L_p compression, traveling salesmen, and stable walks

Dedicated with admiration to the memory of Oded Schramm

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Abstract

We show that if H is a group of polynomial growth whose growth rate is at least quadratic then the L_p compression of the wreath product $\mathbb{Z} \wr H$ equals $\max\left\{\frac{1}{p}, \frac{1}{2}\right\}$. We also show that the L_p compression of $\mathbb{Z} \wr \mathbb{Z}$ equals $\max\left\{\frac{p}{2p-1}, \frac{2}{3}\right\}$ and the L_p compression of $(\mathbb{Z} \wr \mathbb{Z})_0$ (the zero section of $\mathbb{Z} \wr \mathbb{Z}$, equipped with the metric induced from $\mathbb{Z} \wr \mathbb{Z}$) equals $\max\left\{\frac{p+1}{2p}, \frac{3}{4}\right\}$. The fact that the Hilbert compression exponent of $\mathbb{Z} \wr \mathbb{Z}$ equals $\frac{2}{3}$ while the Hilbert compression exponent of $(\mathbb{Z} \wr \mathbb{Z})_0$ equals $\frac{3}{4}$ is used to show that there exists a Lipschitz function $f : (\mathbb{Z} \wr \mathbb{Z})_0 \to L_2$ which cannot be extended to a Lipschitz function defined on all of $\mathbb{Z} \wr \mathbb{Z}$.

1 Introduction

Let G be an infinite group which is generated by a finite symmetric set $S \subseteq G$ and let d_G denote the left-invariant word metric induced by S (formally we should use the notation d_S, but all of our statements below will be independent of the generating set). Assume for the moment that the metric space (G, d_G) does not admit a bi-Lipschitz embedding into Hilbert space[1]. In such a setting the next natural step is to try to measure the extent to which the geometry of (G, d_G) is non-Hilbertian. While one can come up with several useful ways to quantify non-embeddabililty, the present paper is a contribution to the theory of compression exponents: a popular and elegant way of measuring non-bi-Lipschitz embeddability of infinite groups that was introduced by Guentner and Kaminker in [33].

The Hilbert compression exponent of G, denoted $\alpha^*(G)$, is defined as the supremum of those $\alpha \geq 0$ for which there exists a Lipschitz function $f : G \to L_2$ satisfying $\|f(x) - f(y)\|_2 \geq c d_G(x, y)^\alpha$ for every $x, y \in G$ and some constant $c > 0$ which is independent of x, y. More generally, given a target metric space (X, d_X) the compression exponent of G in X, denoted $\alpha^*_X(G)$, is the supremum of those $\alpha \geq 0$ for which there exists a Lipschitz function $f : G \to X$ satisfying $d_X(f(x), f(y)) \geq c d_G(x, y)^\alpha$. When $X = L_p$ for some $p \geq 1$ we shall use the notation $\alpha^*_p(G) = \alpha^*_{L_p}(G)$ (thus $\alpha^*_2(G) = \alpha^*(G)$).

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[1]This assumption is not very restrictive, and in fact it is conjectured that if (G, d_G) does admit a bi-Lipschitz embedding into Hilbert space then G must have an Abelian subgroup of finite index. We refer to [24] for more information on this conjecture and its proof in some interesting special cases.

When $(X, \|\cdot\|_X)$ is a Banach space one can analogously define the equivariant compression exponent of G in X, denoted $\alpha^{\#}_X(G)$, as the supremum over $\alpha \geq 0$ for which there exists a G-equivariant[2] mapping $\psi : G \to X$ satisfying $\|\psi(x) - \psi(y)\|_X \geq c d_G(x,y)^\alpha$. We write as above $\alpha^{\#}_p(G) = \alpha^{\#}_{L_p}(G)$ and $\alpha^{\#}(G) = \alpha^{\#}_2(G)$. Recall that G is said to have the Haagerup property if there exists an equivariant function $\psi : G \to L_2$ such that $\inf\{\|\psi(x) - \psi(y)\|_2 : \ d_G(x,y) \geq t\}$ tends to infinity with t. We refer to the book [18] for more information on the Haagerup property and its applications. Thus the notion of equivariant compression exponent can be viewed as a quantitative refinement of the Haagerup property, and this is indeed the way that bounds on the equivariant compression exponent are usually used.

The parameters $\alpha^*_X(G)$ and $\alpha^{\#}_X(G)$ do not depend on the choice of symmetric generating set S, and are therefore genuine algebraic invariants of the group G. In [33] it was shown that if $\alpha^{\#}(G) > \frac{1}{2}$ then G is amenable. This result was generalized in [47], where it was shown that for $p \geq 1$ if X is a Banach space whose modulus of uniform smoothness has power type p (i.e. for every two unit vectors $x, y \in X$ and $\tau > 0$ we have $\|x + \tau y\|_X + \|x - \tau y\|_X \leq 2 + c\tau^p$ for some $c > 0$ which does not depend on x, y, τ) and $\alpha^{\#}_X(G) > \frac{1}{p}$, then G is amenable. It was also shown in [33] that if $\alpha^*(G) > \frac{1}{2}$ then the reduced C^* algebra of G is exact.

Despite their intrinsic interest and a considerable amount of effort by researchers in recent years, the invariants $\alpha^*_X(G), \alpha^{\#}_X(G)$ have been computed in only a few cases. It was shown in [3] that for any $\alpha \in [0,1]$ there exists a finitely generated group G with $\alpha^*(G) = \alpha$. In light of this fact it is quite remarkable that, apart from a few exceptions, in most of the known cases in which compression exponents have been computed they turned out to be equal to 1 or 0. A classical theorem of Assouad [5] implies that groups of polynomial growth have Hilbert compression exponent 1. On the other hand, Gromov's random groups [32] have Hilbert compression exponent 0. Bourgain's classical metrical characterization of superreflexivity [12] implies that finitely generated free groups have Hilbert compression exponent 1 (this interpretation of Bourgain's theorem was first noted in [33]), and more generally it was shown in [14] that hyperbolic groups have Hilbert compression 1 and in [15] that so does any discrete group acting properly and co-compactly on a finite dimensional CAT(0) cubical complex. In [57] it was shown that co-compact lattices in connected Lie groups, irreducible lattices in semi-simple Lie groups of rank at least 2, polycyclic groups and certain semidirect products with $\mathbb{Z}$ (including wreath products[3] of finite groups with $\mathbb{Z}$ and the Baumslag-Solitar group) all have Hilbert compression exponent 1. The first example of a group with Hilbert compression exponent in $(0,1)$ was found in [4], where it was proved that R. Thompson's group F satisfies $\alpha^*(F) = \frac{1}{2}$. Another well-studied case is the wreath product $\mathbb{Z} \wr \mathbb{Z}$. In [4] and independently in [54] it was shown that $\alpha^*(\mathbb{Z} \wr \mathbb{Z}) \geq \frac{1}{2}$ (see also [31] for a different argument which yields the weaker bound $\alpha^*(\mathbb{Z} \wr \mathbb{Z}) \geq \frac{1}{3}$). Moreover it was shown in [4] that $\alpha^*(\mathbb{Z} \wr \mathbb{Z}) \leq \frac{3}{4}$, and a combination of the later results of [7] and [47], which established sharp upper and lower bounds on $\alpha^*(\mathbb{Z} \wr \mathbb{Z})$, respectively, settles the case of the Hilbert compression exponent of $\mathbb{Z} \wr \mathbb{Z}$ by showing that $\alpha^*(\mathbb{Z} \wr \mathbb{Z}) = \frac{2}{3}$ (nevertheless, the $\frac{3}{4}$ upper bound on $\alpha^*(\mathbb{Z} \wr \mathbb{Z})$ from [4] has a special meaning which is important for our current work—we will return to this topic later in this introduction). More gen-

[2] A mapping $\psi : G \to X$ is called G-equivariant if there exists an action τ of G on X by affine isometries and a vector $v \in X$ such that $\psi(x) = \tau(x)v$ for all $x \in G$. Equivalently there exists an action π on X by linear isometries such that $\varphi := \psi - v$ is a 1-cocycle with respect to π (we denote this by $\varphi \in Z^1(G, \pi)$), i.e., for every $x, y \in G$ we have $\varphi(xy) = \pi(x)\varphi(y) + \varphi(x)$. A key useful point here is that in this case $\|\psi(x) - \psi(y)\|_X$ is an invariant semi-metric on G.

[3] The (restricted) wreath product of G with H, denoted $G \wr H$, is defined as as the group of all pairs (f, x) where $f : H \to G$ has finite support (i.e. $f(z) = e_G$, the identity element of G, for all but finitely many $z \in H$) and $x \in H$, equipped with the product $(f, x)(g, y) := \left(z \mapsto f(z)g(x^{-1}z), xy\right)$. If G is generated by the set $S \subseteq G$ and H is generated by the set $T \subseteq H$ then $G \wr H$ is generated by the set $\{(e_{G^H}, t) : \ t \in T\} \cup \{(\delta_s, e_H) : \ s \in S\}$, where δ_s is the function which takes the value s at e_H and the value e_G on $H \setminus \{e_H\}$. Unless otherwise stated, we will always assume that $G \wr H$ is equipped with the word metric associated with this canonical set of generators (although in most cases our assertions will be independent of the choice of generators).

erally, it was shown in [47] that if we define recursively $\mathbb{Z}_1 = \mathbb{Z}$ and $\mathbb{Z}_{(k+1)} = \mathbb{Z}_{(k)} \wr \mathbb{Z}$ then $\alpha^*(\mathbb{Z}_{(k)}) = \frac{1}{2-2^{1-k}}$. In [47] it was shown that $\alpha^*\left(C_2 \wr \mathbb{Z}^2\right) = \frac{1}{2}$, where C_2 denotes the cyclic group of order 2 (the lower bound of $\frac{1}{2}$ was proved earlier in [57]). Finally, it follows from [22, 47] that $\alpha^*(C_2 \wr F_n) = \alpha^{\#}(C_2 \wr F_n) = \frac{1}{2}$, where F_n is the free group on $n \geq 2$ generators (the upper bound of $\frac{1}{2}$ on $\alpha^*(C_2 \wr F_n)$ is due to [47] while the lower bound on $\alpha^{\#}(C_2 \wr F_n)$ is the key result of [22]). Many of the above results have (at least partial) variants for the L_p compression exponents of the groups in question—-we stated here only the case of Hilbert compression exponents for the sake of simplicity, and we refer to the relevant papers for more information.

The difficulty of evaluating compression exponents is the main reason for our interest in this topic, and our purpose here is to devise new methods to compute them. In doing so we answer questions posed in [57, 47]. One feature of the known methods for computing compression exponents is that they involve a novel interplay between group theory and other mathematical disciplines such as metric geometry, Banach space theory, analysis and probability. It isn't only the case that the latter disciplines are applied to group theory—it turns out that the investigation of compression exponents improved our understanding of issues in analysis and metric geometry as well (e.g. in [47] compression exponents were used to make progress on the theory of non-linear type). In the present paper we apply our new compression exponent calculations to the Lipschitz extension problem, and relate them to the Jones Traveling Salesman problem. These applications will be described in detail presently.

In [57] it was shown that for all $d \in \mathbb{N}$ we have $\alpha^*\left(C_2 \wr \mathbb{Z}^d\right) \geq \frac{1}{d}$. A different embedding yielding this lower bound was obtained in [47], together with the matching upper bound when $d = 2$. Thus, as stated above, $\alpha^*\left(C_2 \wr \mathbb{Z}^2\right) = \frac{1}{2}$. In Section 3 we investigate the value of $\alpha_p^*(G \wr H)$ when G is a general group and H is a group of polynomial growth. A key feature of our result is that we obtain a lower bound on $\alpha_p^*(G \wr H)$ which is independent of the growth rate of H. In combination with the upper bounds obtained in [47], our lower bound implies the following result:

Theorem 1.1. *For every $p \in [1, \infty)$ and every group H of polynomial growth whose growth is at least quadratic, we have:*

$$\alpha_p^*(\mathbb{Z} \wr H) = \alpha_p^*(C_2 \wr H) = \max\left\{\frac{1}{p}, \frac{1}{2}\right\}. \tag{1}$$

As we explain in Remark 3.3 below, the embedding from [47] which yielded the identity $\alpha_2^*\left(C_2 \wr \mathbb{Z}^2\right) = \frac{1}{2}$ was based on the trivial fact, which is special to 2 dimensions, that for every $A \subseteq \mathbb{Z}^2$ of diameter D, the shortest path in $\mathbb{Z}^2$ which covers A has length at most $O\left(D^2\right)$. It therefore turns out that the previous method for bounding $\alpha_p^*\left(C_2 \wr \mathbb{Z}^d\right)$ yields tight bounds only when $p = d = 2$ (this is made precise in Remark 3.3). Hence in order to prove (1) we devise a new embedding which is in the spirit of (but simpler than) the multi-scale arguments used in the proof of the Jones Traveling Salesman Theorem [38] (see also [50] and the survey [53]).

To explain the connection between our proof and the Jones Traveling Salesman Theorem take two elements $(f, x), (g, y)$ in the "planar lamplighter group" $C_2 \wr \mathbb{Z}^2$, i.e., $x, y \in \mathbb{Z}^2$ and $f, g : \mathbb{Z}^d \to \{0, 1\}$ with finite support. The distance between (f, x) and (g, y) in $C_2 \wr \mathbb{Z}^2$ is, up to a factor of 2, the shortest path in the integer grid $\mathbb{Z}^2$ which starts at x, visits all the sites $w \in \mathbb{Z}^2$ at which $f(w)$ and $g(w)$ differ, and terminates at y. Jones [38] associates to every set $A \subseteq \mathbb{R}^2$ of diameter 1 a sequence of numbers, known as the (squares of the) Jones β numbers, whose appropriately weighted sum is (up to universal factors) the length of the shortest Lipschitz curve covering A, assuming such a curve exists. Focusing on our proof of the fact that

$\alpha_1^*\left(C_2 \wr \mathbb{Z}^2\right) = 1$, in our setting we do something similar: we associate to every $(f, x) \in C_2 \wr \mathbb{Z}^2$ a sequence of real numbers such that if we wish to estimate (up to logarithmic terms) the shortest traveling salesman tour starting at x, ending at y, and covering the symmetric difference of the supports of f and g, all we have to do is to compute the ℓ_1 norm of the difference of the sequences associated to (f, x) and (g, y). Since the statement $\alpha_1^*\left(C_2 \wr \mathbb{Z}^2\right) = 1$ does not necessarily imply that $C_2 \wr \mathbb{Z}^2$ admits a bi-Lipschitz embedding into L_1, our result falls short of obtaining a constant-factor approximation of the length of this tour, which, if possible, would be an interesting equivariant version of the Jones Traveling Salesman Theorem (note that if one wishes to estimate the length of the shortest Lipschitz curve covering the symmetric difference $A \triangle B$ for some $A, B \subseteq \mathbb{R}^2$ one cannot "read" this just from the Jones β numbers of A and B without recomputing the Jones β numbers of $A \triangle B$). In view of such a potential strengthening of the Jones Traveling Salesman Theorem, the question whether $C_2 \wr \mathbb{Z}^2$ admits a bi-Lipschitz embedding into L_1 remains an interesting open problem that arises from our work (which currently only yields a "compression 1" version of this statement).

In Section 6 we compute the L_p compression of $\mathbb{Z} \wr \mathbb{Z}$, answering a question posed in [47]:

Theorem 1.2. *For every $p \in [1, \infty)$ we have:*

$$\alpha_p^*(\mathbb{Z} \wr \mathbb{Z}) = \max\left\{\frac{p}{2p-1}, \frac{2}{3}\right\}. \tag{2}$$

The fact that $\alpha_p^*(\mathbb{Z} \wr \mathbb{Z})$ is at least the right-hand side of (2) was proved in [47], so the key issue in (2) is to show that no embedding of $\mathbb{Z} \wr \mathbb{Z}$ can have a compression exponent bigger than the right-hand side of (2). We do so via a non-trivial enhancement of the *Markov type* method for bounding compression exponents that was introduced in [7]. In order to explain the new idea used in proving (2) we first briefly recall the basic bound from [7].

A Markov chain $\{Z_t\}_{t=0}^{\infty}$ with transition probabilities $a_{ij} := \mathbb{P}(Z_{t+1} = j \mid Z_t = i)$ on the state space $\{1, \ldots, n\}$ is *stationary* if $\pi_i := \mathbb{P}(Z_t = i)$ does not depend on t and it is *reversible* if $\pi_i a_{ij} = \pi_j a_{ji}$ for every $i, j \in \{1, \ldots, n\}$. Given a metric space (X, d_X) and $p \in [1, \infty)$, we say that X has *Markov type p* if there exists a constant $K > 0$ such that for every stationary reversible Markov chain $\{Z_t\}_{t=0}^{\infty}$ on $\{1, \ldots, n\}$, every mapping $f : \{1, \ldots, n\} \to X$ and every time $t \in \mathbb{N}$,

$$\mathbb{E}[d_X(f(Z_t), f(Z_0))^p] \le K^p\, t\, \mathbb{E}[d_X(f(Z_1), f(Z_0))^p]. \tag{3}$$

The least such K is called the Markov type p constant of X, and is denoted $M_p(X)$. This important concept was introduced by Ball in [9] and has since found a variety of applications in metric geometry, including applications to the theory of compression exponents [7, 47]. We refer to [48] for examples of spaces which have Markov type p. For our purposes it suffices to mention that Banach spaces whose modulus of uniform smoothness has power type p have Markov type p [48], and therefore the Markov type of L_p, $p \in [1, \infty)$, is $\min\{p, 2\}$.

In [47] a parameter $\beta^*(G)$ is defined to be the supremum over all $\beta \ge 0$ for which there exists a symmetric set of generators S of G and $c > 0$ such that for all $t \in \mathbb{N}$,

$$\mathbb{E}[d_G(W_t, e)] \ge c t^{\beta}, \tag{4}$$

where $\{W_t\}_{t=0}^{\infty}$ is the canonical simple random walk on the Cayley graph of G determined by S, starting at the identity element e_G. The proof in [7] shows that if (X, d_X) has Markov type p and G is amenable then:

$$\alpha_X^*(G) \le \frac{1}{p\beta^*(G)}. \tag{5}$$

In order to prove (2) we establish in Section 5 a crucial strengthening of (5). Given a symmetric probability measure μ on G let $\{g_k\}_{k=1}^{\infty}$ be i.i.d. elements of G which are distributed according to μ. The μ-random walk $\{W_t^{\mu}\}_{t=0}^{\infty}$ is defined as $W_0^{\mu} = e_G$ and $W_t^{\mu} = g_1 g_2 \cdots g_t$ for $t \in \mathbb{N}$. Let ρ be a left-invariant metric on G such that $B_\rho(e_G, r) = \{x \in G : \ \rho(x, e) \leq r\}$ is finite for all $r \geq 0$. Define $\beta_p^*(G, \rho)$ to be the supremum over all $\beta \geq 0$ such that there exists an increasing sequence of integers $\{t_k\}_{k=1}^{\infty}$ and a sequence of symmetric probability measures $\{\mu_k\}_{k=1}^{\infty}$ on G satisfying:

$$\forall k \in \mathbb{N} \quad \int_G \rho(x, e_G)^p d\mu_k(x) < \infty \quad \text{and} \quad \lim_{k\to\infty} \left(t_k \mu_k \left(G \setminus \{e_G\}\right)\right) = \infty, \tag{6}$$

such that for all $k \in \mathbb{N}$,

$$\mathbb{E}_{\mu_k}\left[\rho\left(W_{t_k}^{\mu_k}, e_G\right)\right] \geq t_k^{\beta}\left(\mathbb{E}_{\mu_k}\left[\rho\left(W_1^{\mu_k}, e_G\right)^p\right]\right)^{\beta}.$$

In Section 5 we prove the following result:

Theorem 1.3. *If G is amenable, ρ is a left-invariant metric on G with respect to which all balls are finite, and (X, d_X) has Markov type p, then:*

$$\alpha_X^*(G, \rho) \leq \frac{1}{p\beta_p^*(G, \rho)}. \tag{7}$$

In (7), $\alpha_X^*(G, \rho)$ denotes the supremum of those $\alpha \geq 0$ for which there exists a ρ-Lipschitz map $f : G \to X$ which satisfies $d_X(f(x), f(y)) \geq c\rho(x, y)^{\alpha}$ (we previously defined this parameter only when $\rho = d_G$). We refer to the discussion in Section 5 for more information on the parameter $\beta_p^*(G, \rho)$. At this point it suffices to note that $\beta_p^*(G, d_G) \geq \beta^*(G)$, and therefore (7) is stronger than (5), since we now consider a variant of (4) where the walk can be induced by an arbitrary symmetric probability measure, and the measure itself is allowed to depend on the time t. It turns out that (7) is a crucial *strict* improvement over (5), and we require the full force of this strengthening: we shall use non-standard random walks (i.e., not only the canonical walk on the Cayley graph of G), as well as an adaptation of the walk to the time t in (4), in addition to invariant metrics ρ other than the word metric d_G.

We establish (2) by showing that for every $p \in [1, 2)$ we have $\beta_p^*(\mathbb{Z} \wr \mathbb{Z}, d_{\mathbb{Z}\wr\mathbb{Z}}) = \frac{2p-1}{p^2} > \frac{3}{4} = \beta^*(\mathbb{Z} \wr \mathbb{Z})$ (it follows in particular that (7) is indeed strictly stronger than (5). Note that $\mathbb{Z} \wr \mathbb{Z}$ is amenable and L_p has Markov type p, so we are allowed to use (7)). This is achieved by considering a random walk induced on $\mathbb{Z} \wr \mathbb{Z}$ from a random walk on $\mathbb{Z}$ whose increments are discrete versions of q-stable random variables for every $q > p$. We refer to Section 6 for the details. We believe that there is a key novel feature of our proof which highlights the power of random walk techniques in embedding problems: we adapt the random walk on G to the target space L_p. Previously [44, 10, 48, 7, 47] Markov type was used in embedding problems by considering a Markov chain on the space we wish to embed which arises intrinsically, and "ignored" the intended target space: such chains are typically taken to be the canonical random walk on some graph, but a different example appears in [10], where embeddings of arbitrary subsets A of the Hamming cube $(\{0, 1\}^n, \|\cdot\|_1)$ are investigated via a construction of a special random walk on A which captures the "largeness" of A. Nevertheless, in all known cases the geometric object which was being embedded dictated the study of some natural random walk, while in our computation of $\alpha_p^*(\mathbb{Z} \wr \mathbb{Z})$ the target space L_p influences the choice of the random walk.

Recall that we mentioned above that prior to [7] the best known upper bound [4] on $\alpha^*(\mathbb{Z} \wr \mathbb{Z})$ was $\frac{3}{4}$. An inspection of the proof of this bound in [4] reveals that it considered only points in the normal subgroup of

$\mathbb{Z} \wr \mathbb{Z}$ consisting of all configurations where the lamplighter is at 0, i.e., the *zero section* of $\mathbb{Z} \wr \mathbb{Z}$:

$$(\mathbb{Z} \wr \mathbb{Z})_0 := \{(f, x) \in \mathbb{Z} \wr \mathbb{Z} : \ x = 0\} \lhd \mathbb{Z} \wr \mathbb{Z}.$$

Thus [4] actually establishes the bound $\alpha^* ((\mathbb{Z} \wr \mathbb{Z})_0, d_{\mathbb{Z}\wr\mathbb{Z}}) \leq \frac{3}{4}$. More generally, an obvious variant of the proof of this fact in [4] (see Lemma 7.8 in [47]) shows that for $p \in [1, 2]$ we have $\alpha_p^* ((\mathbb{Z} \wr \mathbb{Z})_0, d_{\mathbb{Z}\wr\mathbb{Z}}) \leq \frac{p+1}{2p}$. Here we obtain the following result:

Theorem 1.4. *For every $p \in [1, \infty)$ we have:*

$$\alpha_p^* ((\mathbb{Z} \wr \mathbb{Z})_0, d_{\mathbb{Z}\wr\mathbb{Z}}) = \max \left\{ \frac{p+1}{2p}, \frac{3}{4} \right\}. \tag{8}$$

An alternative proof of the fact that the right-hand side of (8) is greater than $\alpha_p^* ((\mathbb{Z} \wr \mathbb{Z})_0, d_{\mathbb{Z}\wr\mathbb{Z}})$, which belongs to the framework of (7), is given in Section 7, where we show that for every $p \in [1, 2]$ we have $\beta_p^* ((\mathbb{Z} \wr \mathbb{Z})_0, d_{\mathbb{Z}\wr\mathbb{Z}}) = \frac{2}{p+1}$. The heart of (8) is the construction of an embedding into L_p of the zero section $(\mathbb{Z} \wr \mathbb{Z})_0$ which achieves the claimed compression exponent. This turns out to be quite delicate: a Fourier analytic argument establishing this fact is presented in Section 4.

It is worthwhile to note at this point that in all of our new compression computations, namely (1), (2) and (8), we claim that for some group G equipped with an invariant metric ρ and for every $p \in [2, \infty)$ we have $\alpha_p^*(G, \rho) = \alpha_2^*(G, \rho)$. This is true since because L_2 is isometric to a subset of L_p we obviously have $\alpha_p^*(G, \rho) \geq \alpha_2^*(G, \rho)$. In the reverse direction, all of our upper bounds on L_p compression exponents are based on (7), and since both L_2 and L_p have Markov type 2 [48] the resulting upper bound for L_p coincides with the upper bound for L_2. For this reason it will suffice to prove all of our results when $p \in [1, 2]$.

In Section 8 we apply the fact that $\alpha^* ((\mathbb{Z} \wr \mathbb{Z})_0, d_{\mathbb{Z}\wr\mathbb{Z}}) \neq \alpha^*(\mathbb{Z} \wr \mathbb{Z})$ to the Lipschitz extension problem. This classical problem asks for geometric conditions on a pair of metric spaces (X, d_X) and (Y, d_Y) which ensure that for any subset $A \subseteq X$, any Lipschitz mapping $f : A \to Y$ can be extended to all of X. Among the motivating themes for research on the Lipschitz extension problem is the belief that many classical extension theorems for linear operators between Banach spaces have Lipschitz analogs. Two examples of this phenomenon are the non-linear Hahn-Banach theorem (see for example [59, 11]), which corresponds to extension of real valued functions while preserving their Lipschitz constant, and the non-linear version of Maurey's extension theorem [9, 48]. It turns out that our investigation of the Hilbert compression exponent of the zero section of $\mathbb{Z} \wr \mathbb{Z}$ implies the existence of a Lipschitz function $f : (\mathbb{Z} \wr \mathbb{Z})_0 \to L_2$ which cannot be extended to a Lipschitz function defined on all of $\mathbb{Z} \wr \mathbb{Z}$:

Theorem 1.5. *There exists a Lipschitz function $F : (\mathbb{Z} \wr \mathbb{Z})_0 \to L_2$ which cannot be extended to a Lipschitz function from $\mathbb{Z} \wr \mathbb{Z}$ to L_2.*

For those who believe in the above analogy between the Lipschitz extension problem and the extension problem for linear operators, Theorem 1.5 might seem somewhat surprising: after all $H = (\mathbb{Z} \wr \mathbb{Z})_0$ is a normal subgroup of $G = \mathbb{Z} \wr \mathbb{Z}$ with $G/H \cong \mathbb{Z}$, so it resembles a non-commutative version of a subspace of co-dimension 1 in a Banach space, for which the Lipschitz extension problem is trivial (again by the Hahn-Banach theorem). Nevertheless, the analogy with Banach spaces stops here, as our result shows that the normal subgroup H sits in G in an "entangled" way which makes it impossible to extend certain Lipschitz functions while preserving the Lipschitz property.

To explain the connection with the Lipschitz extension problem take $\psi : (\mathbb{Z} \wr \mathbb{Z})_0 \to L_2$ which is 1-Lipschitz and $\|\psi(x) - \psi(y)\|_2 \geq c d_{\mathbb{Z}\wr\mathbb{Z}}(x,y)^{3/4}$ for all $x, y \in (\mathbb{Z} \wr \mathbb{Z})_0$, where $c > 0$ is a universal constant[4]. We claim that ψ cannot be extended to a Lipschitz function Ψ defined on all of $\mathbb{Z} \wr \mathbb{Z}$, so assume for the sake of contradiction that Ψ extends ψ and is Lipschitz. To arrive at a contradiction we need to contrast the $\frac{3}{4}$ lower bound on the compression exponent of ψ with the Markov type 2 proof of the fact that Ψ cannot have compression larger than $\frac{2}{3}$ from [7]. Let $\{W_t\}_{t=0}^{\infty}$ be the canonical random walk on $\mathbb{Z} \wr \mathbb{Z}$ starting at the identity element. Writing $W_t = (f_t, x_t) \in \mathbb{Z} \wr \mathbb{Z}$ one can see that with high probability $|x_t| \lesssim \sqrt{t}$, while the distance between W_t and the identity element is $\gtrsim t^{3/4}$. The fact that L_2 has Markov type 2 and Ψ is Lipschitz says that we expect $\|\Psi(W_t) - \Psi(W_0)\|_2$ to be $\lesssim \sqrt{t}$. But, if we move W_t to its closest point in the zero section $(\mathbb{Z} \wr \mathbb{Z})_0$ then the image under Ψ will (using the Lipschitz condition) move $\lesssim \sqrt{t}$. Using the compression inequality for ψ we deduce that for large enough t we have $\sqrt{t} \gtrsim \|\Psi(W_t) - \Psi(W_0)\|_2 \gtrsim \left(t^{3/4}\right)^{3/4} = t^{9/16}$, which is a contradiction. This argument is, of course, flawed, since we are allowed to use the fact that L_2 has Markov type 2 only for Markov chains which are stationary and reversible, and this is not the case for the canonical random walk starting at the identity element. Nevertheless, this proof can be salvaged using the same intuition: in Section 8 we consider a certain finite subset of $\mathbb{Z} \wr \mathbb{Z}$ which lies within a narrow tubular neighborhood of $(\mathbb{Z} \wr \mathbb{Z})_0$. We then apply the same ideas to the random walk obtained by choosing a point in this subset uniformly at random and preforming a random walk on the subset with appropriate boundary conditions. We refer to Section 8 for the full details. It is perhaps somewhat amusing to note here that while the notion of Markov type was introduced by Ball [9] in order to prove an extension theorem (Ball's extension theorem), here we use Markov type for the opposite purpose—to prove a non-extendability result.

Thus far we did not discuss the relation between the parameters $\alpha_X^*(G)$ and $\alpha_X^{\#}(G)$ for some Banach space X. This is, in fact, a subtle issue: it is unclear when $\alpha_X^*(G) = \alpha_X^{\#}(G)$. Since for every $p \in [1, \infty)$ the free group F_n on $n \geq 2$ generators satisfies $\alpha_p^*(F_n) = 1$ yet $\alpha_p^{\#}(F_n) = \max\left\{\frac{1}{p}, \frac{1}{2}\right\}$ (see [33, 47]) it follows that the compression exponent and equivariant compression exponent can be different from each other, while in many cases we know that these two invariants coincide: for example $\alpha_p^*(C_2 \wr F_n) = \alpha_p^{\#}(C_2 \wr F_n) = \max\left\{\frac{1}{p}, \frac{1}{2}\right\}$ (see [22, 47]). A useful result of Aharoni, Maurey and Mityagin [1] for Abelian groups, and Gromov (see [24]) for general amenable groups, says that for any amenable group G we have $\alpha_2^*(G) = \alpha_2^{\#}(G)$. This is an obviously useful fact (examples of applications can be found in [24, 8]): for example in [47] it was shown that if X is a Banach space whose modulus of uniform smoothness has power type p then for every finitely generated group G we have:

$$\alpha_X^{\#}(G) \leq \frac{1}{p\beta^*(G)}. \tag{9}$$

The bound (9) implies the bound (5) when G is amenable and X is Hilbert space due to the above reduction to equivariant mappings for amenable groups and Hilbertian targets. At the time of writing of [47] it was unclear whether (9) implies (5) in general, since an Aharoni-Maurey-Mityagin/Gromov type result was not known in non-Hilbertian settings. In Section 5 we further improve (9) by showing that if X is a Banach space whose modulus of uniform smoothness has power type p then:

$$\alpha_X^{\#}(G) \leq \frac{1}{p\beta_p^*(G)}. \tag{10}$$

In Section 9 we prove the following theorem:

[4]It isn't quite accurate that the fact that $\alpha^*\left((\mathbb{Z} \wr \mathbb{Z})_0, d_{\mathbb{Z}\wr\mathbb{Z}}\right) = \frac{3}{4}$ implies the existence of such a function ψ, since all we are assured is a compression exponent lower bound of $\frac{3}{4} - \varepsilon$ for all $\varepsilon > 0$. This is immaterial for the sake of the argument here in the introduction—a precise proof is given in Section 8

Theorem 1.6. *For every $p \in [1,\infty)$, if G is an amenable group and X is a Banach space, then there exists a Banach space Y which is finitely representable*[5] *in $\ell_p(X)$ and:*

$$\alpha_Y^{\#}(G) \geq \alpha_X^{*}(G). \tag{11}$$

Moreover, if $X = L_p$ then we can also take $Y = L_p$ in (11)*, and thus $\alpha_p^{*}(G) = \alpha_p^{\#}(G)$ when G is amenable.*

Note also that if X has modulus of uniform smoothness of power type p then so does $\ell_p(X)$, and hence so does the space Y of Theorem 1.6. Therefore, by virtue of (11), the inequalities (9) and (10) are indeed stronger than the inequalities (5) and (7) in full generality.

We end this introduction by commenting on why so much of the literature (and also the present paper) focused on compression exponents of wreath products. The obvious answer is that groups such as $\mathbb{Z} \wr \mathbb{Z}$ are among the simplest examples of groups for which it was unknown for a long time how to compute their compression exponents. As it turns out, understanding such groups in this context required new ideas and new connections between geometric group theory and other mathematical disciplines. But, there is also a deeper reason for our interest in embeddings of wreath products. Každan's example [40] (see also [25]) of $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ shows that there can be two groups, each of which has positive equivariant compression exponent, yet their semidirect product fails to have a positive equivariant compression exponent, and even fails the Haagerup property. It seems challenging to characterize which semidirect products preserve the property of having positive compression exponents, and wreath products, as examples of semidirect products, are a good place to start trying to understand this fundamental question. The literature on compression exponents of wreath products shows that in many cases this operation preserves the property of having positive compression exponent, but we do not know if this is always true, even for amenable groups: the simplest such example is the groups $C_2 \wr (C_2 \wr \mathbb{Z})$ for which we do not know if it has positive Hilbert compression exponent, even though both C_2 and $C_2 \wr \mathbb{Z}$ have Hilbert compression exponent 1.

2 Preliminaries

In what follows we fix two groups G and H, which are generated by the symmetric finite sets S_G and S_H, respectively. The corresponding left invariant word metrics will be denoted d_G and d_H, respectively. The canonical generating set of the wreath product $G \wr H$ is

$$\{(\mathbf{e_G}, x) : \ x \in S_H\} \cup \left\{\left(\delta_y, e_H\right) : \ y \in S_G\right\},$$

where $\mathbf{e_G} : H \to G$ denotes the constant e_G function and for $y \in G$ the function $\delta_y : H \to G$ takes the value y at e_H and the value e_H elsewhere.

Given a function $f : H \to G$ we denote its support by $\mathbf{supp}(f) \coloneqq \{x \in H : \ f(x) \neq e_G\}$. For a finite subset $A \subseteq H$ and $x, y \in H$ we let $\mathrm{TSP}(A; x, y)$ denote the length of the shortest path in H which starts at x, covers A, and terminates at y, i.e.,

$$\mathrm{TSP}(A; x, y) \coloneqq \inf\left\{\sum_{j=0}^{k-1} d_H(x_j, x_{j+1}) : \ k \in \mathbb{N}, \ \ x = x_0, \ldots, x_k = y \in H \ \wedge \ A \subseteq \{x_0, \ldots, x_k\}\right\}.$$

[5] A Banach space U is said to be finitely representable in a Banach space V if for every $\varepsilon > 0$ and every finite dimensional subspace $F \subseteq U$ there is a linear operator $T : F \to V$ such that for every $x \in F$ we have $\|x\|_U \leq \|Tx\|_V \leq (1+\varepsilon)\|x\|_U$.

Thus

$$|A| + \mathrm{TSP}(A, x, y) = d_{C_2 \wr H}\left(\left(\mathbf{1}_{y^{-1}A}, y^{-1}x\right), (\mathbf{0}, 0)\right),$$

where $\mathbf{0} : H \to C_2$ denotes the constant 0 function. Following [47] we let $\mathscr{L}_G(H)$ denote the wreath product of G with H where the set of generators of G is taken to be $G \setminus \{e_G\}$ (i.e. any two distinct elements of G are at distance 1 from each other). In other words, the difference between $\mathscr{L}_G(H)$ and the classical lamplighter group $C_2 \wr H$ is that we allow the "lamps" to have G types of different "lights", where the cost of switching from one type of light to another is 1. Thus, with this definition it is immediate that for every $(f, x), (g, y) \in \mathscr{L}_G(\mathbb{Z})$ we have

$$d_{\mathscr{L}_G(\mathbb{Z})}((f, x), (g, y)) = d_{C_2 \wr H}\left((\mathbf{1}_{y^{-1}\mathbf{supp}(fg^{-1})}, y^{-1}x), (\mathbf{0}, 0)\right) = \left|\mathbf{supp}\left(fg^{-1}\right)\right| + \mathrm{TSP}\left(\mathbf{supp}\left(fg^{-1}\right); x, y\right). \quad (12)$$

Moreover, distances in the wreath product $G \wr H$, equipped with the canonical generating set, can be computed as follows:

$$d_{G \wr H}((f, x), (g, y)) = \mathrm{TSP}\left(\mathbf{supp}\left(fg^{-1}\right); x, y\right) + \sum_{x \in H} d_G(f(x), g(x)). \quad (13)$$

The following lemma generalizes Lemma 3.1 in [47], which deals with the special case $H = \mathbb{Z}$ (in which case the proof is easier).

Lemma 2.1. *Assume that G contains at least two elements. Then for any $p \geq 1$ we have*

$$\alpha_p^*(\mathscr{L}_G(H)) = \alpha_p^*(C_2 \wr H).$$

Proof. Obviously $\alpha_p^*(\mathscr{L}_G(H)) \leq \alpha_p^*(C_2 \wr H)$, since $\mathscr{L}_G(H)$ contains an isometric copy of $C_2 \wr H$. To prove the reverse direction we may assume that $\alpha_p^*(C_2 \wr H) > 0$. Fix $0 < \alpha < \alpha_p^*(C_2 \wr H)$ and a mapping $\theta : C_2 \wr \mathbb{Z} \to L_p$ satisfying

$$(f, x), (g, y) \in C_2 \wr H \implies d_{C_2 \wr H}((f, x), (g, y))^\alpha \lesssim \|\theta(f, x) - \theta(g, y)\|_p \lesssim d_{C_2 \wr H}((f, x), (g, y)). \quad (14)$$

Let $\{\varepsilon_z\}_{z \in G \setminus \{e_G\}}$ be i.i.d. $\{0, 1\}$-valued Bernoulli random variables, defined on some probability space $(\Omega, \mathbb{P})$. For every $f : H \to G$ define a random mapping $\varepsilon_f : H \to C_2$ by

$$\varepsilon_f(z) \coloneqq \begin{cases} \varepsilon_{f(z)} & \text{if } f(z) \neq e_G, \\ 0 & \text{if } f(z) = e_G. \end{cases}$$

We now define an embedding $F : \mathscr{L}_G(H) \to L_p(\Omega, L_p)$ by

$$F(f, x) \coloneqq \theta(\varepsilon_f, x).$$

Given $(f, x), (g, y) \in G \wr H$ denote $A \coloneqq \mathbf{supp}\left(fg^{-1}\right) = \{z \in H : f(z) \neq g(z)\}$. We also denote by $A_\varepsilon \subseteq H$ the random subset $\mathbf{supp}(\varepsilon_f - \varepsilon_g)$. By definition $A_\varepsilon \subseteq A$, so that $\mathrm{TSP}(A_\varepsilon; x, y) \leq \mathrm{TSP}(A; x, y)$. Hence:

$$\begin{aligned} \|F(f, x) - F(g, y)\|_{L_p(\Omega, L_p)}^p &= \mathbb{E}\left[\left\|\theta(\varepsilon_f, x) - \theta(\varepsilon_g, y)\right\|_p^p\right] \overset{(14)}{\lesssim} \mathbb{E}\left[d_{C_2 \wr H}((\varepsilon_f, x), (\varepsilon_g, y))^p\right] \\ &= \mathbb{E}\left[\mathrm{TSP}(A_\varepsilon; x, y)^p\right] \leq \mathbb{E}\left[\mathrm{TSP}(A; x, y)^p\right] \overset{(12)}{=} d_{\mathscr{L}_G(\mathbb{Z})}((f, x), (g, y))^p. \end{aligned}$$

In the reverse direction, observe that

$$\mathrm{TSP}(A;x,y) \le 2\mathrm{TSP}(A_\varepsilon;x,y) + \mathrm{TSP}(A\setminus A_\varepsilon;x,y), \tag{15}$$

since given a path γ that starts at x, ends at y, and covers A_ε, and a path δ that starts at x, ends at y, and covers $A\setminus A_\varepsilon$, we can consider the path that starts as γ, retraces γ's steps from y back to x, and then continues as δ from x to y. Hence,

$$d_{\mathscr{L}_G(\mathbb{Z})}((f,x),(g,y))^{p\alpha} \overset{(12)}{=} \mathrm{TSP}(A;x,y)^{p\alpha} \overset{(15)}{\lesssim} \mathrm{TSP}(A_\varepsilon;x,y)^{p\alpha} + \mathrm{TSP}(A\setminus A_\varepsilon;x,y)^{p\alpha}. \tag{16}$$

But by the symmetry of our construction the random subsets A_ε and $A\setminus A_\varepsilon$ are identically distributed. So, taking expectation in (16) we see that

$$\begin{aligned} d_{\mathscr{L}_G(\mathbb{Z})}((f,x),(g,y))^{p\alpha} \lesssim \mathbb{E}\left[\mathrm{TSP}(A_\varepsilon;x,y)^{p\alpha}\right] &= \mathbb{E}\left[d_{C_2\wr H}((\varepsilon_f,x),(\varepsilon_g,y))^{p\alpha}\right] \\ &\overset{(14)}{\lesssim} \mathbb{E}\left[\|\theta(\varepsilon_f,x)-\theta(\varepsilon_g,y)\|_p^p\right] = \|F(f,x)-F(g,y)\|_{L_p(\Omega,L_p)}^p. \end{aligned}$$

Thus $G\wr H$ embeds into $L_p(\Omega,L_p)$ with compression α, as required. □

A combination of Lemma 2.1 and Theorem 3.3 in [47] yields the following corollary:

Corollary 2.2. *Let G,H be nontrivial groups and $p\ge 1$. Then*

$$\min\left\{\alpha_p^*(G),\alpha_p^*(C_2\wr H)\right\} \ge \frac{1}{p} \implies \alpha_p^*(G\wr H) \ge \frac{p\alpha_p^*(G)\alpha_p^*(C_2\wr H)}{p\alpha_p^*(G)+p\alpha_p^*(C_2\wr H)-1},$$

and

$$\min\left\{\alpha_p^*(G),\alpha_p^*(C_2\wr H)\right\} \le \frac{1}{p} \implies \alpha_p^*(G\wr H) \ge \min\left\{\alpha_p^*(G),\alpha_p^*(C_2\wr H)\right\}.$$

We end this section with a simple multi-scale estimate for the length of traveling salesmen tours (see for example [55] for a similar estimate). For $r\ge 0$ and $x\in H$ we let $B_H(x,r) := \{y\in H : d_H(x,y)\le r\}$ be the closed ball centered at x with radius r. For a bounded set $A\subseteq H$ and $r>0$ we let $N(A,r)$ be the smallest integer $n\in\mathbb{N}$ such that there exists $x_1,\ldots,x_n\in H$ for which $A\subseteq\bigcup_{m=1}^n B_H(x_m,r)$. Finally, for $\ell\ge 0$ let $\mathrm{TSP}_\ell(A)$ denote the length of the shortest path starting from e_H, coming within a distance of at most $2^{\ell-1}$ from every point in A, and returning to e_H, i.e.

$$\mathrm{TSP}_\ell(A) := \inf\left\{\sum_{j=0}^{k-1} d_H(x_j,x_{j+1}) : k\in\mathbb{N},\ e_H=x_0,\ldots,x_k=e_H\in H,\ A\subseteq\bigcup_{j=0}^{k} B_H\left(x_j,2^{\ell-1}\right)\right\}.$$

Thus $\mathrm{TSP}(A) := \mathrm{TSP}(A;e_H,e_H) = \mathrm{TSP}_0(A) = d_{C_2\wr H}((\mathbf{1}_A,e_H),(\mathbf{0},e_H))$ is the length of the shortest path starting from e_H, covering A, and returning to e_H. We shall use the following easy bound, which holds for every $k,\ell\in\mathbb{N}\cup\{0\}$:

$$A\subseteq B_H\left(e_H,2^k\right) \implies \mathrm{TSP}_\ell(A) \le 3\sum_{j=\ell}^{k} 2^j N\left(A,2^{j-1}\right). \tag{17}$$

The inequality (17) is valid when $\ell \geq k+1$ since in that case $\mathrm{TSP}_\ell(A) = 0$. Now (17) follows by induction from the inequality $\mathrm{TSP}_{\ell-1}(A) \leq \mathrm{TSP}_\ell(A) + 3 \cdot 2^{\ell-1} N\left(A, 2^{\ell-2}\right)$. This inequality holds true since we can take a set $C \subseteq H$ of size $N\left(A, 2^{\ell-2}\right)$ such that $\bigcup_{x \in C} B_H\left(x, 2^{\ell-2}\right) \supseteq A$, and also take a path $\Gamma \subseteq H$ of length $\mathrm{TSP}_\ell(A)$ which starts from e_H, comes within a distance of at most $2^{\ell-1}$ from every point in A, and returns to e_H. If we append to Γ a shortest path from each $x \in C$ to its closest neighbor in Γ (and back) we obtain a new path of length at most $\mathrm{TSP}_\ell(A) + 2\left(2^{\ell-1} + 2^{\ell-2}\right)|C| \leq \mathrm{TSP}_\ell(A) + 3 \cdot 2^{\ell-1}|C|$ which starts from e_H, comes within a distance of at most $2^{\ell-2}$ from every point in A, and returns to e_H, as required.

3 Wreath products of groups with polynomial growth

The goal of this section is to prove the following theorem:

Theorem 3.1. *Let G, H be nontrivial finitely generated groups, and assume that H has polynomial growth. Then for every $p \in [1,2]$ we have:*

$$\alpha_p^*(G \wr H) \geq \min\left\{\frac{1}{p}, \alpha_p^*(G)\right\}. \tag{18}$$

In particular, if the growth rate of H is at least quadratic then for every $p \in [1,2]$ we have

$$\alpha_p^*(\mathbb{Z} \wr H) = \alpha_p^*(C_2 \wr H) = \frac{1}{p}. \tag{19}$$

Proof. We shall first explain how to deduce the identity (19). The lower bound $\alpha_p^*(\mathbb{Z} \wr H) = \alpha_p^*(C_2 \wr H) \geq \frac{1}{p}$ is a consequence of (18). Since for $p \in [1,2]$ the Banach space L_p has Markov type p (see [9]), the result of Austin, Naor and Peres [7] implies that $\alpha_p^*(G \wr H) \leq \frac{1}{p\beta^*(G \wr H)}$. But, as we proved in [47], since the growth of H is at least quadratic we have $\beta^*(G \wr H) = 1$.

To prove (18) note that by Corollary 2.2 it is enough to show that

$$\alpha_p^*(C_2 \wr H) \geq \frac{1}{p}. \tag{20}$$

Recall that for $r \geq 0$ and $x \in H$ we let $B_H(x,r) := \{y \in H : d_H(x,y) \leq r\}$ be the closed ball centered at x with radius r. Assume that H has polynomial growth d, i.e., that for every $r \geq 1$ we have

$$ar^d \leq |B_H(e,r)| \leq br^d \tag{21}$$

for some $a, b > 0$ which do not depend on r. We shall show that for every $1 < p \leq 2$ and $\varepsilon \in (0, 1/p)$ there is a function $F : C_2 \wr H \to L_p$ such that for all $(f,x), (g,y) \in C_2 \wr H$ we have

$$d_{C_2 \wr H}((f,x),(g,y))^{\frac{1}{p}-\varepsilon} \lesssim \|F(f,x) - F(g,y)\|_p \lesssim d_{C_2 \wr H}((f,x),(g,y)), \tag{22}$$

where here, and in the remainder of the proof of Theorem 3.1, the implied constants depend only on a, b, p, d, ε. Moreover, we will show that we can take $\varepsilon = 0$ in (22) if (H, d_H) admits a bi-Lipschitz embedding into L_p. Note that (22) implies also the case $p = 1$ of Theorem 3.1 since L_p is isometric to a subspace of L_1 for all $p \in (1,2]$ (see e.g. [59]).

Let Ω be the disjoint union of the sets of functions $f : A \to C_2$ where A ranges over all finite subsets of H, i.e.,

$$\Omega \coloneqq \bigcup_{\substack{A \subseteq H \\ |A|<\infty}} C_2^A.$$

We will work with the Banach space $\ell_\infty(\Omega)$, and denote its standard coordinate basis by

$$\Big\{v_f : \ f : A \to C_2, \ A \subseteq H, \ |A| < \infty\Big\}.$$

Fix a 1-Lipschitz function $\varphi : [0,\infty) \to [0,1]$ which equals 0 on $[0,1]$ and equals 1 on $[2,\infty)$. For every $(f,x) \in C_2 \wr H$ define a function $\Psi_0(f,x) \in \ell_\infty(\Omega)$ by

$$\Psi_0(f,x) \coloneqq \sum_{k=0}^{\infty} 2^{-(d-1)k/p} \sum_{y \in H} \varphi\left(\frac{d_H(x,y)}{2^k}\right) v_{f\restriction_{B_H(y,2^k)}}. \tag{23}$$

We shall first check that $\Psi_0 - \Psi_0(\mathbf{0}, e_H) \in Z^1(H,\pi)$ for an appropriately chosen action π of $C_2 \wr H$ on $\ell_p(\Omega)$. Recall that the product on $C_2 \wr H$ is given by $(f,x)(g,y) = (f + T_x(g), xy)$, where $T_x(g)(z) \coloneqq g\left(x^{-1}z\right)$. Given $(f,x) \in C_2 \wr H$ and a finite subset $A \subseteq H$ define a bijection $\tau^A_{(f,x)} : C_2^A \to C_2^{xA}$ by $\tau^A_{(f,x)}(h) \coloneqq f + T_x(h)$. Note that for all $(f,x),(g,y) \in C_2 \wr H$ and every finite $A \subseteq H$ we have

$$\tau^A_{(f,x)(g,y)} = \tau^{yA}_{(f,x)} \circ \tau^A_{(g,y)}. \tag{24}$$

Hence if we define

$$\pi(f,x)\left(\sum_{\substack{A \subseteq H \\ |A|<\infty}} \sum_{h \in C_2^A} \alpha_h v_h\right) \coloneqq \sum_{\substack{A \subseteq H \\ |A|<\infty}} \sum_{h \in C_2^A} \alpha_h v_{\tau^A_{(f,x)}(h)},$$

then π is a linear isometric action of $C_2 \wr H$ on $\ell_p(\Omega)$ for all $p \in [1,\infty]$ ($\pi(f,x)$ corresponds to a permutation of the coordinates and hence is an isometry. The fact that $\pi((f,x)(g,y)) = \pi(f,x)\pi(f,y)$ is an immediate consequence of (24)). The definition (23) ensures that for every $(f,x),(g,y) \in C_2 \wr H$ we have $\Psi_0((f,x)(g,y)) = \pi(f,x)\Psi_0(g,y)$. Hence, if we define $\Psi(f,x) \coloneqq \Psi_0(f,x) - \Psi_0(\mathbf{0}, e_H)$ then $\Psi \in Z^1(H,\pi)$.

Note that $\Psi(\mathbf{0}, e_H) = 0$ and

$$\Psi(\mathbf{1}_{\{e_H\}}, e_H) = \sum_{k=0}^{\infty} 2^{-(d-1)k/p} \sum_{y \in B_H(e_H, 2^k)} \varphi\left(\frac{d_H(e_H,y)}{2^k}\right)\left(v_{\delta_{e_H}\restriction_{B_H(y,2^k)}} - v_{\mathbf{0}\restriction_{B_H(y,2^k)}}\right) = 0,$$

where we used the fact that $\varphi(t) = 0$ for $t \in [0,1]$. Moreover, for every $s \in S_H$ we have:

$$\begin{aligned}
\|\Psi(\mathbf{0},s)\|_p^p &= \sum_{k=0}^{\infty} 2^{-(d-1)k} \sum_{y \in H} \left|\varphi\left(\frac{d_H(s,y)}{2^k}\right) - \varphi\left(\frac{d_H(e_H,y)}{2^k}\right)\right|^p \\
&= \sum_{k=0}^{\infty} 2^{-(d-1)k} \sum_{\substack{y \in H \\ 2^k - 1 \le d_H(e_H,y) \le 2^{k+1}+1}} \left|\varphi\left(\frac{d_H(s,y)}{2^k}\right) - \varphi\left(\frac{d_H(e_H,y)}{2^k}\right)\right|^p \\
&\le \sum_{k=0}^{\infty} 2^{-(d-1)k} \cdot 2^{-kp} \left|\left\{y \in H : \ 2^k - 1 \le d_H(e_H,y) \le 2^{k+1}+1\right\}\right| \\
&\le \sum_{k=0}^{\infty} 2^{-(d-1)k} \cdot 2^{-kp} \cdot b\left(2^{k+1}+1\right)^d \le 4^d b \sum_{k=0}^{\infty} 2^{-k(p-1)} \lesssim 1,
\end{aligned}$$

Where we used the fact that $p > 1$. Since Ψ is equivariant and the set $\{(\mathbf{1}_{\{e_H\}}, e_H)\} \cup \{(\mathbf{0}, s) : \ s \in S_H\}$ generates $C_2 \wr H$, we deduce that

$$\|\Psi\|_{\mathrm{Lip}} \lesssim 1. \tag{25}$$

Suppose now that $f : H \to C_2$ and let $m \in \mathbb{N}$ be the minimum integer such that $\mathbf{supp}(f) \subseteq B_H(e_H, 2^m)$. Then

$$\begin{aligned}\|\Psi(f, e_H)\|_p^p &\geq \sum_{k=0}^{\infty} 2^{-(d-1)k} \sum_{\substack{y \in H \\ f\restriction_{B_H(y,2^k)} \neq \mathbf{0}\restriction_{B_H(y,2^k)}}} \varphi\left(\frac{d_H(e_H, y)}{2^k}\right)^p \\ &\geq \sum_{k=0}^{\infty} 2^{-(d-1)k} \left|\left\{y \in H : \ d_H(e_H, y) \geq 2^{k+1} \ \wedge \ \mathbf{supp}(f) \cap B_H(y, 2^k) \neq \emptyset\right\}\right|. \end{aligned} \tag{26}$$

Fix $k \leq m - 3$ and denote $n = N\left(\mathbf{supp}(f), 2^{k-1}\right)$. Let $x_1, \ldots, x_n \in H$ satisfy

$$\mathbf{supp}(f) \subseteq \bigcup_{i=1}^{n} B_H\left(x_i, 2^{k-1}\right). \tag{27}$$

By the minimality of n we are ensured that the balls $\left\{B_H\left(x_i, 2^{k-2}\right)\right\}_{i=1}^n$ are disjoint and that there exists $y_i \in B_H\left(x_i, 2^{k-1}\right) \cap \mathbf{supp}(f)$. Write

$$I \coloneqq \left\{i \in \{1, \ldots, n\} : \ d_H(y, e_H) \geq 2^{k+1} \ \forall y \in B_H\left(x_i, 2^{k-2}\right)\right\}.$$

Note that if $i \in I$ and $y \in B_H\left(x_i, 2^{k-2}\right)$ then $d_H(y_i, y) \leq d_H(y_i, x_i) + d_H(y, x_i) \leq 2^{k-1} + 2^{k-2} < 2^k$. Thus in this case $\mathbf{supp}(f) \cap B_H(y, 2^k) \neq \emptyset$, and therefore

$$\left|\left\{y \in H : \ d_H(e_H, y) \geq 2^{k+1} \ \wedge \ \mathbf{supp}(f) \cap B_H(y, 2^k) \neq \emptyset\right\}\right| \geq |I| \left|B_H\left(e_H, 2^{k-2}\right)\right| \gtrsim 2^{kd}|I|. \tag{28}$$

We shall now bound $|I|$ from below. By the minimality of m there exists $z \in \mathbf{supp}(f)$ such that $d_H(e_H, z) > 2^{m-1}$. By (27) there is some $i \in \{1, \ldots n\}$ for which $d_H(z, x_i) \leq 2^{k-1}$. If $y \in B_H\left(x_i, 2^{k-2}\right)$ then

$$d_H(y, e_H) \geq d_H(e_H, z) - d_H(z, x_i) - d_H(x_i, y) > 2^{m-1} - 2^{k-1} - 2^{k-2} \geq 2^{k+1},$$

since by assumption $k \leq m - 3$. This shows that $|I| \geq 1$. Write $J \coloneqq \{1, \ldots, n\} \setminus I$. For each $i \in J$ there is some $y \in B_H\left(x_i, 2^{k-2}\right)$ for which $d_H(e_H, y) < 2^{k+1}$. Hence $B_H\left(x_i, 2^{k-2}\right) \subseteq B_H\left(e_H, 2^{k+2}\right)$. Since the balls $\left\{B_H\left(x_i, 2^{k-2}\right)\right\}_{i=1}^n$ are disjoint it follows that

$$|J| a 2^{(k-2)d} \overset{(21)}{\leq} |J| \left|B_H\left(e_H, 2^{k-2}\right)\right| \leq \left|B_H\left(e_H, 2^{k+2}\right)\right| \overset{(21)}{\leq} b 2^{(k+2)d}.$$

Thus $n - |I| = |J| \lesssim 1$, which implies that $|I| \gtrsim n$. Plugging this bound into (28) we see that for every $k \leq m - 3$ we have

$$\left|\left\{y \in H : \ d_H(e_H, y) \geq 2^{k+1} \ \wedge \ \mathbf{supp}(f) \cap B_H(y, 2^k) \neq \emptyset\right\}\right| \gtrsim 2^{kd} N\left(\mathbf{supp}(f), 2^{k-1}\right).$$

In combination with (26) we see that

$$\|\Psi(f,e_H)\|_p^p \gtrsim \sum_{k=0}^{m-3} 2^{-(d-1)k}\cdot 2^{kd} N\left(\mathbf{supp}(f),2^{k-1}\right) = \sum_{k=0}^{m-3} 2^k N\left(\mathbf{supp}(f),2^{k-1}\right). \tag{29}$$

We claim that

$$|\mathbf{supp}(f)| + \sum_{k=0}^{m-3} 2^k N\left(\mathbf{supp}(f),2^{k-1}\right) \gtrsim d_{C_2\wr H}((f,e_H),(\mathbf{0},e_H)). \tag{30}$$

Indeed, by combining (17) (with $\ell = 0$) and (13) we see that

$$|\mathbf{supp}(f)| + \sum_{k=0}^{m} 2^k N\left(\mathbf{supp}(f),2^{k-1}\right) \gtrsim d_{C_2\wr H}((f,e_H),(\mathbf{0},e_H)). \tag{31}$$

To check that (31) implies (30) note that is is enough to deal with the case $\mathbf{supp}(f) \neq \emptyset$, and that the fact that $\mathbf{supp}(f) \subseteq B_H(e_H,2^m)$, combined with the doubling condition for (H,d_H), implies that for $k \in \{m-2,m-1,m\}$ we have $N\left(\mathbf{supp}(f),2^{k-1}\right) \lesssim 1$. Thus (31) implies (30) by inspecting the cases $m<3$ and $m \geq 3$ separately.

Fix $\varepsilon \in (0,1)$. By Assouad's theorem [5] (see also the exposition of this theorem in [35]), since H has polynomial growth, and hence is a doubling metric space, there is a function $\theta : H \to L_p$ such that for all $x,y \in H$ we have

$$d_H(x,y)^{1-\varepsilon} \leq \|\theta(x)-\theta(y)\|_p \lesssim d_H(x,y)^{1-\varepsilon} \leq d_H(x,y). \tag{32}$$

By translation we may assume that $\theta(e_H)=0$. We can now define our embedding

$$F : C_2 \wr H \to \ell_p(\Omega)\oplus \ell_p(H)\oplus L_p$$

by $F = \Psi \oplus f \oplus \theta$ (here we identify a finitely supported function $f : H \to C_2$ as a member of $\mathbb{R}^H$, and hence a member of $\ell_p(H)$). Then $\|F\|_{\mathrm{Lip}} := L \lesssim 1$. Thus in order to prove (22), and hence to complete the proof of Theorem 3.1, it remains to show that for all $(f,x) \in C_2 \wr H$ we have

$$d_{C_2\wr H}((f,x),(\mathbf{0},e_H))^{(1-\varepsilon)/p} \lesssim \|F(f,x)-F(\mathbf{0},e_H)\|_p = \left(\|\Psi(f,x)\|_p^p + |\mathbf{supp}(f)| + \|\theta(x)\|_p^p\right)^{1/p}. \tag{33}$$

A combination of (29) and (30) implies that there exists $\eta>0$ which depends only on a,b,d,p,ε such that

$$\eta d_{C_2\wr H}((f,e_H),(\mathbf{0},e_H))^{1/p} \leq \left(\|\Psi(f,e_H)\|_p^p + |\mathbf{supp}(f)|\right)^{1/p} = \|F(f,e_H)-F(\mathbf{0},e_H)\|_p.$$

Hence

$$\begin{aligned}
\|F(f,x)-F(\mathbf{0},e_H)\|_p &\geq \|F(f,e_H)-F(\mathbf{0},e_H)\|_p - \|F(f,x)-F(f,e_H)\|_p \\
&\geq \eta d_{C_2\wr H}((f,e_H),(\mathbf{0},e_H))^{1/p} - L d_H(x,e_H) \\
&\geq \eta\left[\max\left\{0, d_{C_2\wr H}((f,x),(\mathbf{0},e_H)) - d_{C_2\wr H}((f,x),(f,e_H))\right\}\right]^{1/p} - L d_H(x,e_H) \\
&= \eta\left[\max\left\{0, d_{C_2\wr H}((f,x),(\mathbf{0},e_H)) - d_H(x,e_H)\right\}\right]^{1/p} - L d_H(x,e_H) \\
&\geq \frac{\eta}{4} d_{C_2\wr H}((f,x),(\mathbf{0},e_H))^{1/p} \\
&\geq \frac{\eta}{4} d_{C_2\wr H}((f,x),(\mathbf{0},e_H))^{(1-\varepsilon)/p},
\end{aligned}$$

provided that

$$d_H(x, e_H) \le \min\left\{\frac{\eta}{4L} d_{C_2 \wr H}((f,x),(\mathbf{0},e_H))^{1/p}, \frac{1}{2} d_{C_2 \wr H}((f,x),(\mathbf{0},e_H))\right\}. \tag{34}$$

But if (34) fails then $d_H(x, e_H) \gtrsim d_{C_2 \wr H}((f,x),(\mathbf{0},e_H))^{1/p}$, in which case we can use (32) to deduce that

$$\|\theta(x)\|_p \ge d_H(e_H, x)^{1-\varepsilon} \gtrsim d_{C_2 \wr H}((f,x),(\mathbf{0},e_H))^{(1-\varepsilon)/p},$$

which implies (33) and concludes the proof of (22). □

Remark 3.2. Since the only reason for the loss of ε in (22) is the use of Assouad's embedding in (32) we see that if $p > 1$ and (H, d_H) admits a bi-Lipschitz embedding into L_p and has at least quadratic growth then $\alpha_p^*(C_2 \wr H) = \frac{1}{p}$ is attained. ◁

Remark 3.3. In [47] it was shown that $\alpha_2^*(C_2 \wr \mathbb{Z}^2) \ge \frac{1}{2}$ via an embedding which we now describe. We are doing so for several reasons. First of all there are some typos in the formulae given for the embedding in [47] and we wish to take this opportunity to publish a correct version. Secondly, the embedding was given in [47] without a detailed proof of its compression bounds, and since it is based on a different and simpler approach than our proof of Theorem 3.1 it is worthwhile to explain it here. Most importantly, there are several "coincidences" which allow this approach to yield sharp bounds on $\alpha_p^*(C_2 \wr \mathbb{Z}^d)$ only when $p = 2$ and $d = 2$, and we wish to explain these subtleties here. We will therefore first describe the embedding scheme in [47] for general $p \in [1, 2]$ and $d \ge 2$ and then specialize to the case $p = d = 2$.

Let $\left\{v_{y,r,g} : y \in \mathbb{Z}^d,\ r \in \mathbb{N} \cup \{0\},\ g : y + [-r, r]^d \to \{0, 1\}\right\}$ be a system of disjoint unit vectors in L_p. Fix a parameter $\gamma > 0$ which will be determined later and define for every $(f, x) \in C_2 \wr \mathbb{Z}^d$ a vector $F(f, x) = F_0(f, x) - F_0(\mathbf{0}, 0) \in L_p$, where

$$F_0(f,x) := \sum_{y \in \mathbb{Z}^d} \sum_{r=0}^{\infty} \frac{\max\left\{1 - \frac{2r}{1+\|x-y\|_\infty}, 0\right\}}{1 + \|x-y\|_\infty^\gamma} v_{y,r,f\restriction_{y+[-r,r]^d}}$$

One checks as in the proof of Theorem 3.1 that F is equivariant with respect to an appropriate action of $C_2 \wr \mathbb{Z}^d$ on L_p. Moreover, one checks that $\|F(\mathbf{1}_0, 0)\|_p \lesssim 1$ and that for $x \in \{(\pm 1, 0), (0, \pm 1)\}$ we have

$$\begin{aligned}\|F(0,x)\|_p^p &\lesssim \sum_{y \in \mathbb{Z}^d} \sum_{r \in [0, 1+\|y\|_\infty/2]} \left(\frac{1+r}{(1+\|y\|_\infty)^{2+\gamma}}\right)^p \lesssim \sum_{r=0}^{\infty} \sum_{\substack{k \ge 0 \\ k \ge 2(r-1)}} \sum_{\|y\|_\infty = k} \frac{(1+r)^p}{(1+k)^{(3+\gamma)p}} \\ &\lesssim \sum_{r=1}^{\infty} r^p \sum_{k \ge r} \frac{k^{d-1}}{(1+k)^{(2+\gamma)p}} \lesssim \sum_{r=0}^{\infty} \frac{1}{r^{p+\gamma p-d}} < \infty, \end{aligned} \tag{35}$$

where in (35) we need to assume that

$$\gamma > \frac{d+1-p}{p}. \tag{36}$$

It follows that as long as (36) holds true F is Lipschitz.

For the lower bound fix $(f, x) \in C_2 \wr \mathbb{Z}^d$ such that $f \neq \mathbf{0}$ and let $R \ge 0$ be the smallest integer for which there exists $z \in \mathbf{supp}(f)$ such that $\|z - x\|_\infty = R$, i.e., R is the smallest integer such that $\mathbf{supp}(f) \subseteq x + [-R, R]^d$. Note that for every $y \in \mathbb{Z}^d$ such that $\|y - z\|_\infty \in [0, R]$ and every $r \in [\|y - z\|_\infty, (1 + R - \|y - z\|_\infty)/4]$ we have

$z \in y + [-r,r]^d$, and hence $\mathbf{supp}(f) \cap \left(y + [-r,r]^d\right) \neq \emptyset$, and $\|y-x\|_\infty \geq R - \|y-z\|_\infty$, which implies that $\frac{2r}{1+\|y-x\|_\infty} \leq \frac{1}{2}$. Thus:

$$\|F(f,x)\|_p^p \gtrsim \sum_{k=0}^{R} \sum_{\substack{y\in\mathbb{Z}^d \\ \|y-z\|_\infty = k}} \sum_{r\in[k,(1+R-k)/4]} \frac{1}{(1+(k+R)^\gamma)^p}$$

$$\gtrsim \sum_{k\in[0,(1+R)/5]} \left(1+k^{d-1}\right) \cdot \frac{1+R-5k}{4} \cdot \frac{1}{(1+(k+R)^\gamma)^p} \gtrsim R^{d+1-\gamma p}. \quad (37)$$

Note the trivial bound:

$$\mathrm{TSP}(\mathbf{supp}(f); x, x) \leq \mathrm{TSP}\left(x + [-R,R]^d; x, x\right) \lesssim R^d. \quad (38)$$

Assuming also that $\gamma < \frac{d+1}{p}$ we see that a combination of (37) and (38) implies that:

$$\|F(f,x)\|_p \gtrsim \mathrm{TSP}(\mathbf{supp}(f); x, x)^{\frac{d+1-\gamma p}{dp}}. \quad (39)$$

Hence if we define $\Psi(x) = x \oplus F(x) \in \ell_p^d \oplus L_p$ we get the lower bound

$$\|\Psi(f,x)\|_p \gtrsim \|x\|_1 + \mathrm{TSP}(\mathbf{supp}(f); x, x)^{\frac{d+1-\gamma p}{dp}} \gtrsim \left(d_{\mathbb{Z}^d}(x,0) + \mathrm{TSP}(\mathbf{supp}(f); x, x)\right)^{\frac{d+1-\gamma p}{dp}}$$

$$\gtrsim \left(d_{C_2 \wr \mathbb{Z}^d}((f,x),(\mathbf{0},0))\right)^{\frac{d+1-\gamma p}{dp}}. \quad (40)$$

Letting γ tend from above to $\frac{d+1-p}{p}$ in (40) we get the lower bound

$$\alpha_p^*\left(C_2 \wr \mathbb{Z}^d\right) \geq \frac{1}{d}. \quad (41)$$

While (41) reproduces the result of [57], it yields the sharp bound $\alpha_p^*\left(C_2 \wr \mathbb{Z}^d\right) = \frac{1}{p}$ only when $p = d = 2$, in which case the above embedding coincides with the embedding used in [47]. This is why we needed to use a new argument in our proof of Theorem 3.1. Note that if one attempts to use the above reasoning while replacing the group $\mathbb{Z}^d$ by a general group H of growth rate d one realizes that it used the bound

$$|B_H(e_H, r+1)| - |B_H(e_H, r)| \asymp r^{d-1}. \quad (42)$$

Unfortunately the validity of (42) is open for general groups H of growth rate d. To the best of our knowledge the best known general upper bound on the growth rate of spheres is the following fact: there exists $\beta > 0$ (depending on the group H and the choice of generators) such that for every $r \in \mathbb{N}$ we have

$$|B_H(e_H, r+1)| - |B_H(e_H, r)| \lesssim r^{d-\beta}. \quad (43)$$

This is an immediate corollary of a well known (simple) result in metric geometry: since $|B_H(e,r)| \asymp r^d$ the metric space (H, d_H) is doubling (moreover, the counting measure on H is Ahlfors-David d-regular. See [35] for a discussion of these notions). By Lemma 3.3 in [19] (see also Proposition 6.12 in [16]) if (X,d,μ) is a geodesic doubling metric measure space then for all $x \in X$, $r > 0$ and $\delta \in (0,1)$ we have

$$\mu\left(B_X(x,r) \setminus B_X(x,(1-\delta)r)\right) \leq (2\delta)^\beta \mu\left(B_X(x,r)\right), \quad (44)$$

where $\beta > 0$ depends only on the doubling constant of the measure μ (see [19, 16] for a bound on β. In [49] it is shown that the bound on β from [19, 16] is asymptotically sharp as the doubling constant tends to ∞). Clearly (44) implies (43) if we let μ be the counting measure on H and $\delta = \frac{1}{r}$. While it is natural to conjecture that it is possible to take $\beta = 1$ in (43), this has been proved when H is a 2-step nilpotent group [56], but it is unknown in general. ◁

4 The zero section of $\mathbb{Z} \wr \mathbb{Z}$

This section is devoted to the proof of the following theorem:

Theorem 4.1. *Let $(\mathbb{Z} \wr \mathbb{Z})_0$ be the zero section of $\mathbb{Z} \wr \mathbb{Z}$, i.e. the subset of $\mathbb{Z} \wr \mathbb{Z}$ consisting of all $(f, x) \in \mathbb{Z} \wr \mathbb{Z}$ with $x = 0$, with the metric inherited from $\mathbb{Z} \wr \mathbb{Z}$. Then for all $p \in [1, 2]$ we have:*

$$\alpha_p^* ((\mathbb{Z} \wr \mathbb{Z})_0, d_{\mathbb{Z}\wr\mathbb{Z}}) = \frac{p+1}{2p}.$$

Proof. The fact that $\alpha_p^* ((\mathbb{Z} \wr \mathbb{Z})_0, d_{\mathbb{Z}\wr\mathbb{Z}}) \leq \frac{p+1}{2p}$ follows from a variant of an argument from [4]—see Lemma 7.8 in [47]. We present an alternative proof of this fact in Section 7 below.

Fix $\varepsilon \in (0, 1)$. In [47] we have shown that there exists a function $F_0 : \mathbb{Z} \wr \mathbb{Z} \to L_p$ such that the metric $\|F_0(f,x_1) - (f_2, x_2)\|_p$ is $\mathbb{Z} \wr \mathbb{Z}$-invariant and for all $(f, x) \in Z \wr Z$ we have

$$|x|^{(1-\varepsilon)p} + \sum_{j\in\mathbb{Z}} |f(j)|^p + \max\left\{|j|^{(1-\varepsilon)p} : \ f(x+j) \neq 0\right\} \lesssim \|F_0(f, x) - F_0(\mathbf{0}, 0)\|_p^p \lesssim d_{Z\wr Z}((f, x), (\mathbf{0}, 0))^p, \quad (45)$$

where here, and in what follows, the implied constants depend only on p and ε. We note that while (45) was not stated as a separate result in [47], it is contained in the proof of Theorem 3.3 there—see equation (28) in [47] with $a = 1$ and $b = 1 - \varepsilon$. Alternatively (45) is explained in detail for the case $p = 2$ in Remark 2.2 of [7]—the same argument works when we replace in that proof L_2 by L_p and let α be arbitrarily close to $(p - 1)/p$ (instead of arbitrarily close to $1/2$).

Let $\{e_{j,k,\ell} : \ j, k, \ell \in \mathbb{Z}\}$ be the standard basis of $\ell_p(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})$. For every $(f, 0) \in (\mathbb{Z} \wr \mathbb{Z})_0$ define

$$\Phi(f, 0) = \sum_{\ell=1}^{\infty} \sum_{k=0}^{\infty} \sum_{\substack{j\in\mathbb{Z} \\ |j|\in[2^{\ell-1}-1, 2^\ell-1)}} \frac{2^{(k+(p-1)\ell)/p}}{k+1} \exp\left(\frac{2\pi i f(j)}{2^k}\right) e_{j,k,\ell}.$$

Our embedding of $(\mathbb{Z} \wr \mathbb{Z})_0$ will be

$$F \coloneqq F_0 \oplus \Phi \in \ell_p(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}) \oplus L_p.$$

Observe that for every $(f, 0), (g, 0) \in (\mathbb{Z} \wr \mathbb{Z})_0$ we have $\|\Phi(f, 0) - \Phi(g, 0)\|_p = \|\Phi(f - g, 0) - \Phi(\mathbf{0}, 0)\|_p$, so it will suffice to prove the required compression bounds for $\|F(f, 0) - F(g, 0)\|_p$ when $g = 0$.

¿From now on we shall fix $(f, 0) \in (\mathbb{Z} \wr \mathbb{Z})_0$. For every $\ell, m \in \mathbb{Z}$ denote

$$E(\ell, m) = \left\{j :\in \mathbb{Z} : \ |j| \in [2^{\ell-1} - 1, 2^\ell - 1) \ \wedge \ |f(j)| \in [2^m, 2^{m+1})\right\}.$$

We also write $M \coloneqq \max\{|j| : \ f(j) \neq 0\}$, so that

$$d_{\mathbb{Z}\wr\mathbb{Z}}((f, 0), (\mathbf{0}, 0)) \asymp M + \|f\|_1 = M + \sum_{j\in\mathbb{Z}} |f(j)| \asymp M + \sum_{\ell=1}^{\infty} \sum_{m=0}^{\infty} 2^m |E(\ell, m)|. \quad (46)$$

Now,

$$\|\Phi(f,0)-\Phi(\mathbf{0},0)\|_p^p = \sum_{\ell=1}^{\infty}\sum_{k=0}^{\infty}\sum_{\substack{j\in\mathbb{Z}\\ |j|\in[2^{\ell-1}-1,2^\ell-1)}} \frac{2^{k+(p-1)\ell}}{(k+1)^p}\left|1-\exp\left(\frac{2\pi i f(j)}{2^k}\right)\right|^p$$
$$= \sum_{\ell=1}^{\infty}\sum_{k=0}^{\infty}\frac{2^{k+(p-1)\ell}}{(k+1)^p}\sum_{m=0}^{\infty}\sum_{j\in E(\ell,m)}\left|1-\exp\left(\frac{2\pi i f(j)}{2^k}\right)\right|^p. \quad (47)$$

Note that

$$m\le k-2 \implies \sum_{j\in E(\ell,m)}\left|1-\exp\left(\frac{2\pi i f(j)}{2^k}\right)\right|^p \asymp 2^{p(m-k)}|E(\ell,m)|. \quad (48)$$

and for all $m,k\in\mathbb{Z}$,

$$\sum_{j\in E(\ell,m)}\left|1-\exp\left(\frac{2\pi i f(j)}{2^k}\right)\right|^p \lesssim |E(\ell,m)|. \quad (49)$$

Plugging (48) and (49) into (47) we see that

$$\|\Phi(f,0)-\Phi(\mathbf{0},0)\|_p^p \lesssim \sum_{\ell=1}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m+1}\frac{2^{k+(p-1)\ell}}{(k+1)^p}|E(\ell,m)| + \sum_{k=m+2}^{\infty}\frac{2^{k+(p-1)\ell}}{(k+1)^p}2^{p(m-k)}|E(\ell,m)|\right)$$
$$\lesssim \sum_{\ell=1}^{\infty}\sum_{m=0}^{\infty}\frac{2^{m+(p-1)\ell}}{(m+1)^p}|E(\ell,m)| \le \left(\sum_{\ell=1}^{\infty}\sum_{m=0}^{\infty}\left(\frac{2^{m+(p-1)\ell}}{(m+1)^p}|E(\ell,m)|\right)^{1/p}\right)^p. \quad (50)$$

Using the fact that for all $a,b\ge 0$ we have $ab^{p-1}\le\left(\frac{a+b}{2}\right)^p$ we can bound the summands in (50) as follows:

$$\left(\frac{2^{m+(p-1)\ell}}{(m+1)^p}|E(\ell,m)|\right)^{1/p} \lesssim \begin{cases} 2^m|E(\ell,m)| + \frac{2^\ell}{(m+1)^{p/(p-1)}} & \text{if } E(\ell,m)\neq\emptyset,\\ 2^m|E(\ell,m)| & \text{otherwise.}\end{cases} \quad (51)$$

Note that if $E(\ell,m)\neq\emptyset$ then there exists $j\in\mathbb{Z}$ with $|j|\in[2^{\ell-1}-1,2^\ell-1)$ such that $f(j)\neq 0$. By the definition of M this implies that $2^\ell < M$. Using this observation while substituting the the estimates (51) in (50) we see that

$$\|\Phi(f,0)-\Phi(\mathbf{0},0)\|_p \lesssim \sum_{m=0}^{\infty}\sum_{\ell=1}^{\infty}2^m|E(\ell,m)| + \sum_{\ell=1}^{\lfloor\log_2 M\rfloor}2^\ell\sum_{m=0}^{\infty}\frac{1}{(m+1)^{p/(p-1)}}$$
$$\lesssim \sum_{\ell=1}^{\infty}\sum_{m=0}^{\infty}2^m|E(\ell,m)| + M \overset{(46)}{\asymp} d_{\mathbb{Z}\wr\mathbb{Z}}((f,0),(\mathbf{0},0)). \quad (52)$$

This shows that $\|F\|_{\mathrm{Lip}}\lesssim 1$.

In the reverse direction write

$$D := d_{\mathbb{Z}\wr\mathbb{Z}}((f,0),(\mathbf{0},0)) \overset{(46)}{\asymp} M + \sum_{2^\ell<M}\sum_{|j|\in[2^{\ell-1}-1,2^\ell-1)}|f(j)| \asymp \sum_{2^\ell<M}\left(2^\ell + \sum_{|j|\in[2^{\ell-1}-1,2^\ell-1)}|f(j)|\right)$$

It follows that there exists an integer $\ell < \log_2 M$ such that

$$D \lesssim \log(M+1) \cdot \left(2^\ell + \sum_{|j| \in [2^{\ell-1}-1, 2^\ell - 1)} |f(j)| \right). \tag{53}$$

We shall fix this ℓ from now on. Observe that

$$\sum_{|j| \in [2^{\ell-1}-1, 2^\ell - 1)} |f(j)| \asymp \sum_{2^{m+1} < \|f\|_1} 2^m |E(\ell, m)|.$$

Hence there exists an integer $m \lesssim \log(1 + \|f\|_1)$ such that

$$\sum_{|j| \in [2^{\ell-1}-1, 2^\ell - 1)} |f(j)| \lesssim 2^m |E(\ell, m)| \cdot \log(1 + \|f\|_1). \tag{54}$$

We shall fix this m form now on. Combining (53) with (54) yields the bound:

$$D \lesssim \log(M+1) \cdot \left(2^\ell + 2^m |E(\ell, m)| \cdot \log(1 + \|f\|_1) \right) \overset{(46)}{\lesssim} (\log(D+1))^2 \cdot \left(2^\ell + 2^m |E(\ell, m)| \right). \tag{55}$$

Substitute (48) into (47) to get the lower bound

$$\|\Phi(f,0) - \Phi(\mathbf{0},0)\|_p^p \gtrsim \sum_{k=m+2}^{\infty} \frac{2^{k+(p-1)\ell}}{(k+1)^p} 2^{p(m-k)} |E(\ell, m)| \gtrsim \frac{2^{m+(p-1)\ell}}{(m+1)^p} |E(\ell, m)| \gtrsim \frac{2^{m+(p-1)\ell}}{(\log(D+1))^p} |E(\ell, m)|.$$

Also (45) implies that

$$\|F_0(f,0) - F_0(\mathbf{0},0)\|_p^p \gtrsim M^{(1-\varepsilon)p} + 2^{mp} |E(\ell, m)| \gtrsim 2^{(1-\varepsilon)\ell p} + 2^{mp} |E(\ell, m)|.$$

Thus

$$\begin{aligned} \|F(f,0) - F(\mathbf{0},0)\|_p &\gtrsim 2^{(1-\varepsilon)\ell} + 2^m |E(\ell, m)|^{1/p} + \frac{1}{\log(D+1)} \cdot 2^{m/p} |E(\ell, m)|^{1/p} 2^{\ell(p-1)/p} \\ &\gtrsim \frac{1}{\log(D+1)} \cdot \left(2^\ell + 2^m |E(\ell, m)|^{1/p} + 2^{m/p} |E(\ell, m)|^{1/p} 2^{\ell(p-1)/p} \right)^{1-\varepsilon}. \end{aligned} \tag{56}$$

We claim that

$$2^\ell + 2^m |E(\ell, m)|^{1/p} + 2^{m/p} |E(\ell, m)|^{1/p} 2^{\ell(p-1)/p} \geq \frac{2^\ell + (2^m |E(\ell, m)|)^{\frac{p+1}{2p}}}{2}. \tag{57}$$

Indeed, if $(2^m |E(\ell, m)|)^{\frac{p+1}{2p}} \leq 2^\ell$ then (57) is trivial, so assume that $a := (2^m |E(\ell, m)|)^{\frac{p+1}{2p}} \geq 2^\ell$. Since $|E(\ell, m)| = 2^{-m} \cdot a^{2p/(p+1)}$ we see that

$$2^\ell + 2^m |E(\ell, m)|^{1/p} + 2^{m/p} |E(\ell, m)|^{1/p} 2^{\ell(p-1)/p} \geq 2^{(p-1)m/p} a^{2/(p+1)} + a^{2/(p+1)} 2^{\ell(p-1)/p}. \tag{58}$$

Note that by definition $2^{-m} \cdot a^{2p/(p+1)} = |E(\ell, m)| \leq 2^\ell$, so $2^m \geq 2^{-\ell} \cdot a^{2p/(p+1)}$. Substituting this bound into (58) we see that

$$\begin{aligned} 2^\ell + 2^m |E(\ell, m)|^{1/p} + 2^{m/p} |E(\ell, m)|^{1/p} 2^{\ell(p-1)/p} &\geq 2^{-\ell(p-1)/p} \cdot a^{2p/(p+1)} + a^{2/(p+1)} 2^{\ell(p-1)/p} \\ &\geq 2a = 2 \left(2^m |E(\ell, m)| \right)^{\frac{p+1}{2p}}, \end{aligned}$$

where we used the arithmetic mean/geometric mean inequality. This completes the proof of (57).

A combination of (55), (56) and (57) yields

$$\begin{aligned}\|F(f,0)-F(\mathbf{0},0)\|_p &\gtrsim \frac{1}{\log(D+1)}\cdot\left(2^{\ell}+(2^m|E(\ell,m)|)^{\frac{p+1}{2p}}\right)^{1-\varepsilon}\\ &\gtrsim \frac{1}{\log(D+1)}\cdot\left(2^{\ell}+2^m|E(\ell,m)|\right)^{(1-\varepsilon)\frac{p+1}{2p}} \gtrsim \frac{D^{(1-\varepsilon)\frac{p+1}{2p}}}{(\log(D+1))^{1+2(1-\varepsilon)\frac{p+1}{2p}}} \gtrsim D^{(1-2\varepsilon)\frac{p+1}{2p}}.\end{aligned}$$

This completes the proof of Theorem 4.1. □

5 General compression upper bounds for amenable groups

Let Γ be a group which is generated by the finite symmetric set $S \subseteq \Gamma$. Let ρ be a left-invariant metric on Γ such that $B_\rho(e_\Gamma, r) = \{x \in \Gamma : \ \rho(x,e) \le r\}$ is finite for all $r \ge 0$. In most of our applications of the ensuing arguments the metric ρ will be the word metric induced by S, but we will also need to deal with other invariant metrics (see Section 7).

Given a symmetric probability measure μ on Γ let $\{g_k\}_{k=1}^{\infty}$ be i.i.d. elements of Γ which are distributed according to μ. The μ-random walk $\{W_t^\mu\}_{t=0}^{\infty}$ is defined as $W_0^\mu = e_\Gamma$ and $W_t^\mu = g_1 g_2 \cdots g_t$ for $t \in \mathbb{N}$. Fix $p \ge 1$ and assume that

$$\int_\Gamma \rho(x,e_\Gamma)^p d\mu(x) = \mathbb{E}_\mu\left[\rho\left(W_1^\mu, e_\Gamma\right)^p\right] < \infty. \tag{59}$$

Let $\{\mu_t\}_{t=1}^{\infty}$ be a sequence of symmetric probability measures satisfying the integrability condition (59) and define

$$\beta_p^*\left(\{\mu_t\}_{t=1}^{\infty},\rho\right) := \limsup_{t\to\infty} \frac{\log\left(\mathbb{E}_{\mu_t}\left[\rho\left(W_t^{\mu_t}, e_\Gamma\right)\right]\right)}{\log\left(t\mathbb{E}_{\mu_t}\left[\rho\left(W_1^{\mu_t}, e_\Gamma\right)^p\right]\right)}. \tag{60}$$

Finally we let $\beta_p^*(\Gamma,\rho)$ be the supremum of $\beta_p^*\left(\{\mu_t\}_{t=1}^{\infty},\rho\right)$ over all sequences of symmetric probability measures $\{\mu_t\}_{t=1}^{\infty}$ on Γ satisfying

$$\forall t \in \mathbb{N} \quad \int_\Gamma \rho(x,e_\Gamma)^p d\mu_t(x) < \infty \quad \text{and} \quad \lim_{t\to\infty}\left(t\mu_t\left(\Gamma \setminus \{e_\Gamma\}\right)\right) = \infty. \tag{61}$$

When ρ is the word metric induced by the symmetric generating set S we will use the simplified notation $\beta_p^*(\Gamma,\rho) = \beta_p^*(\Gamma)$. This convention does not create any ambiguity since clearly $\beta_p^*(\Gamma,\rho)$ does not depend on the choice of the finite symmetric generating set S (this follows from the fact that due to (61) the denominator in (60) tends to ∞ with t—we establish this fact below).

To better explain the definition (60) we shall make some preliminary observations before passing to the main results of this section. We first note that

$$\beta_p^*(\Gamma,\rho) \le 1. \tag{62}$$

Indeed, since we are assuming that all the ρ-balls are finite there exists $\rho_0 > 0$ such that for every distinct $x, y \in \Gamma$ we have $\rho(x,y) \ge \rho_0$. Hence for every symmetric probability measure μ on Γ which satisfies (59) we have

$$\mathbb{E}_\mu\left[\rho\left(W_1^\mu, e_\Gamma\right)^p\right] \ge \rho_0^p \mu\left(\Gamma \setminus \{e_\Gamma\}\right). \tag{63}$$

Hölder's inequality therefore implies that:

$$\mathbb{E}_\mu\left[\rho\left(W_1^\mu, e_\Gamma\right)\right] = \mathbb{E}_\mu\left[\rho\left(W_1^\mu, e_\Gamma\right)\mathbf{1}_{\Gamma\setminus\{e_\gamma\}}\right]$$
$$\leq \mu\left(\Gamma\setminus\{e_\Gamma\}\right)^{(p-1)/p}\left(\mathbb{E}_\mu\left[\rho\left(W_1^\mu, e_\Gamma\right)^p\right]\right)^{1/p} \overset{(63)}{\leq} \frac{1}{\rho_0^{p-1}}\mathbb{E}_\mu\left[\rho\left(W_1^\mu, e_\Gamma\right)^p\right]. \quad (64)$$

On the other hand, by the triangle inequality we have:

$$\mathbb{E}_\mu\left[\rho\left(W_t^\mu, e_\Gamma\right)\right] \leq \sum_{i=1}^{t}\mathbb{E}_\mu\left[\rho\left(W_i^\mu, W_{i-1}^\mu\right)\right] = t\mathbb{E}_\mu\left[\rho\left(W_1^\mu, e_\Gamma\right)\right] \overset{(64)}{\leq} \frac{t}{\rho_0^{p-1}}\mathbb{E}_\mu\left[\rho\left(W_1^\mu, e_\Gamma\right)^p\right]. \quad (65)$$

It follows that if $\{\mu_t\}_{t=1}^\infty$ are symmetric probability measures on Γ satisfying (61) then

$$\limsup_{t\to\infty}\frac{\log\left(\mathbb{E}_{\mu_t}\left[\rho\left(W_t^{\mu_t}, e_\Gamma\right)\right]\right)}{\log\left(t\mathbb{E}_{\mu_t}\left[\rho\left(W_1^{\mu_t}, e_\Gamma\right)^p\right]\right)} \overset{(63)\wedge(65)}{\leq} \limsup_{t\to\infty}\left(1 - \frac{(p-1)\log\rho_0}{\log\left(t\mu_t\left(\Gamma\setminus\{e_\Gamma\}\right)\right) + p\log\rho_0}\right) \overset{(61)}{=} 1,$$

implying (62).

We also claim that if $1 \leq q \leq p < \infty$ then

$$\beta_p^*(\Gamma,\rho) \leq \beta_q^*(\Gamma,\rho). \quad (66)$$

Indeed, let $\{\mu_t\}_{t=1}^\infty$ be symmetric probability measures on Γ satisfying (61) and note that

$$\mathbb{E}_{\mu_t}\left[\rho\left(W_1^{\mu_t}, e_\Gamma\right)^q\right] \leq \mu_t(\Gamma\setminus\{e_\Gamma\})^{(p-q)/p}\left(\mathbb{E}_{\mu_t}\left[\rho\left(W_1^{\mu_t}, e_\Gamma\right)^p\right]\right)^{q/p} \overset{(63)}{\leq} \frac{1}{\rho_0^{p-q}}\mathbb{E}_{\mu_t}\left[\rho\left(W_1^{\mu_t}, e_\Gamma\right)^p\right]. \quad (67)$$

Hence,

$$\limsup_{t\to\infty}\frac{\log\left(\mathbb{E}_{\mu_t}\left[\rho\left(W_t^{\mu_t}, e_\Gamma\right)\right]\right)}{\log\left(t\mathbb{E}_{\mu_t}\left[\rho\left(W_1^{\mu_t}, e_\Gamma\right)^p\right]\right)} \overset{(67)}{\leq} \limsup_{t\to\infty}\frac{\log\left(\mathbb{E}_{\mu_t}\left[\rho\left(W_t^{\mu_t}, e_\Gamma\right)\right]\right)}{\log\left(t\mathbb{E}_{\mu_t}\left[\rho\left(W_1^{\mu_t}, e_\Gamma\right)^q\right]\right)}\cdot\frac{1}{1+\frac{(p-q)\log\rho_0}{\log\left(t\mathbb{E}_{\mu_t}\left[\rho\left(W_1^{\mu_t}, e_\Gamma\right)^q\right]\right)}}$$
$$\overset{(63)}{\leq} \limsup_{t\to\infty}\frac{\log\left(\mathbb{E}_{\mu_t}\left[\rho\left(W_t^{\mu_t}, e_\Gamma\right)\right]\right)}{\log\left(t\mathbb{E}_{\mu_t}\left[\rho\left(W_1^{\mu_t}, e_\Gamma\right)^q\right]\right)}\cdot\frac{1}{1+\frac{(p-q)\log\rho_0}{\log(t\mu_t(\Gamma\setminus\{e_\Gamma\}))+q\log\rho_0}} \overset{(61)}{\leq} \beta_q^*(\Gamma,\rho),$$

implying (66).

The main result of this section is the following theorem:

Theorem 5.1. *Assume that Γ is amenable and that X is a metric space with Markov type p. Then for every left-invariant metric ρ on Γ such that $|B_\rho(e_\Gamma, r)| < \infty$ for all $r \geq 0$ we have:*

$$\alpha_X^*(\Gamma,\rho) \leq \frac{1}{p\beta_p^*(\Gamma,\rho)}.$$

Remark 5.2. In [7, 47] it was essentially shown that the bound in Theorem 5.1 holds true with $\beta_p^*(\Gamma,\rho)$ replaced by $\beta_\infty^*(\Gamma,\rho)$, which is a weaker bound due to (66). More precisely [7, 47] dealt with the case when

all the measures μ_t equal a fixed measure μ, in which case the second requirement of (61) is simply that μ is not supported on $\{e_\Gamma\}$. If we restrict to this particular case we can define an analogous parameter by

$$\widetilde{\beta}_p^*(\mu,\rho) = \beta_p^*\left(\{\mu,\mu,\mu,\ldots\},\rho\right) \coloneqq \limsup_{t\to\infty} \frac{\log\left(\mathbb{E}_\mu\left[\rho\left(W_t^\mu, e_\Gamma\right)\right]\right)}{\log t}.$$

and similarly by taking the supremum over all symmetric probability measures measures μ satisfying (61) we can define the parameter $\widetilde{\beta}_p^*(\Gamma,\rho)$. An inspection of the results in [7, 47] shows that a variant of Theorem 5.1 is established there with $\beta_p^*(\Gamma,\rho)$ replaced by $\widetilde{\beta}_\infty^*(\Gamma,\rho)$. Thus Theorem 5.1 is formally stronger than the results of [7, 47]. As we shall see in Section 6, this is a strict improvement which is crucial for our proof of the bound $\alpha_p^*(\mathbb{Z}\wr\mathbb{Z}) \leq \frac{p}{2p-1}$, and in Section 7 we will also need to use a family of non-identical measures $\{\mu_t\}_{t=1}^\infty$. ◁

Proof of Theorem 5.1. Let $\{F_n\}_{n=0}^\infty$ be a Følner sequence for Γ, i.e., for every $\varepsilon > 0$ and any finite $K \subseteq \Gamma$, we have $|F_n \triangle (F_n K)| \leq \varepsilon |F_n|$ for large enough n. Fix $\beta < \beta_p^*(\Gamma,\rho)$. Then there exists a sequence of symmetric probability measures $\{\mu_t\}_{t=1}^\infty$ on Γ which satisfy (61) and $\beta < \beta_p^*\left(\{\mu_t\}_{t=1}^\infty,\rho\right)$. This implies that there exists an increasing sequence of integers $\{t_k\}_{k=1}^\infty$ for which

$$\mathbb{E}_{\mu_{t_k}}\left[\rho\left(W_{t_k}^{\mu_{t_k}}, e_\Gamma\right)\right] \geq t_k^\beta \left(\mathbb{E}_{\mu_{t_k}}\left[\rho\left(W_1^{\mu_{t_k}}, e_\Gamma\right)^p\right]\right)^\beta,$$

for all k. For every $t, r \in \mathbb{N}$ consider the event

$$\Lambda_t(r) \coloneqq \bigcap_{j=1}^t \left\{W_j^{\mu_t} \in B_\rho(e_\Gamma, r)\right\}.$$

By the monotone convergence theorem for every $k \in \mathbb{N}$ there exists $r_k \in \mathbb{N}$ such that

$$\mathbb{E}_{\mu_{t_k}}\left[\rho\left(W_{t_k}^{\mu_{t_k}}, e_\Gamma\right)\mathbf{1}_{\Lambda_{t_k}(r_k)}\right] \geq \frac{1}{2} t_k^\beta \left(\mathbb{E}_{\mu_{t_k}}\left[\rho\left(W_1^{\mu_{t_k}}, e_\Gamma\right)^p\right]\right)^\beta. \tag{68}$$

Since $|B_\rho(e_\Gamma, r_k)| < \infty$ for every $k \in \mathbb{N}$ we can find $n_k \in \mathbb{N}$ such that if we denote $A \coloneqq F_{n_k} B_\rho(e_\Gamma, r_k) \supseteq F_{n_k}$ then we have (by the Følner condition with $\varepsilon = 1$),

$$|A \setminus F_{n_k}| \leq |F_{n_k}| \implies |F_{n_k}| \geq \frac{1}{2}|A|. \tag{69}$$

Fix $k \in \mathbb{N}$ and let $\{g_i\}_{i=1}^\infty \subseteq \Gamma$ be i.i.d. group elements distributed according to μ_{t_k} such that $W_t^{\mu_{t_k}} = g_1 g_2 \cdots g_t$ for every $t \in \mathbb{N}$. Let Z_0 be uniformly distributed over A and independent of $\{g_i\}_{i=1}^\infty$. For $t \in \mathbb{N}$ define

$$Z_t \coloneqq \begin{cases} Z_{t-1} g_t & \text{if } Z_{t-1} g_t \in A, \\ Z_{t-1} & \text{otherwise.} \end{cases}$$

Consider the event $\Omega \coloneqq \{Z_0 \in F_{n_k}\} \cap \Lambda_{t_{n_k}}(r_k)$. By construction when Ω occurs we have $Z_{t_k} = Z_0 W_{t_k}^{\mu_{t_k}}$. Hence

$$\begin{aligned}\mathbb{E}_{\mu_{t_k}}\left[\rho\left(Z_{t_k}, Z_0\right)\right] &\geq \mathbb{E}_{\mu_{t_k}}\left[\rho\left(Z_0 W_{t_k}^{\mu_{t_k}}, Z_0\right)\mathbf{1}_\Omega\right] \\ &\overset{(*)}{=} \mathbb{P}\left[Z_0 \in F_{n_k}\right] \cdot \mathbb{E}_{\mu_{t_k}}\left[\rho\left(W_{t_k}^{\mu_{t_k}}, e_\Gamma\right)\mathbf{1}_{\Lambda_{t_k}(r_k)}\right] \overset{(68)\wedge(69)}{\geq} \frac{1}{4} t_k^\beta \left(\mathbb{E}_{\mu_{t_k}}\left[\rho\left(W_1^{\mu_{t_k}}, e_\Gamma\right)^p\right]\right)^\beta, \end{aligned} \tag{70}$$

where in $(*)$ we used the independence of Z_0 and $\{g_i\}_{i=1}^{\infty}$ and the left-invariance of ρ.

On the other hand fix $\alpha \in (0,1)$ and assume that there exists an embedding $f : \Gamma \to X$ and $c, C \in (0,\infty)$ such that

$$x, y \in \Gamma \implies c\rho(x,y)^{\alpha} \leq d_X(f(x), f(x)) \leq C\rho(x,y). \tag{71}$$

Our goal is to show that $\alpha \leq \frac{1}{p\beta}$. Since $\beta < 1$ this inequality is vacuous if $p\alpha < 1$. We may therefore assume that $p\alpha \geq 1$. Since $\{Z_t\}_{t=0}^{\infty}$ is a stationary reversible Markov chain, for every $M > M_p(X)$ and $k \in \mathbb{N}$ we have

$$\begin{aligned}\mathbb{E}\left[d_X(f(Z_{t_k}), f(Z_0))^p\right] &\leq M^p t_k \mathbb{E}\left[d_X(f(Z_1), f(Z_0))^p\right] \\ &\overset{(71)}{\leq} M^p C^p t_k \mathbb{E}\left[\rho(Z_1, Z_0)^p\right] \overset{(**)}{\leq} M^p C^p t_k \mathbb{E}_{\mu_{t_k}}\left[\rho\left(W_1^{\mu_{t_k}}, e_\Gamma\right)^p\right], \end{aligned} \tag{72}$$

Where in $(**)$ we used the point-wise inequality $\rho(Z_1, Z_0) \leq \rho(g_1, e_\Gamma) = \rho\left(W_1^{\mu_{t_k}}, e_\Gamma\right)$. On the other hand,

$$\begin{aligned}\mathbb{E}\left[d_X(f(Z_{t_k}), f(Z_0))^p\right] &\overset{(71)}{\geq} c^p \mathbb{E}\left[\rho\left(Z_{t_k}, fZ_0\right)^{\alpha p}\right] \\ &\overset{\alpha p \geq 1}{\geq} c^p \left(\mathbb{E}\left[\rho(Z_{t_k}, Z_0)\right]\right)^{\alpha p} \overset{(70)}{\geq} \frac{c^p t_k^{\alpha\beta p}}{4^{\alpha p}} \left(\mathbb{E}_{\mu_{t_k}}\left[\rho\left(W_1^{\mu_{t_k}}, e_\Gamma\right)^p\right]\right)^{\alpha\beta p}. \end{aligned} \tag{73}$$

Combining (72) and (73) we deduce that

$$\left(\rho_0^p t_k \mu_{t_k}(\Gamma \setminus \{e_\Gamma\})\right)^{\alpha\beta p - 1} \overset{(63)}{\leq} \left(t_k \mathbb{E}_{\mu_{t_k}}\left[\rho\left(W_1^{\mu_{t_k}}, e_\Gamma\right)^p\right]\right)^{\alpha\beta p - 1} \leq \frac{4^{\alpha p} M^p C^p}{c^p}. \tag{74}$$

Taking $k \to \infty$ in (74) while using the assumption (61) we conclude that $\alpha\beta p \leq 1$, as required. □

The following theorem is a variant of Theorem 5.1 which deals with equivariant embeddings of general groups (not necessarily amenable) into uniformly smooth Banach spaces. Its proof is an obvious modification of the proof of Theorem 2.1 in [47]: one just has to notice that in that proof the i.i.d. group elements $\{\sigma_k\}_{k=1}^{\infty}$ need not be uniformly distributed over a symmetric generating set $S \subseteq \Gamma$—the argument goes through identically if they are allowed to be distributed according to any symmetric probability measure μ satisfying the integrability condition (59).

Theorem 5.3. *Let Γ be a group and ρ a left-invariant metric on Γ such that $|B_\rho(e_\Gamma, r)| < \infty$ for all $r \geq 0$. Assume that X is a Banach space whose modulus of uniform smoothness has power-type $p \in [1,2]$. Then:*

$$\alpha_X^{\#}(\Gamma, \rho) \leq \frac{1}{p\beta_p^*(\Gamma, \rho)}.$$

By the results of Section 9 Theorem 5.3 implies Theorem 5.1 when X is a Banach space whose modulus of uniform smoothness has power-type p rather than a general metric space with Markov type p. Note that the former assumption implies the latter assumption as shown in [48].

6 Stable walks and the L_p compression of $\mathbb{Z} \wr \mathbb{Z}$

This section is devoted to the proof of the following theorem:

Theorem 6.1. *For every $p \in (1,2)$ we have*

$$\beta_p^*(\mathbb{Z} \wr \mathbb{Z}) = \frac{2p-1}{p^2}.$$

Note that since in [47] we proved that $\alpha_p^*(\mathbb{Z} \wr \mathbb{Z}) \geq \frac{p}{2p-1}$, Theorem 5.1 implies that $\beta_p^*(\mathbb{Z} \wr \mathbb{Z}) \leq \frac{2p-1}{p^2}$. Thus in order to prove Theorem 6.1 it suffices to show that $\beta_p^*(\mathbb{Z} \wr \mathbb{Z}) \geq \frac{2p-1}{p^2}$, which would also imply that $\alpha_p^*(\mathbb{Z} \wr \mathbb{Z}) = \frac{p}{2p-1}$. In order to establish this lower bound on $\beta_p^*(\mathbb{Z} \wr \mathbb{Z})$ we will analyze certain symmetric random walks on $\mathbb{Z} \wr \mathbb{Z}$ which arise from discrete approximations of q-stable random variables for some $q \in (p,2)$.

6.1 Some general properties of symmetric walks on $\mathbb{Z}$

Let X be a $\mathbb{Z}$-valued symmetric random variable and let $X_1, X_2, \ldots$ be i.i.d. copies of X. For each $n \in \mathbb{N}$ define $S_n = X_1 + \cdots + X_n$ (and set $S_0 = 0$). We also define $S_{[o,n]}$ to be the random set $\{S_0, \ldots, S_n\}$. We will record here for future use some general properties of the walk S_n. These are simple facts which appeared in various guises in the literate (though we did not manage to pinpoint cleanly stated references for them). We include this brief discussion for the sake of completeness.

Lemma 6.2. *For S_n as above we have*

$$\mathbb{E}\left[|S_n|\right] \geq \frac{1}{4}\mathbb{E}\left[\left|S_{[0,n]}\right|\right]. \tag{75}$$

Proof. Fix $R \geq 0$ and denote $\tau := \min\{t \geq 0 : \ |S_t| \geq R\}$. Note the following inclusion of events:

$$\{|S_n| \geq R\} \supseteq \left\{\tau \leq n \ \wedge \ \mathbf{sign}\left(\sum_{k=\tau+1}^{n} X_k\right) = \mathbf{sign}(S_\tau)\right\}.$$

It follows that:

$$\begin{aligned}\mathbb{P}\left[|S_n| \geq R\right] \geq \sum_{m=0}^{n} \mathbb{P}\left[\tau = m \ \wedge \ \mathbf{sign}\left(\sum_{k=m+1}^{n} X_k\right) = \mathbf{sign}(S_m)\right] &= \sum_{m=0}^{n} \mathbb{P}\left[\tau = m\right] \cdot \mathbb{P}\left[S_{n-m} \geq 0\right] \\ &\overset{(\star)}{\geq} \frac{1}{2}\sum_{m=0}^{n} \mathbb{P}\left[\tau = m\right] = \frac{1}{2}\mathbb{P}\left[\tau \leq n\right], \end{aligned} \tag{76}$$

where in $(\star)$ we used the symmetry of S_{n-m}. Note that if $\left|S_{[0,n]}\right| \geq 2R$ then one of the numbers $\{|S_0|, \ldots, |S_n|\}$ must be at least R. Thus

$$\mathbb{P}\left[\tau \leq n\right] \geq \mathbb{P}\left[\left|S_{[0,n]}\right| \geq 2R\right]. \tag{77}$$

It follows that:

$$\begin{aligned}\mathbb{E}\left[|S_n|\right] = \sum_{R=0}^{\infty} \mathbb{P}\left[|S_n| \geq R\right] &\overset{(76)\wedge(77)}{\geq} \frac{1}{2}\sum_{R=0}^{\infty} \mathbb{P}\left[\left|S_{[0,n]}\right| \geq 2R\right] \\ &\geq \frac{1}{2}\sum_{R=0}^{\infty} \frac{\mathbb{P}\left[\left|S_{[0,n]}\right| \geq 2R\right] + \mathbb{P}\left[\left|S_{[0,n]}\right| \geq 2R+1\right]}{2} = \frac{1}{4}\mathbb{E}\left[\left|S_{[0,n]}\right|\right], \end{aligned}$$

as required. □

The proof of the following lemma is a slight variant of the argument used to prove the first assertion of Lemma 6.3 in [47].

Lemma 6.3. *Let S_n be as above and denote $R_n \coloneqq |\{k \in \{0,\dots,n\} : \ S_k = 0\}|$. Then*

$$\mathbb{P}\left[R_n \geq \frac{1}{2}\sum_{\ell=0}^{n}\mathbb{P}[S_\ell = 0]\right] \geq \frac{1}{8}. \tag{78}$$

Proof. Since $R_n = \sum_{\ell=0}^{n}\mathbf{1}_{\{S_\ell=0\}}$ we have $\mathbb{E}[R_n] = \sum_{\ell=0}^{n}\mathbb{P}[S_\ell = 0]$ and:

$$\begin{aligned}\mathbb{E}\left[R_n^2\right] &= \sum_{\ell=0}^{n}\mathbb{P}[S_\ell = 0] + 2\sum_{\substack{i,j\in\{0,\dots,n\}\\ i<j}}\mathbb{P}\left[S_i = S_j = 0\right]\\ &= \mathbb{E}\left[R_n\right] + 2\sum_{\substack{i,j\in\{0,\dots,n\}\\ i<j}}\mathbb{P}\left[S_i = 0\right]\cdot\mathbb{P}\left[S_{j-i} = 0\right] \leq \mathbb{E}\left[R_n\right] + \left(\mathbb{E}\left[R_n\right]\right)^2 \leq 2\left(\mathbb{E}\left[R_n\right]\right)^2.\end{aligned}$$

Since for every nonnegative random variable Z we have $\mathbb{P}\left[Z \geq \frac{1}{2}\mathbb{E}[Z]\right] \geq \frac{1}{4}\frac{(\mathbb{E}[Z])^2}{\mathbb{E}[Z^2]}$ (which is an easy consequence of the Cauchy-Schwartz inequality—see [51, 2]) we deduce that $\mathbb{P}\left[R_n \geq \frac{1}{2}\mathbb{E}[R_n]\right] \geq \frac{1}{8}$, as required. □

The proof of the following lemma is a slight variant of the argument used to prove the second assertion of Lemma 6.3 in [47].

Lemma 6.4. *For S_n as above we have:*

$$\mathbb{E}\left[\left|S_{[0,n]}\right|\right] \geq \frac{n+1}{2\sum_{\ell=0}^{n}\mathbb{P}\left[S_\ell = 0\right]}. \tag{79}$$

Proof. Fix $k \in \{1,\dots,n+1\}$ and denote $\widetilde{k} \coloneqq \min\left\{k, \left|S_{[0,n]}\right|\right\}$. Let $V_1,\dots,V_{\widetilde{k}}$ be the first distinct $\widetilde{k}$ integers that were visited by the walk $S_0, S_1, \dots, S_n$. For simplicity of notation we also set $V_j = n+1$ when $j \in \{\widetilde{k}+1,\dots,n\}$. Write

$$\tau_j \coloneqq \begin{cases}\min\{0 \leq \tau \leq n : \ S_\tau = V_j\} & j \leq \widetilde{k},\\ n+1 & j > \widetilde{k}.\end{cases}$$

Denote $Y_k \coloneqq \left|\left\{0 \leq j \leq n : \ S_j \in \{V_1,\dots,V_{\widetilde{k}}\}\right\}\right|$. Then

$$\begin{aligned}\mathbb{E}\left[Y_k\right] &= \sum_{j=1}^{k}\mathbb{E}\left[\left|\left\{0 \leq \ell \leq n : \ S_\ell = V_j\right\}\right|\right] = \sum_{j=1}^{k}\mathbb{E}\left[\sum_{\ell=0}^{n}\mathbf{1}_{\{S_\ell=V_j\}}\right] = \sum_{j=1}^{k}\mathbb{E}\left[\sum_{\ell=\tau_j}^{n}\mathbb{P}\left[S_\ell = S_{\tau_j}\right]\middle|\tau_j\right]\\ &= \sum_{j=1}^{k}\mathbb{E}\left[\sum_{\ell=\tau_j}^{n}\mathbb{P}\left[S_{\ell-\tau_j} = 0\right]\middle|\tau_j\right] \leq k\sum_{\ell=0}^{n}\mathbb{P}\left[S_\ell = 0\right]. \qquad (80)\end{aligned}$$

Hence

$$\mathbb{P}\left[\left|S_{[0,n]}\right| \leq k\right] \leq \mathbb{P}\left[Y_k \geq n+1\right] \leq \frac{\mathbb{E}[Y_k]}{n+1} \overset{(80)}{\leq} \frac{k}{n+1}\sum_{\ell=0}^{n}\mathbb{P}\left[S_\ell = 0\right]. \tag{81}$$

It follows that if we denote $m = \frac{n+1}{\sum_{\ell=0}^{n} \mathbb{P}[S_\ell = 0]}$ then

$$\mathbb{E}\left[\left|S_{[0,n]}\right|\right] = \sum_{k=1}^{n+1} \mathbb{P}\left[\left|S_{[0,n]}\right| \geq k\right] = \sum_{k=1}^{n+1} \left(1 - \mathbb{P}\left[\left|S_{[0,n]}\right| \leq k-1\right]\right) \overset{(81)}{\geq} \sum_{k=1}^{\lceil m \rceil} \left(1 - \frac{k-1}{m}\right)$$
$$= \lceil m \rceil - \frac{\lceil m \rceil (\lceil m \rceil - 1)}{2m} \geq \frac{\lceil m \rceil}{2} \geq \frac{n+1}{2\sum_{\ell=0}^{n} \mathbb{P}\left[S_\ell = 0\right]},$$

as required. □

6.2 An analysis of a particular discrete stable walk on $\mathbb{Z}$

In this section we will analyze a specific random walk on $\mathbb{Z}$ which will be used in estimating $\beta_p^*(\mathbb{Z} \wr \mathbb{Z})$. Similar bounds are known to hold in great generality for arbitrary walks which are in the domain of attraction of q-stable random variables, and not only for the walk presented below. Specifically, such general results can be deduced from Gnedenko's local central limit theorem for convergence to stable laws (see Theorem 4.2.1 in [37]), in combination with some estimates on such walks from [29] (see section IX.8, Theorem 1 there). Since for the purpose of proving compression bounds all we need is to construct one such walk, we opted for the sake of concreteness to present here a simple self-contained proof of the required properties of a particular walk which is perfectly suited for the purpose of our applications to embedding theory.

In what follows fix $q \in (p, 2)$. Define $a_1 = a_{-1} = 0$ and for $n \in (\mathbb{N} \setminus \{1\}) \cup \{0\}$,

$$a_n = a_{-n} = \frac{(-1)^n}{2q}\binom{q}{n} = \frac{(-1)^n}{2q} \cdot \frac{q(q-1)\cdots(q-n+1)}{n!}. \tag{82}$$

Note that since $q \in (1, 2)$ the definition (82) implies that for $n \neq 1$ we have $a_n > 0$. Since we defined $a_{\pm 1}$ to be equal 0 it follows that $\{a_n\}_{n \in \mathbb{Z}} \subseteq [0, \infty)$. An application of Stirling's formula implies that as $n \to \infty$ we have

$$a_n = \frac{1}{2q}\binom{n-q-1}{n} \asymp \frac{1}{n^{q+1}}, \tag{83}$$

where the implicit constants depend only on q (and are easily estimated if so desired). Note in particular that since $q > p$, (83) implies that

$$\sum_{n \in \mathbb{Z}} a_n |n|^p < \infty, \tag{84}$$

and

$$\varphi(\theta) \coloneqq \sum_{n \in \mathbb{Z}} a_n e^{in\theta} \tag{85}$$

converges uniformly on $[-\pi, \pi]$. Moreover it is easy to compute $\varphi(\theta)$ explicitly:

$$\varphi(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} + \frac{1}{2q}\sum_{n=0}^{\infty} (-1)^n \binom{q}{n}\left(e^{in\theta} + e^{-in\theta}\right) = \cos\theta + \frac{\left(1 - e^{i\theta}\right)^q + \left(1 - e^{-i\theta}\right)^q}{2q}$$
$$= \cos\theta + \frac{2^{q/2}}{q}(1 - \cos\theta)^{q/2} \cos\left(\frac{q(\pi - \theta)}{2}\right) \in \mathbb{R}. \tag{86}$$

An immediate consequence of (86) is that $\sum_{n\in\mathbb{Z}} a_n = \varphi(0) = 1$. Thus we can define a symmetric random variable X on $\mathbb{Z}$ by $\mathbb{P}[X = n] = a_n$. With this notation (84) becomes $\mathbb{E}|X|^p < \infty$. Another corollary of the identity (86) is that there exits $\varepsilon = \varepsilon(q) \in (0,1)$ and $c = c(q) > 0$ such that for every $\theta \in [-\varepsilon, \varepsilon]$ we have $\mathbb{E}\left[e^{i\theta X}\right] = \varphi(\theta) \in \left[e^{-2c|\theta|^q}, e^{-c|\theta|^q}\right]$. Note also that since for every $\theta \neq 0$ we have $|\varphi(\theta)| < \sum_{n\in\mathbb{Z}} a_n = 1$ there exists some $\delta = \delta(q) \in (0,1)$ such that for every $\theta \in [-\pi, -\varepsilon] \cup [\varepsilon, \pi]$ we have $|\varphi(\theta)| \leq 1 - \delta$.

Now let $X_1, X_2, \ldots$ be i.i.d. copies of X. Denote $S_n = X_1 + \cdots + X_n$. Then the above bounds imply that

$$\begin{aligned}\mathbb{P}[S_n = 0] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\mathbb{E}\left[e^{i\theta S_n}\right]\right)d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(\theta)^n d\theta \\ &\in \frac{1}{2\pi}\left[\int_{-\varepsilon}^{\varepsilon} e^{-2cn|\theta|^q}d\theta - \int_{[-\pi,-\varepsilon]\cup[\varepsilon,\pi]}(1-\delta)^n d\theta,\ \int_{-\varepsilon}^{\varepsilon} e^{-cn|\theta|^q}d\theta + \int_{[-\pi,-\varepsilon]\cup[\varepsilon,\pi]}(1-\delta)^n d\theta\right].\end{aligned}$$

This implies that as $n \to \infty$ we have

$$\mathbb{P}[S_n = 0] \asymp \frac{1}{n^{1/q}}. \tag{87}$$

Substituting (87) into (79) we see that

$$\mathbb{E}\left[\left|S_{[0,n]}\right|\right] = \mathbb{E}\left[|\{S_0, \ldots, S_n\}|\right] \gtrsim n^{1/q}. \tag{88}$$

In combination with (79) it follows that

$$\mathbb{E}[|S_n|] \gtrsim n^{1/q}. \tag{89}$$

Additionally, if we let R_n be as in Lemma 6.3 (for the particular symmetric walk S_n studied here) then by plugging (87) into (78) we get the bound

$$\mathbb{E}\left[R_n^{1/q}\right] \gtrsim n^{(q-1)/q^2}. \tag{90}$$

6.3 The induced walk on $\mathbb{Z} \wr \mathbb{Z}$ and the lower bound on $\beta_p^*(\mathbb{Z} \wr \mathbb{Z})$

In this section we will conclude the proof of Theorem 6.1. Modulo the previous preparatory sections, the argument below closely follows the proof of Theorem 6.2 in [47].

For every $n_1, n_2, n_3 \in \mathbb{Z}$ define $f_{n_1,n_2}^{n_3} : \mathbb{Z} \to \mathbb{Z}$ by

$$f_{n_1,n_2}^{n_3}(k) := n_1\mathbf{1}_{\{0\}} + n_2\mathbf{1}_{\{n_3\}} = \begin{cases} n_1 & \text{if } k = 0, \\ n_2 & \text{if } k = n_3 \ \wedge\ n_3 \neq 0, \\ n_1 + n_2 & \text{if } k = 0 = n_3, \\ 0 & \text{otherwise.}\end{cases}$$

Denote

$$x_{n_1,n_2,n_3} := \left(f_{n_1,n_2}^{n_3}, n_3\right) \in \mathbb{Z} \wr \mathbb{Z}. \tag{91}$$

To better understand the meaning of this group element, note that for every $(g, \ell) \in \mathbb{Z}\wr\mathbb{Z}$ we have $(g,\ell)x_{n_1,n_2,n_3} = (h, \ell + n_3)$ where

$$h(k) = \begin{cases} g(k) + n_1 & \text{if } k = \ell, \\ g(k) + n_2 & \text{if } k = \ell + n_3 \ \wedge\ n_3 \neq 0, \\ g(\ell) + n_1 + n_2 & \text{if } k = \ell \ \wedge\ n_3 = 0, \\ g(k) & \text{otherwise.}\end{cases}$$

Thus if we let μ be the symmetric probability measure on $\mathbb{Z} \wr \mathbb{Z}$ given by $\mu(\{x_{n_1,n_2,n_3}\}) = a_{n_1} a_{n_2} a_{n_3}$, where $\{a_n\}_{n\in\mathbb{Z}}$ are the coefficients from Section 6.2, then the walk $\{W_t^\mu\}_{t=0}^\infty$ can be described in words as follows: start at $(\mathbf{0}, 0)$ and at each step choose three i.i.d. numbers $n_1, n_2, n_3 \in \mathbb{Z}$ distributed according to the random variable X from Section 6.2. Add n_1 to the current location of the lamplighter, move the lamplighter n_3 units and add n_2 to the new location of the lamplighter.

Write $W_t^\mu = (f_t, m_t)$. By the above description m_t has the same distribution as the walk S_t from Section 6.2. Fix $n \in \mathbb{N}$ and for $m \in \mathbb{Z}$ denote $T_m := |\{t \in \{0, \ldots, n\} : \ m_t = m\}|$. The above description of the walk W_t^μ ensures that conditioned on $\{T_m\}_{m\in\mathbb{Z}}$ and on "terminal point" m_n, if $k \in \mathbb{Z} \setminus \{0, m_n\}$ then $f_n(m)$ has the same distribution as S_{2T_m}, if $m \in \{0, m_n\}$ and $m_n \neq 0$ then $f_n(m)$ has the same distribution as $S_{\max\{2T_m - 1, 0\}}$, and if $m \in \{0, m_n\}$ and $m_n = 0$ then $f_n(m)$ has the same distribution as S_{2T_m}. Thus using (89) we see that

$$\mathbb{E}\left[|f_n(m)|\right] \gtrsim \mathbb{E}\left[T_m^{1/q}\right]. \tag{92}$$

Fix $m \in \mathbb{Z}$ and for $t \in \{0, \ldots, n\}$ define the event $A_t := \{m_t = m \ \wedge \ m \notin \{0, \ldots, m_{\ell-1}\}$. Note that conditioned on A_t the random variable T_m has the same distribution as R_{T_m}, where $\{R_k\}_{k=0}^n$ is as in (90). Hence,

$$\mathbb{E}\left[T_m^{1/q}\right] \geq \sum_{t=0}^{\lfloor n/2 \rfloor} \mathbb{P}(A_t) \cdot \mathbb{E}\left[T_m^{1/q} \middle| A_t\right] \gtrsim \sum_{t=0}^{\lfloor n/2 \rfloor} \mathbb{P}(A_t) \cdot n^{(q-1)/q^2} = n^{(q-1)/q^2} \mathbb{P}\left[m \in \{m_0, \ldots, m_{\lfloor n/2 \rfloor}\}\right]. \tag{93}$$

It follows that

$$\begin{aligned}\mathbb{E}\left[d_{\mathbb{Z}\wr\mathbb{Z}}\left(W_n^\mu, (\mathbf{0}, 0)\right)\right] &\gtrsim \sum_{m\in\mathbb{Z}} \mathbb{E}\left[|f_n(m)|\right] \overset{(92)}{\gtrsim} \sum_{m\in\mathbb{Z}} \mathbb{E}\left[T_m^{1/q}\right] \overset{(93)}{\gtrsim} n^{(q-1)/q^2} \sum_{m\in\mathbb{Z}} \mathbb{P}\left[m \in \{m_0, \ldots, m_{\lfloor n/2 \rfloor}\}\right] \\ &= n^{(q-1)/q^2} \mathbb{E}\left[\left|\{S_0, \ldots, S_{\lfloor n/2 \rfloor}\}\right|\right] \overset{(88)}{\gtrsim} n^{(q-1)/q^2} \cdot n^{1/q} = n^{(2q-1)/q^2}. \end{aligned} \tag{94}$$

On the other hand it follows from (84) that $\mathbb{E}\left[d_{\mathbb{Z}\wr\mathbb{Z}}\left(W_1^\mu, (\mathbf{0}, 0)\right)^p\right] < \infty$ so we deduce from the definition of $\beta_p^*(\mathbb{Z} \wr \mathbb{Z})$ that

$$\beta_p^*(\mathbb{Z} \wr \mathbb{Z}) \geq \frac{2q-1}{q^2}.$$

Letting $q \to p^+$ we deduce Theorem 6.1. □

Remark 6.5. The same argument as above actually shows that for every finitely generated group G and every $p \in (1, 2]$ we have

$$\beta_p^*(G \wr \mathbb{Z}) \geq \frac{1}{p} + \left(1 - \frac{1}{p}\right) \beta_p^*(G). \tag{95}$$

This implies Theorem 6.1 since the computations in Section 6.2 show that $\beta_p^*(\mathbb{Z}) \geq \frac{1}{p}$. Note of course that due to Theorem 5.1 we actually know that $\beta_p^*(\mathbb{Z}) = \frac{1}{p}$. We also observe that if H is a finitely generated group whose growth is at least quadratic then $\beta_p^*(G \wr H) = 1$. Indeed we have established the fact that $\beta_p^*(G \wr H) \leq 1$ in (62), while the lower bound follows from Theorem 6.1 in [47] which states that $\beta^*(G \wr H) = 1$, combined with the obvious fact that $\beta^*(G \wr H) \leq \beta_p^*(G \wr H)$. ◁

Remark 6.6. Define inductively $\mathbb{Z}_{(1)} = \mathbb{Z}$ and $\mathbb{Z}_{k+1} = \mathbb{Z}_{(k)} \wr \mathbb{Z}$. Then for $p \in (1, 2]$ we have $\beta_p^*(Z_{(1)}) = \frac{1}{p}$ and (95) implies that $\beta_p^*(\mathbb{Z}_{(k+1)}) \geq \frac{1}{p} + \left(1 - \frac{1}{p}\right) \beta_p^*(\mathbb{Z}_{(k)})$. It follows by induction that for all $k \in \mathbb{N}$ we have

$$\beta_p^*(\mathbb{Z}_{(k)}) \geq 1 - \left(1 - \frac{1}{p}\right)^k. \tag{96}$$

Note that $\alpha_p^*(\mathbb{Z}_{(1)}) = 1$ and by [57] $\alpha_p^*(C_2 \wr \mathbb{Z}) = 1$ (see also the different proof of this fact in [47]). Thus Corollary 2.2 implies that

$$\alpha_p^*(\mathbb{Z}_{(k+1)}) \geq \frac{p\alpha_p^*(\mathbb{Z}_{(k)})}{p\alpha_p^*(\mathbb{Z}_{(k)}) + p - 1}.$$

It follows by induction that

$$\alpha_p^*(\mathbb{Z}_{(k)}) \geq \frac{1}{p\left(1 - \left(1 - \frac{1}{p}\right)^k\right)}. \tag{97}$$

By combining (96) and (97) with Theorem 5.1 we see that

$$\alpha_p^*(\mathbb{Z}_{(k)}) = \frac{1}{p\left(1 - \left(1 - \frac{1}{p}\right)^k\right)} \quad \text{and} \quad \beta_p^*(\mathbb{Z}_{(k)}) = 1 - \left(1 - \frac{1}{p}\right)^k.$$

For $p \in (2, \infty)$ the same reasoning (using the fact that L_p has Markov type 2 [48]) shows that $\alpha_p^*(\mathbb{Z}_{(k)}) = \alpha_2^*(\mathbb{Z}_{(k)})$ and $\beta_p^*(\mathbb{Z}_{(k)}) = \beta_2^*(\mathbb{Z}_{(k)})$. ◁

7 A computation of $\beta_p^*((\mathbb{Z} \wr \mathbb{Z})_0, d_{\mathbb{Z}\wr\mathbb{Z}})$

The purpose of this section is to prove the following result:

Theorem 7.1. *Let G, H be infinite groups generated by the finite symmetric sets $S_G \subseteq G$ and $S_H \subseteq H$, respectively. Let $(G \wr H)_0 = \{(f, x) \in G \wr H : \ x = e_H\}$ be the zero section of $G \wr H$. Then for all $p \in [1, 2]$ we have*

$$\beta_p^*((G \wr H)_0, d_{G\wr H}) \geq \frac{2}{p+1}. \tag{98}$$

Specializing to the case $G = H = \mathbb{Z}$ we can apply Theorem 5.1 when ρ is the metric induced from $\mathbb{Z} \wr \mathbb{Z}$ on the amenable group $(Z \wr \mathbb{Z})_0$ to deduce that

$$\frac{p+1}{2p} \geq \frac{1}{p\beta_p^*((\mathbb{Z} \wr \mathbb{Z})_0, d_{\mathbb{Z}\wr\mathbb{Z}})} \geq \alpha_p^*((\mathbb{Z} \wr \mathbb{Z})_0, d_{\mathbb{Z}\wr\mathbb{Z}}) \overset{\text{(Thm. 4.1)}}{=} \frac{p+1}{2p}. \tag{99}$$

Thus in particular there is equality in (98) when $G = H = \mathbb{Z}$.

Proof of Theorem 7.1. For every $k \in \mathbb{N}$ let $g_k \in G$ and $h_k \in H$ be elements satisfying $d_G(g_k, e_G) = k$ and $d_H(h_k, e_H) = k$. Such elements exists since G, H are assumed to be infinite. We shall write below $h_k^{-1} = h_{-k}$. Fix an even integer $n \in \mathbb{N}$. For every $k \in [1, n/2] \cup [-n/2, -1]$ and $\varepsilon, \delta \in \{-1, 1\}$ define $f_{k,\varepsilon,\delta} : H \to G$ by

$$f_{k,\varepsilon,\delta}(x) \coloneqq \begin{cases} g_n^\varepsilon & \text{if } x = e_H, \\ g_n^\delta & \text{if } x = h_k, \\ e_G & \text{otherwise.} \end{cases}$$

Let μ_n be the symmetric measure on $(G \wr H)_0$ which is uniformly distributed on the $4n$ elements

$$\{(f_{k,\varepsilon,\delta}, e_H) : \ k \in [1, n/2] \cup [-n/2, -1], \ \varepsilon, \delta \in \{-1, 1\}\} \subseteq (G \wr H)_0.$$

Then the following point-wise inequality holds true:

$$0 < d_{G\wr H}\left(W_1^{\mu_n}, e_{G\wr H}\right) \leq 3n. \tag{100}$$

It follows in particular that the conditions in (61) hold true for the sequence $\{\mu_n\}_{n=1}^{\infty}$. Moreover, for each $k \in [1, n/2] \cup [-n/2, -1]$ the probability that in exactly one of the first n steps of the walk $\left\{W_t^{\mu_n}\right\}_{t=0}^{\infty}$ the h_k coordinate was altered is $\left(1 - \frac{1}{n}\right)^{n-1} > \frac{1}{3}$. Therefore the expected number of of coordinates h_k that were altered exactly once is greater than $n/3$. Each such coordinate contributes n to the distance between $W_n^{\mu_n}$ and $e_{G\wr H}$. Hence

$$\mathbb{E}_{\mu_n}\left[d_{G\wr H}\left(W_n^{\mu_n}, e_{G\wr H}\right)\right] \geq \frac{n^2}{3}. \tag{101}$$

It follows from the definition (60) that

$$\beta_p^*\left((G \wr H)_0, d_{G\wr H}\right) \geq \beta_p^*\left(\{\mu_n\}_{n=1}^{\infty}, d_{G\wr H}\right) \overset{(100)\wedge(101)}{\geq} \limsup_{n\to\infty} \frac{\log(n^2/3)}{\log(3^p n^{1+p})} = \frac{2}{p+1},$$

as required. □

8 An application to the Lipschitz extension problem

The purpose of this section is to prove the following theorem:

Theorem 8.1. *There exists a Lipschitz function $F : (\mathbb{Z} \wr \mathbb{Z})_0 \to L_2$ which cannot be extended to a Lipschitz function from $\mathbb{Z} \wr \mathbb{Z}$ to L_2.*

The key step in the proof of Theorem 8.1 is the use of the function constructed in Theorem 4.1. The other fact that we will need is Lemma 8.2 below. Recall that a Markov chain $\{Z_t\}_{t=0}^{\infty}$ is called a symmetric Markov chain on $\mathbb{Z} \wr \mathbb{Z}$ if there exists an N-point subset $\{z_1, \ldots, z_N\} \subseteq \mathbb{Z} \wr \mathbb{Z}$ and an $N \times N$ symmetric stochastic matrix $A = (a_{ij})$ such that $\mathbb{P}[Z_0 = z_i] = \frac{1}{N}$ for all $i \in \{1, \ldots, N\}$ and for all $i, j \in \{1, \ldots, N\}$ and $t \in \mathbb{N}$ we have $\mathbb{P}[Z_{t+1} = x_j | Z_t = z_i] = a_{ij}$.

The following lemma asserts that there is a fast-diverging symmetric Markov chain on $\mathbb{Z} \wr \mathbb{Z}$ which remains within a relatively narrow tubular neighborhood around the zero section $(\mathbb{Z} \wr \mathbb{Z})_0$.

Lemma 8.2. *For every $\varepsilon > 0$ there exists an integer $n_0(\varepsilon) \in N$ such that for all $n \geq n_0(\varepsilon)$ there is a symmetric Markov chain $\{Z_t\}_{t=0}^{\infty}$ on $\mathbb{Z} \wr \mathbb{Z}$ which satisfies the following conditions:*

1. *$d_{\mathbb{Z}\wr\mathbb{Z}}(Z_1, Z_0) \leq 4$ (point-wise),*
2. *$d_{\mathbb{Z}\wr\mathbb{Z}}\left(Z_t, (\mathbb{Z} \wr \mathbb{Z})_0\right) \leq 2n^{(1+\varepsilon)/2}$ for all $t \geq 0$ (point-wise),*
3. *$\mathbb{E}\left[d_{\mathbb{Z}\wr\mathbb{Z}}\left(Z_n, Z_0\right)\right] \gtrsim n^{3/4}$.*

Assuming Lemma 8.2 for the moment we shall prove Theorem 8.1.

Proof of Theorem 8.1. Fix $\varepsilon \in (0, 1/11)$. By Theorem 4.1 there exists a function $F : (\mathbb{Z} \wr \mathbb{Z})_0 \to L_2$ and $c = c(\varepsilon) > 0$ such that $\|F\|_{\mathrm{Lip}} = 1$ and for every $x, y \in (\mathbb{Z} \wr \mathbb{Z})_0$ we have

$$\|F(x) - F(y)\|_2 \geq c d_{\mathbb{Z}\wr\mathbb{Z}}(x, y)^{(3-\varepsilon)/4}. \tag{102}$$

Assume for the sake of contradiction that there exists a function $\widetilde{F} : \mathbb{Z} \wr \mathbb{Z} \to L_2$ such that $\widetilde{F} \restriction_{(\mathbb{Z}\wr\mathbb{Z})_0} = F$ and $\left\|\widetilde{F}\right\|_{\mathrm{Lip}} = L < \infty$.

Let $n_0(\varepsilon)$ and $\{Z_t\}_{t=0}^{\infty}$ be as in Lemma 8.2 and fix $n \geq n_0(\varepsilon)$. Write $Z_t = (f_t, k_t)$ and define $Z_t^0 = (f_t, 0) \in (\mathbb{Z} \wr \mathbb{Z})_0$. The second assertion of Lemma 8.2 implies that for all $t \geq 0$ we have

$$d_{\mathbb{Z}\wr\mathbb{Z}}\left(Z_t, Z_t^0\right) \leq 2n^{(1+\varepsilon)/2}. \tag{103}$$

Using the Markov type 2 property of L_2 [9] (with constant 1) and the first assertion of Lemma 8.2 we see that:

$$\mathbb{E}\left[\left\|\widetilde{F}(Z_n) - \widetilde{F}(Z_0)\right\|_2^2\right] \leq n\mathbb{E}\left[\left\|\widetilde{F}(Z_1) - \widetilde{F}(Z_0)\right\|_2^2\right] \leq nL^2\mathbb{E}\left[d_{\mathbb{Z}\wr\mathbb{Z}}(Z_1, Z_0)^2\right] \leq 16nL^2. \tag{104}$$

Note the following elementary corollary of the triangle inequality which holds for every metric space (X, d), every $p \geq 1$ and every $a_1, a_2, b_1, b_2 \in X$:

$$d(a_1, b_1)^p \geq \frac{1}{3^{p-1}} d(a_2, b_2)^p - d(a_1, a_2)^p - d(b_1, b_2)^p. \tag{105}$$

Hence we have the following point-wise inequality:

$$\begin{aligned}
\left\|\widetilde{F}(Z_n) - \widetilde{F}(Z_0)\right\|_2^2 &\overset{(105)}{\geq} \frac{1}{3}\left\|F\left(Z_n^0\right) - F\left(Z_0^0\right)\right\|_2^2 - \left\|\widetilde{F}(Z_n) - \widetilde{F}\left(Z_n^0\right)\right\|_2^2 - \left\|\widetilde{F}(Z_0) - \widetilde{F}\left(Z_0^0\right)\right\|_2^2 \\
&\overset{(102)}{\geq} \frac{c^2}{3} d_{\mathbb{Z}\wr\mathbb{Z}}\left(Z_n^0, Z_0^0\right)^{(3-\varepsilon)/2} - L^2 d_{\mathbb{Z}\wr\mathbb{Z}}\left(Z_n, Z_n^0\right)^2 - L^2 d_{\mathbb{Z}\wr\mathbb{Z}}\left(Z_0, Z_0^0\right)^2 \\
&\overset{(105)\wedge(103)}{\geq} \frac{c^2}{3}\left(\frac{1}{3} d_{\mathbb{Z}\wr\mathbb{Z}}(Z_n, Z_0)^{(3-\varepsilon)/2} - d_{\mathbb{Z}\wr\mathbb{Z}}\left(Z_n, Z_n^0\right)^{(3-\varepsilon)/2} - d_{\mathbb{Z}\wr\mathbb{Z}}\left(Z_0, Z_0^0\right)^{(3-\varepsilon)/2}\right) - 8L^2 n^{1+\varepsilon} \\
&\overset{(103)}{\geq} \frac{c^2}{9} d_{\mathbb{Z}\wr\mathbb{Z}}(Z_n, Z_0)^{(3-\varepsilon)/2} - 10L^2 n^{1+\varepsilon}.
\end{aligned} \tag{106}$$

Taking expectation in (106) and using the third assertion of Lemma 8.2 we see that:

$$\begin{aligned}
16nL^2 \geq \mathbb{E}\left[\left\|\widetilde{F}(Z_n) - \widetilde{F}(Z_0)\right\|_2^2\right] &\geq \frac{c^2}{9}\mathbb{E}\left[d_{\mathbb{Z}\wr\mathbb{Z}}(Z_n, Z_0)^{(3-\varepsilon)/2}\right] - 10L^2 n^{1+\varepsilon} \\
&\geq \left(\mathbb{E}\left[d_{\mathbb{Z}\wr\mathbb{Z}}(Z_n, Z_0)\right]\right)^{(3-\varepsilon)/2} - 10L^2 n^{1+\varepsilon} \gtrsim n^{3(3-\varepsilon)/8} - 10L^2 n^{1+\varepsilon},
\end{aligned}$$

which is a contradiction for large enough n since the assumption $\varepsilon < 1/11$ implies that $\frac{3(3-\varepsilon)}{8} > 1 + \varepsilon$. □

It remains to prove Lemma 8.2.

Proof of Lemma 8.2. Fix an integer $n \in \mathbb{N}$ and $\varepsilon \in (0, 1/4)$. Define two subsets $U_n, V_n \subseteq \mathbb{Z} \wr \mathbb{Z}$ by

$$U_n := \left\{(f, k) \in \mathbb{Z} \wr \mathbb{Z} : \ \mathbf{supp}(f) \subseteq [-n, n]\,,\ |k| \leq 2n^{(1+\varepsilon)/2},\ |f(\ell)| \leq n^2 \ \ \forall\ \ell \in \mathbb{Z}\right\},$$

$$V_n := \left\{(f, k) \in \mathbb{Z} \wr \mathbb{Z} : \ \mathbf{supp}(f) \subseteq [-n, n]\,,\ |k| \leq n^{(1+\varepsilon)/2}, |f(\ell)| \leq n^2 - 2n \ \forall\ \ell \in \mathbb{Z}\right\}.$$

Then $|U_n| \asymp \left(2n^2+1\right)^{2n+1}\left(4n^{(1+\varepsilon)/2}+1\right)$ and $|V_n| \asymp \left(2n^2-4n+1\right)^{2n+1}\left(2n^{(1+\varepsilon)/2}+1\right)$ so that

$$\frac{|V_n|}{|U_n|} \gtrsim 1. \tag{107}$$

Consider the set $S = \{x_{n_1,n_2,n_3} : \ n_1,n_2,n_3 \in \{-1,1\}\}$, where x_{n_1,n_2,n_3} are as defined in (91). Then S is a symmetric generating set of $\mathbb{Z} \wr \mathbb{Z}$ consisting of 8 elements. Let $g_1, g_2, \ldots$ be i.i.d. elements of $\mathbb{Z} \wr \mathbb{Z}$ which are uniformly distributed over S and denote $W_m := g_1 \cdots g_m = (f_m, k_m)$. Then by construction the sequence $\{k_m\}_{m=1}^{\infty}$ has the same distribution as the standard random walk on $\mathbb{Z}$, i.e., the same distribution as $\{S_m = \varepsilon_1 + \cdots + \varepsilon_m\}_{m=1}^{\infty}$ where $\varepsilon_1, \varepsilon_2, \ldots$ are i.i.d. Bernoulli random variables (this fact was explained in greater generality in Section 6.3). Also, as shown by Èrschler [28], we have

$$\mathbb{E}\left[d_{\mathbb{Z}\wr\mathbb{Z}}\left(W_n, (\mathbf{0},0)\right)\right] \geq cn^{3/4}, \tag{108}$$

where $c > 0$ is a universal constant. Note that since $d_{\mathbb{Z}\wr\mathbb{Z}}(x_{n_1,n_2,n_3}, (\mathbf{0},0)) \leq 4$ for every $n_1, n_2, n_3 \in \{-1,1\}$ we have point-wise bound

$$d_{\mathbb{Z}\wr\mathbb{Z}}\left(W_n, (\mathbf{0},0)\right) \leq 4n. \tag{109}$$

Now let Z_0 be uniformly distributed over U_n and independent of $\{g_i\}_{i=1}^{\infty}$. For $t \in \mathbb{N}$ define

$$Z_t := \begin{cases} Z_{t-1}g_t & \text{if } Z_{t-1}g_t \in U_n, \\ Z_{t-1} & \text{otherwise.} \end{cases}$$

The first two assertions of Lemma 8.2 hold true by construction. It remains to establish the third assertion of Lemma 8.2.

Consider the events $\mathcal{E} := \{Z_0 \in V_n\}$ and $\mathcal{F} := \left\{\max_{m\leq n}|k_m| \leq n^{(1+\varepsilon)/2}\right\}$. Note that if the event $\mathcal{E}\cap\mathcal{F}$ occurs then $Z_n = Z_0W_n$ since by design in this case $Z_0 \in V_n$ and therefore Z_0W_t cannot leave U_n for all $t \leq n$. It follows that

$$\begin{aligned}\mathbb{E}\left[d_{\mathbb{Z}\wr\mathbb{Z}}\left(Z_n, Z_0\right)\right] &\geq \mathbb{E}\left[d_{\mathbb{Z}\wr\mathbb{Z}}\left(W_n, (\mathbf{0},0)\right)\mathbf{1}_{\mathcal{E}\cap\mathcal{F}}\right] = \mathbb{P}\left[\mathcal{E}\right]\left(\mathbb{E}\left[d_{\mathbb{Z}\wr\mathbb{Z}}\left(W_n, (\mathbf{0},0)\right)\right] - \mathbb{E}\left[d_{\mathbb{Z}\wr\mathbb{Z}}\left(W_n, (\mathbf{0},0)\right)\mathbf{1}_{\mathcal{F}^c}\right]\right) \\ &\overset{(108)\wedge(109)}{\geq} \frac{|V_n|}{|U_n|}\left(cn^{3/4} - 4n(1-\mathbb{P}[\mathcal{F}])\right) \overset{(107)}{\gtrsim} cn^{3/4} - 4n(1-\mathbb{P}[\mathcal{F}]). \end{aligned} \tag{110}$$

For large enough n (depending on ε) we have

$$4n(1-\mathbb{P}[\mathcal{F}]) \leq \frac{c}{2}n^{3/4}, \tag{111}$$

since Doob's maximal inequality (see e.g. [27]) implies that for every $p > 1$ we have

$$1-\mathbb{P}[\mathcal{F}] = \mathbb{P}\left[\max_{m\leq n}|k_m| > n^{(1+\varepsilon)/2}\right] \leq \left(\frac{p}{p-1}\right)^p \frac{\mathbb{E}\left[|\varepsilon_1+\cdots+\varepsilon_n|^p\right]}{n^{p(1+\varepsilon)/2}} \overset{(\clubsuit)}{\lesssim} \left(\frac{p}{p-1}\right)^p \frac{(10np)^{p/2}}{n^{p(1+\varepsilon)/2}} = \frac{C(p)}{n^{p\varepsilon/2}}, \tag{112}$$

where in (♣) we used Khinchine's inequality (see e.g. [46]) and $C(p)$ depends only on p. Hence choosing p large enough in (112) (depending on ε) implies (111). Combining (110) and (111) implies that

$$\mathbb{E}\left[d_{\mathbb{Z}\wr\mathbb{Z}}\left(Z_n, Z_0\right)\right] \gtrsim n^{3/4},$$

which completes the proof of Lemma 8.2. □

9 Reduction to equivariant embeddings

Recall that a Banach space $(X, \|\cdot\|_X)$ is said to be finitely representable in a Banach space $(Y, \|\cdot\|_Y)$ if for every $\varepsilon > 0$ and every finite dimensional subspace $F \subseteq X$ there is a linear operator $T : F \to Y$ such that for every $x \in F$ we have $\|x\|_X \leq \|Tx\|_Y \leq (1+\varepsilon)\|x\|_X$.

Theorem 9.1. *Let Γ be an amenable group which is generated by a finite symmetric set $S \subseteq \Gamma$. Fix $p \geq 1$, two functions $\omega, \Omega : [0,\infty) \to [0,\infty)$ and a Banach space $(X, \|\cdot\|_X)$ such that there is a mapping $\psi : \Gamma \to X$ which satisfies:*

$$g, h \in \Gamma \implies \omega\left(d_\Gamma(g,h)\right) \leq \|\psi(g) - \psi(h)\|_X \leq \Omega\left(d_\Gamma(g,h)\right). \tag{113}$$

Then there exists a Banach space Y which is finitely representable in $\ell_p(X)$ and an equivariant mapping $\Psi : \Gamma \to Y$ such that

$$g, h \in \Gamma \implies \omega\left(d_\Gamma(g,h)\right) \leq \|\Psi(g) - \Psi(h)\|_Y \leq \Omega\left(d_\Gamma(g,h)\right). \tag{114}$$

Moreover, if $X = L_p(\mu)$ for some measure μ then Y can be taken to be isometric to L_p.

Note that as a special case of Theorem 9.1 we conclude that for every $p \geq 1$ if Γ is an amenable group then $\alpha_p^*(\Gamma) = \alpha_p^\#(\Gamma)$.

In what follows given a Banach space X we denote by $\mathrm{Isom}(X)$ the group of all linear isometric automorphims of X. We shall require the following lemma in the proof of Theorem 9.1:

Lemma 9.2. *Fix $p \in [1,\infty)$. Let G be a finitely generated group and $(\Omega, \mathcal{F}, \mu)$ be a measure space (thus Ω is a set, $\mathcal{F}$ is a σ algebra, and μ is a measure on $\mathcal{F}$). Assume that $\pi_0 : G \to \mathrm{Isom}\left(L_p(\mu, \mathcal{F})\right)$ is a homomorphism and that $f_0 \in Z^1(G, \pi_0)$ a 1-cocycle. Then there exists a homomorphism $\pi : G \to \mathrm{Isom}\left(L_p\right)$ and a 1-cocycle $f \in Z^1(G, \pi)$ such that $\|f(x)\|_{L_p} = \|f_0(x)\|_{L_p(\mu,\mathcal{F})}$ for all $x \in G$.*

Proof. Given $A \subseteq L_p(\mu, \mathcal{F})$ we denote as usual the smallest sub-σ algebra of $\mathcal{F}$ with respect to which all the elements of A are measurable by $\sigma(A)$. Define inductively a sequence $\{\mathcal{F}_n\}_{n=1}^\infty$ of sub-σ algebras of $\mathcal{F}$ and two sequences $\{U_n\}_{n=1}^\infty$, $\{V_n\}_{n=1}^\infty$ of linear subspaces of $L_p(\mu, \mathcal{F})$ as follows:

$$U_1 = \mathrm{span}\left(\bigcup_{x \in G} \pi_0(x) f_0(G)\right), \quad \mathcal{F}_1 \coloneqq \sigma\left(U_1\right), \quad V_1 = L_p(\mu, \mathcal{F}_1),$$

and inductively

$$U_{n+1} \coloneqq \mathrm{span}\left(\bigcup_{x \in G} \pi_0(x) V_n\right), \quad \mathcal{F}_{n+1} \coloneqq \sigma(U_{n+1}), \quad V_{n+1} = L_p(\mu, \mathcal{F}_{n+1}).$$

By construction for each $n \in \mathbb{N}$ we have $U_n \subseteq V_n \subseteq U_{n+1}$, the measure space $(\Omega, \mathcal{F}_n, \mu)$ is separable (since G is countable) and $\mathcal{F}_{n+1} \supseteq \mathcal{F}_n$. Let $\mathcal{F}_\infty$ be the σ-algebra generated by $\bigcup_{n=1}^\infty \mathcal{F}_n$. Note that for every $\varepsilon > 0$ and every $A \in \mathcal{F}_\infty$ there is some $n \in \mathbb{N}$ and $B \in \mathcal{F}_n$ such that $\mu(A \triangle B) \leq \varepsilon$ (this is because the set of all such $A \in \mathcal{F}$ forms a σ algebra, and therefore contains $\mathcal{F}_\infty$). By considering approximations by simple functions we deduce that

$$L_p(\mu, \mathcal{F}_\infty) = \overline{\bigcup_{n=1}^\infty V_n}, \tag{115}$$

where the closure is taken in $L_p(\mu,\mathcal{F})$. We claim that for each $x \in G$ we have $\pi_0(x) \in \mathrm{Isom}\left(L_p(\mu,\mathcal{F}_\infty)\right)$. Indeed, by construction $\pi_0(x)U_n = U_n$ for all $n \in \mathbb{N}$, and therefore $V_n \subseteq \pi_0(x)V_{n+1} \subseteq V_{n+2}$, which implies that $\pi_0(x)L_p(\mu,\mathcal{F}_\infty) = L_p(\mu,\mathcal{F}_\infty)$, as required. Note also that $f(G) \subseteq L_p(\mu,\mathcal{F}_\infty)$.

Since $L_p(\mu,\mathcal{F}_\infty)$ is separable it is isometric to one of the spaces:

$$L_p, \quad \ell_p, \quad ,\left\{\ell_p^n\right\}_{n=1}^\infty, \quad L_p \oplus \ell_p, \quad \left\{L_p \oplus \ell_p^n\right\}_{n=1}^\infty, \tag{116}$$

where the direct sums in (116) are ℓ_p direct sums (see [60]). In what follows we will slightly abuse notation by saying that $L_p(\mu,\mathcal{F}_\infty)$ is equal to one of the spaces listed in (116). The standard fact (116) follows from decomposing the measure $\mu \restriction_{\mathcal{F}_\infty}$ into a non-atomic part and a purely atomic part, and noting that the purely atomic part can contain at most countably many atoms while the non-atomic part is isomorphic to $[0,1]$ (equipped with the Lebesgue measure) by Lebesgue's isomorphism theorem (see [34]).

If $L_p(\mu,\mathcal{F}_\infty) = L_p$ then we are done, since we can take $\pi = \pi_0 \restriction_{L_p(\mu,\mathcal{F}_\infty)}$, so assume that $L_p(\mu,\mathcal{F}_\infty)$ is not isometric to L_p. We may therefore also assume that $p \neq 2$. If $L_p(\mu,\mathcal{F}_\infty) = \ell_p$ then by Lamperti's theorem [41] (see also Chapter 3 in [30]) for every $x \in G$, since $\pi_0(x)$ is a linear isometric automorphism of ℓ_p (and $p \neq 2$) we have $\pi_0(x)e_i = \theta_i^x e_{\tau^x(i)}$ for all $i \in \mathbb{N}$, where $\{e_i\}_{i=1}^\infty$ is the standard coordinate basis of ℓ_p, the function $\tau^x : \mathbb{N} \to \mathbb{N}$ is one-to-one and onto and $|\theta^x| \equiv 1$. Define $\pi(x) \in \mathrm{Isom}\left(L_p\right)$ and $f : G \to L_p$ by setting for $h \in L_p$ and $t \in \left[2^{-i}, 2^{-i+1}\right]$,

$$\pi(x)h(t) \coloneqq \theta_i^x h\left(2^{i-\tau^x(i)}t\right) \quad \text{and} \quad f(x)(t) = 2^{i/p}\langle f_0(x), e_i\rangle.$$

It is immediate to check that π, f satisfy the assertion of Lemma 9.2.

It remains to deal with the case $L_p(\mu,\mathcal{F}_\infty) = L_p \oplus \ell_p(S)$ where S is a nonempty set which is finite or countable. In this case we use Lamperti's theorem once more to deduce that for each $x \in G$ the linear isometric automorphism $\pi_0(x)$ maps disjoint functions to disjoint functions, and therefore it maps indicators of atoms to indicators of atoms. Hence $\pi_0(x)L_p = L_p$ and $\pi_0(x)\ell_p(S) = \ell_p(S)$. Now, as above $\pi_0(x) \restriction_{\ell_p(S)}$ must correspond (up to changes of sign) to a permutation of the coordinates. Hence, denoting the projection from $L_p \oplus \ell_p(S)$ onto L_p by Q, the same reasoning as above shows that there exists a homomorphism $\pi' : G \to L_p$ and $f' \in Z^1(G,\pi')$ such that for all $x \in G$ we have $\|f'(x)\|_{L_p} = \|f_0(x) - Qf_0(x)\|_{\ell_p(S)}$. It follows that if we define $\pi(x) \in \mathrm{Isom}\left(L_p \oplus L_p\right)$ by $\pi(x) = \pi_0(x) \restriction_{L_p} \oplus \pi'$ and $f : G \to L_p \oplus L_p$ by $f(x) = (Qf) \oplus f'$ then (using the fact that $L_p \oplus L_p$ is isometric to L_p) the assertion of Lemma 9.2 follows in this case as well. □

Proof of Theorem 9.1. Let $\{F_n\}_{n=0}^\infty$ be a Følner sequence for Γ and let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$. Define $\mathscr{M} : \ell_\infty(\Gamma) \to \mathbb{R}$ by

$$\mathscr{M}(f) = \lim_{\mathscr{U}} \frac{1}{|F_n|} \sum_{x \in F_n} f(x). \tag{117}$$

It follows immediately from the Følner condition that $\mathscr{M}$ is an invariant mean on Γ, i.e., a linear functional $\mathscr{M} : \ell_\infty(\Gamma) \to \mathbb{R}$ which maps the constant 1 function to 1, assigns non-negative values to non-negative functions and $\mathscr{M}(R_y f) = \mathscr{M}(f)$ for every $y \in \Gamma$, where $R_y f(x) = f(xy)$ (we refer to [58] for proofs and more information on this topic). Define a semi-norm $\|\cdot\|_{\mathscr{M},p}$ on $\ell_\infty(\Gamma, X)$ (the space of all X-valued bounded functions on Γ) by:

$$f \in \ell_\infty(\Gamma, X) \implies \|f\|_{\mathscr{M},p} \coloneqq \left(\mathscr{M}\left(\|f\|_X^p\right)\right)^{1/p}.$$

This is indeed a semi-norm since invariant means satisfy Hölder's inequality (see for example Lemma 2 on page 119 of Section III.3 in [26]). Hence if we let $W = \{f \in \ell_\infty(\Gamma, X) : \|f\|_{\mathscr{M},p} = 0\}$ then W is a linear subspace and $Y_0 \coloneqq \ell_\infty(\Gamma, X)/W$ is a normed space. Let Y be the completion of Y_0.

By a slight abuse of notation we denote for $y \in \Gamma$ and $f \in \ell_\infty(\Gamma, X)$, $R_y(f + W) \coloneqq R_y f + W$, which is a well defined linear isometric automorphism of Y_0 since $\|\cdot\|_{\mathscr{M},p}$ is R_y-invariant. Moreover R is an action of Γ on Y_0 by linear isometric automorphisms, and it therefore extends to such an action on Y as well.

Note that by virtue of the upper bound in (113) for every $g, x \in \Gamma$ we have $\|\psi(xg) - \psi(x)\|_X \leq \Omega\left(d_\Gamma(g, e_\Gamma)\right)$. Thus $R_g\psi - \psi \in \ell_\infty(\Gamma, X)$ and we can define $\Psi(g) \in Y$ by $\Psi(g) = (R_g\psi - \psi) + W$. Then $\Psi \in Z^1(\Gamma, R)$. Moreover $\Psi(e_\Gamma) = 0$ and for every $g_1, g_2 \in \Gamma$ we have

$$\|\Psi(g_1) - \Psi(g_2)\|_Y = \left(\mathscr{M}\left(\left\|R_{g_1}\psi - R_{g_2}\psi\right\|_X^p\right)\right)^{1/p} \overset{(113)}{\in} \left[\omega\left(d_\Gamma(g_1, g_2)\right), \Omega\left(d_\Gamma(g_1, g_2)\right)\right].$$

This establishes (114), so it remains to prove the required properties of Y, i.e., that it is finitely representable in $\ell_p(X)$ and that it is an $L_p(\nu)$ space if X is an $L_p(\mu)$ space.

Up to this point we did not use the fact that $\mathscr{M}$ was constructed as an ultralimit of averages along Følner sets as in (117) and we could have taken $\mathscr{M}$ to be any invariant mean on Γ. But now we will use the special structure of $\mathscr{M}$ to relate the space Y to a certain ultraproduct of Banach spaces. We do not know whether the properties required of Y hold true for general invariant means on Γ. We did not investigate this question since it is irrelevant for our purposes.

For each $n \geq 0$ let X_n be the Banach space X^{F_n} equipped with the norm:

$$\psi : F_n \to X \implies \|\psi\|_{X_n} = \left(\frac{1}{|F_n|}\sum_{h \in F_n}\|\psi(h)\|_X^p\right)^{1/p}.$$

Let $\widetilde{X}$ be the ultraproduct $\left(\prod_{n=0}^\infty X_n\right)_{\mathscr{U}}$. We briefly recall the definition of $\widetilde{X}$ for the sake of completeness (see [20, 21, 36] for more details and complete proofs of the ensuing claims). Let Z be the space $\left(\prod_{n=0}^\infty X_n\right)_\infty$, i.e., the space of all sequences $x = (x_0, x_1, x_2, \ldots)$ where $x_n \in X_n$ for each n and $\|x\|_Z \coloneqq \sup_{n\geq 0}\|x_n\|_{X_n} < \infty$. Let $N \subseteq Z$ be the subspace consisting of sequences $(x_n)_{n=0}^\infty$ for which $\lim_{\mathscr{U}}\|x_n\|_{X_n} = 0$. Then N is a closed subspace of Z and $\widetilde{X}$ is the quotient space Z/N, equipped with the usual quotient norm. We shall denote an element of $\widetilde{X}$, which is an equivalence class of elements in Z, by $[x_n]_{n=0}^\infty$. The norm on $\widetilde{X}$ is given by the concrete formula $\left\|[x_n]_{n=0}^\infty\right\|_{\widetilde{X}} = \lim_{\mathscr{U}}\|x_n\|_{X_n}$.

Since by construction each of the spaces X_n embeds isometrically into $\ell_p(X)$, by classical ulraproduct theory (see [36]) $\widetilde{X}$ is finitely representable in $\ell_p(X)$. Moreover, if $X = L_p(\mu)$ for some measure μ then, as shown in [20, 21, 36], $\widetilde{X} = L_p(\tau)$ for some measure τ.

Define $T : Y_0 \to \widetilde{X}$ by $T(f + W) = [f\restriction_{F_n}]_{n=0}^\infty$. Then by construction (and the definition of W) T is well defined and is an isometric embedding of Y_0 into $\widetilde{X}$. Hence also Y embeds isometrically into $\widetilde{X}$, and for ease of notation we will identify Y with $\overline{T(Y_0)} \subseteq \widetilde{X}$. It follows in particular that Y is finitely representable in $\ell_p(X)$.

It remains to show that if $X = L_p(\mu)$ then $Y = L_p(\nu)$ for some measure ν since once this is achieved we can apply Lemma 9.2 in order to replace Y by L_p. We know that in this case $\widetilde{X} = L_p(\tau)$ but we need to recall the lattice structure on $\widetilde{X}$ in order to proceed (since we do not know whether the action of Γ on Y extends

to an action of Γ on $\widetilde{X}$ by isometric linear automorphisms). Since each X_n is of the form $L_p(\mu_n)$ for some measure μ_n, the ultraproduct $\widetilde{X}$ has a Banach lattice structure whose positive cone is $\left\{[x_n]_{n=0}^{\infty} : \ x_n \geq 0 \ \forall n\right\}$ and $[x_n]_{n=0}^{\infty} \wedge [y_n]_{n=0}^{\infty} = [x_n \wedge y_n]_{n=0}^{\infty}$, $[x_n]_{n=0}^{\infty} \vee [y_n]_{n=0}^{\infty} = [x_n \vee y_n]_{n=0}^{\infty}$ (all of this is discussed in detail in [36]). The explicit embedding of Y_0 into $\widetilde{X}$ ensures that $x \wedge y, x \vee y \in Y_0$ for all $x, y \in Y_0$. Moreover if $x, y \in Y_0$ are disjoint, i.e., $|x| \wedge |y| = 0$, then $\|x + y\|_{\widetilde{X}} = \left(\|x\|_{\widetilde{X}}^p + \|y\|_{\widetilde{X}}^p\right)^{1/p}$. These identities pass to the closure Y of Y_0 (since, for example, we know that $\widetilde{X} = L_p(\tau)$ and therefore convergence in $\widetilde{X}$ implies almost everywhere convergence along a subsequence). This shows that the Banach space Y is an abstract L_p space, and therefore by Kakutani's representation theorem [39] (see also the presentation in [43]) $Y = L_p(\nu)$ for some measure ν. □

10 Open problems

We list below several of the many interesting open questions related to the computation of compression exponents.

Question 10.1. *Does $C_2 \wr \mathbb{Z}^2$ admit a bi-Lipschitz embedding into L_1?*

The significance of Question 10.1 was explained in the introduction. Since we know that $\alpha_1^*\left(C_2 \wr \mathbb{Z}^2\right) = 1$ the following question is more general then 10.1:

Question 10.2. *For which finitely generated groups G and $p \geq 1$ is $\alpha_p^*(G)$ attained?*

Somewhat less ambitiously than Question 10.2 one might ask for meaningful conditions on G which imply that $\alpha_p^*(G)$ is attained. As explained in Remark 3.2, this holds true if $p > 1$ and $G = C_2 \wr H$ where H is a finitely generated group with super-linear polynomial growth which admits a bi-Lipschitz embedding into L_p. In particular this holds true for $G = C_2 \wr \mathbb{Z}^2$ and $p > 1$. Note that not every group of polynomial growth H admits a bi-Lipschitz embedding into L_1, as shown by Cheeger and Kleiner [17] when H is the discrete Heisenberg group, i.e. the group of 3×3 matrices generated by the following symmetric set $S \subseteq GL_3(\mathbb{Q})$ and equipped with the associated word metric:

$$S = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Similarly to Question 7.1 in [47] one might ask the following question:

Question 10.3. *Is it true that for every finitely generated amenable group G and every $p \in [1, 2]$ we have $\alpha_p^*(G) = \frac{1}{p\beta_p^*(G)}$?*

It was shown in [3] the for every $\alpha \in [0, 1]$ there exists a finitely generated group G such that $\alpha_2^*(G) = \alpha$. Since there are only countably many finitely presented groups the set

$$\Omega_p^* \coloneqq \{\alpha_p^*(G) : \ G \text{ finitely presented}\} \subseteq [0, 1]$$

is at most countable for every $p \in [1, \infty)$ (though it seems to be unknown whether or not it is infinite). One can similarly define the set $\Omega_p^{\#}$ of possible equivariant compression exponents of finitely presented groups. Several restrictions on the relations between these sets follow from the following inequalities which hold for every finitely generated group G: for every $p \geq 1$ we have $\alpha_p^*(G) \geq \alpha_2^*(G)$ since L_2 embeds isometrically into L_p (see e.g. [60]). Similarly Lemma 2.3 in [47] states that $\alpha_p^{\#}(G) \geq \alpha_2^{\#}(G)$. Since L_q embeds isometrically into L_p for $1 \leq p \leq q \leq 2$ (see [59]) we also know that in this case $\alpha_p^*(G) \geq \alpha_q^*(G)$. For every $1 \leq p \leq q$ the metric space $\left(L_p, \|x-y\|_p^{p/q}\right)$ embeds isometrically into L_q (for $1 \leq p \leq q \leq 2$ this follows from [13, 59] and for the remaining range this is proved in Remark 5.10 of [45]). Hence if $p \in [1,2]$ and $p \leq q$ then $\alpha_q^*(G) \geq \max\left\{\frac{p}{q}, \frac{p}{2}\right\} \cdot \alpha_p^*(G)$ and if $2 \leq p \leq q$ then $\alpha_q^*(G) \geq \frac{p}{q}\alpha_p^*(G)$.

Question 10.4. *Evaluate the (at most countable) sets $\Omega_p^*, \Omega_p^{\#}$. Is Ω_p^* finite or infinite? How do the sets $\Omega_p^*, \Omega_p^{\#}$ vary with p? Is it true that $\Omega_p^* = \Omega_p^{\#}$?*

In this paper we computed $\alpha_p^*((\mathbb{Z} \wr \mathbb{Z})_0, d_{\mathbb{Z}\wr\mathbb{Z}})$. Note that the metric on the zero section $(\mathbb{Z}\wr\mathbb{Z})_0$ is not equivalent to a geodesic metric. This fact makes it meaningful to consider embeddings of $((\mathbb{Z} \wr \mathbb{Z})_0, d_{\mathbb{Z}\wr\mathbb{Z}})$ into L_p which are not necessarily Lipschitz, leading to the following question:

Question 10.5. *For every $\alpha_1 > 0$ evaluate the supremum over $\alpha_2 \geq 0$ such that there exists an embedding $f : (\mathbb{Z} \wr \mathbb{Z})_0 \to L_p$ which satisfies*

$$x, y \in (\mathbb{Z} \wr \mathbb{Z})_0 \implies c d_{\mathbb{Z}\wr\mathbb{Z}}(x,y)^{\alpha_2} \leq \|f(x) - f(y)\|_p \leq d_{\mathbb{Z}\wr\mathbb{Z}}(x,y)^{\alpha_1},$$

for some constant c.

We believe that the methods of the present paper can be used to answer Question 10.5 at least for some additional values of α_1 (we dealt here only with $\alpha_1 = 1$), but we did not pursue this research direction.

Question 10.6. *The present paper contributes methods for evaluating compression exponents of wreath products $G \wr H$ in terms of the compression exponents of G and H. This continues the lines of research that originated in [4] and continued in [31, 57, 54, 7, 47, 22]. It would be of great interest (and probably quite challenging) to design such methods for more general semi-direct products $G \rtimes H$.*

In Theorem 3.1 we computed $\alpha_p^*(C_2 \wr H)$ when H has polynomial growth. It seems likely that our methods yield non-trivial compression bounds also when H has intermediate growth. But, it would be of great interest to design methods which deal with the case when H has exponential growth. A simple example of this type is the group $C_2 \wr (C_2 \wr \mathbb{Z})$, for which we do not even know whether the Hilbert compression exponent is positive.

Question 10.7. *In our definition of L_p compression we considered embeddings into L_p because it contains isometrically all separable $L_p(\mu)$ spaces. Nevertheless, the embeddings that we construct take values in the sequence space ℓ_p. Does there exist a finitely generated group G for which $\alpha_p^*(G) \neq \alpha_{\ell_p}^*(G)$? Is the ℓ_p compression exponent of a net in L_p equal to 1? Note that for $p \neq 2$ the function space L_p does not admit a bi-Lipschitz embedding into the sequence space ℓ_p—this follows via a differentiation argument (see [11]) from the corresponding statement for linear isomorphic embeddings (see [52]).*

The subtlety between embeddings into L_p and embeddings into ℓ_p which is highlighted in Question 10.7 was pointed out to us by Marc Bourdon.

Subsequent developments

Since the posting of this article, several interesting results were posted on problems that were discussed here. In the breakthrough result [23], de Cornulier, Stalder and Valette proved that if G and H have the Haagerup property then so does $G \wr H$. In [42], Li proved that for $p \in [1, 2]$, if $\alpha_p^*(G) > 0$ and $\alpha_p^*(H) > 0$ then also $\alpha_p^*(G \wr H) > 0$. In [6], Austin constructed an example of a finitely generated amenable group G such that $\alpha_p^*(G) = 0$ for all $p \in [1, \infty)$.

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