Overview of Integration Techniques
MAT 104 – Frank Swenton, Summer 2000

Fundamental integrands (see table, page 400 of the text)

- *Know well* the antiderivatives of basic terms–everything reduces to them in the end.
- Remember to think of $\dfrac{1}{x^a}$ as x^{-a} when antidifferentiating with the power rule.
- A few other useful integrals to know:

  $\int \tan\theta\, d\theta = -\ln|\cos\theta| + C$ (Don't forget the minus),

  $\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$, and

  $\displaystyle\int \frac{dw}{a^2+w^2} = \frac{1}{a}\tan^{-1}\left(\frac{w}{a}\right) + C$ (Note the a's vs. a^2's, and that the w^2 has coefficient 1)

Substitution (make sure you substitute for all components, including the dx)

- When using substitution for definite integrals, be very careful with the *limits of integration*!
- Be sure to account for *each term* in the integral when substituting, especially the "dx".
- For indefinite integrals, be sure that your final answer is in terms of the variable that was originally given.
- When making substitutions involving fractional powers, it's often easier to reverse the substitution (e.g., instead of $w = \sqrt{x}$, $dw = \frac{1}{2\sqrt{x}}dx$; use $x = w^2$, $dx = 2w\,dw$).

Integration by parts: $\int u\,dv = uv - \int v\,du$

- Factor the integrand so that one factor (the u) becomes simpler when differentiated and what's left (the dv) is not too bad to integrate.
- Don't forget that "$dv = dx, v = x$" is sometimes useful (e.g., for $\int \arctan x\,dx$)

Trig Substitution: To get rid of square roots and half-powers of certain quadratic terms, when nothing easier applies (e.g., substitution)

a. $\sqrt{a^2-x^2}$ or $(a^2-x^2)^{k/2}$: Use $x = a\sin\theta$
b. $\sqrt{a^2+x^2}$ or $(a^2+x^2)^{k/2}$: Use $x = a\tan\theta$
c. $\sqrt{x^2-a^2}$ or $(x^2-a^2)^{k/2}$: Use $x = a\sec\theta$

- Remember the equations $\sin^2\theta + \cos^2\theta = 1$ and $\tan^2\theta + 1 = \sec^2\theta$.
- When finished, be sure to simplify "trigs of inverse-trigs"; draw a right-triangle and use it. (e.g. $\tan(\cos^{-1}(x)) = \sqrt{1-x^2}/x$)

Integrals Involving Trig Functions

- Make the integrand simpler using known identities, e.g., $\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$, $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$, $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$, as well as those for $\sin mx \cos nx$, $\sin mx \sin nx$, and $\cos mx \cos nx$.
- If there is an odd total power of $\sin\theta$ in the integrand, substitute $w = \cos\theta$ and use the identity $\sin^2\theta = 1 - \cos^2\theta$ (or vice-versa if there is an odd total power of $\cos\theta$). If there are *even* total powers of *both* $\sin\theta$ and $\cos\theta$, then use the formulae for $\sin^2\theta$ and $\cos^2\theta$ to reduce the integral to one involving powers of $\cos 2\theta$ and continue.

Partial Fractions

- Use this method for rational functions–but **only** when nothing easier applies (e.g., substitution).

0. If the degree of the numerator is not *strictly smaller* than the degree of the denominator, use **long division** first to split the rational function into the sum of a polynomial and a proper rational function.
1. **Factor the denominator** completely, into linear factors $(ax+b)^n$ and *irreducible* quadratic factors $(ax^2+bx+c)^n$ [$\longleftarrow$ must have $b^2-4ac<0$, otherwise the quadratic term splits into linear terms].
2. **Write the** partial fraction **equation** for the (proper!) rational function:
   - constant unknowns over linear powers, and linear unknowns over quadratic powers.
   - the total number of unknowns in the equation should equal the degree of the denominator.
   - the denominators on the right should all be distinct and divide the denominator on the left.
3. **Solve** the partial fraction equation.
4. **Split the integrand** into partial fractions as given by the solution of the equation, and rewrite the integral.
5. **Integrate** the smaller integrals and come to a final answer.
   - For an integral of the form $\int \frac{dx+f}{(ax^2+bx+c)^n}dx$ (with the denominator irreducible), start the substitution you'd like to do, then split up the numerator into the sum of a multiple of $2ax+b$ and a constant.

Miscellaneous

- Remember that $e^{kx}=(e^x)^k$, $\left(\pm x^2\pm a^2\right)^{k/2}=\left(\sqrt{\pm x^2\pm a^2}\right)^k$, and $(ax+b)^{1/n}=\sqrt[n]{ax+b}$.
  - When you see a fractional power of something, think "root."
- To integrate something like $\int e^{ax}\sin bx\,dx$ or $\int e^{ax}\cos bx\,dx$, use integration by parts *twice* and solve the resulting equation.
- If you see $\sqrt[n]{ax+b}$ or even just $\sqrt[n]{x}$, you may *sometimes* want to try substituting $w=\sqrt[n]{ax+b}$, i.e. $w^n=ax+b$.
  - In general, you want to get rid of roots one way or another.
- Remember that any quadratic ax^2+bx+c can be put into the form $\pm w^2\pm k^2$ by completing the square and substituting.
  - e.g., Using $x^2+4x+5=(x+2)^2+1$ and $w=x+2$, $\int \frac{1}{\sqrt{x^2+4x+5}}dx$ becomes $\int \frac{1}{\sqrt{w^2+1}}dw$.

Overall Tips

- *Work carefully* and neatly. It's faster in the long-run to do it right the first time than to do it over if you make a mistake, and neat, organized work is easier for both you and the grader to check.
- *Check your work* when you can. Don't lose points on algebra mistakes and incorrect indefinite integrals when you don't have to. Differentiate antiderivatives, multiply out squares you've completed, add up partial fractions, etc. (when time allows).
- *Don't Panic!* If you need to take an integral and you don't know immediately how to go about it, then try use the things you've learned (identities, formulae, methods of integration, etc.) to make the integral simpler or make it look more like something you know how to integrate. Any integral that you'll be asked to compute outright (as opposed to just saying whether it's convergent or divergent) *will be workable* using some method or combination of methods that you've learned. There is no penalty for choosing the wrong means first, if you find the right one eventually. Be patient and persistent, and some application of the methods you've learned will work.