1. (10 points) Determine whether the following converge or diverge. If they converge, evaluate.

(a) \[
\sum_{n=1}^{\infty} \frac{3^n + 2^{n+1}}{5^n}
\]

\[
\sum_{n=1}^{\infty} \left( \frac{3}{5} \right)^n + \sum_{n=1}^{\infty} 2 \left( \frac{2}{5} \right)^n = \frac{3/5}{1 - 3/5} + \frac{2(2/5)}{1 - 2/5} = \frac{3/5}{2/5} + \frac{4/5}{3/5} = \frac{3}{2} + \frac{4}{3} = \frac{17}{6}
\]

(b) \[
\int_{0}^{1} x^2 \ln x \, dx
\]

First use integration by parts to find an antiderivative:

\[
\int x^2 \ln x \, dx = \frac{x^3 \ln x}{3} - \int \frac{x^2}{3} \, dx = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C
\]

So

\[
\int_{0}^{1} x^2 \ln x \, dx = \left[ \frac{x^3 \ln x}{3} - \frac{x^3}{9} \right]_{0}^{1} = \frac{1}{3} \ln 1 - \frac{1}{9} - \lim_{t \to 0} \frac{x^3 \ln x}{3} + 0 = -\frac{1}{3} \lim_{t \to 0} \frac{t^3 \ln t}{t^3} = -\frac{1}{9}.
\]

Here we need to show that the limit is 0. We use L’Hôpital’s Rule:

\[
\lim_{t \to 0} t^3 \ln t = \lim_{t \to 0} \frac{\ln t}{1/t^3} = \lim_{t \to 0} \frac{1/t}{-3/t^4} = \lim_{t \to 0} \frac{t^3}{3} = 0.
\]

2. (15 points) Determine whether the following improper integrals converge or diverge. Justify your answers.

(a) \[
\int_{1}^{\infty} \frac{\sqrt{x^7 + 100x}}{x^5} \, dx
\]

The numerator \(\sqrt{x^7 + 100x}\) is dominated by the highest power of \(x\), in other words \(\sqrt{x^7 + 100x} \sim x^{7/2}\) as \(x\) goes to \(\infty\). So the quotient will be asymptotic to \(x^{7/2}/x^5 = 1/x^{3/2}\) as \(x\) goes to \(\infty\). Since \(\int_{1}^{\infty} \frac{dx}{x^{3/2}}\) converges by the \(p\)-test with \(p = 3/2\), we can conclude that the original integral converges by the limit comparison test.

(b) \[
\int_{0}^{\infty} \frac{\sqrt[3]{x}}{\sqrt{x} + x^4} \, dx
\]

We need to split the integral, say at \(x = 1\) since both endpoints are problematic.

First consider \(\int_{0}^{1} \frac{\sqrt[3]{x}}{\sqrt{x} + x^4} \, dx\). As \(x\) goes to zero, \(x^4\) dies out much faster than \(\sqrt{x}\), so the denominator will behave more and more like \(\sqrt{x}\). In other words we can say that

\[
\frac{\sqrt[3]{x}}{\sqrt{x} + x^4} \sim \frac{\sqrt[3]{x}}{\sqrt{x}} = \frac{1}{x^{1/2-1/6}} = \frac{1}{x^{1/3}} \text{ as } x \to 0.
\]
Since \( \frac{1}{x^{1/3}} \) converges (by the \([0,1]\)-version of the \(p\) test with \( p = 1/3 \)) we can see that
\[
\int_0^1 \frac{\sqrt{x}}{\sqrt{x} + x^4} \, dx \text{ converges by the limit comparison test.}
\]
Now as \( x \to \infty \) both \( \sqrt{x} \) and \( x^4 \) go to infinity, but \( x^4 \) goes much faster. So as \( x \) goes to infinity,
\[
\frac{\sqrt{x}}{\sqrt{x} + x^4} \, dx \sim \frac{\sqrt{x}}{x^4} = \frac{1}{x^{4-1/6}} \text{ as } x \to \infty.
\]
Using the other \( p \)-test we see that
\[
\int_1^\infty \frac{\sqrt{x}}{\sqrt{x} + x^4} \, dx \text{ also converges by limit comparison.}
\]

(c) \( \int_1^\infty \frac{(\ln x)^2}{x^3 + 2} \, dx \).

Since \( 0 \leq \sin^2 x \leq 1 \) we conclude that
\[
\int_1^\infty \frac{(\ln x)^2}{x^3 + 2} \, dx \leq \int_1^\infty \frac{\ln x}{x^3 + 2} \, dx
\]
For \( x \) in \([1, \infty)\) we know that \( \ln x < x \) and so
\[
\int_1^\infty \frac{\ln x}{x^3 + 2} \, dx < \int_1^\infty \frac{x}{x^3 + 2} \, dx
\]
Since for a rational function the highest powers of \( x \) dominate as \( x \) goes to \( \infty \) we have \( x/(x^3 + 2) \sim 1/x^2 \) as \( x \to \infty \). By the \( p \)-test we know that \( \int_1^\infty \frac{1}{x^2} \, dx \) converges, so by the limit comparison test we know that \( \int_1^\infty \frac{x}{x^3 + 2} \, dx \) also converges and by the comparison test we conclude that the original integral also converges.

3. (25 points) For each of the series below determine whether it converges or diverges. Justify your answers.

(a) \( \sum_{n=1}^\infty \frac{n^2 + 5n}{(n+1)(n+2)(n+3)} \)

This series diverges. For rational functions the highest power dominates as we go to infinity. So the \( n \)th term is asymptotic to \( n^2/n^3 = 1/n \) as \( n \) goes to infinity. Since \( \sum_{n=1}^\infty \frac{1}{n} \) diverges by the \( p \)-test with \( p = 1 \), we conclude that the original series diverges by the limit comparison test.

(b) \( \sum_{n=2}^\infty \frac{1}{n \ln n} \)

In this case we use the integral test. Observe that \( \int_2^\infty \frac{dx}{x \ln x} = \ln(\ln x) + C \) and \( \lim_{t \to \infty} \ln(\ln t) = \infty \) since \( \lim_{t \to \infty} \ln t = \infty \). Therefore the improper integral \( \int_2^\infty \frac{dx}{x \ln x} \) diverges. By the integral test the series diverges as well.
In this case the ratio or root test works well. For example, with the ratio test we have
\[
\lim_{n \to \infty} \frac{(n + 1)^2}{3^{n+1}} \cdot \frac{3^n}{n^2} = \lim_{n \to \infty} \left( \frac{n + 1}{n} \right)^2 \left( \frac{1}{3} \right) = \frac{1}{3}
\]

Since this ratio is less than 1, the ratio test says that the series converges.

Here the easiest solution uses the root test. The \(n\)th root of \(a_n\) is simply \(\frac{n + 1}{3n + 6}\) and as \(n\) goes to infinity this approaches \(\frac{1}{3}\). Since the \(n\)th root of \(a_n\) goes to \(\frac{1}{3}\) and \(\frac{1}{3}\) is less than 1, we conclude that the series behaves more and more like a geometric series with \(r = \frac{1}{3}\) and so it converges.

Here we can argue that \(n^3 + 1\) is larger than \(n^3\). Since the square root function is monotonically increasing we can say that \(\sqrt{n^3 + 1} > \sqrt{n^3} > 0\) and taking reciprocals reverses inequalities on \((0, \infty)\) so
\[
\frac{1}{\sqrt{n^3 + 1}} < \frac{1}{\sqrt{n^3}}
\]

The sum \(\sum \frac{1}{n^{3/2}}\) converges by the \(p\)-test with \(p = 3/2\), so the original series also converges, by the comparison test.