Mat104 Solutions to Problems on Complex Numbers from Old Exams

(1) Solve $z^5 = 6i$. Let $z = r(\cos\theta + i\sin\theta)$. Then $z^5 = r^5(\cos 5\theta + i\sin 5\theta)$. This has modulus r^5 and argument 5θ. We want this to match the complex number $6i$ which has modulus 6 and infinitely many possible arguments, although all are of the form $\pi/2, \pi/2 \pm 2\pi, \pi/2 \pm 4\pi, \pi/2 \pm 6\pi, \pi/2 \pm 8\pi, \pi/2 \pm 10\pi, \dots$. (We will see that we don't really lose anything if we drop the $\pm$ in our list of possible arguments for $6i$.) So we choose

$$r^5 = 6 \quad \text{and} \quad 5\theta = \frac{\pi}{2} \text{ or } \frac{\pi}{2} + 2\pi \text{ or } \frac{\pi}{2} + 4\pi \text{ or } \frac{\pi}{2} + 6\pi \text{ or } \frac{\pi}{2} + 8\pi \text{ or } \frac{\pi}{2} + 10\pi \text{ or } \dots$$

In other words, in order to have $z^5 = 6i$ we should take z of the form $r(\cos\theta + i\sin\theta)$ where

$$r = \sqrt[5]{6} \quad \text{and} \quad \theta = \frac{\pi}{10} \text{ or } \frac{\pi}{10} + \frac{2\pi}{5} \text{ or } \frac{\pi}{10} + \frac{4\pi}{5} \text{ or } \frac{\pi}{10} + \frac{6\pi}{5} \text{ or } \frac{\pi}{10} + \frac{8\pi}{5} \text{ or } \frac{\pi}{10} + \frac{10\pi}{5} \text{ or } \dots$$

Notice that there are only really 5 choices for theta. The sixth choice $\theta = \pi/10 + 10\pi/5 = \pi/10 + 2\pi$ gives the same complex number as the first choice, where we simply take $\theta = \pi/10$. So there are exactly 5 solutions to $z^5 = 6i$ corresponding to $r = \sqrt[5]{6}$ and $\theta = \pi/10, \pi/10 + 2\pi/5, \pi/10 + 4\pi/5, \pi/10 + 6\pi/5$ and $\pi/10 + 8\pi/5$.

If we sketch these complex numbers we would see that they all lie on the circle of radius $\sqrt[5]{6} \approx 1.43$ and they are separated from each other by an angle of $2\pi/5$.

(2) Find the real part of $(\cos 0.7 + i\sin 0.7)^{53}$. This is the same as

$$(e^{0.7i})^{53} = e^{53\cdot 0.7i} = e^{37.1i} = \cos(37.1) + i\sin(37.1).$$

So the real part is simply $\cos(37.1)$.

(3) Find all complex numbers z in rectangular form such that $(z-1)^4 = -1$.

Solve $w^4 = -1$ first and then $z = w + 1$. The complex number -1 has modulus 1 and argument of the form $\pm\pi, \pm 3\pi, \pm 5\pi, \pm 7\pi, \dots$. If $w = r(\cos\theta + i\sin\theta)$ then $w^4 = r^4(\cos 4\theta + i\sin 4\theta)$. So

$$r^4 = 1 \quad \text{and} \quad 4\theta = \pm\pi, \pm 3\pi, \pm 5\pi, \pm 7\pi, \pm 9\pi, \dots$$

So take $r = 1$ and $\theta = \dfrac{\pi}{4}$ or $\theta = \dfrac{3\pi}{4}$ or $\theta = \dfrac{5\pi}{4}$ or $\theta = \dfrac{7\pi}{4}$. This is a complete list of the four distinct fourth roots of -1. (The next choice of theta in the sequence is nothing new since $9\pi/4 = \pi/4 + 2\pi$ which corresponds to the same complex number we get from taking $\theta = \pi/4$.)

So

$$w = \cos\theta + i\sin\theta \quad \text{where } \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \text{ or } \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$$

and since $z = w + 1$ we have

$$z = \left(1 + \frac{\sqrt{2}}{2}\right) + i\frac{\sqrt{2}}{2} \quad \text{from } \theta = \frac{\pi}{4}$$

or

$$z = \left(1 - \frac{\sqrt{2}}{2}\right) + i\frac{\sqrt{2}}{2} \quad \text{from } \theta = \frac{3\pi}{4}$$

or

$$z = \left(1 - \frac{\sqrt{2}}{2}\right) - i\frac{\sqrt{2}}{2} \quad \text{from } \theta = \frac{5\pi}{4}$$

or

$$z = \left(1 + \frac{\sqrt{2}}{2}\right) - i\frac{\sqrt{2}}{2} \quad \text{from } \theta = \frac{7\pi}{4}$$

(4) Write $(\sqrt{3} + i)^{50}$ in polar and in Cartesian form. First put $\sqrt{3} + i$ into polar form. Its modulus is

$$|\sqrt{3} + i| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2.$$

Its argument θ must satisfy $\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$. So $\theta = \frac{\pi}{6}$. Thus in polar form we have

$$(\sqrt{3} + i)^{50} = (2e^{i\pi/6})^{50} = 2^{50} \cdot e^{\frac{50\pi}{6}i}.$$

This can be simplified since $\frac{50\pi}{6} = 8\pi + \frac{2\pi}{6}$. Thus

$$(\sqrt{3} + i)^{50} = 2^{50} \cdot e^{\frac{\pi}{3}i} = 2^{50}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

So this is our answer in polar form. In Cartesian form we have

$$(\sqrt{3} + i)^{50} = 2^{50}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2^{49} + 2^{49}\sqrt{3}i.$$

(5) Find all fifth roots of -32. As usual, let $z = r(\cos\theta + i\sin\theta)$. Then $z^5 = r^5(\cos 5\theta + i\sin 5\theta)$. To match up with -32 which has modulus $32 = 2^5$ and argument of the form $\pi + 2\pi k$ where k can be any integer we take

$$r = 2 \quad \text{and} \quad \theta = \frac{\pi}{5}, \frac{3\pi}{5}, \frac{5\pi}{5}, \frac{7\pi}{5}, \text{ or } \frac{9\pi}{5}$$

to get a complete list of the fifth roots of -32. (As usual, note that the next angle in the sequence would be $11\pi/5 = \pi/5 + 2\pi$ and so gives the same complex number as does choosing $\theta = \pi/5$.)

(6) (a)

$$\frac{1}{1+i} + \frac{1}{1-i} = \frac{1 - i + 1 + i}{(1+i)(1-i)} = \frac{2}{1 - i^2} = \frac{2}{2} = 1 = 1 + 0i.$$

(b)

$$e^{2+i\pi/3} = e^2 \cdot e^{i\pi/3} = e^2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = e^2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{e^2}{2} + \frac{\sqrt{3}e^2}{2}i.$$

(7) If $z^3 = 8i$ then z has modulus $\sqrt[3]{8} = 2$ and its argument θ will be one third of the argument of $8i$. In other words, we should choose

$$\theta = \frac{1}{3}\frac{\pi}{2} = \frac{\pi}{6} \text{ or } \frac{1}{3}\left(\frac{\pi}{2} + 2\pi\right) = \cdots = \frac{5\pi}{6} \text{ or } \frac{1}{3}\left(\frac{\pi}{2} + 4\pi\right) = \cdots = \frac{3\pi}{2}$$

Thus if we denote the three cube roots by z_1,z_2 and z_3 we get

$$z_1 = 2e^{i\pi/6} = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i$$

$$z_2 = 2e^{i5\pi/6} = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\sqrt{3} + i$$

$$z_3 = 2e^{i3\pi/2} = 2(-i) = -2i.$$

(8) $1+i$ has modulus $\sqrt{2}$ and argument $\pi/4, \pi/4+2\pi, \pi/4+4\pi, \ldots$. So z will have modulus r so that $r^5 = \sqrt{2}$, that is $r = \sqrt[10]{2}$. The argument θ of z will be one fifth of the argument of $1+i$, so the five fifth roots will correspond to $\theta = \pi/20, \pi/20 + 2\pi/5, \pi/20 + 4\pi/5, \pi/20 + 6\pi/5$ and $\pi/20 + 8\pi/5$.

(9) The imaginary part is $1/2$ since

$$\frac{2+i}{3-i} = \frac{2+i}{3-i}\frac{3+i}{3+i} = \frac{6+5i+i^2}{9-i^2} = \frac{5+5i}{10} = \frac{1}{2} + \frac{1}{2}i$$

(10) Since $1-i$ has argument $-\pi/4$ and modulus $\sqrt{2}$ we know that

$$(1-i)^{1999} = \left(\sqrt{2}e^{-i\pi/4}\right)^{1999} = (\sqrt{2})^{1999}e^{-i(1999\pi/4)}.$$

But $1999\pi/4 = 499\pi + 3\pi/4 = 498\pi + 7\pi/4$ and so $-1999\pi/4 = -498\pi - 7\pi/4 = -500\pi + \pi/4$. Therefore $(1-i)^{1999}$ is a complex number in the first quadrant, with argument $\pi/4$.

(11) $e^{iz} = 3i$. Let $z = a + ib$. Then $iz = ai - b$. So $e^{iz} = e^{-b}e^{ai}$. Thus e^{iz} will have modulus e^{-b} and argument a. On the other hand, $3i$ has modulus 3 and argument $\pi/2 + 2\pi k$, where k can be any integer. So there will be infinitely many solutions, but we must choose a and b so that $e^{-b} = 3$ and $a = \pi/2 + 2\pi k$ with k an integer. So

$$z = \left(\frac{\pi}{2} + 2\pi k\right) - i\ln 3, \qquad \text{where } k \in \mathbb{Z}.$$

(12) Write $(1-i)^{100}$ as $a + ib$ where a and b are real.

The complex number $1-i$ has modulus $\sqrt{2}$ and argument $-\pi/4$. That is

$$\begin{aligned} 1-i &= \sqrt{2}(\cos(-\pi/4) + i\sin(-\pi/4)) \\ \implies (1-i)^{100} &= (\sqrt{2})^{100}(\cos(-100\pi/4) + i\sin(-100\pi/4)) \\ &= 2^{50}(\cos(-25\pi) + i\sin(-25\pi)) \\ &= 2^{50}(\cos(-\pi) + i\sin(-\pi)) = 2^{50}(-1+0i) = -2^{50} \end{aligned}$$

(13) The real part of $e^{(5+12i)x}$ where x is real is $e^{5x}\cos 12x$ since
$$e^{(5+12i)x} = e^{5x}e^{12ix} = e^{5x}(\cos 12x + i\sin 12x).$$

(14) $z^6 = 8$ where $z = r(\cos\theta + i\sin\theta)$. As usual, $r^6 = 8$ and θ is one sixth of the argument of the complex number 8, that is θ is one sixth of an integer multiple of 2π. Thus
$$r = (2^3)^{1/6} = 2^{1/2} = \sqrt{2} \text{ and } \theta = 0, \frac{2\pi}{6}, \frac{4\pi}{6}, \frac{6\pi}{6}, \frac{8\pi}{6}, \frac{10\pi}{6}, \ldots.$$
In other words we get the 6 distinct sixth roots of 8 if $z = r(\cos\theta + i\sin\theta)$ where
$$r = \sqrt{2} \text{ and } \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3} \text{ or } \frac{5\pi}{3}$$

(15) Summing this series is very similar to the problem of computing the sum $\displaystyle\sum_{n=0}^{\infty}\frac{\cos n\theta}{n!}$, worked out in detail as Example 4 on page 669 of Stein & Barcellos. In this case
$$\sum_{n=0}^{\infty}\left(\frac{\cos n\theta}{n!} + i\frac{\sin n\theta}{n!}\right) = \sum_{n=0}^{\infty}\frac{e^{in\theta}}{n!} = \sum_{n=0}^{\infty}\frac{(e^{i\theta})^n}{n!}$$
Since $\displaystyle\sum_{n=0}^{\infty}\frac{\cos n\theta}{n!}$ and $\displaystyle\sum_{n=0}^{\infty}\frac{\sin n\theta}{n!}$ both converge we can break this up as
$$\sum_{n=0}^{\infty}\left(\frac{\cos n\theta}{n!} + i\frac{\sin n\theta}{n!}\right) = \sum_{n=0}^{\infty}\frac{\cos n\theta}{n!} + i\sum_{n=0}^{\infty}\frac{\sin n\theta}{n!}$$
From this we can conclude that $\displaystyle\sum_{n=0}^{\infty}\frac{\sin n\theta}{n!}$ is just the imaginary part of $\displaystyle\sum_{n=0}^{\infty}\frac{(e^{i\theta})^n}{n!}$.

Since $e^z = \displaystyle\sum_{n=0}^{\infty}\frac{z^n}{n!}$ for any complex number z we have
$$\sum_{n=0}^{\infty}\frac{(e^{i\theta})^n}{n!} = e^{e^{i\theta}} = e^{\cos\theta + i\sin\theta} = e^{\cos\theta}\cdot e^{i\sin\theta} = e^{\cos\theta}\cdot(\cos(\sin\theta) + i\sin(\sin\theta))$$
Taking the imaginary part we get
$$\sum_{n=0}^{\infty}\frac{\sin n\theta}{n!} = e^{\cos\theta}\sin(\sin\theta)$$

(16) $\displaystyle\sum_{0}^{\infty}\frac{\cos(n\theta)}{2^n}$ is the real part of a complex geometric series since
$$(e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta \text{ and } \left(\frac{e^{i\theta}}{2}\right)^n = \frac{\cos n\theta}{2^n} + i\frac{\sin n\theta}{2^n}$$
Both $\displaystyle\sum_{0}^{\infty}\frac{\cos(n\theta)}{2^n}$ and $\displaystyle\sum_{0}^{\infty}\frac{\sin(n\theta)}{2^n}$ converge absolutely by comparison to the real geometric series $\displaystyle\sum\frac{1}{2^n}$. The same arguments we used for ordinary geometric series tell us that $\sum_0^{\infty} r^n$ converges to $1/(1-r)$ whenever $|r| < 1$, even if r is complex.

So $\sum_{0}^{\infty}\left(\frac{e^{i\theta}}{2}\right)^n$ converges to $1/(1-e^{i\theta}/2)$ and all we have to do is find the real part of this complex number.

$$\begin{aligned}
\frac{1}{1-e^{i\theta}/2} &= \frac{1}{1-\left(\frac{\cos\theta}{2}+i\frac{\sin\theta}{2}\right)} \\
&= \frac{2}{(2-\cos\theta)-i\sin\theta} \\
&= \frac{2}{(2-\cos\theta)-i\sin\theta}\cdot\left(\frac{(2-\cos\theta)+i\sin\theta}{(2-\cos\theta)+i\sin\theta}\right) \\
&= \frac{4-2\cos\theta+2i\sin\theta}{4-4\cos\theta+\cos^2\theta+\sin^2\theta} \\
&= \frac{4-2\cos\theta+2i\sin\theta}{5-4\cos\theta}
\end{aligned}$$

Conclusion:

$$\sum_{0}^{\infty}\frac{\cos n\theta}{2^n}=\frac{4-2\cos\theta}{5-4\cos\theta} \text{ and } \sum_{0}^{\infty}\frac{\sin n\theta}{2^n}=\frac{2\sin\theta}{5-4\cos\theta}$$