

## Fall 04 Midterm Answers

$$1. \int_0^1 \frac{\sin(\tan^{-1} x) \tan^{-1} x}{x^2 + 1} dx = \frac{1}{\sqrt{2}} \left(1 - \frac{\pi}{4}\right)$$

$$u = \tan^{-1} x$$

$$du = \frac{dx}{x^2 + 1}$$

$$\begin{aligned} \int_0^1 \frac{\sin(\tan^{-1} x) \tan^{-1} x}{x^2 + 1} dx &= \int_0^{\pi/4} u \sin u \, du \\ &= -u \cos u + \sin u \Big|_0^{\pi/4} \\ &= \frac{1}{\sqrt{2}} \left(1 - \frac{\pi}{4}\right) \end{aligned}$$

$$2. \int \frac{x^2 - x + 2}{x^3 + x} dx = 2 \ln |x| - 1/2 \ln(x^2 + 1) - \tan^{-1} x + C$$

$$\frac{x^2 - x + 2}{x^3 + x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

$$x^2 - x + 2 = A(x^2 + 1) + (Bx + C)x$$

$$A = 2 \quad C = -1 \quad B = -1$$

$$\begin{aligned} \int \frac{x^2 - x + 2}{x^3 + x} dx &= \int \frac{2}{x} dx - \int \frac{x}{x^2 + 1} dx - \int \frac{dx}{x^2 + 1} \\ &= 2 \ln |x| - 1/2 \ln(x^2 + 1) - \tan^{-1} x + C \end{aligned}$$

$$3. \int e^{\frac{\sqrt{1-x^2}}{x}} \cdot \frac{dx}{x^2 \sqrt{1-x^2}} = -e^{\frac{\sqrt{1-x^2}}{x}} + C$$

$$x = \sin \theta$$

$$dx = \cos \theta \, d\theta.$$

$$\begin{aligned} \int e^{\frac{\sqrt{1-x^2}}{x}} \cdot \frac{dx}{x^2 \sqrt{1-x^2}} &= \int e^{\frac{\cos \theta}{\sin \theta}} \cdot \frac{\cos \theta \, d\theta}{\sin^2 \theta \cos \theta} \\ &= \int e^{\cot \theta} \cdot \csc^2 \theta \, d\theta \\ &= -e^{\cot \theta} + C \\ &= -e^{\frac{\sqrt{1-x^2}}{x}} + C \end{aligned}$$

4.  $\int_0^\pi \frac{\sin \sqrt{x}}{x(1+x)} dx$  Converges.

As  $x \rightarrow 0$ ,  $\sin x \sim x$ , so  $\sin \sqrt{x} \sim \sqrt{x}$ .

$$\frac{\sin \sqrt{x}}{x(1+x)} \sim \frac{\sqrt{x}}{x \cdot 2} = \frac{1}{2x^{1/2}}.$$

$\int_0^\pi \frac{dx}{2x^{1/2}}$  converges, so by limit comparison test, the original integral converges.

5.  $\int_0^\infty \frac{\ln(5 + \cos x)}{x^2} dx$  Diverges.

$$\int_0^\infty \frac{\ln(5 + \cos x)}{x^2} = \int_0^1 \frac{\ln(5 + \cos x)}{x^2} + \int_1^\infty \frac{\ln(5 + \cos x)}{x^2}$$

As  $x \rightarrow 0$ ,  $\frac{\ln(5 + \cos x)}{x^2} \sim \frac{\ln 6}{x^2}$ .  $\int_0^1 \frac{\ln 6}{x^2} dx$  diverges, so first integral diverges by limit comparison test. Since one part of the original integral diverges, the whole integral must diverge.

6.  $\sum_{n=3}^\infty \frac{7^n \ln n}{n!}$  Converges.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{7^{n+1} \ln(n+1)}{(n+1)!} \cdot \frac{n!}{7^n \ln n} \\ &= \lim_{n \rightarrow \infty} \frac{7}{n+1} \cdot \frac{\ln(n+1)}{\ln n} \\ &= 0 < 1 \end{aligned}$$

so series converges by Ratio Test.

7.  $\sum_{n=1}^\infty \sin\left(\frac{1}{n^2}\right) \ln n$  Converges.

As  $n \rightarrow \infty$ ,  $1/n^2 \rightarrow 0$ , so  $\sin(1/n^2) \sim 1/n^2$ .  $\ln n < n^{1/2}$ , so

$$\sum_{n=1}^\infty \frac{\ln n}{n^2} < \sum_{n=1}^\infty \frac{n^{1/2}}{n^2} = \sum_{n=1}^\infty \frac{1}{n^{3/2}}.$$

The right hand series converges, so the left hand series does as well. (Direct Comparison.) Then the original series converges, by Limit Comparison.

$$8. \sum_{n=2}^{\infty} \frac{2^n + 3^{n+1}}{5^n} = \frac{89}{30}$$

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{2^n + 3^{n+1}}{5^n} &= \sum_{n=2}^{\infty} \frac{2^n}{5^n} + \sum_{n=2}^{\infty} \frac{3^{n+1}}{5^n} \\ &= \left(\frac{4}{25} + \frac{8}{125} + \dots\right) + \left(\frac{27}{25} + \frac{81}{125} + \dots\right) \\ &= \frac{4}{25} \left(1 + \frac{2}{5} + \frac{4}{25} + \dots\right) + \frac{27}{25} \left(1 + \frac{3}{5} + \frac{9}{25} + \dots\right) \\ &= \frac{4}{25} \cdot \frac{1}{1 - 2/5} + \frac{27}{25} \cdot \frac{1}{1 - 3/5} \\ &= \frac{89}{30} \end{aligned}$$

$$9. \sum_{n=1}^{\infty} \left(1 + \frac{x}{n}\right)^{n^2} \text{ converges when } x < 0.$$

Use the root test.

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n^{1/n} &= \lim_{n \rightarrow \infty} (1 + x/n)^{n^2/n} \\ &= \lim_{n \rightarrow \infty} (1 + x/n)^n \\ &= e^x. \end{aligned}$$

When this limit is  $< 1$ , the series converges. Thus the series converges when  $e^x < 1$ , *i.e.* when  $x < 0$ . When this limit is  $> 1$ , the series diverges. Thus the series diverges when  $e^x > 1$ , *i.e.* when  $x > 0$ . When the limit is equal to 1 *i.e.* when  $x = 0$ , the test doesn't tell us anything, but in this case, the series is

$$\sum_{n=1}^{\infty} (1)^{n^2} = \sum_{n=1}^{\infty} 1$$

which obviously diverges.