Mat104 Solutions to Taylor and Power Series Problems from Old Exams

(1) (a). This is a 0/0 form. We can use Taylor series to understand the limit.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \ldots \\ e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \cdots + \frac{(-1)^n x^n}{n!} + \ldots \\ \text{Thus } e^x - e^{-x} &= 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \ldots \end{aligned}$$

From this we find that

$$e^x - e^{-x} - 2x = \frac{2x^3}{3!} + \text{ higher degree terms}$$

As x approaches 0, the lowest power of x will dominate because the higher degree terms vanish much more rapidly. We can say that

$$e^x + e^{-x} - 2x \sim \frac{2x^3}{3!} \text{ as } x \to 0.$$

Next we consider the denominator.

$$x \ln(1 + x) = x(x - x^2/2 + x^3/3 - x^4/4 + \ldots) = x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \ldots.$$

Thus the denominator $x^2 - x \ln(1 + x)$ will be dominated by its lowest degree term $\frac{x^3}{2}$ as we let $x \to 0$ and so

$$\frac{e^x + e^{-x} - 2x}{x^2 - x \ln(1 + x)} \sim \frac{2x^3/3!}{x^3/2} = \frac{4}{6} = \frac{2}{3} \text{ as } x \to 0.$$

(1b) Again we have a 0/0 form. In a similar manner we manipulate Taylor series to determine what power of x the numerator and denominator resemble as x approaches 0. First recall that

$$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots \text{ and } \sin x = x - x^3/3! + x^5/5! - x^7/7! + \ldots$$

Then we can easily compute that

$$\cos x^2 - 1 + x^4/2 = x^8/4! \text{ plus higher degree terms}$$

$$x^2(x - \sin x)^2 = x^8/(3!3!) \text{ plus higher degree terms}$$

Thus

$$\frac{\cos x^2 - 1 + x^4/2}{x^2(x - \sin x)^2} \sim \frac{x^8/4!}{x^8/(3!3!)} = \frac{3!3!}{4!} = \frac{3}{2} \text{ as } x \to 0$$

(2) Rewrite $n \tan(1/n)$ as $\frac{\tan(1/n)}{1/n}$. This is a 0/0 form and we can use L'Hôpital's Rule to show that the limit is 1.

(3) Use the Taylor series for $\sin x$ and e^x to understand how the numerator behaves near $x = 0$.

$$\begin{aligned}(\sin x)(e^{x^2}) &= (x - x^3/3! + x^5/5! - \dots)(1 + x^2 + x^4/2 + \dots) \\ &= (x + x^3 - x^3/3! + \text{ higher degree terms })\end{aligned}$$

So

$$\sin x \cdot e^{x^2} - x = 5x^3/6 + \text{ higher degree terms.}$$

Now for the denominator.

$$\ln(1 + x^3) = x^3 - (x^3)^2/2 + (x^3)^3/3 - \dots = x^3 + \text{ higher degree terms.}$$

We conclude that the quotient will go to 5/6 as x goes to 0.

(4) Here we use the Taylor series for $\cos x$.

$$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \dots \implies 1 - \cos x = x^2/2! - x^4/4! + x^6/6! - \dots.$$

So when x is close to 0, $1 - \cos x \sim x^2/2!$. When n is large, then $1/n$ will be close to 0, so $1 - \cos(1/n) \sim 1/2n^2$. Thus $n^2(1 - \cos(1/n)) \sim 1/2$ as n goes to infinity.

(5) Here it is useful to combine the fractions

$$\frac{1}{\sin x} - \frac{1}{1 - e^{-x}} = \frac{(1 - e^{-x}) - \sin x}{(\sin x)(1 - e^{-x})}$$

Again we use power series to understand how the numerator and denominator behave near $x = 0$.

$$\begin{aligned}1 - e^{-x} - \sin x &= -x^2/2 + \text{ higher order terms} \\ (\sin x)(1 - e^{-x}) &= (x - x^3/3! + x^5/5! + \dots)(x - x^2/2 + x^3/3! + \dots) \\ &= x^2 + \text{ higher order terms.}\end{aligned}$$

So the quotient will behave like $\dfrac{-x^2/2}{x^2}$ and go to $-1/2$ as x goes to 0.

(6) Use the Taylor series for $\cos(x)$, substitute x^3 instead of x. Thus we find that

$$\cos(x^3) - 1 = -x^6/2 + \text{ higher order terms}$$

Similarly,

$$\sin(x^2) - x^2 = -\frac{x^6}{3!} + \text{ higher order terms}$$

and so the quotient $\dfrac{\cos x^3 - 1}{\sin x^2 - x^2}$ goes to $\dfrac{-x^6/2}{-x^6/6} = 3$ as x goes to 0.

(7) Using the Taylor series for $\sin x$, $\cos x$ and for e^x:

$$\begin{aligned}\sin x - x &= -x^3/3! + \text{higher order terms} \\ (\cos x - 1)(e^{2x} - 1) &= (-x^2/2! + x^4/4! - x^6/6! + \dots)(2x + (2x)^2/2! + (2x)^3/3! + \dots) \\ &= -x^3 + \text{ higher order terms}\end{aligned}$$

So $\dfrac{\sin x - x}{(\cos x - 1)(e^{2x} - 1)} = \dfrac{-x^3/6}{-x^3} \to \dfrac{1}{6}$ as $x \to 0$.

(8) Use the absolute ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{|x|^n} = |x| \left(\frac{n^2+1}{n^2+2n+2}\right) \to |x| \text{ as } n \to \infty$$

Therefore the series converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. If $|x| = 1$, the ratio test gives no information, so we have to look at the endpoints separately:

$$x = 1 \quad \Longrightarrow \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} \text{ an absolutely convergent series by comparison to } \frac{1}{n^2}$$

$$x = -1 \quad \Longrightarrow \quad \sum_{n=0}^{\infty} \frac{1}{n^2+1} \text{ an absolutely convergent series by comparison to } \frac{1}{n^2}$$

Conclusion: This power series is absolutely convergent on $[-1, 1]$ and diverges everywhere else.

(9) Use the absolute ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{e^{n+1}|x-1|^{n+1}}{2^{n+1}(n+1)} \cdot \frac{2^n \cdot n}{e^n \cdot |x-1|^n} = \frac{e}{2} \cdot \frac{n}{n+1} \cdot |x-1| \to \frac{e}{2}|x-1| \text{ as } n \to \infty$$

The series converges absolutely if this limit is less than 1, diverges if this limit is greater than 1 and must be checked when the limit is equal to 1. Since $\frac{e}{2} \cdot |x-1|$ is less than 1 whenever $|x-1| < \frac{2}{e}$, so the series is absolutely convergent on $(1-\frac{2}{e}, 1+\frac{2}{e})$ and divergent on $(-\infty, 1-\frac{2}{e})$ and on $(1+\frac{2}{e}, \infty)$. Now we check the endpoints:

$$x - 1 = \frac{2}{e} \text{ gives the series } \sum_{n=1}^{\infty} \left(\frac{e}{2}\right)^n \cdot \left(\frac{2}{e}\right)^n \cdot \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ a divergent series}$$

$$x - 1 = \frac{-2}{e} \text{ gives } \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ a conditionally convergent (alternating) series.}$$

Conclusion: This power series is absolutely convergent on $(1-\frac{2}{e}, 1+\frac{2}{e})$, conditionally convergent at $1-\frac{2}{e}$ and divergent everywhere else.

(10) Since

$$\sin(t^2) = t^2 - (t^2)^3/3! + (t^2)^5/5! - (t^2)^7/7! + \cdots + \frac{(-1)^k t^{4k+2}}{(2k+1)!} + \ldots$$

when we integrate we get

$$f(x) = x^3/3 - x^7/(7\cdot 3!) + x^{11}/(11\cdot 5!) - x^{15}/(15\cdot 7!) + \cdots + \frac{(-1)^k x^{4k+3}}{(4k+3)(2k+1)!} + \ldots$$

The coefficient of x^{100} in the Taylor expansion is, by definition, $\frac{f^{(100)}(0)}{100!}$. But our computation shows that x^{100} appears with coefficient 0. Conclusion: $f^{(100)}(0) = 0$.

(11) $1-\cos(2x^2)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\cdot 4^k\cdot x^{4k}}{(2k)!}$. Dividing through by x we find

$$\frac{1-\cos(2x^2)}{x}=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\cdot 4^k\cdot x^{4k-1}}{(2k)!}=\frac{4x^3}{2!}-\frac{16x^7}{4!}+\frac{64x^{11}}{6!}-\dots$$

Since the coefficient of x^8 is zero, we conclude that $f^{(8)}(0)=0$. Since the coefficient of x^7 is $-16/4!=-2/3$ we conclude that $f^{(7)}(0)/7!=-2/3$ and thus $f^{(7)}(0)=-\frac{2\cdot 7!}{3}$.

(12)

$$\text{(a) } \ln(1+x^3)=x^3-\frac{x^6}{2}+\frac{x^9}{3}-\frac{x^{12}}{4}+\dots+\frac{(-1)^{n-1}x^{3n}}{n}+\dots$$

and this will be valid if $x^3\in(-1,1]$, that is, if x is in $(-1,1]$.

$$\text{(b) } \frac{1}{1+x^2} = 1-x^2+x^4-x^6+x^8-\dots=\sum_{k=0}^{\infty}(-1)^k x^{2k} \text{ valid on } (-1,1).$$

$$\Longrightarrow \frac{x}{1+x^2} = x-x^3+x^5-x^7+x^9-\dots=\sum_{k=0}^{\infty}(-1)^k x^{2k+1} \text{ also valid on } (-1,1).$$

(13)

(a) $e^{x^2}=1+x^2+x^4/2!+x^6/3!+\dots+x^{2n}/n!+\dots$ valid on $(-\infty,\infty)$

(b) $\frac{1}{1-x^3}=1+x^3+x^6+x^9+\dots+x^{3n}+\dots$ valid if $|x^3|<1$ that is, on $(-1,1)$

(c) $(1+x^2)=1+2x+x^2=1+2x+x^2+0\cdot x^3+0\cdot x^4+0\cdot x^5+\dots+0\cdot x^n+\dots$

(13d) Find the first three terms of the Taylor series at $x=1$ for $f(x)=\frac{x}{1+x}$. We need to compute the first two derivatives and evaluate at $x=1$. First $f(1)=1/2$. Next

$$f'(1) = \left.\frac{1}{(1+x)^2}\right|_{x=1}=\frac{1}{4}$$

$$f''(1) = \left.(-2)(1+x)^{-3}\right|_{x=1}=\frac{-2}{2^3}=-\frac{1}{4}$$

$$\Longrightarrow \text{Taylor expansion} = \frac{1}{2}+\frac{x-1}{4}-\frac{1}{4}\frac{(x-1)^2}{2!}+\dots$$

$$= \frac{1}{2}+\frac{x-1}{4}-\frac{(x-1)^2}{8}+\dots$$

(14) Normally we cannot substitute $\sqrt{x}$ into a power series and still get a power series, but in this case we are OK because the Taylor series for cosine contains only even terms:

$$\begin{aligned} x\cos\sqrt{x} &= x\left(1 - x/2! + x^2/4! - x^3/6! + \cdots + \frac{(-1)^k x^k}{(2k)!} + \dots\right) \\ &= x - x^2/2! + x^3/4! - x^4/6! + \cdots + \frac{(-1)^k x^{k+1}}{(2k)!} + \dots \end{aligned}$$

(15)

$$\text{(a)} \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \dots$$

$$\text{(b)} \qquad \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots + (-1)^n x^n + \dots \quad \text{valid on } (-1,1)$$

$$\begin{aligned} \text{(c)} \qquad \frac{\cos x}{1+x} &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)(1 - x + x^2 - x^3 + x^4 - \dots) \\ &= 1 - x + x^2(1 - 1/2!) + x^3(-1 + 1/2!) + \dots \\ &= 1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \text{ higher order terms} \end{aligned}$$

(15d) The coefficient of x^3 is on the one hand $\frac{f'''(0)}{3!}$ and on the other hand we found by multiplying power series that it was $-1/2$. Thus $f'''(0) = -3!/2 = -3$.

(16)

$$\begin{aligned} \sin t &= t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots + \frac{(-1)^k t^{2k+1}}{(2k+1)!} + \dots \\ \Longrightarrow \frac{\sin t}{t} &= 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \cdots + \frac{(-1)^k t^{2k}}{(2k+1)!} + \dots \end{aligned}$$

This series is absolutely convergent on $(-\infty, \infty)$ since

$$\lim_{k\to\infty}\left|\frac{t^{2k+2}}{(2k+3)!} \cdot \frac{(2k+1)!}{t^{2k}}\right| = \lim_{k\to\infty} \frac{|t|^2}{(2k+3)(2k+2)} = 0.$$

We can integrate this series to get the series expansion for $F(x)$:

$$F(x) = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)(2k+1)!} + \dots$$

Absolute convergence is guaranteed on the same interval $(-\infty, \infty)$. (Basic principle – you can't ruin absolute convergence by integrating or differentiating). Finally,

$$\begin{aligned} F^{(20)}(0) &= (20!)(\text{coefficient of } x^{20}) = 0 \Longrightarrow F^{(20)}(0) = 0. \\ F^{(21)}(0) &= (21!)(\text{coefficient of } x^{21}) = (21!) \cdot \frac{(-1)^{10}}{21 \cdot 21!} \Longrightarrow F^{(21)}(0) = \frac{1}{21} \end{aligned}$$

(17)

$$\frac{e^x - 1}{x} = \frac{1}{x}\left(x + x^2/2! + x^3/3! + \cdots + x^n/n! + \ldots\right) = 1 + x/2! + x^2/3! + \cdots + x^{n-1}/n! + \ldots$$

This is absolutely convergent on $(-\infty, \infty)$ since

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^n}{(n+1)!} \cdot \frac{n!}{x^{n-1}}\right| = \frac{|x|}{n+1} \to 0 \text{ as } n \to \infty.$$

$$\frac{f^{(100)}(0)}{100!} = \text{coefficient of } x^{100} = \frac{1}{101!} \implies f^{(100)}(0) = 100!/101! = 1/101.$$

(18)

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} + \ldots \\ xe^x &= x + x^2 + \frac{x^3}{2!} + \cdots + \frac{x^n}{(n-1)!} + \frac{x^{n+1}}{n!} + \ldots \\ (1+x)e^x &= 1 + 2x + x^2(1/2! + 1) + x^3(1/3! + 1/2!) + \cdots + x^n(1/n! + 1/(n-1)!) + \ldots \\ &= \sum_{n=0}^{\infty} \frac{1+n}{n!} x^n. \end{aligned}$$

We are also asked to find the first four terms of the Taylor expansion of $1/\sqrt{x^2+1}$ about $x = 0$. First compute the expansion for $(1+u)^{-1/2}$ and then make a substitution.

$f(0) = 1$

$f'(u) = (-1/2)(1+u)^{-3/2} \implies f'(0) = -1/2.$

$f''(u) = (-1/2)(-3/2)(1+u)^{-5/2} \implies f''(0) = 3/4.$

$f'''(u) = (-1/2)(-3/2)(-5/2)(1+u)^{-7/2} \implies f'''(0) = -15/8$

$\implies (1+u)^{-1/2} = 1 - u/2 + (3/4)u^2/2! - (15/8)u^3/3! + \ldots$

$\implies (1+u)^{-1/2} = 1 - u/2 + 3u^2/8 - 5u^3/16 + \ldots$

$\implies (1+x^2)^{-1/2} = 1 - x^2/2 + 3x^4/8 - 5x^6/16 + \ldots$

(19) Use the absolute ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \cdots = \frac{(n+1)^2+1}{n^2+1} \cdot \frac{n+1}{n+2} \cdot \frac{1}{4} \cdot |x-3| \to \frac{|x-3|}{4} \text{ as } n \to \infty.$$

So the series is absolutely convergent if the limit $|x-3|/4$ is less than 1, that is if $|x-3| < 4$. The series is divergent if $|x-3|$ is bigger than 4. We must check the endpoints. If $x - 3 = 4$, that is, if $x = 7$ the series becomes

$$\sum_{n=0}^{\infty} \frac{n^2+1}{n+1}$$

which diverges because its nth term grows without bound as n goes to infinity. If $x-3=-4$, that is, if $x=-1$, then the series becomes

$$\sum_{n=0}^{\infty}(-1)^n\frac{n^2+1}{n+1}$$

which also diverges by the nth term test.

Conclusion: This series is absolutely convergent on $(-1,7)$ and divergent everywhere else.

(20) Again use the absolute ratio test. In this case

$$\left|\frac{a_{n+1}}{a_n}\right| = \cdots = |2x-1|\cdot\frac{n}{n+1}\cdot\frac{\ln n}{\ln(n+1)}$$

By L'Hôpital's Rule, both fractions go to 1 as n goes to infinity. So the series is absolutely convergent if $|2x-1|<1$ and divergent if $|2x-1|>1$. Check endpoints:

$2x-1=1$ gives $\sum_2^\infty \dfrac{1}{n\ln n}$ divergent by the integral test

$2x-1=-1$ gives $\sum_2^\infty \dfrac{(-1)^n}{n\ln n}$ conditionally convergent by Alternating Series Test

Conclusion: absolutely convergent on (0,1). conditionally convergent at $x=0$. Divergent everywhere else.

(21) Use the absolute ratio test. This series converges absolutely when $|x-2|<1$ (that is, for x in $(1,3)$). The series diverges if $|x-2|>1$. If $x-2=1$ the series diverges by comparison to $\sum\frac{1}{n}$. If $x-2=-1$ the series converges (conditionally) by the alternating series test.

$$\frac{f^{(17)}(2)}{17!} = \text{ coefficient of } (x-2)^{17} = \frac{(17+1)^2}{17^3} \implies f^{(17)}(2) = 17!(18)^2/17^3 = \frac{16!\cdot 18^2}{17^2}.$$

(22) The absolute ratio test give

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{n^n}|x| = (n+1)\cdot\left(\frac{n+1}{n}\right)^n\cdot|x|.$$

Recall that $\left(\dfrac{n+1}{n}\right)^n$ goes to e as n goes to ∞. So the quotient $|a_{n+1}/a_n|$ equals 0 if $x=0$, but for any other choice of x it goes to ∞ as n does. This power series diverges except at its center $x=0$.

(23) In this case $|a_{n+1}/a_n|$ approaches $|x|/5$ as n goes to ∞. The series is absolutely convergent on $(-5,5)$. The radius of convergence is 5.

(24) Here $|a_{n+1}/a_n|$ approaches $|x-1|/2$ as n goes to ∞. So we have absolute convergence on $(-1,3)$, divergence if $x>3$ or if $x<-1$. Check endpoints.

$x=3 \implies x-1=2 \implies$ the series is $\sum\dfrac{n+1}{2n+1}$,divergent by the nth term test.

$x=-1 \implies x-1=-2 \implies$ the series is $\sum\dfrac{n+1}{2n+1}(-1)^n$,again divergent by the nth term test

Conclusion: Absolutely convergent on $(-1,3)$. Divergent elsewhere.

(25) To estimate $\sqrt{11}$ we use the Taylor series for $f(x) = \sqrt{x}$ centered at the point $x = 9$. First we compute the derivatives:

$$f(x) = \sqrt{x} \implies f(9) = 3$$

$$f'(x) = (1/2)x^{-1/2} = \frac{1}{2\sqrt{x}} \implies f'(9) = \frac{1}{6}$$

$$f''(x) = (-1/2)(1/2)x^{-3/2} = -\frac{1}{4}\left(\frac{1}{\sqrt{x}}\right)^3 \implies f''(9) = -\frac{1}{4}\cdot\frac{1}{27} = -\frac{1}{108}$$

$$f'''(x) = (-3/2)(-1/4)x^{-5/2} = \frac{3}{8}\left(\frac{1}{\sqrt{x}}\right)^5 \implies f'''(9) = \frac{3}{8}\cdot\frac{1}{3^5} = \frac{1}{8\cdot 3^4}$$

We can see a pattern emerging in these derivatives, but for us it is enough to notice that the numbers $f^{(n)}(9)$ will alternate in sign, and the Taylor series will be an alternating series (after the first term, and as long as we choose x bigger than 9.)

$$\begin{aligned} f(x) &\approx f(9) + f'(9)(x-9) + \frac{f''(9)(x-9)^2}{2!} + \frac{f'''(9)(x-9)^3}{3!} + \dots \\ &= 3 + \frac{x-9}{6} - \frac{(x-9)^2}{2(108)} + \frac{(x-9)^3}{8\cdot 3^4\cdot 3!} - \dots \end{aligned}$$

Thus, taking $x = 11$,

$$\sqrt{11} \approx 3 + \frac{2}{6} - \frac{4}{2(108)} + \frac{8}{8\cdot 3^4\cdot 3!} - \dots = 3 + \frac{1}{3} - \frac{1}{54} + \frac{1}{6\cdot 81} - \dots.$$

This series converges to $\sqrt{11}$ and after the first term it becomes an alternating series. We conclude that

$$\begin{aligned} \text{First order (tangent line) approx to } \sqrt{11} &= 3 + \frac{1}{3} = \frac{10}{3} \\ \text{Second order approx. to } \sqrt{11} &= 3 + \frac{1}{3} - \frac{1}{54} \\ \text{Third order approx. to } \sqrt{11} &= 3 + \frac{1}{3} - \frac{1}{54} + \frac{1}{486} \end{aligned}$$

Once the series alternates we know that the actual value $\sqrt{11}$ lies between any two partial sums. So

$$3 + \frac{1}{3} - \frac{1}{54} < \sqrt{11} < 3 + \frac{1}{3} - \frac{1}{54} + \frac{1}{486}$$

or in other words, $\sqrt{11} \approx 3 + \frac{1}{3} - \frac{1}{54}$ and the error is at most $\frac{1}{486}$.