding of a meaning attached to their calculations. But the question here is the foundational role, if any, of semantics in mathematics.

Let us make a case study: infinitesimals. The ontological question might be phrased as: do infinitesimals exist?

It was Eudoxus who first observed the necessity of postulating that, given two line segments, if the shorter is repeated sufficiently often it will cover the longer—that there are no infinitesimal line segments. This is known as the axiom of Archimedes, though Archimedes gives credit to Eudoxus. So infinitesimals were banished from Greek geometry, only to arise again with the invention of the calculus in the 1600s.

When computing the derivative, or fluxion, of a function $f(x)$, the practice was to assume that $h \neq 0$, to simplify the difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

and then to set $h = 0$. It was George Berkeley [12] who pointed out the logical fallacy of this procedure. He derided fluxions as the "ghosts of departed quantities" and of course he was quite right. But the infinitesimal calculus flourished for several centuries—because it was deep, beautiful, and powerful—with no rational understanding of it, much as quantum mechanics flourishes today. Then in the 1800s infinitesimals were banished from the calculus, with this definition: the derivative of $f(x)$ equals L in case for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all h with $|h| < \delta$ and $h \neq 0$,

$$\left| \frac{f(x+h) - f(x)}{h} - L \right| < \varepsilon.$$

This is a static formulation of the notion, and the original semantics of motion into an infinitesimal region was abandoned.

But in the 1900s infinitesimals rose again, phoenix-like, thanks to the genius of Abraham Robinson, the creator of nonstandard analysis [13]. He concluded his epoch-making book with the prophetic words:

> Returning now to the theory of this book, we observe that it is presented, naturally, within the framework of contemporary Mathematics, and thus appears to affirm the existence of all sorts of infinitary entities. However, from a formalist point of view we may look at our theory syntactically and may consider that what we have done is to introduce *new deductive procedures* rather than new mathematical entities.

His untimely death prevented him from developing this point of view, but what Robinson really created was a new logic.

The central idea of Robinson's new logic can be explained as follows. One of the customary logical symbols is $\forall$ "for all". Introduce a new logical symbol $\forall^{\mathrm{st}}$ "for all fixed". Letting the variables range over the strictly positive real or rational numbers, we can then define

$$x \simeq 0 \;\leftrightarrow\; \forall^{\mathrm{st}}\varepsilon \; x < \varepsilon$$

"x is infinitesimal if and only if for all fixed ε, $x < \varepsilon$". This is a new *concept* (with free variable x), not expressible in classical mathematics. But the *statement* (with no free variables)

$$\exists x \; x \simeq 0$$

"there exists x such that x is infinitesimal" can be expressed in classical mathematics: by the rules of Robinson's new logic, nonstandard analysis, it is equivalent to

$$\forall\varepsilon \, \exists x \; x < \varepsilon$$

"for all ε there exists x such that $x < \varepsilon$", which of course is a trivial theorem of classical mathematics.

The mental image—semantics in its poetic role—behind the concept "x is infinitesimal" is the 17th century image of a number x so small that it is indeed the ghost of a departed quantity. But this mental image plays no ontological role; the existential statement "there exists an infinitesimal x" reduces to a commonplace about strictly positive numbers.

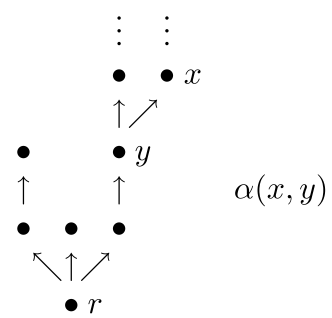
The moral of this is that the role of syntax in mathematics is not to express semantic truths (because there are no semantic truths in mathematics to express). Mathematics is syntax, and the syntax of mathematics is mathematics itself. Syntax can be put to new and creative uses.

Nonstandard analysis is a powerful addition to classical mathematics. Statements and proofs in it can be reduced to classical statements and proofs, and often the nonstandard proofs are much shorter and easier to understand. But the new concepts are in general new, not reducible to classical mathematics.

So do infinitesimals exist or not? This is the wrong question. The question is, as Humpty Dumpty said to Alice [14], which is to be master—that's all. Mathematics is our invention, and we can have infinitesimals or not, as we choose. The only constraint is consistency.

But what a constraint that is! I began this talk by asserting that the present foundational situation is bleak. And indeed, we have no reason to assume that the axiom systems we use in mathematics are consistent. For all we know, they may lead to a contradiction. Platonists believe otherwise, but to a formalist their arguments carry no conviction.

Is there any hope for a future mathematics that is demonstrably consistent and yet powerful? Perhaps so. I am engaged in a syntactical extension of Robinson's new logic, and I shall roughly indicate the idea by means of a beautiful theorem known as König's lemma. Consider a tree such that each node has only finitely many branches rising from it. Denote the root of the tree by r and let $\alpha(x, y)$ express that the node x lies immediately above the node y.



The hypothesis is that for each number n, it is possible to climb up the tree through n nodes:

$$\forall n \exists \psi \big[\, \psi(0) = r \;\&\; \forall j{<}n \; \alpha\big(\psi(j+1), \psi(j)\big) \,\big].$$

This hypothesis is finitary; it can be expressed in a demonstrably consistent theory. The conclusion is that it is possible to climb forever:

$$\exists \phi \big[\, \phi(0) = r \;\; \& \; \forall j \; \alpha\big(\phi(j+1), \phi(j)\big) \,\big].$$

This conclusion is infinitary, and is expressed in set theory, a theory that may or may not be consistent. The proof of König's lemma is a typical nonconstructive argument of classical mathematics, and goes as follows. Only finitely many branches issue from the root. Therefore at least one of the branches must admit the possibility of arbitrarily long climbs from it, since by hypothesis there is the possibility of an arbitrarily long climb from the root. Choose such a branch and repeat the argument. Doing this repeatedly, one climbs forever. Now the idea is to introduce a new logical symbol $\exists^{\mathrm{pf}}$ "there exists a progressive function" and state the conclusion with $\exists\phi$ replaced by $\exists^{\mathrm{pf}}\phi$:

$$\exists^{\mathrm{pf}} \phi \big[\, \phi(0) = r \;\; \& \; \forall j \; \alpha\big(\phi(j+1), \phi(j)\big) \,\big].$$

The mental image of the concept—semantics in its poetic role—is the same, that of an infinitely long climb. But the syntax of the new logical symbol $\exists^{\mathrm{pf}}$ is so constructed that the existential statement reduces to the finitary hypothesis. In this way we avoid the ontology of a completed infinity, a notion that Gauss objected to almost 200 years ago but that is at the core of contemporary set-theoretic mathematics.

So here is a prospect for the new millennium: a new logic for a new mathematics, demonstrably consistent yet powerful.

In conclusion, I summarize my position on the foundations of mathematics (with the explicit avowal that contrary views are cogently held by many mathematicians). The present axiom systems are unsatisfactory because we have no assurance that they are consistent. It is worthwhile to try to construct a demonstrably consistent powerful mathematics. In this endeavor, semantics can only obfuscate, and all notions of a completed infinity must be avoided. What is real in mathematics is simply the formulas and proofs themselves, as strings of symbols.

In mathematics, the ontological quest is misconceived and should be abandoned.

References

[1] Kurt Gödel, "Über formal unentscheidbare Sätze der Principia Mathematica und verwanter Systeme I", *Monatshefte für Mathematik und Physik*, vol. 37, 173-198, 1931.

[2] David Hilbert, "Über das Unendliche", Mathematische Annalen, vol. 95, 1926.

[3] Kurt Gödel, "Zur intuitionistischen Aritmetik und Zahlentheorie", *Ergebnisse eines math. Koll.*, vol. 4, 34-38, 1933.

[4] L. E. J. Brouwer, "Die onbetrouwbaarheid der logische principes", Tijdschrift voor wijsbegeerte, vol. 2, 1908.

[5] Werner Heisenberg, "The Physical Principles of the Quantum Theory", translated by Carl Eckhart and Frank C. Hoyt, The University of Chicago Press, Chicago, Illinois, 1930.

[6] Erwin Schrödinger, "Abhandlungen zur Wellenmechanik", Leipzig, J. A. Barth, 1927.

[7] Richard Hough, "Captain James Cook", W. W. Norton & Co., New York, 1995.

[8] Jack Golson, ed. "Polynesian Navigation: A Symposium on Andrew Sharp's Theory of Accidental Voyages", Third Edition, A. H. & A. W. Reed Ltd., Wellington, 1972.

[9] E. S. Craighill Handy, "Polynesian Religion", The Museum, Honolulu, 1927.

[10] Euclid, "Elements", translated with introduction and commentary by Thomas L. Heath, Dover, New York, 1956.

[11] David Hilbert, "Grundlagen der Geometrie", Leipzig, B. G. Teubner, 1899.

[12] George Berkeley, "The analyst; or, a discourse addressed to an infidel mathematician. Wherein it is examined whether the object, principles, and inferences of the modern analysis are more distinctly conceived, or more evidently deduced, than religious mysteries and points of faith", London, Printed for J. Tonson, 1734.

[13] Abraham Robinson, "Non-Standard Analysis", Revised edition, North-Holland, 1974.

[14] Lewis Carroll, "Through the looking-glass, and what Alice found there", New York, Macmillan & Co., 1875.