MAT104—Calculus. Spring 2003. Solutions to Final Exam.

1. [10 points] Find the following integrals.

(a) $\displaystyle\int e^{2x}\sin e^x\,\mathrm{d}x$

(b) $\displaystyle\int_0^1 \frac{x^2}{(\sqrt{4-x^2})^3}\,\mathrm{d}x$

Solution.

(a) Substituting $z = e^x$, $\mathrm{d}z = e^x\,\mathrm{d}x$, we get

$$\int e^{2x}\sin e^x\,\mathrm{d}x = \int z\sin z\,\mathrm{d}z.$$

On this latter integral we apply integration by parts to get

$$\begin{aligned}\int e^{2x}\sin e^x\,\mathrm{d}x = \int z\sin z\,\mathrm{d}z &= \int -z\,\mathrm{d}(\cos z)\\ &= -z\cos z + \int \cos z\,\mathrm{d}z\\ &= -z\cos z + \sin z + C\\ &= -e^x\cos e^x + \sin e^x + C.\end{aligned}$$

(b) Let's use trigonometric substitution $x = 2\sin\theta$. That yields

$$\begin{aligned}\int_0^1 \frac{x^2}{(\sqrt{4-x^2})^3}\,\mathrm{d}x &= \int_0^{\pi/6}\frac{8\sin^2\theta\cos\theta\,\mathrm{d}\theta}{8\cos^3\theta}\\ &= \int_0^{\pi/6}\frac{\sin^2\theta}{\cos^2\theta}\,\mathrm{d}\theta = \int_0^{\pi/6}\tan^2\theta\,\mathrm{d}\theta\\ &= \int_0^{\pi/6}(\sec^2\theta - 1)\,\mathrm{d}\theta = (\tan\theta-\theta)\Big|_0^{\pi/6}\\ &= \frac{\sqrt{3}}{3} - \frac{\pi}{6} = \frac{2\sqrt{3}-\pi}{6}.\end{aligned}$$

2. [12 points] (a) Let R be the region bounded by the curve $y = x^3$, the x-axis and two vertical lines $x = 1$ and $x = 2$. Find the volume of the region obtained by rotating R around the line $x = 3$.

(b) Let C be the portion of the curve $y = x^3$ between the points $(1, 1)$ and $(2, 8)$. Find the area of the surface generated by rotating C about the x-axis.

Solution.

(a) We can apply the Shell Method to find that

$$\begin{aligned}\text{Volume} = 2\pi\int_1^2 x^3(3-x)\,\mathrm{d}x &= 2\pi\int_1^2(3x^3 - x^4)\,\mathrm{d}x\\ &= 2\pi\left(\frac{3}{4}x^4 - \frac{1}{5}x^5\right)\Big|_1^2 = 2\pi\left(\frac{45}{4} - \frac{31}{5}\right) = \frac{101\pi}{10}.\end{aligned}$$

(b) Surface Area $= 2\pi \int_1^2 y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = 2\pi \int_1^2 x^3\sqrt{1+(3x^2)^2}\,\mathrm{d}x = 2\pi \int_1^2 x^3\sqrt{1+9x^4}\,\mathrm{d}x.$

In this integral it is wholesome to notice that a radical substitution $z = 1+9x^4$, $\mathrm{d}z = 36x^3\,\mathrm{d}x$ yields a simple integral

$$\frac{\pi}{18}\int_{10}^{145}\sqrt{z}\,\mathrm{d}z = \frac{\pi}{18}\cdot\frac{2}{3}\cdot z^{3/2}\Big|_{10}^{145} = \frac{[(\sqrt{145})^3-(\sqrt{10})^3]\pi}{27}.$$

3. **[15 points]** Determine whether the given improper integrals converge or diverge. Justify your answers.

(a) $\displaystyle\int_2^\infty \frac{\sin^2 x}{x(\ln x)^2}\,\mathrm{d}x$ (b) $\displaystyle\int_0^1 \frac{\sin(x^2)}{x^{5/2}}\,\mathrm{d}x$ (c) $\displaystyle\int_1^\infty \frac{\tan^{-1}(x^2)}{x^3+\sqrt{x}}\,\mathrm{d}x$

(d) $\displaystyle\int_0^1 \frac{x^{3/2}}{\ln(1+x^2)}\,\mathrm{d}x$ (e) $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{x^2-1}$.

Solution.

(a) The only problem spot of this improper integral is at ∞. Note that

$$\int_2^\infty \frac{\sin^2 x}{x(\ln x)^2}\,\mathrm{d}x < \int_2^\infty \frac{\mathrm{d}x}{x(\ln x)^2} = \int_{\ln 2}^\infty \frac{\mathrm{d}z}{z^2} < \infty,$$

where we introduced the substitution $z = \ln x$. Therefore, by Comparison Test, our integral converges.

(b) The only problem spot of this improper integral is at 0. There we have

$$\frac{\sin(x^2)}{x^{5/2}} \sim \frac{x^2}{x^{5/2}} = \frac{1}{x^{1/2}} \quad (x \to 0).$$

The improper integral $\displaystyle\int_0^1 \frac{\mathrm{d}x}{x^{1/2}}$ converges by p-test. Therefore, by Limit Comparison Test for improper integrals, our integral converges as well.

(c) The only problem spot of this improper integral is at ∞. There we have

$$\frac{\tan^{-1}(x^2)}{x^3+\sqrt{x}} \sim \frac{\pi/2}{x^3} \quad (x \to \infty).$$

The improper integral $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{x^3}$ converges by p-test. Therefore, by Limit Comparison Test for improper integrals, our integral converges as well.

(d) The only problem spot of this improper integral is at 0. There we have (use, for example, Maclaurin series for $\ln(1+x)$)

$$\frac{x^{3/2}}{\ln(1+x^2)} \sim \frac{x^{3/2}}{x^2} = \frac{1}{x^{1/2}} \quad (x \to 0).$$

The improper integral $\displaystyle\int_0^1 \frac{\mathrm{d}x}{x^{1/2}}$ converges by p-test. Therefore, by Limit Comparison Test for improper integrals, our integral converges as well.

(e) This improper integral has two problem spots: at 1 and at ∞. At ∞ we have

$$\frac{1}{x^2-1} \sim \frac{1}{x^2} \quad (x \to \infty),$$

while at 1 we have

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} \sim \frac{1/2}{x-1} \quad (x \to 1).$$

Let's write our integral as

$$\int_1^\infty \frac{\mathrm{d}x}{x^2-1} = \int_1^2 \frac{\mathrm{d}x}{x^2-1} + \int_2^\infty \frac{\mathrm{d}x}{x^2-1}.$$

By p-tests for improper integrals, $\displaystyle\int_1^2 \frac{\mathrm{d}x}{x-1} = \int_0^1 \frac{\mathrm{d}z}{z}$ diverges, while $\displaystyle\int_2^\infty \frac{\mathrm{d}x}{x^2}$ converges. By Limit Comparison Test for improper integrals, that means that the first integral above diverges, while the second converges. Hence our improper integral diverges.

4. [15 points] Write **AC** or **CC** or **D** to indicate whether the given series is **A**bsolutely **C**onvergent, or **C**onditionally **C**onvergent or **D**ivergent. Justify your answers.

(a) $\displaystyle\sum_{n=1}^\infty \frac{2^{2n}+(-5)^n}{5^n}$ (b) $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n 2^n}{\sqrt[3]{n!}}$ (c) $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n}{\sin(1/n)(\sqrt[n]{e}-1)}$

(d) $\displaystyle\sum_{n=2}^\infty \frac{(-1)^n}{n\sqrt{\ln n}}$ (e) $\displaystyle\sum_{n=1}^\infty \frac{\left(1+\frac{1}{\sqrt{n}}\right)^n}{n^{n/2}}$.

Solution.

(a) **D** This series is a sum of two series,

$$\sum_{n=1}^\infty \frac{2^{2n}+(-5)^n}{5^n} = \sum_{n=1}^\infty \frac{2^{2n}}{5^n} + \sum_{n=1}^\infty \frac{(-5)^n}{5^n} = \sum_{n=1}^\infty \left(\frac{4}{5}\right)^n + \sum_{n=1}^\infty (-1)^n.$$

Both of these series are geometric. However, first of them converges as its quotient is $4/5 < 1$, while the second diverges by n^{th} Term Test. Hence their sum has to diverge.

(b) **AC** Here absolute convergence can be easily established using the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}2^{n+1}}{\sqrt[3]{(n+1)!}}}{\dfrac{(-1)^n 2^n}{\sqrt[3]{n!}}} \right| = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \sqrt[3]{\frac{n!}{(n+1)!}} = \lim_{n\to\infty} \frac{2}{\sqrt[3]{n+1}} = 0 < 1.$$

(c) **D** Note that, as $n \to \infty$, $\sin\dfrac{1}{n} \to 0$ and $\sqrt[n]{e} = e^{1/n} \to 1$, whence

$$\lim_{n\to\infty} \frac{1}{\sin(1/n)(\sqrt[n]{e}-1)} = +\infty.$$

This means that the n^{th} terms of our series fall into two subsequences (those with odd n's and those with even n's), one of which diverges to $-\infty$ and the other to $+\infty$. In any case, this series miserably fails n^{th} Term Test and therefore diverges.

(d) **CC** This series does not converge absolutely. Indeed, to the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ we can apply Integral Test to see that it is equiconvergent with the improper integral

$$\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \int_{\ln 2}^{\infty} \frac{dz}{\sqrt{z}} = \infty.$$

On the other hand, our series does converge conditionally because it converges by Alternating Series Test. Indeed, terms of this series have alternating signs and numbers $\frac{1}{n\sqrt{\ln n}}$ form a monotonically decreasing sequence of positive numbers with limit 0.

(e) **AC** We can use Root Test; indeed,

$$\lim_{n\to\infty} \left[\frac{\left(1+\frac{1}{\sqrt{n}}\right)^n}{n^{n/2}} \right]^{1/n} = \lim_{n\to\infty} \frac{1+\frac{1}{\sqrt{n}}}{\sqrt{n}} = 0 < 1.$$

5. [10 points] Let $0 \leqslant \theta \leqslant 2\pi$ and consider the series $\sum_{n=1}^{\infty}(-1)^{n+1}\tan^{2n}(\theta)$. Determine the values of θ for which the series converges and compute the sum. Simplify your answer.

Solution.

If we write the first couple of terms

$$\tan^2\theta - \tan^4\theta + \tan^6\theta - \tan^8\theta + \cdots$$

it becomes apparent that this is a geometric series with first term $a = \tan^2\theta$ and quotient $q = -\tan^2\theta$. Remember that geometric series converges if and only if $|q| < 1$, which in this case reads as

$$|-\tan^2\theta| < 1, \quad \text{i.e.} \quad -1 < \tan\theta < 1.$$

In the given interval for θ, this condition is satisfied for

$$0 \leqslant \theta < \frac{\pi}{4}, \quad \frac{3\pi}{4} < \theta < \frac{5\pi}{4}, \quad \text{and} \quad \frac{7\pi}{4} < \theta \leqslant 2\pi.$$

Within the interval of convergence, the sum of the infinite geometric series is given by

$$\frac{a}{1-q} = \frac{\tan^2\theta}{1+\tan^2\theta} = \frac{\tan^2\theta}{\sec^2\theta} = \sin^2\theta.$$

6. [9 points] Find the first three nonzero terms of the Taylor series at 0 for the function $f(x) = \frac{\sin x}{1+x^3}$.

Solution.

The formula for the sum of the infinite geometric series gives at once the Maclaurin expansion

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = 1 - x^3 + x^6 - x^9 + \cdots .$$

We will obtain the required Maclaurin expansion by multiplication of known expansions as follows:

$$\begin{aligned}
f(x) = \frac{\sin x}{1+x^3} = \sin x \cdot \frac{1}{1+x^3} &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)(1 - x^3 + x^6 - \cdots) \\
&= x - \frac{x^3}{6} + \frac{x^5}{120} - x^4 + \frac{x^6}{6} + \cdots \\
&= x - \frac{x^3}{6} - x^4 + \frac{x^5}{120} + \frac{x^6}{6} + \cdots ,
\end{aligned}$$

where dots stand for terms with powers of x^7 and higher. These are already first five nonzero terms in the Maclaurin series for $f(x)$.

7. [10 points] Find $\displaystyle\lim_{x\to 0} \frac{\left(e^{2x^2} - 1 - 2x^2\right)(\cos x - 1)}{(\sin 3x - \ln(1+3x))\ x^4}$.

Solution.

We can compute this limit by inserting the Maclaurin expansions of all functions appearing in this limit. We obtain

$$\begin{aligned}
&\lim_{x\to 0} \frac{\left(e^{2x^2} - 1 - 2x^2\right)(\cos x - 1)}{(\sin 3x - \ln(1+3x))\ x^4} \\
&= \lim_{x\to 0} \frac{\left(1 + 2x^2 + \frac{(2x^2)^2}{2} + \frac{(2x^2)^3}{6} + \cdots - 1 - 2x^2\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots - 1\right)}{\left(3x - \frac{(3x)^3}{6} + \cdots - 3x + \frac{(3x)^2}{2} - \frac{(3x)^3}{3} + \cdots\right)\ x^4} \\
&= \lim_{x\to 0} \frac{\left(2x^4 + \frac{4x^6}{3} + \cdots\right)\left(-\frac{x^2}{2} + \frac{x^4}{24} + \cdots\right)}{\left(\frac{9x^2}{2} - \frac{27x^3}{2} + \cdots\right)\ x^4} = \lim_{x\to 0} \frac{2x^4 \cdot \left(-\frac{x^2}{2}\right)}{\frac{9x^2}{2} \cdot x^4} = -\frac{2}{9}.
\end{aligned}$$

8. [9 points] Let $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and let w be a complex number whose modulus is 2 and whose argument is $\pi/3$. (**Note:** The modulus of a complex number is the same as the magnitude.) Write each of the quantities below in the form $a + ib$ where and a and b are real numbers.

(a) $\frac{1}{z}$ (b) z^{80} (c) $z^2 \cdot w$.

Solution.

Write z and w in polar form. These are easily seen to be

$$z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}, \quad w = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).$$

In the following we then simply apply rules for multiplying and dividing complex numbers in polar form and in particular DeMoivre's Law.

(a) $\dfrac{1}{z} = z^{-1} = \cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i.$

(b) $z^{80} = \cos\dfrac{80\pi}{4} + i\sin\dfrac{80\pi}{4} = \cos(20\pi) + i\sin(20\pi) = 1.$

(c) $z^2 w = \left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right) \cdot 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right) = 2\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right) = -\sqrt{3} + i.$

9. [10 points] Find all complex numbers z satisfying the equation $(2z-1)^4 = -16$. Express your answers in the form $a + ib$, where a and b are real.

Solution.

For clarity let's write $w = 2z - 1$ for the moment. We need to solve the equation

$$w^4 = -16 = 16(\cos\pi + i\sin\pi).$$

All complex roots of this equation are given by a general formula

$$w = \sqrt[4]{16}\left(\cos\frac{\pi + 2k\pi}{4} + i\sin\frac{\pi + 2k\pi}{4}\right), \quad k = 0, 1, 2, 3.$$

This gives four solutions

$$w_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2} + \sqrt{2}i, \qquad w_2 = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -\sqrt{2} + \sqrt{2}i,$$
$$w_3 = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -\sqrt{2} - \sqrt{2}i, \quad w_4 = 2\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \sqrt{2} - \sqrt{2}i.$$

The corresponding values of z can be found from an equation $z = (w+1)/2$. We find that they are

$$z_1 = \frac{\sqrt{2}+1}{2} + \frac{\sqrt{2}}{2}i, \quad z_2 = \frac{-\sqrt{2}+1}{2} + \frac{\sqrt{2}}{2}i, \quad z_3 = \frac{-\sqrt{2}+1}{2} - \frac{\sqrt{2}}{2}i, \quad z_4 = \frac{\sqrt{2}+1}{2} - \frac{\sqrt{2}}{2}i.$$