# Integration by substitution

The best general advice is this: substitute for the troublesome part of the integral but don't be too greedy. For example, in $\int \frac{e^{\sqrt{t}}dt}{\sqrt{t}}$, the $e^{\sqrt{t}}$ is troublesome, but you should substitute just for the $\sqrt{t}$; let $y = \sqrt{t}$.

There is a natural tendency immediately to differentiate after a substitution is made, but this is not always wise. In the example above, we can get away with this: $\frac{d\sqrt{t}}{dt} = \frac{1}{2\sqrt{t}}$ and we were lucky, since $\frac{1}{\sqrt{t}}$ occurs in the integral. But a safer procedure is to express the old variable in terms of the new and then differentiate. In our example, $y = \sqrt{t}$ so $t = y^2$ and $dt = 2y\,dy$.

Similarly, in $\int \frac{dx}{x - \sqrt{x}}$ the troublesome part is $\sqrt{x}$. Let $y = \sqrt{x}$, so $x = y^2$ and $dx = 2y\,dy$.

## Trigonometric functions and cofuntions

The following pairs of trigonometric functions are called **co**functions of each other, because each function in a pair is the other function of the **co**mplementary angle:

$$\begin{array}{ll} \sin\theta = \cos(\pi/2 - \theta) & \cos\theta = \sin(\pi/2 - \theta) \\ \tan\theta = \cot(\pi/2 - \theta) & \cot\theta = \tan(\pi/2 - \theta) \\ \sec\theta = \csc(\pi/2 - \theta) & \csc\theta = \sec(\pi/2 - \theta) \end{array}$$

Notice that $d(\pi/2 - \theta) = -d\theta$. This has the following consequence:

*In any integration or differentiation formula involving trigonometric functions of $\theta$ alone, we can replace all trigonometric functions by their cofunctions and change the overall sign.*

The use of this rule cut memorization in half.

For example, $\int \cot\theta\,d\theta = \int \frac{\cos\theta\,d\theta}{\sin\theta} = \ln|\sin\theta| + C$, so $\int \tan\theta\,d\theta = -\ln|\cos\theta| + C$.

If we remember that $\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$, then we automatically have $\int \csc\theta\,d\theta = -\ln|\csc\theta + \cot\theta| + C$.

And if we remember that $d\sec\theta = \sec\theta\tan\theta\,d\theta$, then we know that $d\csc\theta = -\csc\theta\cot\theta\,d\theta$.