MATH 104 QUIZ # 3 Spring 2003 Covers Sections 8.8, 10.1-10.6 of the textbook

1. *(10 points)* Determine whether the following converge or diverge. If they converge, evaluate.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + 2^{n+1}}{5^n}$

$$\sum_1^{\infty}\left(\frac{3}{5}\right)^n + \sum_1^{\infty} 2\left(\frac{2}{5}\right)^n = \frac{3/5}{1-3/5} + \frac{2(2/5)}{1-2/5} = \frac{3/5}{2/5} + \frac{4/5}{3/5} = \frac{3}{2} + \frac{4}{3} = \frac{17}{6}$$

   (b) $\displaystyle\int_0^1 x^2 \ln x\, dx$ First use integration by parts to find an antiderivative:

$$\int x^2 \ln x\, dx = \frac{x^3 \ln x}{3} - \int \frac{x^2}{3}\, dx = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C$$

   So

$$\int_0^1 x^2 \ln x\, dx = \lim_{t\to 0} \frac{x^3 \ln x}{3} - \frac{x^3}{9}\Big|_t^1 = \frac{1 \ln 1}{3} - \frac{1}{9} - \lim_{t\to 0} \frac{x^3 \ln x}{3} + \frac{0}{9} = -\frac{1}{9} - \frac{1}{3}\lim_{t\to 0} t^3 \ln t = -\frac{1}{9}.$$

   Here we need to show that the limit is 0. We use L'Hôpital's Rule:

$$\lim_{t\to 0} t^3 \ln t = \lim_{t\to 0} \frac{\ln t}{1/t^3} = \lim_{t\to 0} \frac{1/t}{-3/t^4} = \lim_{t\to 0} -\frac{t^3}{3} = 0.$$

2. *(15 points)* Determine whether the following improper integrals converge or diverge. Justify your answers.

   (a) $\displaystyle\int_1^{\infty} \frac{\sqrt{x^7 + 100x}}{x^5}\, dx$ converges.
   The numerator $\sqrt{x^7 + 100x}$ is dominated by the highest power of x, in other words $\sqrt{x^7 + 100x} \sim x^{7/2}$ as x goes to ∞. So the quotient will be asymptotic to $x^{7/2}/x^5 = 1/x^{3/2}$ as x goes to ∞. Since $\displaystyle\int_1^{\infty} \frac{dx}{x^{3/2}}$ converges by the p-test with $p = 3/2$, we can conclude that the original integral converges by the limit comparison test.

   (b) $\displaystyle\int_0^{\infty} \frac{\sqrt[6]{x}}{\sqrt{x} + x^4}\, dx$ We need to split the integral, say at $x = 1$ since both endpoints are problematic.
   First consider $\displaystyle\int_0^1 \frac{\sqrt[6]{x}}{\sqrt{x} + x^4}\, dx$. As x goes to zero, x^4 dies out much faster than $\sqrt{x}$, so the denominator will behave more and more like $\sqrt{x}$. In other words we can say that

$$\frac{\sqrt[6]{x}}{\sqrt{x} + x^4} \sim \frac{\sqrt[6]{x}}{\sqrt{x}} = \frac{1}{x^{1/2-1/6}} = \frac{1}{x^{1/3}} \text{ as } x \to 0.$$

Since $\dfrac{1}{x^{1/3}}$ converges (by the [0,1]-version of the p test with $p = 1/3$) we can see that

$$\int_0^1 \frac{\sqrt[6]{x}}{\sqrt{x}+x^4}\,dx \text{ converges by the limit comparison test.}$$

Now as $x \longrightarrow \infty$ both $\sqrt{x}$ and x^4 go to infinity, but x^4 goes much faster. So as x goes to infinity,

$$\frac{\sqrt[6]{x}}{\sqrt{x}+x^4}\,dx \sim \frac{\sqrt[6]{x}}{x^4} = \frac{1}{x^{4-1/6}} \text{ as } x \longrightarrow \infty.$$

Using the other p-test we see that

$$\int_1^\infty \frac{\sqrt[6]{x}}{\sqrt{x}+x^4}\,dx \text{ also converges by limit comparison.}$$

(c) $\displaystyle\int_1^\infty \frac{(\ln x)\sin^2 x}{x^3+2}\,dx.$

Since $0 \le \sin^2 x \le 1$ we conclude that

$$\int_1^\infty \frac{(\ln x)\sin^2 x}{x^3+2}\,dx \le \int_1^\infty \frac{\ln x}{x^3+2}\,dx$$

For x in $[1,\infty)$ we know that $\ln x < x$ and so

$$\int_1^\infty \frac{\ln x}{x^3+2}\,dx < \int_1^\infty \frac{x}{x^3+2}\,dx$$

Since for a rational function the highest powers of x dominate as x goes to ∞ we have $x/(x^3+2) \sim 1/x^2$ as $x \longrightarrow \infty$. By the p-test we know that $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges, so by the limit comparison test we know that $\displaystyle\int_1^\infty \frac{x}{x^3+2}\,dx$ also converges and by the comparison test we conclude that the original integral also converges.

3. *(25 points)* For each of the series below determine whether it converges or diverges. Justify your answers.

(a) $\displaystyle\sum_{n=1}^\infty \frac{n^2+5n}{(n+1)(n+2)(n+3)}$

This series diverges. For rational functions the highest power dominates as we go to infinity. So the nth term is asymptotic to $n^2/n^3 = 1/n$ as n goes to infinity. Since $\displaystyle\sum_1^\infty \frac{1}{n}$ diverges by the p-test with $p = 1$, we conclude that the original series diverges by the limit comparison test.

(b) $\displaystyle\sum_{n=2}^\infty \frac{1}{n\ln n}$

In this case we use the integral test. Observe that $\displaystyle\int \frac{dx}{x\ln x} = \ln(\ln x)+C$ and $\lim_{t\to\infty} \ln(\ln t) = \infty$ since $\lim_{t\to\infty} \ln t = \infty$. Therefore the improper integral $\displaystyle\int_2^\infty \frac{dx}{x\ln x}$ diverges. By the integral test the series diverges as well.

(c) $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

In this case the ratio or root test works well. For example, with the ratio test we have

$$\lim_{n\to\infty} \frac{(n+1)^2}{3^{n+1}} \cdot \frac{3^n}{n^2} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \left(\frac{1}{3}\right) = \frac{1}{3}$$

Since this ratio is less than 1, the ratio test says that the series converges.

(d) $\sum_{n=1}^{\infty} \left(\frac{n+1}{3n+6}\right)^n$

Here the easiest solution uses the root test. The nth root of a_n is simply $\frac{n+1}{3n+6}$ and as n goes to infinity this approaches 1/3. Since the nth root of a_n goes to 1/3 and 1/3 is less than 1, we conclude that the series behaves more and more like a geometric series with $r = 1/3$ and so it converges.

(e) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

Here we can argue that $n^3 + 1$ is larger than n^3. Since the square root function is monotonically increasing we can say that $\sqrt{n^3+1} > \sqrt{n^3} > 0$ and taking reciprocals reverses inequalities on $(0, \infty)$ so

$$\frac{1}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3}}$$

The sum $\sum \frac{1}{n^{3/2}}$ converges by the p-test with $p = 3/2$, so the original series also converges, by the comparison test.