Euler's Formula

Where does Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

come from? How do we even *define*, for example, e^i? We can't multiple e by itself the square root of minus one times.

The answer is to use the Taylor series for the exponential function. For any complex number z we define e^z by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Since $|z^n| = |z|^n$, this series converges absolutely: $\sum_{n=0}^{\infty} \frac{|z|^n}{n!}$ is a real series that we already know converges.

If we multiply the series for e^z term-by-term with the series for e^w, collect terms of the same total degree, and use a certain famous theorem of algebra, we find that the law of exponents

$$e^{z+w} = e^z \cdot e^w$$

continues to hold for complex numbers.

Now for Euler's formula:

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

The special case $\theta = 2\pi$ gives

$$e^{2\pi i} = 1.$$

This celebrated formula links together three numbers of totally different origins: e comes from analysis, π from geometry, and i from algebra.

Here is just one application of Euler's formula. The addition formulas for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$ are somewhat hard to remember, and their geometric proofs usually leave something to be desired. But it is impossible to forget that

$$e^{i(\alpha+\beta)} = e^{i\alpha} \cdot e^{i\beta}.$$

Now use Euler's formula thrice:

$$\begin{aligned}
\cos(\alpha+\beta) + i\sin(\alpha+\beta) &= [\cos\alpha + i\sin\alpha] \cdot [\cos\beta + i\sin\beta] \\
&= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\sin\beta).
\end{aligned}$$

Equate the real and imaginary parts and presto! we have

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\sin(\alpha+\beta) &= \cos\alpha\sin\beta + \sin\alpha\cos\beta.
\end{aligned}$$