Final Exam Solutions - MAT 104

Problem 1 (8 points). Compute the following integrals:

(a) $\displaystyle\int \frac{x}{(1-x^2)^{3/2}}dx$

Solution:

$$\begin{aligned}\int \frac{x}{(1-x^2)^{3/2}}dx &= -\frac{1}{2}\int (1-x^2)^{-3/2}(-2xdx)\\ &= -\frac{1}{2}\times\frac{1}{-1/2}(1-x^2)^{-1/2}+C = \frac{1}{\sqrt{1-x^2}}+C.\end{aligned}$$

□

(b) $\displaystyle\int x\ln(x+1)dx$

Solution: We use integration by parts, taking $u=\ln(x+1)$ and $dv = xdx$. Then

$$\begin{aligned}\int x\ln(x+1)dx &= \frac{1}{2}x^2\ln(x+1) - \frac{1}{2}\int\frac{x^2}{x+1}dx\\ &= \frac{1}{2}x^2\ln(x+1) - \frac{1}{2}\int\left(x-1+\frac{1}{x+1}\right)dx\\ &= \frac{1}{2}x^2\ln(x+1) - \frac{1}{4}x^2+\frac{1}{2}x-\frac{1}{2}\ln(x+1)+C.\end{aligned}$$

□

Problem 2 (12 points). (a) Let R be the region bounded by the x-axis and the graph of $y=1/(x^4+1)$ as x runs from 0 to ∞. Find the volume of the solid of revolution obtained by revolving R about the y-axis.

Solution: We use the shell method. The radius of each shell is $r=x$, and the height is $h=y=1/(x^4+1)$. Hence

$$\begin{aligned}\text{Volume} = 2\pi\int_0^\infty \frac{x}{x^4+1}dx &= 2\pi\lim_{b\to\infty}\frac{1}{2}\int_0^b\frac{2xdx}{(x^2)^2+1}\\ &= \lim_{b\to\infty}\pi\arctan(x^2)\Big|_0^b = \frac{\pi^2}{2},\end{aligned}$$

since $\lim_{x\to\infty}\arctan x = \pi/2$. □

(b) Calculate the area of the surface obtained by revolving the graph of $y=e^x$ between the points (0,1) and $(1,e)$ around the x-axis.

Solution: We have to add the area of thin strips of width

$$ds = \sqrt{1 + (dy/dx)^2} = \sqrt{1 + e^{2x}}$$

and length $2\pi r = 2\pi e^x$. Then

$$\text{Surface} = 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}} = 2\pi \int_1^e \sqrt{1 + u^2} du,$$

where we have applied the change of variables $u = e^x$. To find the antiderivative of $\sqrt{1 + u^2}$, we apply the trigonometric substitution $u = \tan z$, so $du = \sec^2 z$, and hence

$$\int \sqrt{1 + u^2} du = \int \sec z \sec^2 z dz = \int \sec^3 z dz.$$

We now integrate by parts with $u = \sec z$ and $dv = \sec^2 z dz$, to get

$$\begin{aligned}
\int \sec^3 z dz &= \sec z \tan z - \int \tan z \sec z \tan z dz \\
&= \sec z \tan z - \int \tan^2 z \sec z dz \\
&= \sec z \tan z - \int (\sec^2 z - 1) \sec z dz \\
&= \sec z \tan z + \int \sec z dz - \int \sec^3 z dz.
\end{aligned}$$

Thus

$$\begin{aligned}
\int \sec^3 z dz &= \frac{1}{2}\big(\sec z \tan z + \int \sec z dz \big) \\
&= \frac{1}{2} \sec z \tan z + \frac{1}{2} \ln |\tan z + \sec z| + C,
\end{aligned}$$

whence

$$\int \sqrt{1 + u^2} du = \frac{1}{2} u \sqrt{1 + u^2} + \frac{1}{2} \ln |u + \sqrt{1 + u^2}| + C.$$

Therefore

$$\begin{aligned}
\text{Surface} &= \pi \big(u\sqrt{1 + u^2} + \ln |u + \sqrt{1 + u^2}| \big) \Big|_1^e \\
&= \pi \Big(e\sqrt{1 + e^2} - \sqrt{2} + \ln \Big| \frac{e + \sqrt{1 + e^2}}{1 + \sqrt{2}} \Big| \Big).
\end{aligned}$$

$\square$

Problem 3 (16 points). Determine whether the following integrals converge or diverge. Give your reasons.

(a) $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}+x^3}$

Solution: **Converges.** We write

$$\int_0^\infty \frac{dx}{\sqrt{x}+x^3} = \int_0^1 \frac{dx}{\sqrt{x}+x^3} + \int_1^\infty \frac{dx}{\sqrt{x}+x^3}.$$

The first integral converges since $\dfrac{1}{\sqrt{x}+x^3} \le \dfrac{1}{\sqrt{x}}$ and $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ converges (p-test). Likewise, the second integral converges since $\dfrac{1}{\sqrt{x}+x^3} \le \dfrac{1}{x^3}$ and $\displaystyle\int_1^\infty \frac{dx}{x^3}$ converges (p-test, at ∞). $\square$

(b) $\displaystyle\int_0^1 \frac{\tan\sqrt{x}}{x+x^2}dx$

Solution: **Converges.** We have that, for small x, $\tan\sqrt{x} \sim \sqrt{x}$, so $\dfrac{\tan\sqrt{x}}{x+x^2} \sim \dfrac{\sqrt{x}}{x+x^2} \sim \dfrac{1}{\sqrt{x}}$, since x^2 is much smaller than x if x is small. The conclusion follows since $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ converges (p-test). $\square$

(c) $\displaystyle\int_0^1 \frac{\ln(1+x)}{x^3}dx$

Solution: **Diverges.** For small x, $\ln(1+x) \sim x$, so $\dfrac{\ln(1+x)}{x^3} \sim \dfrac{1}{x^2}$, and $\displaystyle\int_0^1 \frac{1}{x^2}$ diverges (p-test). $\square$

(d) $\displaystyle\int_1^\infty \frac{dx}{x\ln x}$

Solution: **Diverges.** We have that

$$\int_1^\infty \frac{dx}{x\ln x} = \int_1^2 \frac{dx}{x\ln x} + \int_2^\infty \frac{dx}{x\ln x},$$

and both of these two integrals diverge, since $\displaystyle\int \frac{dx}{x\ln x} = \ln\ln x$ and none of the limits $\lim_{x\to 1}\ln\ln x$ and $\lim_{x\to\infty}\ln\ln x$ exist. $\square$

Problem 4 (16 points). Determine whether the following series converge or diverge. Give your reasons.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{\sqrt{n^5+1}}$

Solution: **Diverges.** We have, for large n, $\dfrac{n^2}{\sqrt{n^5+1}} \sim \dfrac{1}{n^{1/2}}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges. □

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{n^3+1}$

Solution: **Converges conditionally.** $\dfrac{n^2}{n^3+1}$ decreases to zero, so the series converges by the alternating series test. It doesn't converge absolutely since $\left|\dfrac{(-1)^n n^2}{n^3+1}\right| \sim \dfrac{1}{n}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. □

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 \cdot 3^n}{n!}$

Solution: **Converges.** Let $a_n = \dfrac{n^2 \cdot 3^n}{n!}$. We have

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)^2 \cdot 3^{n+1}}{(n+1)!}}{\dfrac{n^2 \cdot 3^n}{n!}} = \frac{3(n+1)}{n^2} \to 0 < 1,$$

so the series converges by the ratio test. □

(d) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{n+1}{n+3}\right)^{n^2}$

Solution: **Converges.** Let $a_n = \left(\dfrac{n+1}{n+3}\right)^{n^2}$. Then

$$(a_n)^{1/n} = \left(\frac{n+1}{n+3}\right)^n = \left(1 - \frac{2}{n+3}\right)^n \to e^{-2} < 1,$$

so the series converges by the root test. □

Problem 5 (12 points). Let $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n+2}\left(\frac{x-2}{3}\right)^n$.

(a) For what values of x does the series converge?

Solution: By the ratio test, the power series converges for $\left|\dfrac{x-2}{3}\right| < 1$, i. e. $-1 < x < 5$. For $x = -1$, we obtain the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}$, which converges by the alternating series test. For $x = 5$, we obtain $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+2}$, which diverges. Therefore the series converges for $-1 \le x < 5$. □

(b) Find $f^{(50)}(2)$.

Solution: Let $a_n = \dfrac{1}{(n+2)3^n}$. Thus $f(x) = \sum_{n=0}^{\infty} a_n (x-2)^n$. Therefore $f^{(50)}(2) = 50! \cdot a_n = \dfrac{50!}{52 \cdot 3^{50}}$. □

Problem 6 (12 points).

(a) Use Taylor series to compute $\displaystyle\lim_{x\to 0} \frac{(e^x - 1 - x)^2 \cos x}{x(\sin x - x)}$.

Solution: The first few terms of the Taylor series of each of e^x, $\sin x$, and $\cos x$ are $1 + x + x^2/2$, $x - x^3/6$, and $1 - x^2/2$, respectively. Hence

$$\lim_{x\to 0} \frac{(e^x - 1 - x)^2 \cos x}{x(\sin x - x)} = \lim_{x\to 0} \frac{(x^2/2)^2(1 - x^2/2)}{x(-x^3/6)} = \frac{-6}{4} = -\frac{3}{2}.$$

□

(b) Find the Taylor series of $F(x) = \displaystyle\int_0^{\infty} \frac{dt}{1+t^4} dt$. For what values of x does it converge?

Solution: Since $\dfrac{1}{1+t^4} = \displaystyle\sum_{n=0}^{\infty} (-t^4)^n$ (geometric series), and the fact that power series can be integrated "term by term" within its interval of convergence ($|t| < 1$ in this case), we have that

$$F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{4n+1},$$

for $|x| < 1$. The series also converges for $x = 1$ by the alternating series test. □

Problem 7 (12 points). For the questions below express your answers in the form $a + ib$, where a and b are real numbers. Simplify your expressions for a and b.

(a) Simplify $\left(\dfrac{7+i}{3+4i}\right)^{43}$.

Solution: First,

$$\frac{7+i}{3+4i} = \frac{(7+i)(3-4i)}{3^2+4^2} = \frac{25-25i}{25} = 1 - i = \sqrt{2}e^{-\frac{\pi}{4}i}.$$

Thus

$$\left(\frac{7+i}{3+4i}\right)^{43} = 2^{\frac{43}{2}}e^{-\frac{43\pi}{4}i} = 2^{\frac{43}{2}}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -2^{21} - 2^{21}i.$$

□

(b) Solve $z^4 = -8iz$.

Solution: This is a four-degree polynomial equation, so it has four solutions. One is $z_1 = 0$, and the other three are the solutions of $z^3 = -8i = 8e^{-\frac{\pi}{2}i}$. These are

$$\begin{aligned} z_2 &= 2e^{-\frac{\pi}{6}i} = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i; \\ z_3 &= 2e^{(-\frac{\pi}{6}+\frac{2\pi}{3})i} = 2e^{\frac{\pi}{2}i} = 2i; \text{ and} \\ z_4 &= 2e^{(-\frac{\pi}{6}+\frac{4\pi}{3})i} = 2e^{\frac{7\pi}{6}i} = -\sqrt{3} - i. \end{aligned}$$

□

Problem 8 (12 points). Find all real solutions to the following differential equations.

(a) $y'' + 2y' + 10y = 0$

Solution: The solutions of the quadratic equation $\lambda^2 + 2\lambda + 10 = 0$ are $\dfrac{-2 \pm \sqrt{4-40}}{2} = -1 \pm 3i$. Hence, the solutions of the equation are

$$y = C_1 e^{-x}\cos 3x + C_2 e^{-x}\sin 3x.$$

□

(b) $2y'' + y' - 3y = 0$

Solution: The solutions of the quadratic equation $2\lambda^2 + \lambda - 3 = 0$ are 1 and $-3/2$. Thus, the solutions of the equation are

$$y = C_1 e^{x} + C_2 e^{-\frac{3}{2}x}.$$

□