Mat104 Taylor Series and Power Series from Old Exams

(1) Use MacLaurin polynomials to evaluate the following limits:

(a) $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x^2 - x\ln(1+x)}$ (b) $\displaystyle\lim_{x\to 0} \frac{\cos(x^2) - 1 + x^4/2}{x^2(x-\sin x)^2}$

(2) Evaluate or show that $\lim_{n\to\infty} n\tan\left(\frac{1}{n}\right)$

(3) Find $\displaystyle\lim_{x\to 0} \frac{\sin x \cdot e^{x^2} - x}{\ln(1+x^3)}$.

(4) Find $\displaystyle\lim_{n\to\infty} n^2\left(1 - \cos\frac{1}{n}\right)$ or show that it does not exist.

(5) Find $\displaystyle\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{1-e^{-x}}\right)$

(6) Find $\displaystyle\lim_{x\to 0} \frac{\cos(x^3) - 1}{\sin(x^2) - x^2}$.

(7) Find $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{(\cos x - 1)(e^{2x} - 1)}$.

(8) For what real values of x does $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n^2+1}$ converge? Give your reasons.

(9) For what real values of x does $\displaystyle\sum_{n=1}^{\infty} \frac{e^n (x-1)^n}{2^n \cdot n}$ converge? Give your reasons.

(10) Let $f(x) = \displaystyle\int_0^x \sin(t^2)\,dt$. Find the Taylor series at 0 (i.e., the Taylor series about $a = 0$) of $f(x)$, giving enough terms to make the pattern clear. Also, find the 100th derivative of $f(x)$ at 0.

(11) Find the Taylor series at 0 of $f(x) = \dfrac{1-\cos(2x^2)}{x}$ and find $f^{(7)}(0)$ and $f^{(8)}(0)$.

(12) Find the Taylor series about 0 of

(a) $\ln(1+x^3)$ (b) $\dfrac{x}{1+x^2}$

(13) (a) Find the Taylor series at $x = 0$ for e^{x^2}.

(b) Find the Taylor series at $x = 0$ for $\dfrac{1}{1-x^3}$.

(c) Find the Taylor series at $x = 0$ for $(1+x)^2$.

(d) Find the first three terms of the Taylor series at $x = 1$ for $\dfrac{x}{1+x}$.

(14) Find the Taylor series at $x = 0$ (McLaurin series) of $f(x) = x\cos\sqrt{x}$.

(15) Find the Taylor series about 0 for each of the following functions. Give the expansion up to and including terms involving x^3.

(a) $\cos x$

(b) $\dfrac{1}{1+x}$

(c) $\dfrac{\cos x}{1+x}$

From your answer to part (c), give the value of $f'''(0)$ where $f(x) = \dfrac{\cos x}{1+x}$.

(16) Let $F(x)$ be the function defined by

$$F(x) = \int_0^x \frac{\sin(t)}{t}\,dt.$$

Find the MacLaurin series of the function F and compute its radius of convergence. Find $F^{(20)}(0)$ and $F^{(21)}(0)$.

(17) Let $f(x) = \dfrac{e^x - 1}{x}$. Find the Taylor series at 0 (McLaurin series) for $f(x)$. For what values of x does the series converge? Give your reasons. Find the 100^{th} derivative of f at 0.

(18) Find the Taylor-MacLaurin Series about $x = 0$ for $(x+1)e^x$. Find the first four terms in the Taylor expansion about $x = 0$ for $\dfrac{1}{\sqrt{x^2+1}}$.

(19) Find all values of x for which $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+1}{n+1} \cdot \frac{1}{4^n} \cdot (x-3)^n$ converges.

(20) For what x does the series $\displaystyle\sum_{n=2}^{\infty} \frac{(2x-1)^n}{n \ln n}$ converge? Give your reasons.

(21) For what x does $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^3} \cdot (x-2)^n$ converge? Find $f^{(17)}(2)$.

(22) For what x does $\displaystyle\sum_{n=0}^{\infty} (nx)^n$ converge?

(23) Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{5^n} \cdot x^n$.

(24) For which values of x does the power series

$$\sum_{n=1}^{\infty} \frac{n+1}{2n+1} \cdot \frac{(x-1)^n}{2^n}$$

converge absolutely? conditionally?

(25) Find a good approximation of $\sqrt{11}$ using some Taylor polynomial of degree 2, and estimate the error.