MAT 104 Midterm March 2003

Problem	Possible score	Average score
1	12	9.5
2	12	4.9
3	12	8.9
4	12	5.4
5	12	7.5
6	12	8.1
7	12	6.8
8	6	4.8
9	10	4.3
Total	100	60.2

1. *(12 points)* Find $\int \frac{\sin^3(\ln x)\cos^2(\ln x)}{x}\,dx$.

   Let $u = \ln x$. Then the integral becomes

$$\int \sin^3 u \cos^2 u \, du = \int \sin^2 u \cos^2 u \sin u \, du = \int (1-\cos^2 u)\cos^2 u \sin u \, du.$$

   Substitute $t = \cos u$ to get

$$-\int (1-t^2)t^2\,dt = \int (t^4 - t^2)\,dt = \frac{t^5}{5} - \frac{t^3}{3} + C = \frac{\cos^5(\ln x)}{5} - \frac{\cos^3(\ln x)}{3} + C.$$

2. *(12 points)* Find $\int \frac{\ln(x^2+x+1)}{x^2}\,dx$.

   Do integration by parts with $u = \ln(x^2+x+1)$ and $dv = dx/x^2$. Then $du = (2x+1)\,dx/(x^2+x+1)$ and $v = -1/x$. So

$$\int \frac{\ln(x^2+x+1)}{x^2}\,dx = -\frac{1}{x}\ln(x^2+x+1) + \int \frac{2x+1}{x(x^2+x+1)}$$

   Now use partial fractions to compute the integral on the right.

$$\frac{2x+1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1} = \frac{Ax^2+Ax+A+Bx^2+Cx}{x(x^2+x+1)}$$

   This leads to $0 = A+B$, $2 = A+C$, $1 = A$ and we conclude that $B = -1$ and $C = 1$. So now we have

$$\int \frac{2x+1}{x(x^2+x+1)}\,dx = \int \frac{dx}{x} + \int \frac{1-x}{x^2+x+1}\,dx = \ln|x| + \int \frac{1-x}{(x+1/2)^2+3/4}\,dx.$$

In this integral we make the substitution $u = x + 1/2$ and then $-u + 3/2 = -x + 1$. So we get

$$\begin{aligned}\ln|x| + \int \frac{1-x}{(x+1/2)^2 + 3/4}\,dx &= \ln|x| - \frac{1}{2}\int \frac{2u}{u^2+3/4}\,du + \frac{3}{2}\int \frac{du}{u^2+3/4} \\ &= \ln|x| - \frac{1}{2}\ln(u^2+3/4) + \frac{2}{\sqrt{3}}\frac{3}{2}\arctan\left(\frac{2u}{\sqrt{3}}\right) + C\end{aligned}$$

Returning to the original variable x and combining all the pieces gives

$$-\frac{1}{x}\ln(x^2+x+1) + \ln|x| - \frac{1}{2}\ln(x^2+x+1) + \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

3. *(12 points)* Find $\int e^{3x}\arctan(e^x)\,dx$.

Start with a substitution $w = e^x$. Then $dw = e^x\,dx$. We get

$$\int w^3\arctan(w)\frac{dw}{w} = \int w^2\arctan(w)\,dw.$$

Now do integration by parts, with $u = \arctan w$ and $dv = w^2\,dw$. So $du = dw/(1+w^2)$ and $v = w^3/3$. Thus our integral becomes

$$\frac{w^3}{3}\arctan(w) - \frac{1}{3}\int \frac{w^3}{1+w^2}\,dw$$

Long division gives $w^3/(1+w^2) = w - w/(1+w^2)$ so we have

$$\frac{w^3}{3}\arctan(w) - \frac{1}{3}\frac{w^2}{2} + \frac{1}{3}\frac{1}{2}\ln(1+w^2) + C = \frac{e^{3x}\arctan(e^x)}{3} - \frac{e^{2x}}{6} + \frac{\ln(1+e^{2x})}{6} + C$$

4. *(12 points)* Find $\int_0^2 \frac{dx}{\sqrt{x^2+4x}}$.

Complete the square: $x^2 + 4x = (x+2)^2 - 4$ and make the substitution $u = x + 2$ to get

$$\int_0^2 \frac{dx}{\sqrt{x^2+4x}} = \int_2^4 \frac{du}{\sqrt{u^2-4}}$$

Now make the substitution $u = 2\sec\theta$. The integral becomes

$$\int_0^{\pi/3} \sec\theta\,d\theta$$

since $du = 2\sec\theta\tan\theta\,d\theta$, $u^2 - 4 = 4\tan^2\theta$. In addition $u = 2$ corresponds to $\sec\theta = 1$ or $\theta = 0$ and $u = 4$ corresponds to $\sec\theta = 2$ or $\cos\theta = 1/2$ or $\theta = \pi/3$.

Continuing the integration

$$\int_0^{\pi/3} \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|\Big|_0^{\pi/3} = \ln|2+\sqrt{3}| - \ln|1+0| = \ln|2+\sqrt{3}|.$$

5. *(12 points)* Find $\int \sin^2\theta\cos 3\theta\,d\theta$.

Replace $\sin^2\theta$ by $(1-\cos 2\theta)/2$ to get

$$\frac{1}{2}\int \cos 3\theta\,d\theta - \frac{1}{2}\int \cos 2\theta\cos 3\theta\,d\theta = \frac{1}{2}\frac{\sin 3\theta}{3} - \frac{1}{2}\int \cos 2\theta\cos 3\theta\,d\theta.$$

Either using the formulas in the text or using the addition formula for cosine we find that

$$\cos 2\theta \cos 3\theta = \frac{1}{2}(\cos 5\theta + \cos\theta)$$

Thus

$$\frac{1}{2}\int \cos 2\theta \cos 3\theta \, d\theta = \frac{1}{4}\int (\cos 5\theta + \cos\theta)\, d\theta = \frac{\sin 5\theta}{20} + \frac{\sin\theta}{4} + C$$

Putting all the pieces together our answer is

$$\frac{\sin 3\theta}{6} - \frac{\sin 5\theta}{20} - \frac{\sin\theta}{4} + C$$

6. *(12 points)* Set up an integral for the area of the region enclosed between the curve $y = x^3 - 2$ and its tangent line at $x = -1$. **JUST SET UP THE INTEGRAL. DO NOT COMPUTE A NUMERICAL VALUE.**

First we have to find the tangent line at the point $(-1, -3)$ on the cubic curve. The slope is $dy/dx = 3x^2$ evaluated when $x = -1$. So the slope is 3. The line of slope 3 through the point $(-1, -3)$ turns out to be the line $y = 3x$.

If we sketch the graphs, it is clear that the tangent line lies above the cubic. So our top curve is $y = 3x$ and the bottom curve will be $y = x^3 - 2$. We need to find the points where the tangent and the cubic intersect. Of course one intersection is at $x = -1$. To find the other one we set the two curves equal, and using the fact that $x = -1$ is a root we know that $x + 1$ should be a factor of the cubic equation we get. Long division then lets us find the other root.

$$x^3 - 2 = 3x \Rightarrow x^3 - 3x - 2 = 0 \Rightarrow (x+1)(x^2 - x - 2) = 0 \Rightarrow (x+1)(x+1)(x-2) = 0$$

So the other intersection is at $x = 2$. So the integral we want is

$$\int_{-1}^{2} (3x - x^3 + 2)\, dx$$

7. *(12 points)* The region R is bounded by the curves $y = \ln x$, $y = 0$ and $x = e$. The solid S is obtained by revolving R around the y-axis.

   (a) Set up an integral for the volume of S using the shell method.
   We slice the region vertically as x runs from 1 where the graph of $y = \ln x$ crosses the x-axis up to $x = e$, a given boundary curve for the region. The slice at x has length $\ln x$. We rotate about the y axis, so the radius is x. Thus

$$V = \int_1^e 2\pi x \ln x \, dx$$

   (b) Set up an integral for the volume of S using the disk or washer method.
   Now we have to slice horizontally. The slice at height y runs from the graph of $y = \ln x$ to the vertical line $x = e$. This horizontal slice will generate a washer. The outer radius will be e and the inner radius (the radius of the hole in the center) will be $x = e^y$. We slice as y runs from 0 to 1. So our integral is

$$V = \pi \int_0^1 (e^2 - (e^y)^2) \, dy = \pi \int_0^1 (e^2 - e^{2y}) \, dy$$

   (c) Compute the volume of S. If we use the shell integral we compute using integration by parts.

$$\int x \ln x \, dx = \frac{x^2 \ln x}{2} - \int \frac{x}{2} \, dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C.$$

   Evaluating between 1 and e we eventually get

$$\pi \cdot \frac{e^2 + 1}{2}$$

   using the fact that $\ln e = 1$ and $\ln 1 = 0$.

8. *(6 points)* Sketch the curve given in polar coordinates by $r = 1 + \sin\theta$.

As θ runs from 0 to $\pi/2$ we move through the first quadrant in a counterclockwise direction and r increases steadily from 1 to 2. As we move through the second quadrant, r decreases in a completely symmetric fashion from 2 back to 1. As we swing through the third quadrant, the r values continue to steadily decrease, reaching the value 0 as θ reaches $3\pi/2$. So the curve curls into the origin, tangent to the the $\theta = 3\pi/2$ ray (the negative y-axis). In a symmetric way, as we swing through the fourth quadrant, the r values increase from 0 to 1, so the curve comes out of the origin, again tangent to the negative y-axis.

The resulting curve is a cardioid that crosses the positive x-axis at 1, the positive y-axis at 2, the negative x-axis at -1 and has a cusp at the origin, centered on the negative y-axis.

9. *(10 points)* Find the length of the curve $y = \dfrac{e^x + e^{-x}}{2}$ as x runs from 0 to 1.

Parametrize the curve as $(x, y) = (t, \dfrac{e^t + e^{-t}}{2})$.

$$\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = \frac{e^t - e^{-t}}{2}$$

So we need to compute $1 + (dy/dt)^2$:

$$1 + \left(\frac{e^t - e^{-t}}{2}\right)^2 = 1 + \frac{e^{2t} - 2 + e^{-2t}}{4} = \frac{e^{2t} + 2 + e^{-2t}}{4} = \frac{(e^t + e^{-t})^2}{4}$$

Thus the speed is

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{e^t + e^{-t}}{2}$$

So the arclength is

$$\int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \frac{e^t + e^{-t}}{2}\, dt = \frac{e^t - e^{-t}}{2}\bigg|_0^1 = \frac{e - 1/e}{2} - \frac{1-1}{2} = \frac{e - 1/e}{2}$$