Resonant leading term geometric optics expansions with boundary layers for quasilinear hyperbolic boundary problems

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Abstract
We construct and justify leading order weakly nonlinear geometric optics expansions for nonlinear hyperbolic initial value problems, including the compressible Euler equations. The technique of simultaneous Picard iteration is employed to show approximate solutions tend to the exact solutions in the small wavelength limit. Recent work [2] by Coulombel, Gues, and Williams studied the case of reflecting wave trains whose expansions involve only real phases. We treat generic boundary frequencies by incorporating into our expansions both real and nonreal phases. Nonreal phases introduce difficulties such as approximately solving complex transport equations and result in the addition of boundary layers with exponential decay. This also prevents us from doing an error analysis based on almost periodic profiles as in [2].

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1 Introduction
In this paper we consider quasilinear hyperbolic fixed boundary problems on the domain \( \mathbb{R}_+^{d+1} = \{ x = (x', x_d) = (t, y, x_d) : x_d \geq 0 \} \) for a class of equations which includes, in particular, the compressible Euler equations. The class consists of systems of the following form:

\[
\sum_{j=0}^{d} A_j(v_\epsilon) \partial_{x_j} v_\epsilon = f(v_\epsilon),
\]

(1.1)

\[
b(v_\epsilon) |_{x_d = 0} = g_0 + \epsilon G \left( \frac{x'}{\epsilon}, \frac{x' \cdot \beta}{\epsilon} \right),
\]

\[
v_\epsilon = u_0 \text{ in } t < 0.
\]

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Here, $A_0 = I$, and we assume $A_j \in C^\infty(\mathbb{R}^N, \mathbb{R}^{N^2})$, $f \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$, and $b \in C^\infty(\mathbb{R}^N, \mathbb{R}^p)$. For the boundary data, we take $G(x', \theta_0) \in C^\infty(\mathbb{R}^d \times T^1, \mathbb{R}^p)$ periodic in $\theta_0$ and supported in $\{x_0 \geq 0\}$ and the boundary frequency $\beta = (\beta_0, \ldots, \beta_{d-1}) \in \mathbb{R}^d \setminus 0$. We assume that $\beta$ belongs to the set of regular boundary frequencies\(^1\), a subset of $\mathbb{R}^d \setminus 0$ whose complement has zero measure.

The goal of this paper is to obtain qualitative information about the exact solution to (1.1) by explicitly constructing an approximate solution with a leading order geometric optics expansion that exhibits the qualitative information, and then showing that this approximate solution tends to the exact solution as $\epsilon \to 0$. In [2], this problem was solved under the stronger assumption that $\beta$ belongs to the hyperbolic region\(^2\). There, the authors constructed approximate solutions in the form of leading order expansions of highly oscillatory wavefronts with real phase functions. Our expansions include similar wavefronts with real phases, but in order to handle more general $\beta$, we are required to add to our expansions terms of a different kind, which we call elliptic boundary layers. These are highly oscillatory with envelopes of rapid exponential decay, the result of incorporating nonreal phase functions into our construction. Indeed, working with nonreal phase functions is forced on us in the case that $\beta$ lies outside of the hyperbolic region. It appears that this paper is the first to rigorously justify leading order expansions involving multiple real and nonreal phase functions, and thus both hyperbolic and elliptic profiles, in quasilinear hyperbolic boundary problems.

### 1.1 The statement of the main theorem and the form of the approximate solution

The precise statement of the main theorem is given in this section and is followed by a formal construction of the approximate solution to aid in interpreting the theorem and discussing its consequences. Observing the features of the approximate solution gives an idea of the qualitative information we obtain in the paper on the exact solution.

The system (1.1) is first reformulated in terms of a perturbation of a constant state solution, and the theorem is stated in terms of the solution of the resulting system. We set $v_\epsilon = u_0 + \epsilon u_\epsilon$, a perturbation of a constant state $u_0$ with $f(u_0) = 0$ and $b(u_0) = g_0$, and we translate (1.1) to the following problem for $u_\epsilon$ (where the $A_j$ have been altered accordingly):

\[
\begin{align*}
(a) \ P(\epsilon u_\epsilon, \partial_x)u_\epsilon & := \sum_{j=0}^d A_j(\epsilon u_\epsilon)\partial_{x_j} u_\epsilon = \mathcal{F}(\epsilon u_\epsilon)u_\epsilon, \\
(b) \ B(\epsilon u_\epsilon)u_\epsilon|_{x_\epsilon = 0} & = G\left( x', \frac{\beta \cdot x'}{\epsilon} \right), \\
(c) \ u_\epsilon & = 0 \mbox{ in } t < 0.
\end{align*}
\]

where $B(v)$ and $\mathcal{F}(v)$ denote the $p \times N$ and $N \times N$ real matrices, $C^\infty$ in $v$, defined by

\[
B(v)v = b(u_0 + v) - b(u_0), \quad \mathcal{F}(v)v = f(u_0 + v).
\]

The following theorem verifies that the approximate solution is in fact close to the exact solution $u_\epsilon(x)$ for small $\epsilon$. Our proposed approximate solution, the leading order geometric optics expansion, is denoted by $u_\epsilon^0(x)$.

**Theorem 1.1.** Suppose the assumptions hold that $L(\partial_x)$ is hyperbolic with characteristics of constant multiplicity, the boundary $\{x_\epsilon = 0\}$ is noncharacteristic, $(L(\partial_x), B(0))$ satisfies the uniform stability condition, and $\beta$ is a regular boundary frequency (i.e. Assumptions 2.1, 2.2, 2.5, and 2.7 hold, resp.) Then given the exact solution $u_\epsilon \in C^1((\infty, T] \times \mathbb{R}^d_\epsilon)$ of (1.2), defined on the time interval $(-\infty, T]$ independent of $\epsilon$, for $u_\epsilon^0$ as in (1.14), where $v$ and the $\sigma_{m,k}$ satisfy\(^3\) the profile equations, defined on a time interval $(-\infty, T']$

---

\(^1\)See Assumption 2.7. In particular, restricting $\beta$ to the set of regular boundary frequencies prevents the case of glancing $\beta$.

\(^2\)For a comprehensive discussion and treatment of the case of glancing $\beta$ for the semilinear problem, we direct the reader to [12].

\(^3\)It is worth noting that the hyperbolic region is not generally a set of full measure, typically having a cone (though possibly the empty set) as its boundary. For a description, see Remark 2.9 (iii).

\(^4\)The profile equations are not solved exactly. Here we refer to a kind of approximate solution whose error must satisfy the hypotheses of Proposition 4.6.
independent of $\epsilon$, we have
\begin{equation}
|u_\epsilon(x) - u_\epsilon^0(x)|_{L^\infty((-\infty, T_0) \times \mathbb{R}^d)} \to 0 \text{ as } \epsilon \to 0,
\end{equation}
where $T_0$ is the minimum of $T$ and $T'$.

To begin with, this kind of result requires the existence of the exact solution $u_\epsilon$ on a time interval independent of $\epsilon$, a result which is proven in [13] by M. Williams with the use of the singular system. First we will say a few words about this to provide context, and then we will interpret specific consequences of the main theorem by discussing the form of $u_\epsilon^0$. It is the features of $u_\epsilon^0$ that inform us on the behavior of $u_\epsilon$, by the theorem.

The singular system

Despite the fact that we need to establish an existence time $T$ of the exact solution $u_\epsilon$ to (1.2) which holds uniformly for small $\epsilon$, applying the standard theory to the problem (1.2) yields existence times $T_\epsilon$ which shrink to zero since the Sobolev norms of the initial data blow up in the limit $\epsilon \to 0$. To get estimates uniform in $\epsilon$, one may use a reformulation of the system (1.2) known as the singular system, which is also used in [2]. The singular system is obtained by recasting (1.2) in terms of an unknown $U_\epsilon(x, \theta_0)$ periodic in $\theta_0$ such that the solution of (1.2) is formed by making the substitution
\begin{equation}
U_\epsilon(x) = U_\epsilon \left( x, \frac{\beta \cdot x'}{\epsilon} \right).
\end{equation}

The singular system is written in the form
\begin{equation}
\begin{aligned}
(a) \quad & \frac{\partial_x U_\epsilon}{\epsilon} + \sum_{j=0}^{d-1} A_j(U_\epsilon) \left( \partial_x \beta_j + \frac{\beta \partial_\theta_0}{\epsilon} \right) U_\epsilon = F(\epsilon U_\epsilon) U_\epsilon, \\
(b) \quad & B(\epsilon U_\epsilon)(U_\epsilon)_{|x_d=0} = G(x', \theta_0), \\
(c) \quad & U_\epsilon = 0 \text{ in } t < 0,
\end{aligned}
\end{equation}
where $F = A_d^{-1} F$. In [13], the solution is constructed and the necessary estimates uniform in $\epsilon$ for $U_\epsilon$ are proven, estimates which are crucial to our analysis as well as the analysis of [2]. Under the hypotheses of Theorem 4.7, the existence of the unique solution $U_\epsilon$ is shown in [13] in the space\footnote{The subscript $T$ is used to indicate the corresponding function space with time interval $(-\infty, T]$ instead of $\mathbb{R}$.}
\begin{equation}
E_T^u = C(x_d, H^s_T(x', \theta_0)) \cap L^2(x_d, H^{s+1}_T(x', \theta_0)),
\end{equation}
implying the existence of the solution $u_\epsilon$ to (1.2) in the space $C^1((-\infty, T] \times \mathbb{R}^d)$, on a fixed time interval independent of $\epsilon$ in particular, and so we are able to make sense of the limit which is claimed in the main theorem. The singular system formulation is also used in our proof of the main theorem. We actually prove a stronger result, Theorem 4.7, showing an approximate solution of the singular system tends to $U_\epsilon$ in $E_T^{s-1}$ as $\epsilon \to 0$, and the main theorem follows as a corollary.

The form of the approximate solution

One key feature of the approximate solution is the manner in which it oscillates with linear phase functions. To explain this more precisely, we first point out that the solution restricted to the boundary $u_\epsilon|_{x_d=0}$ oscillates with phase $\phi_0(x')$, where
\begin{equation}
\phi_0(x') := \tau t + \eta \cdot y = \beta \cdot x'.
\end{equation}
Note the appearance of $\phi_0(x')$ in the boundary data (1.2)(b). This results in oscillations of the solution in the whole domain $\mathbb{R}^{d+1}_+$. These oscillations are associated to linear phases $\phi_m(x)$ which have trace on the boundary equal to $\phi_0(x')$ and which are characteristic for the operator $L(\partial_x)$, defined by
\begin{equation}
L(\partial_x) := P(0, \partial_x) = \partial_t + \sum_{j=1}^{d} A_j(0) \partial_{x_j}.
\end{equation}
To find the characteristic phases, we consider the matrix

$$
\frac{1}{i} A(\tau, \eta) = -A_d^{-1}(0) \left( \tau I + \sum_{j=1}^{d-1} \eta_j A_j(0) \right).
$$

Let $\omega_m$, $m = 1, \ldots, M$ denote the distinct eigenvalues of $\frac{1}{i} A(\tau, \eta) = \frac{1}{i} \tau A(\beta)$. We suppose that each $\omega_m$ is a semisimple eigenvalue with multiplicity denoted by $\mu_m$. Thus, the correct phase functions are

$$
\phi_m(x) := \phi_0(x') + \omega_m x_d = (\beta, \omega_m) \cdot x, \quad m = 1, \ldots, M.
$$

We remark that the statement that the solution oscillates with phases $\phi_m$ is a formal one. In particular, this is not precisely the correct statement for a phase which is nonreal, i.e. in the case that $\omega_m$ is nonreal. However, we are still able to handle these cases. In fact, while in [2], restricting to hyperbolic $\beta$ led to all the $\omega_m$ being real, by allowing for $\beta$ outside the hyperbolic region in our study, we allow for nonreal $\omega_m$, and so to handle these cases is the main contribution of the paper.

Now we turn our attention to the natural basis in which to write down our approximate solution, beginning with the following lemma, proved in [1]. Given Assumption 2.1, this provides a decomposition which is key to the construction of our expansion and will serve us later in the construction of the projectors which are used in the derivation of the profile equations.

**Lemma 1.2.** The space $\mathbb{C}^N$ admits the decomposition:

$$
\mathbb{C}^N = \oplus_{m=1}^{M} \text{Ker } L(d\phi_m)
$$

and for each $m$ such that $\omega_m$ is real, the associated vector space in (1.12) admits a basis of real vectors. If we let $P_1, \ldots, P_M$ denote the projectors associated with the decomposition (1.12), then for all $m = 1, \ldots, M$, there holds $\text{Im } A_d^{-1} L(d\phi_m) = \text{Ker } P_m$.

For each $m = 1, \ldots, M$, we choose a basis of $\text{Ker } L(d\phi_m)$:

$$
\{r_{m, k}, k = 1, \ldots, \mu_m, \}
$$

where for real $\omega_m$, we take a basis of real $r_{m, k}$. Now we are in a position to give the ansatz for our approximate solution to (1.2).

$$
u^a (x) = \varphi(x) + \sum_{m=1}^{M} \sum_{k=1}^{\mu_m} \sigma_{m, k} \left( x, \frac{\phi_m(x)}{\epsilon} \right) r_{m, k},
$$

where $\varphi(x)$ and the $\sigma_{m, k}(x, \theta_m)$ are $C^1$ functions defined for $x \in \mathbb{R}^d$ and, at least initially, $\theta_m$ on the real line. Later, some of these will be extended (in the domain for $\theta_m$) to one of the half complex planes $\{ \text{Im } z \geq 0 \}$, $\{ \text{Im } z \leq 0 \}$. Additionally, we require that each of the $\sigma_{m, k}(x, \theta_m)$ is periodic in $\theta_m$ with mean 0 (as $\theta_m$ varies in $\mathbb{R}$) and we will refer to these as the periodic profiles. Equation (1.14) gives a decent picture of the dependence of our approximate solution on the linear phases, though there remain subtleties to be explained in the case that some of the $\phi_m$ are nonreal.

To justify the form (1.14) for our approximate solution, consider the result of plugging (1.14) into (1.2)(a) and expanding the error in ascending powers of $\epsilon$:

$$
P(\epsilon u^a, \partial_x) u^a - \mathcal{F}(\epsilon u^a) u^a = \epsilon^{-1} \left( \sum_{m=1}^{M} \sum_{k=0}^{\mu_m} L(d\phi_m) \partial_{\theta_m} \sigma_{m, k} \left( x, \frac{\phi_m(x)}{\epsilon} \right) r_{m, k} \right) + \epsilon^0 (\ldots) + \ldots.
$$

Observe that since each $r_{m, k}$ belongs to $\text{Ker } L(d\phi_m)$, the term of order $\frac{1}{\epsilon}$ in (1.15) is in fact zero. A higher order geometric optics expansion can be formed by adding to $u^a(x)$ higher order terms, setting the resulting coefficients which appear in the right hand side equal to zero, and ensuring that those differential equations are satisfied.
Expansions with elliptic boundary layers were constructed in [12], treating the semilinear problem for generic $\beta$ and in [8], where a semilinear dispersive problem with maximally dissipative boundary conditions was considered. In both papers, higher order expansions in ascending powers of $\epsilon$ are used to justify the approximate solution as well as construct the exact solution. High order expansions involving surface waves, which arise from a failure of the uniform Lopatinski condition at a frequency $\beta$ in the elliptic region, were constructed and justified in [9] for quasilinear hyperbolic systems. Here, in the spirit of [2], we justify the leading term expansion $u_\epsilon^a(x)$. In this situation it is impossible to construct a high order expansion without a small divisor assumption, an assumption we do not make. While the small divisor assumption would exclude $\beta$ only from a set of measure zero, verifying it for a given $\beta$ can be difficult if not impossible.

We have yet to specify the profiles $v(x)$ and $\sigma_{m,k}(x, \theta_m)$, and a good choice of these is required to achieve the correct leading term. That choice is related to the fact that, while the order $\frac{1}{2}$ error has been eliminated, these profiles still appear nontrivially in the remaining error in the right hand side of (1.15). With simply $u_\epsilon^a(x)$ plugged into (1.15), an $O(1)$ error remains, in particular. This error arises as an obstacle in our proof of the main theorem, though we are able to produce terms which cancel out all but a negligible amount of the error as a consequence of the fact that $v(x)$ and $\sigma_{m,k}(x, \theta_m)$ are approximate solutions of the profile equations. We explain this process in greater detail in sections 1.2 and 1.3.

**Plugging complex phases into profiles and the result: boundary layers and almost periodic profiles**

Now we take some time to elaborate on what we mean in (1.14) when we plug the phases $\phi_m(x)/\epsilon$ into the periodic arguments of the profiles. For each $m$ such that $\omega_m$ is real, we call the $\sigma_{m,k}(x, \theta_m)$, $k = 1, \ldots, \mu_m$, hyperbolic profiles, and we note that in order to make the substitution

$$
\theta_m = \frac{\phi_m(x)}{\epsilon} = \frac{\beta \cdot x'}{\epsilon} + \frac{\omega_m}{\epsilon} x_d,
$$

such as those made in (1.14), a hyperbolic profile needs only to be defined for real $\theta_m$. On the other hand, when $\omega_m$ is nonreal, we call the $\sigma_{m,k}(x, \theta_m)$ elliptic profiles, and making this substitution requires us to evaluate these at nonreal values. Let us introduce the place holders $\theta_0$ for $\beta \cdot x'/\epsilon$ and $\xi_d$ for $x_d/\epsilon$, so that the analogous substitution to (1.16) is

$$
\theta_m = \theta_0 + \omega_m \xi_d
$$

Let us also introduce the functions

$$
\psi_{m,k}(x, \theta_0, \xi_d) := \sigma_{m,k}(x, \theta_0 + \omega_m \xi_d).
$$

Thus, in addition to the representation of $u_\epsilon^a$ given in (1.14), we have

$$
u_\epsilon^a(x) = v(x) + \sum_{m=1}^{M} \sum_{k=1}^{\mu_m} \psi_{m,k} \left( x, \frac{\phi_0(x')}{\epsilon}, \frac{x_d}{\epsilon} \right) r_{m,k}.
$$

Observe that for each elliptic profile, as we vary the parameters $\theta_0 \in \mathbb{R}$ and $\xi_d \geq 0$ (corresponding to varying $x = (x', x_d) \in \mathbb{R}^{d+1}$), the value $\theta_m = \theta_0 + \omega_m \xi_d$ varies throughout only one of the half complex planes $\{\text{Im} \theta_m \geq 0\}$, $\{\text{Im} \theta_m \leq 0\}$. To make sense of the complex substitution in the periodic argument of an elliptic profile $\sigma_{m,k}$, we define the profile first for real $\theta_m$ and then holomorphically extend the $\theta_m$-dependence into the appropriate half complex plane. This is done by making use of the Fourier expansions of the profiles.

Consider a profile $\sigma_{m,k}(x, \theta_m)$, periodic in $\theta_m$ with mean 0, and its expansion of the form

$$
\sigma_{m,k}(x, \theta_m) = \sum_{j \in \mathbb{Z} \setminus 0} a_j(x) e^{ij\theta_m}.
$$

For the moment we proceed formally, assuming that one can evaluate the above expression at $\theta_m = \theta_0 + \omega_m \xi_d$ by substituting $\theta_m = \theta_0 + \omega_m \xi_d$ in the argument of each exponential in the expansion for $\sigma_{m,k}(x, \theta_m)$ and

---

5[12] handles more general $\beta$ for the problem studied there than we do here, treating glancing modes of order two. There it is also shown that higher-order glancing cases can result in blow-up.
yield a convergent expansion for the profile $\psi_{m,k}(x, \theta_0, \xi_d)$ as defined in (1.18). That is, given (1.20), we also have

$$
\psi_{m,k}(x, \theta_0, \xi_d) = \sum_{j \in \mathbb{Z}^+ \setminus 0} a_j(x)e^{ij(\theta_0 + \omega_m \xi_d)}.
$$

We remark that we have yet to make sense of this sum and that the space of convergence must be clarified even when $\omega_m \in \mathbb{R}$. The convergence is made rigorous for any $\omega_m \in \mathbb{C}$ with Proposition 4.5, given that the profiles satisfy criteria to be explained in full later. Let us rewrite such an expansion in the following form:

$$
\psi_{m,k}(x, \theta_0, \xi_d) = \sum_{j \in \mathbb{Z}^+ \setminus 0} a_j(x)e^{ij\theta_0}e^{ij\text{Re}(\omega_m)\xi_d}e^{-j\text{Im}(\omega_m)\xi_d}.
$$

Suppose $\sigma_{m,k}(x, \theta_m)$ is elliptic. We first consider the case that $\text{Im} \, \omega_m > 0$. Then, the terms in the sum in (1.22) with $j < 0$ grow exponentially with $\xi_d$. It is easy to check that such terms result in an unsatisfactory candidate for our approximate solution $u_0^e(x)$, as defined in (1.14), which is nonphysical in the sense that it blows up in $L^\infty$ as $\epsilon \to 0$. Thus we will need to have $\sigma_{m,k}(x, \theta_m)$ such that $a_j(x) = 0$ for $j < 0$, i.e.

$$
\sigma_{m,k}(x, \theta_m) = \sum_{j \in \mathbb{Z}^+ \setminus 0} a_j(x)e^{ij\theta_m}, \quad \text{for } \text{Im} \, \omega_m > 0.
$$

Provided the above is a Fourier series in real $\theta_m$ converging in $H^{4+3}_d(x, \theta_m)$, one can show that it holomorphically extends into $\{\text{Im} \, \theta_m \geq 0\}$. It follows that we can make sense of the corresponding profile $\psi_{m,k}(x, \theta_0, \xi_d) = \sigma_{m,k}(x, \theta_0 + \omega_m \xi_d)$ for the values $(x, \theta_0, \xi_d) \in \mathbb{R}^d \times \mathbb{T} \times \mathbb{R}_+$, and Proposition 4.5 shows

$$
\psi_{m,k}(x, \theta_0, \xi_d) = \sum_{j \in \mathbb{Z}^+ \setminus 0} a_j(x)e^{ij(\theta_0 + \omega_m \xi_d)} = \sum_{j \in \mathbb{Z}^+ \setminus 0} a_j(x)e^{ij\theta_0}e^{ij\text{Re}(\omega_m)\xi_d}e^{-j\text{Im}(\omega_m)\xi_d}, \quad \text{for } \text{Im} \, \omega_m > 0.
$$

As a result, the profile $\psi_{m,k}(x, \theta_0, \xi_d)$ must decay exponentially in $\xi_d$.

In the case that $\text{Im} \, \omega_m < 0$, similar considerations lead us to construct a corresponding elliptic profile of the form

$$
\sigma_{m,k}(x, \theta_m) = \sum_{j \in \mathbb{Z}^+ \setminus 0} a_j(x)e^{ij\theta_m}, \quad \text{for } \text{Im} \, \omega_m < 0,
$$

a sum which holomorphically extends into $\{\text{Im} \, \theta_m \leq 0\}$. This kind of elliptic profile also results in $\psi_{m,k}(x, \theta_0, \xi_d) = \sigma_{m,k}(x, \theta_0 + \omega_m \xi_d)$ which decays exponentially as $\xi_d$ increases. For a hyperbolic profile $\sigma_{m,k}(x, \theta_m)$, which has $\text{Im} \, \omega_m = 0$, we do not make such restrictions on its coefficients $a_j(x)$, so we describe it with (1.20) and $\psi_{m,k}(x, \theta_0, \xi_d)$ with the expansions (1.21) and (1.22). In general, any one of our profiles $\psi_{m,k}(x, \theta_0, \xi_d)$ is periodic in $\theta_0$, and either decays in $\xi_d$ if it is elliptic or is almost periodic in $\xi_d$ if it is hyperbolic. Since the elliptic profiles $\sigma_{m,k}(x, \theta_m)$ result in $\psi_{m,k}(x, \theta_0, \xi_d)$ which decay in $\xi_d$, upon replacing the placeholder $\xi_d$ with $\frac{\xi_d}{\epsilon}$ we see they contribute to a boundary layer with rapid exponential decay in $x_d$ for small $\epsilon$, i.e. the elliptic boundary layer part of the approximate solution described by (1.14).

Another important consequence of taking each elliptic profile either to have only positive or negative spectrum is that it considerably simplifies the couplings between the profiles that can occur in the profile equations in a way which is very useful for our solution. 6 Ultimately, it is perfectly fine that we make such restrictions on the spectra since we are able to solve the profile equations for profiles of this form, which allows us to satisfy a useful solvability condition.

With the following remark, we summarize the different forms of expansions for profiles.

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6 We stress that this restriction of couplings is not a kind of extra condition which we must impose. It is instead implied when we write down the system of profile equations for profiles specifically of this form. Moreover, without this consideration it is not clear that the hyperbolic equations (together with the equation for $u(x)$) form their own sub-system, which is exploited in a critical way in our solution.
Remark 1.3. For \( Z_m = \{ n \in \mathbb{Z} : n \text{Im} \omega_m \geq 0 \} \), i.e.

\[
Z_m = \begin{cases} 
\mathbb{Z} & \text{if } \omega_m \in \mathbb{R}, \\
\mathbb{Z}^+ & \text{if } \text{Im} \omega_m > 0, \\
\mathbb{Z}^- & \text{if } \text{Im} \omega_m < 0,
\end{cases}
\]

each periodic profile has an expansion of the form

\[
\sigma_{m,k}(x, \theta_m) = \sum_{j \in Z_m \backslash 0} a_j(x) e^{ij\theta_m},
\]

and each profile \( \psi_{m,k}(x, \theta_0, \xi_d) \) has

\[
\psi_{m,k}(x, \theta_0, \xi_d) = \sum_{j \in Z_m \backslash 0} a_j(x) e^{ij(\theta_0 + \omega_m \xi_d)},
\]

where the sense of convergence is clarified with Proposition 4.5.

### 1.2 The profile equations

In order to have an approximate solution \( u^*_\epsilon(x) \), as in equation (1.14), such that the main theorem holds, our profiles \( \psi(x) \), \( \sigma_{m,k}(x, \theta_m) \) must approximately satisfy the system of equations which we refer to as the profile equations. To see where this additional criterion comes from, we recall that the idea that \( u^*_\epsilon(x) \) is the leading order term of a geometric optics expansion suggests we may add to \( u^*_\epsilon(x) \) higher order terms which result in solving (1.2)(a) to higher order. In particular, we should be able to find a function, say \( u^1 \), such that when we replace \( u^*_\epsilon(x) \) with \( u^*_\epsilon(x) + \epsilon u^1(x) \) in (1.15), both the \( O(1/\epsilon) \) and \( O(1) \) error terms in the right hand side are annihilated. Ensuring the possibility of constructing such \( u^1 \), which suggests we have the right leading term \( u^*_\epsilon \), requires solvability conditions to hold; these will be translated into conditions on the profiles known as the profile equations.

**Derivation**

We may lift \( u^*_\epsilon(x) \) as defined in (1.14) under the substitution \( \theta = (\theta_1, \ldots, \theta_M) = (\phi_1/\epsilon, \ldots, \phi_M/\epsilon) = \phi/\epsilon \) to get the following function

\[
\nu^0(x, \theta_1, \ldots, \theta_M) := \psi(x) + \sum_{m=1}^{M} \sum_{k=0}^{\mu_m} \sigma_{m,k}(x, \theta_m)r_{m,k}.
\]

An additional useful representation of the leading term \( u^*_\epsilon \) can be found, with (1.19) in mind, if we analogously lift the function \( u^*_\epsilon(x) \) under the substitution \( (\theta_0, \xi_d) = (\phi_0/\epsilon, x_d/\epsilon) \) to get the function

\[
\mathcal{U}^0(x, \theta_0, \xi_d) := \psi(x) + \sum_{m=1}^{M} \sum_{k=0}^{\mu_m} \psi_{m,k}(x, \theta_0, \xi_d)r_{m,k}.
\]

We use only the formulation \( \nu^0(x, \theta) \) for the derivation and solution of the profile equations, though we need the formulation \( \mathcal{U}^0(x, \theta_0, \xi_d) \) in addition to \( \nu^0(x, \theta) \) for the error analysis.

A calculation shows that taking

\[
u^0(x, \theta_0, \xi_d) := \mathcal{U}^0(x, \theta_0, \xi_d)|_{\theta_0 = \omega_0^1, \xi_d = \omega_d^1},
\]
or, equivalently,

\[
u^0(x) := \nu^0(\theta(\theta_0, \xi_d))|_{\theta_0 = \omega_0^1, \xi_d = \omega_d^1},
\]

where we use the notation

\[
\theta(\theta_0, \xi_d) = (\theta_0 + \omega_1 \xi_d, \ldots, \theta_0 + \omega_M \xi_d),
\]

(1.31)
results in the vanishing of the terms of order $\frac{1}{\epsilon}$ when plugging $u^\epsilon$ into $P(\epsilon u^\epsilon, \partial_x)u^\epsilon$ in (1.2)(a). We define the operator

$$\mathcal{L}(\partial \theta) := \sum_{m=1}^{M} \tilde{L}(d\phi_m) \partial \theta_m$$

where we have denoted $\tilde{L}(d\phi_m) = A^{-1}_d L(d\phi_m)$. The $O(1/\epsilon)$ error term in (1.2)(a) can be written in terms of $\mathcal{V}^0$ as

$$\mathcal{L}(\partial \theta) \mathcal{V}^0 |_{\theta = \phi/\epsilon}.$$

Thus the condition $\mathcal{L}(\partial \theta) \mathcal{V}^0 = 0$ implies the cancellation of the $O(1/\epsilon)$ error, and this is satisfied by the definition of $\mathcal{V}^0$ regardless of the choice of our profiles $\tilde{v}$, $\sigma_{m,k}$.

Now let us consider plugging in a corrected approximate solution

$$u_\epsilon^*(x) := (\mathcal{U}^0(x, \theta_0, \xi_d) + \epsilon \mathcal{U}^1(x, \theta_0, \xi_d)) |_{\theta_0 = \frac{\phi}{\epsilon}, \xi_d = \frac{\theta}{\epsilon}} = (\mathcal{V}^0(x, \theta(\theta_0, \xi_d)) + \epsilon \mathcal{V}^1(x, \theta(\theta_0, \xi_d))) |_{\theta_0 = \frac{\phi}{\epsilon}, \xi_d = \frac{\theta}{\epsilon}}.$$

Here we do not necessarily have $\mathcal{V}^1$ of the form (1.29), but we do impose other constraints\(^7\) on its form (and analogously for $\mathcal{U}^1$.) Let us define the operator

$$\mathcal{M}(\mathcal{V}) \partial \theta := \sum_{m=1}^{M} \sum_{j=0}^{d-1} \beta_j \partial \theta A_j(0) \mathcal{V} \partial \theta_m.$$

Then the cancellation of the $O(1)$ error term in (1.15) (with $u^\epsilon$ replaced by $u^*_\epsilon$) is satisfied if

$$\mathcal{L}(\partial \theta) \mathcal{V}^1 + \tilde{L}(\partial_x) \mathcal{V}^0 + \mathcal{M}(\mathcal{V}^0) \partial \theta \mathcal{V}^0 = F(0) \mathcal{V}^0.$$

Observe that the operator $\mathcal{L}(\partial \theta)$ is singular, so that one cannot simply invert it to solve for $\mathcal{V}^1$ in (1.38). One way to proceed is to ensure the existence of a solution $\mathcal{V}^1$ by imposing the following condition on $\mathcal{V}^0$:

$$\tilde{L}(\partial_x) \mathcal{V}^0 + \mathcal{M}(\mathcal{V}^0) \partial \theta \mathcal{V}^0 - F(0) \mathcal{V}^0 \in \text{Im} \mathcal{L}(\partial \theta).$$

This condition turns out to be equivalent to a differential equation in $\mathcal{V}^0$. To get an idea of how this works, recall from Lemma 1.2 that $\text{Im} \tilde{L}(d\phi_m) = \text{Ker} P_m$, where $P_m$ is the projection to the subspace spanned by $r_{m,k}$, $k = 1, \ldots, \mu_m$. Using the fact that $\mathcal{L}(\partial \theta) = \sum_{m=1}^{M} \tilde{L}(d\phi_m) \partial \theta_m$ and the decomposition of Lemma 1.2, we are able to construct a projection operator $\mathbb{E}$ such that if a function of a certain type\(^8\) $\mathcal{V}(x, \theta)$ satisfies $\mathbb{E}^\epsilon \mathcal{V} = 0$, then $\mathcal{V} \in \text{Im} \mathcal{L}(\partial \theta)$. Such a projector $\mathbb{E}^\epsilon$ is also defined in the study [6]. Thus, a sufficient condition for (1.39) is that

$$\mathbb{E}^\epsilon (\tilde{L}(\partial_x) \mathcal{V}^0 + \mathcal{M}(\mathcal{V}^0) \partial \theta \mathcal{V}^0) = \mathbb{E}^\epsilon (F(0) \mathcal{V}^0),$$

with the caveat that, in order for this to work, the functions to which $\mathbb{E}^\epsilon$ is applied must satisfy the hypothesis of Proposition 3.8, a consideration which we will discuss in detail later. This equation is closely related to the profile equations, but it is not quite what we want. The current task is to solve for the correct choices of $\tilde{v}$ and the $\sigma_{m,k}$, but if we simply impose the condition (1.40) on $\mathcal{V}^0$ it is not clear how (or even whether or not) that would determine $\tilde{v}$ and the $\sigma_{m,k}$.

To get the profile equations, we define another projection operator, $\mathbb{E}$, very closely related to $\mathbb{E}^\epsilon$,\(^9\) which satisfies

$$\mathbb{E}^\epsilon (\mathcal{V}) |_{\theta = \phi/\epsilon} = \mathbb{E}(\mathcal{V}) |_{\theta = \phi/\epsilon}.$$

---

\(^7\)We will require that $\mathcal{V}^1(x, \theta) \in H^{s-2}(x, \theta)$, defined in (3.4).

\(^8\)It is required that $\mathcal{V}(x, \theta) \in H^{s-2}(x, \theta)$ and that it is a trigonometric polynomial in $\theta$. See Proposition 3.8.

\(^9\)The relation between the two projectors is clarified in Section 3 and will be used in the error analysis. Both projectors are defined in Definition 3.5.
Thus we have that the following agrees with (1.40) after making the substitution \( \theta = \theta(\theta_0, \xi_d) \):

\[
(1.42) \quad E(L(\partial_x)\mathcal{V}^0 + \mathcal{M}(\mathcal{V}^0)\partial_0 \mathcal{V}^0) = E\left(F(0)\mathcal{V}^0\right).
\]

Together with the appropriate boundary conditions, (1.42) forms the profile equations. The set of equations for \( \mathcal{V}^0 \), including the interior profile equations and the boundary conditions, are given in (3.31). This is indeed equivalent to a system of transport equations in terms of \( \mathcal{V} \) and the \( \sigma_{m,k} \) which we will solve, at least approximately.

We digress to mention one interesting difference in our study from [2]: in that study (in which the boundary frequency is assumed to be in the hyperbolic region) they formulate and solve both a formulation of the profile equations in terms of \( \mathcal{V} \), as we have with (3.31), and another formulation in terms of \( \mathcal{U}^0 \), referred to as the \textit{almost periodic formulation}. For the more general setting we encountered serious difficulties\(^{10}\) to making a derivation in terms of \( \mathcal{U}^0 \), and we show it is possible to solve the general problem by only using a formulation of the profile equations in terms of \( \mathcal{V} \). In fact, we found there were many instances where we were able to do our analysis in one fell swoop in terms of \( \mathcal{V} \), especially by using Proposition 4.5 in the error analysis, whereas in [2], the two settings are addressed separately.

**Large system formulation and its solution**

The profile equations (1.42) can be expressed as a quasilinear differo-integral system of equations with \( \mathcal{V} \) and the profiles \( \sigma_{m,k} \) as the unknowns. This system is explicitly written out in Proposition 3.18 and is derived in the following way: the expression for \( \mathcal{V}^0 \) in the right hand side of (1.29) is regarded as an ansatz, which we plug into (1.42). The projector \( E \), which is designed to project to the characteristic modes for the operator \( L(\partial_y) \), decomposes into a sum of projectors \( E_0, E_{m,k} \), where \( E_0 \) generally projects to a component with nonzero mean, and each \( E_{m,k} \) projects to a component with mean zero. The result is an \( N \times N \) system for \( \mathcal{V}(x) \) (the component of (1.42) in the image of \( E_0 \)) coupled with a set of complex transport equations indexed by \((m,k)\), \( m = 1, \ldots, M \), \( k = 1, \ldots, \nu_{m,n} \), where the \((m,k)\) equation (the component in the image of \( E_{m,k} \)) is the transport equation for \( \sigma_{m,k}(x,\theta_m) \). In this equation we have the corresponding characteristic, complex vector field \( X_{\phi_m}^{11} \) applied to \( \sigma_{m,k}(x,\theta_m) \). More precisely, for each hyperbolic profile, the corresponding vector field is real, and for each elliptic profile, the vector field is nonreal.

The nonlinear quadratic term \( \mathcal{M}(\mathcal{V}^0)\partial_0 \mathcal{V}^0 \) in (1.42) results in couplings between the profiles. These manifest as integrals of two types in the large system. The first type is a mean of the product of two periodic functions, in lines (3.51) and (3.52)(b), and the second type is the interaction integrals, such as \( \int_{q,n}^{p,n} \), \( p,n \), defined in (3.47) and appearing in (3.53) and (3.55). The interaction integrals account for resonances, as described in Proposition 3.16. Resonances occur in products such as \( \sigma_{q,k}(x,\phi_q/\epsilon)\partial_0 \sigma_{r,k'}(x,\phi_r/\epsilon) \) when we have a relation among the phases of the form

\[
(1.43) \quad n\phi_p = n_q\phi_q + n_r\phi_r,
\]

where \( p \in \{1,\ldots,M\} \setminus \{q,r\} \), \( n_q \) and \( n_r \) are in the spectra of \( \sigma_{q,k} \) and \( \sigma_{r,k'} \), respectively, and \( n \) is allowed to be any integer\(^{12}\). The implication of (1.43) is that oscillations with phases \( \phi_q \) and \( \phi_r \) interact to produce oscillations with phase \( \phi_p \). In the equation for \( \sigma_{p,l}(x,\theta_p) \), in which we apply \( E_{p,l} \) to the product \( \sigma_{q,k}(x,\theta_q)\partial_y \sigma_{r,k'}(x,\theta_r)\mathcal{M}(r_{q,k})r_{r,k'} \) (a constituent of the term \( \mathcal{M}(\mathcal{V}^0)\partial_0 \mathcal{V}^0 \) from (1.42)), the projector accounts for these relations and outputs the appropriate function which depends only on \((x,\theta_p)\).

In [2], the authors derive and solve a large system for the profiles which is almost identical to the system we give in Proposition 3.18, except the restrictions on the boundary frequency result in the absence of any elliptic profiles. As a result, the system in that case was a system of real transport equations. In that study, the authors develop an iteration scheme of the form (3.71).\(^{13}\) Equation (3.71)(a) implies that \( \mathcal{V}^0 \) has a decomposition such as (1.29), but with other components, \( \mathcal{V}^0 \), and the profile iterates \( \sigma_{m,k}^{n}(x,\theta) \) in place of \( \mathcal{V}(x) \) and the \( \sigma_{m,k}(x,\theta) \). Corresponding to (3.71)(b) is an equivalent large system in the profile iterates,
essentially the large system (3.51)-(3.52) with respective iterates in place. In [2] the authors prove an energy estimate for this system by combining a Kreiss-type estimate for the nonzero mean system (for $\mathbb{R}^n$) with an estimate for the zero mean system (for the profile iterates $\sigma_{m,k}^n$).

The latter is proven in [2] by taking the $L^2(dx \, dB_m)$ pairing of $\sigma_{m,k}^n(x, \theta_m)$ with its corresponding equation and integrating by parts to transfer derivatives off of the $n$-th iterates. The profiles (which are all hyperbolic in that case) are categorized as either incoming or outgoing according to their corresponding group velocities. This determines whether the corresponding boundary terms appear with a good or a bad sign upon performing the integration by parts on the equation for a given profile. In the result, the $L^2$ norms of the outgoing profiles $\sigma_{m,k}^n$, $m \in \mathcal{O}$ (good sign) and their traces $\sigma_{m,k}^n|_{x_d=0}$ appear with positive sign on the same side (the lower bound,) bounded by the norms of the previous iterates, a source term, and the prescribed function on the boundary $G$, as desired. On the other hand, when writing the bound resulting from the same calculation, but for the incoming profiles $\sigma_{m,k}^n$, $m \in \mathcal{I}$ (bad sign), the norms of the traces fall on the upper bound side, and so they are forced to separately bound the traces of the incoming profiles in order to proceed. Due to the uniform stability assumption\textsuperscript{14}, the authors are able to estimate these terms for the incoming profiles by using the boundary condition to rewrite the trace of incoming $\sigma_{m,k}^n$ in terms of $G$ and the traces of the outgoing profiles, which had already been estimated, having occurred with good sign.

In contrast, for our study we must deal with complex transport equations, and while methods from [2] are used to a degree in handling the large system, since complex transport equations are generally not solvable, a different approach is required. Dealing with the more general complex phases also forced us to handle complex resonances, which determine the ways in which hyperbolic and elliptic profiles are coupled with each other in the large system. Careful examination of these resonances and the possible interactions revealed the following about the system: the subset formed by the equations for $v$ and the hyperbolic profiles is actually a sub-system of real transport equations not dependent on the elliptic profiles, which we refer to as the hyperbolic sub-system, given in (3.69). On the other hand, resonances between hyperbolic and elliptic phases do cause the complex transport equations to depend on the hyperbolic profiles. This does not cause much additional difficulty since we are able to work at first strictly within the hyperbolic sub-system to solve for $v$ and the hyperbolic profiles. The procedure used to solve the full system in [2] can be straightforwardly adapted to solve our real sub-system; our hyperbolic profiles are split into two groups, the incoming and the outgoing, and essentially we use the same method sketched above with the iteration scheme (3.71).

As mentioned before, in contrast with the real transport equations, the complex transport equations for the elliptic profiles are generally not solvable. In [12], a Taylor expansion in $x_d$ is developed to approximately solve the corresponding complex profile equations by satisfying the equations evaluated at $x_d = 0$ along with high order transverse derivatives of the equations at $x_d = 0$. One could use a similar approach for this problem, but using the error bounds of Taylor’s theorem to get an appropriate bound on the error in $H^s$ would require us also to take a large number of derivatives of the equations and require those to hold on the boundary. Taking derivatives of the other profiles appearing in those equations means we would have to require them to be in a higher order Sobolev space. Interestingly, however, in our proof of Theorem 4.7 we were able to show that merely requiring the complex transport equations themselves to hold to at most first order in $x_d$ is sufficient for our purposes. The error term is handled by an application of Proposition 4.6 in a process which we explain further in our discussion on the error analysis.

In the original boundary condition, the value we must prescribe for elliptic $\sigma_{m,k}|_{x_d=0}$ can be straightforwardly read off from linear relations with $G$ and the hyperbolic, outgoing profiles at $x_d = 0$, as a result of the uniform stability condition. The value we must prescribe for $\partial_{x_d} \sigma_{m,k}|_{x_d=0}$ is determined by taking the transport equation for $\sigma_{m,k}$ from the large system, evaluating this at $x_d = 0$, and plugging in the (zeroth order in $x_d$) traces of the other profiles, all of which have already been obtained. The problem reduces to extending elliptic $\sigma_{m,k}$ from the data $\{\sigma_{m,k}|_{x_d=0}, \partial_{x_d} \sigma_{m,k}|_{x_d=0}\} \in H^s \times H^{s-1}$ on the boundary to a function in $H^s$ in the interior, which is also supported in $\{t \geq 0\}$. We accomplish this by solving a wave equation

\begin{equation}
(1.44) \quad \partial_{x_d}^2 S - \Delta_x \sigma_{m,k} = 0,
\end{equation}

with slightly modified initial data, and then translating the solution away from the boundary with the substitution $\sigma_{m,k}(x, \theta_m) = \zeta(t-x_d, x''(\theta_m))^{15}$ taking advantage of finite speed of propagation to ensure the

\textsuperscript{14}We also assume this; see Assumption 2.5.

\textsuperscript{15}Strictly speaking, this is not precisely the definition of $\sigma_{m,k}(x, \theta_m)$, which actually includes multiplication by a smooth cutoff in the variable $x_d$. 

support is contained in \( \{ t \geq 0 \} \). This process is described in detail with Lemma 3.28 and Proposition 3.29.

### 1.3 Proof of the main theorem

Recall the singular system (1.6) discussed in Section 1.1. As we stated there, in order to prove the main theorem, we show a stronger result in terms of the solution of this system, \( U_\varepsilon \). The stronger result we show is the following:

**Theorem 1.4.** Define

\[
U_\varepsilon^0(x, \theta_0) = \mathcal{V}^0 \left( x, \theta_0 + \frac{x_d}{\varepsilon}, \ldots, \theta_0 + \frac{x_M}{\varepsilon} \right)
\]

where \( \mathcal{V}^0 \) is as in (1.29). If \( G(x', \theta_0) \in H^{s+1}_T \) for sufficiently large \( s \), then for the exact solution \( U_\varepsilon \) of the singular system (1.6), we have

\[
|U_\varepsilon(x, \theta_0) - U_\varepsilon^0(x, \theta_0)|_{L^\infty(x_d, H^{s-1}(x', \theta_0))} \to 0 \text{ as } \varepsilon \to 0.
\]

Recall that the exact solution of (1.2) is formed by making the substitution

\[
u_\varepsilon(x) = U_\varepsilon(x, \theta_0) \big|_{\theta_0 = \frac{x_d}{\varepsilon}}.
\]

Our approximate solution \( u_\varepsilon^a(x) \) satisfies

\[
u_\varepsilon^a(x) = U_\varepsilon^0(x, \theta_0) \big|_{\theta_0 = \frac{x_d}{\varepsilon}},
\]

and it follows that Theorem 1.1 is a direct consequence of Theorem 1.4.

Before sketching the proof of Theorem 1.4, we define the following norm which is useful in the error analysis:

\[
\mathcal{E}_T^\varepsilon = \{ U(x, \theta, \xi_d) : \sup_{\xi_d \geq 0} |U(\cdot, \cdot, \xi_d)|_{E_T^\varepsilon} < \infty \}.
\]

Typically, taking functions such as \( \mathcal{V}^0(x, \theta) \in H^{s+1} \) whose dependence on \( \theta \) is extended to the product of complex half-spaces discussed in Remark 3.3 and then substituting in the complex valued function \( \theta = \theta(\theta_0, \xi_d) \) yields a function which is bounded in the \( \mathcal{E}_T^\varepsilon \) norm. It is convenient at times to work in the spaces for such functions, \( \mathcal{E}_T^{\varepsilon;k} \), before evaluating at \( \xi_d = x_d/\varepsilon \) as in (1.45).

**Proof of the main result**

In the proof of Theorem 7.1 of [13], for some \( T_0 > 0 \), the iteration scheme for the singular system, (4.41), is used to produce \( U_\varepsilon(x, \theta_0) \) and iterates \( U_\varepsilon^n(x, \theta_0) \) bounded in \( E_{T_0}^\varepsilon \) uniformly with respect to \( n \) and \( \varepsilon \) and which satisfy

\[
\lim_{n \to \infty} U_\varepsilon^n(x, \theta_0) = U_\varepsilon \in E_{T_0}^{\varepsilon-1} \text{ uniformly with respect to } \varepsilon \in (0, \varepsilon_0],
\]

where \( U_\varepsilon \) solves the singular system (1.6).

As in [2], we use the strategy of simultaneous Picard iteration to prove the main theorem, an idea due to the study [3], whereby we construct iterates \( U_\varepsilon^n(x, \theta_0) \) converging to \( U_\varepsilon^0 \) uniformly as \( n \to \infty \) and show that \( U_\varepsilon^n \) is close to \( U_\varepsilon^n \) for each \( n \). In Section 3.3, Proposition 3.29 of Section 3.4, and Definition 3.31 we construct a function \( \mathcal{V}^0(x, \theta) \in H^s(\mathbb{R}^{d+1} \times T^M) \) which approximately satisfies the profile equations, and iterates \( \mathcal{V}^0,n(x, \theta) \) approximately satisfying the equations of a corresponding iteration scheme. The iterates \( \mathcal{V}^0,n \) are bounded in \( \mathbb{H}^{s+1}_{T_0} \) uniformly with respect to \( n \) and satisfy

\[
\lim_{n \to \infty} \mathcal{V}^0,n = \mathcal{V}^0 \text{ in } \mathbb{H}^s_{T_0}.
\]

\[\text{[16]}\text{This is the product of half-spaces on which } \mathcal{V}^0(x, \theta) \text{ becomes defined once we have extended each of the elliptic profiles } \sigma_{m,k}(x, \theta_m) \text{ to its natural complex half-space, as discussed in the subsection on plugging in complex phases of Section 1.1.}\]
Again, we plug in the function $\theta(\theta_0, \xi_d)$ in the placeholder argument $\theta$ to get

\[(1.52)\]

$$U^{0,n}(x, \theta_0, \xi_d) := \mathcal{V}^{0,n}(x, \theta_0 + \omega_1 \xi_d, \ldots, \theta_0 + \omega_M \xi_d),$$

followed by $\xi_d = \frac{x_d}{\epsilon}$, yielding

\[(1.53)\]

$$U^{0,n}_\epsilon(x, \theta_0) := \mathcal{V}^{0,n}(x, \theta_0 + \omega_1 \frac{x_d}{\epsilon}, \ldots, \theta_0 + \omega_M \frac{x_d}{\epsilon}).$$

Thus, by using (1.51) and applying the estimates given by Proposition 4.5 and Lemma 4.8, we get that

\[(1.54)\]

$$\lim_{n \to \infty} U^{0,n}_\epsilon(\theta) = U^0_\epsilon \text{ in } E^{n-1}_{T_0} \text{ uniformly with respect to } \epsilon \in (0, \epsilon_0),$$

where $U^0_\epsilon$ is as in Theorem 1.4. Therefore, to conclude $\lim_{\epsilon \to 0} U^{0,n}_\epsilon(x, \theta_0) - U_\epsilon(x, \theta_0) = 0$ in $E^{n-1}_{T_0}$ and finish the proof of Theorem 1.4, it is sufficient to show

\[(1.55)\]

$$\lim_{\epsilon \to 0} \left| U^{0,n}_\epsilon - U^{n}_\epsilon \right| \to 0 \text{ for all } n.$$

Indeed, (1.55) is proved as Section 4.2 by induction on $n$. One might try to prove the statement in this way by applying the estimate for the linearized singular system\(^{17}\) of Proposition 4.9 to $(U^{0,n+1}_\epsilon - U^{n+1}_\epsilon)$. The problem with this is that if we take $\mathcal{A}(eU^{n}_\epsilon)$, which we define to be the operator appearing in the left hand side of the equation for $U^{n+1}_\epsilon$, i.e. (4.41)(a), and apply it to the difference $(U^{0,n}_\epsilon - U^{n}_\epsilon)$, the resulting quantity leaves an $O(1)$ error, so we do not get convergence to zero as $\epsilon \to 0$. This happens for the following reason: while $U^{n+1}_\epsilon$ solves this equation exactly, $U^{0,n+1}_\epsilon$ was only designed to eliminate the $O(1/\epsilon)$ order error term. To be more precise, recalling our discussion on the formulations of the analogous error terms in (1.15) in terms of $\mathcal{V}^0$ in the section on the profile equations, one sees that an $O(1)$ error is left in (1.15) when we form $u^\omega = U^0_\epsilon|_{\theta = \theta_0/\epsilon}$.\(^{18}\)

We can amend this argument by using a similar approach to the $O(1)$ error elimination discussed in Section 1.2. The fix lies in constructing a corrector for $V^{0,n+1}$, say $V^{1,n+1}$, analogous to the function $V^1$ which we discussed there. In the following we refer to the corresponding equations\(^{19}\) having the appropriate iterates in place of the original functions. Suppose for the moment that there exists $V^{1,n+1}$ which results in the elimination of the $O(1)$ terms upon the replacement of $V^{0,n+1}$ by $V^{0,n+1} + \epsilon V^{1,n+1}$. Then one can obtain a corrector for $U^{0,n+1}_\epsilon$, say $U^{1,n+1}_\epsilon(x, \theta_0)$, by evaluating $V^{1,n+1}(x, \theta)$ at $\theta = (\theta_0 + \omega_1 x_d/\epsilon, \ldots, \theta_0 + \omega_M x_d/\epsilon)$. Applying $\mathcal{A}(eU^{n}_\epsilon)$ to $(U^{0,n+1}_\epsilon + \epsilon U^{1,n+1}_\epsilon - U^{n+1}_\epsilon)$ results in an error which converges to zero as $\epsilon \to 0$. Then, assuming for the moment that $U^{1,n+1}_\epsilon$ is bounded uniformly in $\epsilon$, an application of the estimate from [13] gives us

\[(1.56)\]

$$\lim_{\epsilon \to 0} \left| U^{0,n+1}_\epsilon - U^{n+1}_\epsilon \right|_{E^{n-1}_{T_0}} = 0,$$

which would allow us to conclude the induction step.

However, there are two major obstacles to constructing such a corrector $V^{1,n+1}$ for $V^{0,n+1}$. To see the first, recall that we mentioned in our discussion on solving the elliptic profile equations that in general they cannot be exactly solved. It is the same situation for the iterated profile equations. We form $V^{0,n+1}$ so that the error $R^{n+1}_\epsilon(x, \theta)$ is zero at the boundary $\{x_d = 0\}$ and elliptically polarized\(^{19}\). This implies that when we make the substitution $\theta = (\theta_0, \xi_d)$ the error decays exponentially in $\xi_d$. We describe these conditions on $R^{n+1}_\epsilon$ precisely in Proposition 4.6 and prove that they imply $R^{n+1}_\epsilon(x, \theta(\theta_0, x_d/\epsilon))$ goes to zero in $E^{n-1}_{T_0}$ as $\epsilon \to 0$. Thus, the solution of the profile equations modulo this error is a sufficient solvability condition.

The second issue is that, since we do not make a small-divisor assumption, we can only guarantee solvability if we are working with finite trigonometric polynomials\(^{20}\), as opposed to general elements of $H^T_0$ which may have infinitely many nonzero Fourier coefficients. This is resolved with an approach similar to

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\(^{17}\)This linear estimate is taken from [13], where it is used to prove the existence of the solution to the singular system.

\(^{18}\)Here, we refer to equations (1.15), (1.38), etc.

\(^{19}\)This means $E_\epsilon R^{n+1}_\epsilon = R^{n+1}_\epsilon$, where $E_\epsilon$ is the elliptic projector defined in (3.13). The good decay properties of $R^{n+1}_\epsilon|_{d = \theta(\theta_0, \xi_d)}$ basically come from the fact that $R^{n+1}_\epsilon$ then has zero mean and only oscillates nontrivially in variables $\theta$ for $j$ such that $\omega_j$ is nonreal.

\(^{20}\)For details on solvability, see Proposition 3.8.
that in [2], based on the idea due to [5] of using trigonometric polynomial approximations of the functions. We approximate each $\mathcal{V}_{p,n+1}$ by a trigonometric polynomial $\mathcal{V}_{p,n+1}^0$, construct a corresponding corrector $\mathcal{V}_{p}^1$, and use Proposition 4.5 and Lemma 4.8 to work with $U_{p,0}^{0,n+1} = \mathcal{V}_{p,n+1}^0|_{\theta=\theta_0,\xi_d}$ and $U_{p,0}^{0,n+1} = U_{p,0}^{0,n+1}|_{\xi_d=x_d/\epsilon}$ approximating $U_{p,0}^{0,n+1}$ and $U_{p,0}^{0,n+1}$, respectively.

We emphasize that for our study we are forced to handle the trigonometric polynomial approximations differently from [2] as a result of dealing with nonreal $\omega_{m}$, since this means the $U$-type functions are no longer almost periodic. Because we construct the corrector in the space for $V$-type functions, rather than in the space for $U$, as done in [2], we must check that taking a trigonometric approximation for $V(x,\theta)$ in $H^{s+1}_v$, and then plugging in $\theta=\theta_0,\xi_d$, yields a good approximation in the space for $U(x,\theta_0,\xi_d)$, with respect to the norm $E^{s}_v$. Establishing an estimate of the form $|U|_{E^{s}_v} \leq C|V|_{H^{s+1}_v}$ is key, and having complex $\omega_m$ introduces new difficulties.

A positive $\delta$ is fixed and each of these approximations has error bounded by $O(\delta)$ in the corresponding norm. By carefully replacing the trigonometric polynomial approximations with respective quantities in the argument sketched above, we are able to solve away all but $O(\delta)$ of the $O(1)$ error.\footnote{To be more precise, the error we are referring to is in the right hand side of (4.60) (note the left hand side.) The first two terms are small due to the approximate solution of the profile equations, and the last term is that which is solved away. By this we mean that after incorporating the corrector, which we refer to here as $U_{p,0}^{1,n+1}$, all that is left is the quantity (4.67) (where the corrector is instead denoted by $U_{p,0}^{1,n+1}$).}

From this we can establish

\begin{equation}
|U_{p,0}^{0,n+1} - U_{p,0}^{0,n+1}|_{E^{s-1}_v} \leq C\delta + c(\epsilon) + K(\delta)\epsilon,
\end{equation}

for all $\delta$, (where the error terms in the right hand side are given in the proof of Proposition 4.7,) so that the proof of the theorem can be concluded.

\section{Assumptions and other preliminaries}

We now take some time to go over the assumptions we make for the problem, some related definitions, and the example of the compressible Euler equations, to which our analysis applies.

We assume that $L(\partial_x)$ is hyperbolic with characteristics of constant multiplicity, making the following:

\begin{assumption}
The matrix $A_0 = I$. For an open neighborhood $\mathcal{O}$ of $0 \in \mathbb{R}^N$, there exists an integer $q \geq 1$, some real functions $\lambda_1, \ldots, \lambda_q$ that are $C^\infty$ on $\mathcal{O} \times \mathbb{R}^d \setminus \{0\}$, homogeneous of degree 1, and analytic in $\xi$, and there exist some positive integers $\nu_1, \ldots, \nu_q$ such that:

\begin{equation}
\det \left[ \tau I + \sum_{j=1}^d \xi_j A_j(u) \right] = \prod_{k=1}^q \left( \tau + \lambda_k(u, \xi) \right)^{\nu_k}
\end{equation}

for $u \in \mathcal{O}$, $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d \setminus \{0\}$. Moreover the eigenvalues $\lambda_1(u, \xi), \ldots, \lambda_q(u, \xi)$ are semi-simple (their algebraic multiplicity equals their geometric multiplicity) and satisfy $\lambda_1(u, \xi) < \cdots < \lambda_q(u, \xi)$ for all $u \in \mathcal{O}$, $\xi \in \mathbb{R}^d \setminus \{0\}$.

Also, for our study we consider a noncharacteristic boundary $\{x_d=0\}$:

\begin{assumption}
For $u \in \mathcal{O}$ the matrix $A_d(u)$ is invertible and the matrix $B(u)$ has maximal rank, its rank $p$ being equal to the number of positive eigenvalues of $A_d(u)$ (counted with their multiplicity).

We now provide some notation regarding frequencies $\tau - i\gamma \in \mathbb{C}$ and $\eta \in \mathbb{R}^{d-1}$ dual to the variables $t$ and $y$, respectively. We define the matrix

\begin{equation}
A(\zeta) := -i A^{-1}_d(0) \left( (\tau - i\gamma)I + \sum_{j=1}^{d-1} \eta_j A_j(0) \right), \quad \zeta := (\tau - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1},
\end{equation}

\end{assumption}
where

\[ \zeta = \zeta \in \mathbb{C} : \tau^2 + \gamma^2 + |\eta|^2 = 1 \}, \quad \Sigma := \{ \zeta \in \mathbb{E} : \tau^2 + \gamma^2 + |\eta|^2 = 1 \}, \]

\[ \Xi_0 := \{ (\tau, \eta) \in \mathbb{R} \times \mathbb{R}^{d-1} \setminus (0,0) \} = \Xi \cap \{ \gamma = 0 \}, \quad \Sigma_0 := \Sigma \cap \Xi_0. \]

With these in mind, we define the symbol

\[ (2.3) \quad L(\tau, \xi) := \tau I + \sum_{j=1}^{d} \xi_j A_j(0). \]

The following is a result due to Kreiss [7] in the case of strict hyperbolicity, i.e. when all the eigenvalues in Assumption 2.1 have multiplicity \( \nu_j = 1 \), and to Métivier [10] in our more general case.

**Proposition 2.3 ([7, 10]).** Let Assumptions 2.1 and 2.2 be satisfied. Then for all \( \zeta \in \Xi \setminus \Xi_0 \), the matrix \( A(\zeta) \) has no purely imaginary eigenvalue and its stable subspace \( \mathbb{E}^s(\zeta) \) has dimension \( p \). Furthermore, \( \mathbb{E}^s \) defines an analytic vector bundle over \( \Xi \setminus \Xi_0 \) that can be extended as a continuous vector bundle over \( \Xi \).

For \( (\tau, \eta) \in \Xi_0 \), we define \( \mathbb{E}^s(\tau, \eta) \) to be the continuous extension obtained in Proposition 2.3 of \( \mathbb{E}^s \) to \( (\tau, \eta) \).

Now we describe our assumption of uniform stability, defined as in [7, 3]:

**Definition 2.4.** The problem (1.2) is uniformly stable at \( u = 0 \) if the linearized operators \( (L(\partial_x), B(0)) \) at \( u = 0 \) are such that

\[ (2.4) \quad B(0) : \mathbb{E}^s(\tau - i\gamma, \eta) \to \mathbb{C}^p \text{ is an isomorphism for all } (\tau - i\gamma, \eta) \in \Sigma. \]

**Assumption 2.5.** The problem (1.2) is uniformly stable at \( u = 0 \).

The following is an important example, the Euler equations, which satisfies the above assumptions so that our main results may be applied.

**Example 2.6 (Euler equations).** The following are the isentropic, compressible Euler equations in three space dimensions on the half space \( \{ x_3 \geq 0 \} \), in the unknowns density \( \rho \) and velocity \( u = (u_1, u_2, u_3) \):

\[ (2.5) \quad \partial_t \left( \begin{array}{c} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \end{array} \right) + \partial_{x_1} \left( \begin{array}{c} \rho u_1 \\ \rho u_1 u_1 + p(\rho) \\ \rho u_1 u_2 \\ \rho u_1 u_3 \end{array} \right) + \partial_{x_2} \left( \begin{array}{c} \rho u_2 \\ \rho u_1 u_2 + p(\rho) \\ \rho u_2 u_2 + p(\rho) \\ \rho u_2 u_3 \\ \rho u_2 u_1 \end{array} \right) + \partial_{x_3} \left( \begin{array}{c} \rho u_3 \\ \rho u_1 u_3 \\ \rho u_2 u_3 + p(\rho) \\ \rho u_3 u_3 + p(\rho) \\ \rho u_3 u_2 \\ \rho u_3 u_1 \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right), \]

where \( p(\rho) \) is the pressure. The hyperbolicity assumption, Assumption 2.1, is satisfied in the region of state space where \( \rho > 0 \), \( c^2 = p'(\rho) > 0 \). In this case the eigenvalues \( \lambda_k(\rho, u, \xi) \) are

\[ (2.6) \quad \lambda_1 = u \cdot \xi - c|\xi|, \quad \lambda_2 = u \cdot \xi, \quad \lambda_3 = u \cdot \xi + c|\xi|, \quad \text{with } (\nu_1, \nu_2, \nu_3) = (1, 2, 1). \]

For this problem, the boundary condition we impose is the natural “residual boundary condition,” which is obtained in the vanishing viscosity limit of the compressible Navier-Stokes equations with Dirichlet boundary conditions. Fixing any constant state \( (\rho, u) \) with \( u_3 \notin \{ -c, 0 \} \) about which to linearize the problem, so that \( (\rho, u) = u_0 \) in (1.1), we have noncharacteristic boundary \( \{ x_3 = 0 \} \) for the system. Consider in particular Assumption 2.5 in each of the following cases:

(a) **Subsonic outflow:** \( u_3 < 0, \ |u_3| < c \). In this case there is exactly one positive eigenvalue of \( A_3(\rho, u) \) \( (p = 1) \), so we need one scalar boundary condition. Taking \( b(\rho, u) = u_3 \) in (1.1), we have \( B(0) = [0 \ 0 \ 0 \ 1] \) and Assumption 2.5 is satisfied.

(b) **Subsonic inflow:** \( 0 < u_3 < c \). For this we get \( p = 3 \) and boundary condition \( b(\rho, u) = (\rho u_3, u_1, u_2) \) and linearized operator

\[ (2.7) \quad B(0)(\dot{\rho}, \dot{u}) = (\dot{\rho} u_3 + \rho \dot{u}_3, \dot{u}_1, \dot{u}_2), \]
which satisfies Assumption 2.5.

(c) **Supersonic inflow:** $0 < c < u_3$. This case is trivial, with $p = 4$ and $B(0)$ the $4 \times 4$ identity matrix, and so Assumption 2.5 holds.

(d) **Supersonic outflow:** $u_3 < 0$, $|u_3| > c$. This is another trivial case, with $p = 0$, where $B(0)$ is absent, meaning Assumption 2.5 holds vacuously.

With clear modifications of the above statements, the same holds for the 2D Euler equations, which are in fact strictly hyperbolic. For complete proofs and discussion verifying these cases, we refer the reader to [4], Section 5.

**Regular boundary frequencies**

We now discuss an assumption regarding the boundary frequency $\beta = (\tau, \eta) \in \mathbb{R} \times \mathbb{R}^{d-1} \setminus (0,0)$ which is relevant to the construction of the approximate solution, allowing us to obtain the phases $\phi_m$ and several other objects used in the construction.

**Assumption 2.7.** For $\zeta = (\tau - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1}$ consider the matrix

$$A = \begin{pmatrix} -A_{d-1}^{-1}(0) & (\tau - i\gamma) I + \sum_{j=1}^{d-1} \eta_j A_j(0) \end{pmatrix}. $$

Let $\omega_m$, $m = 1, \ldots, M$ denote the distinct eigenvalues of $A$. We suppose that each $\omega_m$ is a semisimple eigenvalue with multiplicity denoted $\mu_m$. Moreover, we assume there is a conic neighborhood $O$ of $\beta$ in $\mathbb{C} \times \mathbb{R}^{d-1} \setminus \{0\}$ on which the eigenvalues of $A(\zeta)$ are semisimple and given by functions $\omega_m(\zeta)$, $m = 1, \ldots, M$ analytic in $\zeta_0 = \tau - i\gamma$ and smooth in $\eta$, where $\omega_m = \omega_m(\beta)$ and $\omega_m(\zeta)$ is of constant multiplicity $\mu_m$.

We call $\beta$ a regular boundary frequency provided Assumption 2.7 holds. We also have the complex characteristic vector field associated to $\phi_m$:

$$X_{\phi_m} := \partial_{x_d} + \sum_{j=0}^{d-1} -\partial_{\xi} \omega_m(\beta) \partial_{x_j},$$

which is real for real $\omega_m$. Moreover, for each $\omega_m$ which is real, Assumption 2.1 implies there is a unique $k_m \in \{1, \ldots, q\}$ such that $\tau + \lambda_{k_m}(\eta, \omega_m) = 0$, and in this case we have $\mu_m = \nu_{k_m}$.

With this in mind, we make the following definition:

**Definition 2.8.** (i) For $m$ such that $\omega_m$ is real, we call $(\beta, \omega_m)$ a hyperbolic mode if

$$\partial_{\xi} \lambda_{k_m}(\eta, \omega_m) \neq 0. $$

(ii) For $m$ with nonreal $\omega_m$, we call $(\beta, \omega_m)$ an elliptic mode.

**Remark 2.9.** (i) For each real $\omega_m$, Assumption 2.7 guarantees that $(\beta, \omega_m)$ is a hyperbolic mode. That (2.10) holds is a consequence of the semisimplicity of $\omega_m$. Also, the condition (2.10) and the implicit function theorem imply that $\omega_m$ is real and of multiplicity $\nu_{k_m}$ in a neighborhood of $\beta$. Thus, there is a slight redundancy in Assumption 2.7 when $\omega_m$ is real.

(ii) When $\omega_m$ is nonreal, the fact that the $A_j$ are real implies that $\frac{1}{2} A(\beta)$ also has the complex conjugate $\bar{\omega}_m$ as an eigenvalue, and thus $\bar{\omega}_m = \omega_m^*$ for some $m' \neq m$. Furthermore, if the vector $r \in \mathbb{C}^N$ is an eigenvector of $\frac{1}{2} A(\beta)$ associated to $\omega_m$, then $\overline{r}$ is an eigenvector associated to $\overline{\omega}_m = \omega_m^*$.

(iii) The boundary frequency $\beta$ lies in the hyperbolic region (as defined in [2]) if and only if Assumption 2.7 holds with all $\omega_m$ real (with smoothness in real $\zeta_0$ replacing the analyticity condition.) The elliptic region consists of all $\beta$ such that Assumption 2.7 holds with all $\omega_m$ nonreal.

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22Assumption 2.1 has no immediate implication for the multiplicity of nonreal $\omega_m$.

23If $\omega_m$ is real and $\partial_{\xi} \lambda_{k_m}(\eta, \omega_m) = 0$, we refer to $(\beta, \omega_m)$ as a glancing mode. An explanation of how (2.10) follows from semisimplicity can be found in the proof of Lemma 2.7 of [10].
Example 2.10. We return to the 3D Euler equations, linearizing about \((\rho, u) = (\rho, u_1, u_2, u_3)\), as in Example 2.6.

For \(|u_3| > c\) the set of regular boundary frequencies is all of \(\mathbb{R}^3 \setminus 0\). In the case \(|u_3| < c\), one finds that the set of regular boundary frequencies is
\[
(\tau, \eta) \in \mathbb{R}^3 : |\tau + u_1 \eta_1 + u_2 \eta_2| \neq \sqrt{c^2 - u_3^2} |\eta|.
\]

We remark that in the former case, the hyperbolic region is also \(\mathcal{H} = \mathbb{R}^3 \setminus 0\), and in the latter case \(\mathcal{H} = \{(\tau, \eta) \in \mathbb{R}^3 : |\tau + u_1 \eta_1 + u_2 \eta_2| > \sqrt{c^2 - u_3^2} |\eta|\} \).

Remark 2.11. A key feature related to the assumptions which distinguishes our study from that in [2] is that in [2] it is assumed that \(\beta\). In our study, we allow for any regular boundary frequency \(\beta\) thus also handling cases in which any number of the \(\omega_m\) are nonreal.

The partition of the phases into incoming and outgoing hyperbolic phases and elliptic phases with positive and negative imaginary parts

For real \(\omega_m\) we have the associated real group velocity:
\[
v_m := \nabla \lambda_{k_m}(\eta, \omega_m),
\]
which has the following relationship with the characteristic vector field (2.9):
\[
\partial_{\xi_0} \omega_m(\beta) = -\frac{1}{\partial_{\xi_0} \lambda_{k_m}(\eta, \omega_m)}, \quad \partial_{\xi_j} \omega_m(\beta) = -\frac{\partial_{\xi_j} \lambda_{k_m}(\eta, \omega_m)}{\partial_{\xi_0} \lambda_{k_m}(\eta, \omega_m)}, \quad j = 1, \ldots, d - 1.
\]

Observe that each group velocity \(v_m\) can be thought of as either incoming or outgoing with respect to the interior of the domain \(\mathbb{R}^d_+\), in particular since the last coordinate of \(v_m\) is nonzero, by (2.10). With this in mind, we classify the phases in the following way:

Definition 2.12. For real \(\omega_m\), the phase \(\phi_m\) is incoming if the group velocity \(v_m\) is incoming (that is, \(\partial_{\xi_0} \lambda_{k_m}(\eta, \omega_m) > 0\)), and it is outgoing if the group velocity \(v_m\) is outgoing (\(\partial_{\xi_0} \lambda_{k_m}(\eta, \omega_m) < 0\)).

With this classification of the real phases, we let \(\mathcal{I}\) denote the set of indices \(m \in \{1, \ldots, M\}\) such that \(\phi_m\) is incoming and \(\mathcal{O}\) the set of \(m\) such that \(\phi_m\) is outgoing. We classify the remaining complex phases \(\phi_m\), which correspond to nonreal \(\omega_m\), by the distinction that \(\mathcal{P}\) is the set of \(m\) such that \(\text{Im} \omega_m > 0\) and \(\mathcal{N}\) is the set of \(m\) such that \(\text{Im} \omega_m < 0\). We thus form the partition
\[
\{1, \ldots, M\} = \mathcal{I} \cup \mathcal{O} \cup \mathcal{P} \cup \mathcal{N}.
\]

Recall the \(\omega_m(\tau, \eta)\) are eigenvalues of \(\frac{1}{i}A(\tau, \eta)\), meaning the \(i\omega_m(\tau, \eta)\) are the eigenvalues of \(A(\tau, \eta)\). Consider also the fact that \(\text{Im} \omega_m(\tau, \eta) > 0\) if and only if \(\text{Re} i\omega_m(\tau, \eta) < 0\). From these observations, we see that the stable subspace \(E^s(\tau, \eta)\) of \(A(\tau, \eta)\) must contain each of the eigenspaces corresponding to some \(\omega_m(\tau, \eta)\) with Re \(i\omega_m < 0\), so that \(E^s(\tau, \eta)\) contains the subspaces \(L(d\phi_m)\) for \(m \in \mathcal{P}\). The incoming phases, corresponding to \(m \in \mathcal{I}\), also play a role in setting up a decomposition for the stable subspace \(E^s\) at the boundary, which is established in the following lemma.

Lemma 2.13. The stable subspace \(E^s(\tau, \eta)\) admits the decomposition:
\[
E^s(\tau, \eta) = \oplus_{m \in \mathcal{I} \cup \mathcal{P}} L(d\phi_m),
\]
where, in the decomposition, the vector spaces \(L(d\phi_m)\) for \(m \in \mathcal{I}\) admit a basis of real vectors.

Proof. It is easy to show that the subspaces \(L(d\phi_m)\) for \(m \in \mathcal{P}\) are in \(E^s(\tau, \eta)\). We will show that this is also the case for the subspaces \(L(d\phi_m)\) with \(m \in \mathcal{I}\), from which the result will follow. Since \(E^s(\tau, \eta)\)
is close to $E^s(\tau - i\gamma, \eta)$ for small $\gamma > 0$, in accordance with Proposition 2.3, it will suffice to show for $m \in I$ that $i\omega_m(\tau - i\gamma, \eta)$, close to $i\omega_m$, satisfies

\begin{equation}
\text{Re } i\omega_m(\tau - i\gamma, \eta) < 0. \tag{2.16}
\end{equation}

Recalling (2.13), we see that since $\phi_m$ is incoming, i.e. $m \in I$, we have

\begin{equation}
\partial_{\gamma}\omega_m(\tau, \eta) < 0. \tag{2.17}
\end{equation}

Using analyticity of $\omega_m(\zeta)$ with (2.17), since $\omega_m(\tau - i\gamma, \eta)$ has imaginary part equal to zero, it follows that $\omega_m(\tau - i\gamma, \eta)$ has positive imaginary part for small positive $\gamma$. Therefore (2.16) holds. The statement that each of the vector spaces $\text{Ker } L(d\phi_m)$ for $m \in I$ admits a basis of real vectors follows from Lemma 1.2, which was proved in [1].

3 Profile equations: formulation with periodic profiles

Definition 3.1. (i) We define $Z^M \subset \mathbb{Z}^M$ by

\begin{equation}
Z^M := \{\alpha = (\alpha_i)_{i=1}^M : \alpha_i \in Z_i\}, \tag{3.1}
\end{equation}

where $Z_i$ is defined by

\begin{equation}
Z_i := \begin{cases}
\mathbb{Z} & \text{for } i \in I \cup O, \\
\mathbb{Z}^+ & \text{for } i \in P, \\
\mathbb{Z}^- & \text{for } i \in N,
\end{cases} \tag{3.2}
\end{equation}

where $I$, $O$, $P$, and $N$ are as defined in the comments following Definition 2.1. We also have the equivalent formulation $Z_i = \{n \in \mathbb{Z} : n\text{Im }\omega_i \geq 0\}$.

(ii) We also define $Z^{M,k} := \{\alpha \in Z^M : \text{ at most } k \text{ components of } \alpha \text{ are nonzero}\}$. \tag{3.3}

Definition 3.2. For $k = 1, 2$, we define the following spaces:

\begin{equation}
H^{s,k}(\mathbb{R}^{d+1}_+ \times T^M) = \left\{ V(x, \theta) \in H^s(\mathbb{R}^{d+1}_+ \times T^M) : V(x, \theta) = \sum_{\alpha \in Z^{M,k}} V_\alpha(x) e^{i\alpha \cdot \theta} \right\}. \tag{3.4}
\end{equation}

For $s > (d + 1 + 2)/2$, it is clear that multiplication defines a continuous map

\begin{equation}
H^{s,1}(\mathbb{R}^{d+1}_+ \times T^M) \times H^{s,1}(\mathbb{R}^{d+1}_+ \times T^M) \to H^{s,2}(\mathbb{R}^{d+1}_+ \times T^M). \tag{3.5}
\end{equation}

Remark 3.3. We define

\begin{equation}
C^M := C_1 \times C_2 \times \cdots \times C_M, \tag{3.6}
\end{equation}

where $C_i$ is defined by

\begin{equation}
C_i := \begin{cases}
\mathbb{R} & \text{for } i \in I \cup O, \\
\{n \in \mathbb{Z} : n\text{Im }\omega_i \geq 0\} & \text{for } i \in P, \\
\{n \in \mathbb{Z} : n\text{Im }\omega_i \leq 0\} & \text{for } i \in N.
\end{cases} \tag{3.7}
\end{equation}

For $V \in H^{s,2}(\mathbb{R}^{d+1}_+ \times T^M)$ where $s > d/2 + 3$, one can show that $\text{spec } V \subset Z^{M,2}$ implies $V$ extends holomorphically in $\theta$ to the interior of $C^M$. In particular, this uses the fact that then $\text{Im}(\alpha_i \theta_i) \geq 0$ for $i \in P \cup N$. This allows us to make sense of

\begin{equation}
U(x, \theta_0, \xi_d) := V(x, \theta_0 + \omega_1 \xi_d, \ldots, \theta_0 + \omega_M \xi_d) \tag{3.8}
\end{equation}

for $\theta_0 \in \mathbb{T}$, $\xi_d \geq 0$.\n
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3.1 The solvability condition

Our periodic ansatz has the form

\[ V^0(x, \theta) = \varphi(x) + \sum_{m=1}^{M} \sum_{k=1}^{\mu_m} \sigma_{m,k}(x, \theta_m)r_{m,k}, \]

a function in \( H^{s_1}(\mathbb{R}^{d+1} \times \mathbb{T}^M) \), where \( s \) is to be specified, and which is holomorphic in \( \theta \in \mathbb{C}^M \subset \mathbb{C}^M \) (in each single variable \( \theta_m \)) in particular. To construct \( V^0 \) (we still need to solve for the profiles) and its corrector \( V^1 \), we need to define the projection operators \( E \) and \( E^p \), which are used to get the profile equations to be solved for the profiles.

**Definition 3.4.** Setting \( \phi := (\phi_1, \ldots, \phi_M) \), we call \( \alpha \in \mathbb{Z}^{M:2} \) a characteristic mode provided \( \det L(d(\alpha \cdot \phi)) = 0 \) and write \( \alpha \in \mathcal{C} \). We decompose

\[ \mathcal{C} = \bigcup_{m=1}^{M} \mathcal{C}_m, \]

where \( \mathcal{C}_m = \{ \alpha \in \mathbb{Z}^{M:2} : \alpha \cdot \phi = n_n \phi_m \text{ for some } n_n \in \mathbb{Z} \} \).

Now we are ready to define the projectors which will give us the profile equations and the differential equation for the condition (1.39).

**Definition 3.5.** Let \( V \in H^{s_1+1:2}_T \). The action of \( E \) on \( V \) is defined by

\[ E = E_0 + \sum_{m=1}^{M} E_m, \quad \text{where } E_0V = V_0 \text{ and } E_mV = \sum_{\alpha \in \mathcal{C}_m \setminus \{0\}} P_m V_\alpha(x)e^{i\alpha \cdot \theta_m}, \]

where \( P_m \) denotes the projection onto Ker \( L(d\phi_m) \), the action of \( E^p \) on \( V \) is defined by

\[ E^p = E_0 + \sum_{m=1}^{M} E_m^p, \quad \text{where } E_0^pV = V_0 \text{ and } E_m^pV = \sum_{\alpha \in \mathcal{C}_m \setminus \{0\}} P_m V_\alpha(x)e^{i\alpha \cdot \theta_m}, \]

and we use the notation

\[ E_h = E_0 + \sum_{m \in \mathcal{I} \cup \mathcal{O}} E_m, \quad E_c = \sum_{m \in \mathcal{P} \cup \mathcal{N}} E_m. \]

**Remark 3.6.** (i) It is shown in [2] that \( E \) is a continuous map in the \( H^s \) norm. Here, to complete the statement \( E : H^{s:2} \to H^{s:1} \), we must also have for each \( \alpha \in \mathcal{C}_m \) that \( n_n \in \mathbb{Z}_m \). This is easily verified from the definitions of \( Z_m \), \( C_m \), and the \( n_n \). A similar calculation is done in Remark 3.12. (ii) It is not hard to show the continuity of \( E^p : H^{s:2}_T \to H^{s:2}_T \).

**Remark 3.7.** Denoting evaluation at \( \theta = \theta(\theta_0, \xi_d) \) by \( \Phi \), observe \( \Phi \circ E = \Phi \circ E^p \). The projector \( E \) serves as the main tool for solving for \( \varphi(x) \) and the periodic profiles \( \sigma_{m,k}(x, \theta_m) \) of our ansatz. The projector \( E^p \) is key to describing solvability (such as the condition (1.39)) and is thus used in the error analysis in solving away an \( O(1) \) error output of \( \Phi \circ (I - E) \) written in the form \( \Phi \circ (I - E^p) \), so that we can prove Theorem 4.7. That is the purpose which the following proposition serves.

**Proposition 3.8.** Given \( H \in H^{s:2}_T(x, \theta) \) with finitely many nonzero Fourier coefficients, suppose

\[ E^pH = 0, \]

Then there exists \( V \in H^{s:2}_T(x, \theta) \) such that

\[ \mathcal{L}(\partial_\theta)V = H. \]
Proof. We write out the series of $\mathcal{H}$ as

\begin{equation}
\mathcal{H}(x, \theta) = \sum_{\alpha \in \mathbb{Z}^{M/2}} H_\alpha(x)e^{i\alpha \cdot \theta}.
\end{equation}

Now we search for $V_\alpha(x)$ such that

\begin{equation}
V(x, \theta) := \sum_{\alpha \in \mathbb{Z}^{M/2}} V_\alpha(x)e^{i\alpha \cdot \theta},
\end{equation}

satisfies

\begin{equation}
\mathcal{L}(\partial_\theta)V(x, \theta) = \sum_{m=1}^{M} \sum_{\alpha \in \mathbb{Z}^{M/2}} i\alpha_m \tilde{L}(d\phi_m)V_\alpha(x)e^{i\alpha \cdot \theta} = \sum_{\alpha \in \mathbb{Z}^{M/2}} H_\alpha(x)e^{i\alpha \cdot \theta}.
\end{equation}

This holds if and only if for all $\alpha \in \mathbb{Z}^{M/2}$

\begin{equation}
i\tilde{L} \left( \sum_m \alpha_m \beta, \alpha \cdot \omega \right) V_\alpha(x) = H_\alpha(x),
\end{equation}

where $\omega = (\omega_1, \ldots, \omega_M)$. We handle first the case where $\alpha \in \mathcal{C}_j \setminus 0$ for some $j = 1, 2, \ldots, M$. Note that

\begin{equation}
0 = E_0^j \mathcal{H}(x, \theta) = \sum_{\alpha \in \mathcal{C}_j \setminus 0} P_j H_\alpha(x)e^{i\alpha \cdot \theta},
\end{equation}

which implies for each $\alpha \in \mathcal{C}_j$

\begin{equation}
0 = P_j H_\alpha(x).
\end{equation}

Recalling from Lemma 1.2 that $\text{Im} \ A_d^{-1}L(d\phi) = \text{Ker} \ P_j$, we see there exists $W_\alpha(x) \in H^s_T(x)$ such that

\begin{equation}
H_\alpha(x) = i\tilde{L}(d\phi_j)W_\alpha(x).
\end{equation}

Observe then

\begin{align}
H_\alpha(x) &= i\tilde{L}(n_\alpha \beta, n_\alpha \omega_j) \frac{1}{n_\alpha} W_\alpha(x), \\
&= i\tilde{L} \left( \sum_{m=1}^{M} \alpha_m \beta, \alpha \cdot \omega \right) \frac{1}{n_\alpha} W_\alpha(x).
\end{align}

So we may satisfy (3.19) by defining $V_\alpha(x) = \frac{1}{n_\alpha} W_\alpha(x)$.

For the case $\alpha = 0$, where $H_\alpha = E_0^0 \mathcal{H} = 0$, or for any other $\alpha$ such that $H_\alpha = 0$, we may satisfy (3.19) by taking $V_\alpha = 0$.

For $\alpha \notin \mathcal{C}$ with $H_\alpha \neq 0$, we have $\text{det} \ L(\sum_m \alpha_m \beta, \alpha \cdot \omega) \neq 0$ by definition of $\mathcal{C}$, so then (3.19) can be solved directly for $V_\alpha(x)$. Since there are only finitely many nonzero $H_\alpha$, we have no issues with convergence. The same holds for the $V_\alpha$, and so the sum in (3.17) clearly describes an element of $H^s_T(x, \theta)$.

Remark 3.9. (i) In view of Proposition 3.8, in order to construct a corrector $\mathcal{V}^1$ appropriate for our periodic ansatz $\mathcal{V}^0$ to satisfy (1.38), it is tempting to require something like

\begin{equation}
E^s(\tilde{L}(\partial_x)\mathcal{V}^0 + \mathcal{M}(\mathcal{V}^0)\partial_\theta\mathcal{V}^0) = E^s(F(0)\mathcal{V}^0).
\end{equation}

However, note that to use Proposition 3.8, if the $\mathcal{V}^0$ we seek has infinitely many nonzero Fourier coefficients, (3.25) alone is insufficient; we must use finite trigonometric polynomial approximations. Furthermore, even if $\mathcal{V}^0$ is replaced with a finite trigonometric polynomial approximation, it is not clear how one might solve (3.25) for $\mathcal{V}^0$ of the form (3.9). For an explanation, consider the components $E_m^s$ of the projector $E^s$. The
goal is to obtain a profile, say, \( \sigma_{m,k}(x, \theta_m) \) in (3.9) by examining the \( m \)th component of some projector. However, \( E_m^\sigma \) maps into \( H^{s,2}(x, \theta) \), meaning its output generally varies with more than one component of \( \theta \) as opposed to varying with only the \( \theta_m \) component, and so we have little hope of getting from it an equation for \( \sigma_{m,k}(x, \theta_m) \). Meanwhile, \( E \) has components \( E_m^\sigma \) mapping into \( H^s(x, \theta_m) \), spaces suited to \( \sigma_{m,k}(x, \theta_m) \), and these will give us the equations for the \( \sigma_{m,k}(x, \theta_m) \). In fact, \( V^0 \) having the form (3.9) is equivalent to having

\[
(3.26) \quad E V^0 = V^0.
\]

\( (ii) \) With the considerations made in Remark 3.7, it is natural to require instead of (3.25) that

\[
(3.27) \quad E(\tilde{L}(\partial_x)V^0 + M(V^0)\partial_\theta V^0) = E(F(0)V^0).
\]

Observe that if we satisfy (3.27), we have

\[
(3.28) \quad (I - E)(\tilde{L}(\partial_x)V^0 + M(V^0)\partial_\theta V^0 - F(0)V^0) = \tilde{L}(\partial_x)V^0 + M(V^0)\partial_\theta V^0 - F(0)V^0.
\]

Once \( V^0 \) is determined by imposing (3.27), with Proposition 3.8 one can solve for a corrector \( V^1 \) in

\[
(3.29) \quad \mathcal{L}(\partial_\theta)V^1 = -(I - E)(\tilde{L}(\partial_x)V^0 + M(V^0)\partial_\theta V^0 - F(0)V^0),
\]

(if we assume for the moment that we are working with finite trigonometric polynomials.) Although such \( V^1 \) does not necessarily satisfy (1.38), after evaluating at \( \theta = \theta_0, \zeta_d \), which, recall, is denoted by \( \Phi \), it follows from (3.28) and (3.29) that

\[
(3.30) \quad \Phi \left( \mathcal{L}(\partial_\theta)V^1 \right) = -\Phi \left( \tilde{L}(\partial_x)V^0 + M(V^0)\partial_\theta V^0 - F(0)V^0 \right),
\]

where we have used the property \( \Phi \circ (I - E) = \Phi \circ (I - E) \). Thus (3.30) implies that we have a corrected approximate solution solving (1.2)(a) to order \( O(1) \) (disregarding for now the error introduced by using trigonometric approximations.)

It is from the components of (3.27) that we obtain the components \( \tilde{\nu}_s, \sigma_{m,k}, m = 1, \ldots, M, k = 1, \ldots, \mu_m, \) of \( V^0 \). Now we collect the set of equations which determines \( V^0(x, \theta) \in H^s_+(\mathbb{R}_+^{d+1} \times T^M) \) for \( s \) sufficiently large (specified later):

\[
(3.31) \begin{align*}
\text{a)} \quad & E V^0 = V^0 \\
\text{b)} \quad & E \left( \tilde{L}(\partial_x)V^0 + M(V^0)\partial_\theta V^0 \right) = E(F(0)V^0) \text{ in } x_d \geq 0 \\
\text{c)} \quad & B(0)V^0(x',0,\theta_0,\ldots,\theta_0) = G(x',\theta_0) \\
\text{d)} \quad & V^0 = 0 \text{ in } t < 0.
\end{align*}
\]

To summarize, (3.31)(a) implies \( \mathcal{L}(\partial_\theta)V^0 = 0 \) so that (1.2)(a) is solved to order \( \frac{1}{\epsilon} \). Equation (3.31)(b) represents the solvability conditions for the existence of a corrected approximate solution, and equations (3.31)(c) and (3.31)(d) straightforwardly correspond to (1.2)(b) and (1.2)(c).

### 3.2 The large system for individual profiles.

The next major task is to rephrase (3.31)(b) in terms of \( \tilde{\nu} \) and the \( \sigma_{m,k} \) to get the interior profile equations. Equation (3.31)(b) will be broken down into components with \( E_0 \) and \( E_{m,k}, m = 1, \ldots, M, k = 1, \ldots, \mu_m, \) replacing \( E \), yielding equations for \( \tilde{\nu} \) and the \( \sigma_{m,k} \).

Recall the decomposition of the projector

\[
(3.32) \quad E = E_0 + \sum_{m=1}^M E_m
\]
For \( m = 1, \ldots, M \), we enumerate by \( \{ \ell_{m,k}, k = 1, \ldots, \nu_m \} \) a basis of vectors for the left eigenspace of

\[
i A(\beta) = A_d^{-1}(0)(\tau I + \sum_{j=1}^{d-1} A_j(0)\eta_j)
\]

associated to the eigenvalue \(-\omega_m\), and with the following property:

\[
\ell_{m,k} \cdot r_{m',k'} = \begin{cases} 
1, & \text{if } m = m' \text{ and } k = k' \\
0, & \text{otherwise}
\end{cases}
\]

For \( v \in \mathbb{C}^N \) set

\[
P_{m,k}v = (\ell_{m,k} \cdot v)r_{m,k} \quad \text{(no complex conjugation here)}.
\]

So now we may write

\[
E = E_0 + \sum_{m=1}^{M} \sum_{k=1}^{\nu_m} E_{m,k},
\]

where

\[
E_{m,k}(V_\alpha e^{i\alpha \cdot \theta}) := \begin{cases} 
(P_{m,k}V_\alpha)e^{i\alpha \cdot \theta_m}, & \alpha \in C_m \setminus \{0\} \\
0, & \text{otherwise}
\end{cases} ; \text{i.e., } E_{m,k} = P_{m,k}E_m.
\]

Having defined the \( E_{m,k} \), we can proceed with the explicit derivation of the large system in terms of \( v(x) \) and the \( \sigma_{m,k}(x,\theta_m) \) by applying \( E_0 \) and \( E_{m,k} \) to (3.31)(b). We begin with the identity of the following lemma, whose proof we omit here, as it is almost identical to that of Lemma 2.11 of [2].

**Lemma 3.10.** Suppose \( E V^0 = V^0 \). Then

\[
E_{m,k}(\tilde{L}(\partial_x)V^0) = (X_{\phi_m} \sigma_{m,k})r_{m,k}
\]

where \( X_{\phi_m} \) is the characteristic vector field associated to \( \phi_m \):

\[
X_{\phi_m} := \partial_{x_d} + \sum_{j=0}^{d-1} -\partial_{x_j} \omega_m(\beta) \partial_{x_j}.
\]

**Resonances**

Let us fix \( p \in \{1, \ldots, M\} \) and consider the projector \( E_p \) in an instance where there exists a relation of the form

\[
n_p \phi_p = n_q \phi_q + n_r \phi_r \quad \text{where } q, r \neq p \text{ and } (n_p, n_q, n_r) \in \mathbb{Z} \times \mathbb{Z}_q \times \mathbb{Z}_r.
\]

It takes some effort to evaluate certain terms arising from the quantity \( E_p(M(V^0)\partial_{\phi}V^0) \) which then depend on the profiles \( \sigma_{q,k} \) and \( \sigma_{r,k'} \), in particular. As discussed in Section 1.2, we must account for the ways in which the \( \phi_q \) oscillations interact with the \( \phi_r \) oscillations to produce \( \phi_p \) oscillations. This is the purpose of the interaction integrals, (3.47), appearing in the profile equations. To handle these difficulties we introduce some new definitions regarding resonances and interaction integrals.

**Definition 3.11.** We say \((\phi_p, \phi_q, \phi_r)\) is an ordered triple of resonant phases and that \((\phi_q, \phi_r)\) forms a \( p \)-resonance if

\[
n_p \phi_p = n_q \phi_q + n_r \phi_r
\]

for some \( (n_p, n_q, n_r) \in \mathbb{Z} \times \mathbb{Z}_q \times \mathbb{Z}_r \), each entry nonzero. The triple \((\phi_p, \phi_q, \phi_r)\) is called normalized if we have both (i) \( \gcd(n_p, n_q, n_r) = 1 \) and (ii) if \( p \in I \cup O \), then \( n_p > 0 \).
We explicitly treat the case that the only normalized triples of resonant phases are permutations of \((\phi_p, \phi_q, \phi_r)\) corresponding to one of the six rearrangements of (3.41). It follows from the next remark that the sign of \(n_p\) is determined, so this normalization uniquely determines \(n_p, n_q, n_r\).

**Remark 3.12.** For a triple of resonant phases \((\phi_p, \phi_q, \phi_r)\) with (3.41), one has \((n_p, n_q, n_r) \in Z_p \times Z_q \times Z_r\). To see this, taking the imaginary part of \(n_p \phi_p = n_q \phi_q + n_r \phi_r\), observe

\[
(3.42) \quad n_p \Im \omega_p = n_q \Im \omega_q + n_r \Im \omega_r.
\]

Thus, if \((\phi_q, \phi_r)\) forms a \(p\)-resonance, since \(n_q \Im \omega_q\) and \(n_r \Im \omega_r\) are nonnegative, so is \(n_p \Im \omega_p\), so it follows that \(n_p \in Z_p\).

We now make a related observation with the following proposition which will greatly simplify our solution of the interior equations.

**Proposition 3.13.** (i) A hyperbolic resonance (i.e. a \(p\)-resonance, where \(p \in I \cup O\)) can only be formed by a pair of hyperbolic phases, and (ii) an elliptic resonance can only be formed by a pair including at least one elliptic phase.

**Proof.** (i) Suppose for some \(p \in I \cup O\) that \(n_p \phi_p = n_q \phi_q + n_r \phi_r\) with \((n_p, n_q, n_r) \in Z_p \times Z_q \times Z_r\). Considering the imaginary part gives

\[
(3.43) \quad 0 = n_q \Im \omega_q + n_r \Im \omega_r,
\]

so recalling \(n_i \in Z_i\), \(n_i \Im \omega_i \geq 0\), we see \(\Im \omega_q = \Im \omega_r = 0\). Hence \(q, r \in I \cup O\). The proof of (ii) is similar.

We proceed by defining the interaction integrals which will appear in the interior equations.

**Definition 3.14.** Suppose \((\phi_q, \phi_r)\) forms a normalized \(p\)-resonance, with

\[
(3.44) \quad n_p \phi_p = n_q \phi_q + n_r \phi_r.
\]

For any \(f \in H^\infty_r(x, \theta_q)\) define \(f_{n_q} \in H^\infty_r(x, \theta_q)\) to be the image of \(f\) under the preparation map

\[
(3.45) \quad f(x, \theta_q) = \sum_{k \in \mathbb{Z}} f_k(x)e^{ik\theta_q} \rightarrow \sum_{k \in \mathbb{Z}} f_{kn_q}(x)e^{ikn_q\theta_q}.
\]

Suppose \(s > \frac{d+3}{2} + 1\) and that \(\sigma_{q,k}, \sigma_{r,k} \in H^\infty_r(\mathbb{R}_+^{d+1} \times \mathbb{T})\). We get exactly two normalized \(p\)-resonance formations from the arrangements of (3.44):

\[
(3.46) \quad n_p \phi_p = n_q \phi_q + n_r \phi_r, \quad n_p \phi_p = n_r \phi_r + n_q \phi_q.
\]

To these equations we associate, respectively, the two families of prepared integrals:

\[
\begin{align*}
J_{p,n_q,n_r}^{k,k'}(x, \theta_q) := & \frac{1}{2\pi} \int_0^{2\pi} (\sigma_{q,k})_{n_q} \left( x, \frac{n_p}{n_q} \theta_p - \frac{n_r}{n_q} \theta_r \right) \partial \sigma_{q,k'}(x, \theta_r) d \theta_r, \quad k \in \{1, \ldots, \mu_q\}, k' \in \{1, \ldots, \mu_r\}, \\
J_{p,n_r,n_q}^{k,k'}(x, \theta_q) := & \frac{1}{2\pi} \int_0^{2\pi} (\sigma_{r,k})_{n_r} \left( x, \frac{n_p}{n_r} \theta_p - \frac{n_q}{n_r} \theta_q \right) \partial \sigma_{q,k'}(x, \theta_q) d \theta_q, \quad k \in \{1, \ldots, \mu_r\}, k' \in \{1, \ldots, \mu_q\}.
\end{align*}
\]

**Remark 3.15.** Strictly speaking, \(J_{p,n_q,n_r}^{k,k'}\) and \(J_{p,n_r,n_q}^{k,k'}\) are symbolically \(I_{n_q,n_p,n_r}^{k,k'}\) and \(I_{n_r,-n_q,n_r}^{k,k'}\) (as defined in [2], with respect to the triple \((\phi_q, \phi_p, \phi_r)\) and \(n_q \phi_q = n_p \phi_p + n_r \phi_r\) where \(n'_r = -n_r\)). The difference is that we do not require \(n_q > 0, q < p < r\) — recall, instead, our main requirement is that \((n_p, n_q, n_r) \in Z_p \times Z_q \times Z_r\).
The following proposition shows that the prepared integrals ‘pick out’ the \( p \)-resonances.

**Proposition 3.16.** Suppose \( s > \frac{d+3}{2} + 1 \) and that \( \sigma_{q,k}, \sigma_{r,k}' \in H_T^s(\mathbb{R}^{d+1}_+ \times \mathbb{T}) \) have Fourier series

\[
\sigma_{q,k}(x, \theta_q) = \sum_{j \in \mathbb{Z}_q \setminus 0} a_j(x)e^{ij\theta_q} \quad \text{and} \quad \sigma_{r,k'}(x, \theta_r) = \sum_{j \in \mathbb{Z}_r \setminus 0} b_j(x)e^{ij\theta_r}.
\]

Given \((\phi_q, \phi_r)\) forms a normalized \( p \)-resonance, the prepared integral \( J_{p,n_q,n_r}(x, \theta_p) \) belongs to \( H_T^{s-1}(x, \theta_p) \) and has Fourier series

\[
J_{p,n_q,n_r}(x, \theta_p) = \sum_{j \in \mathbb{Z}} a_{jn_q}(x)b_{jn_r}(x)i \cdot (jn_r)e^{ijn_r\theta_p}.
\]

For the other integral in (3.47), one switches \( n_q \) and \( n_r \) above. Moreover, the functions \( J_{p,n_q,n_r}(x, \theta_p) \) can be described for \( \theta_p \in \mathbb{C}_p \) through analytic extension of the expansions into the complex half plane \( \mathbb{C}_p \) in the same way that we extend \( \mathcal{V}_p(x, \theta) \) to \( \mathbb{C}_M \) and each of the profiles \( \sigma_{m,k}(x, \theta_m) \) into \( \mathbb{C}_m \) in the manner discussed in Remark 3.3.

**Remark 3.17.** We note that for \( p \in \mathcal{P} \cup \mathcal{N} \), in (3.49), we can sum over just \( j \in \mathbb{Z}^+ \setminus 0 \). To see this, observe that then \((\phi_q, \phi_r)\) forms an elliptic resonance, so Proposition 3.13 guarantees one of \( \phi_q, \phi_r \) is elliptic – without loss of generality, say \( \phi_q \). Thus, \( \mathbb{Z}_q \setminus 0 \) consists only of negative integers or positive integers, and the only nonzero Fourier coefficients of \( \sigma_{q,k} \) are those \( a_j(x) \) appearing in (3.48), with \( j \in \mathbb{Z}_q \setminus 0 \); we have \( a_j(x) = 0 \) for all other \( j \). For example, if \( q \in \mathcal{P} \), then \( \mathbb{Z}_q = \mathbb{Z}^+ \), and so, noting \( n_q > 0 \), we have the \( a_{jn_q}(x) \) appearing in (3.49) are only nonzero for \( j > 0 \). Indeed, \( q \in \mathcal{N} \) implies the same of the \( a_{jn_q}(x) \), so we may sum over just \( j \in \mathbb{Z}^+ \setminus 0 \) in either case. We add that, for such \( j \), the \( e^{ijn_r\theta_r} \) factors are bounded as \( \theta_p \) ranges over \( \mathbb{C}_p \).

**Proof of Proposition 3.16.** The main part of the proof is showing that the expansion (3.49) converges to the desired result in \( H_T^{s-1}(x, \theta_p) \), where we have restricted our attention to real \( \theta_p \). The proof of this is very similar to the proof of Proposition 2.13 from [2]. Analytic extension into the complex half plane \( \mathbb{C}_p \) is done in the same way that the expansions for the profiles are extended into their corresponding complex half planes, as in Remark 3.3. \( \square \)

**Interior equations**

By applying \( \mathcal{E}_q \) and the \( \mathcal{E}_{m,k} \) to (3.31)(b), we obtain the system for \( \nu(x) \) and the \( \sigma_{m,k}(x, \theta_m) \) with the following proposition.

**Proposition 3.18.** Suppose \((\phi_p, \phi_r)\) forms a normalized \( p \)-resonance, with

\[
n_q \phi_p = n_q \phi_q + n_r \phi_r.
\]

and that all other normalized triples of resonant phases are permutations of \((\phi_p, \phi_q, \phi_r)\) each corresponding to one of the six rearrangements of (3.50).

(i) Given \( \mathcal{V}_q \) as in (3.26) and represented by (3.9), the equation (3.31)(b) is equivalent to the following system:

\[
\mathcal{L}(\partial_x) \mathcal{V} + \sum_{j=0}^{d-1} \sum_{m=1}^{M} \sum_{k,k'=1}^{\mu_m} \frac{1}{2\pi} \left( \int_0^{2\pi} \sigma_{m,k}(x, \theta_m) \partial_{\theta_m} \sigma_{m,k'}(x, \theta_m) d\theta_m \right) R_{j,m}^{k,k'} = F(0) \mathcal{V}.
\]
The constant scalars $R^{k,k'}_{3.56}$

(3.54)

(a) $X_{\phi_p} \sigma_{p,l}(x, \theta_p) + \sum_{j=0}^{d-1} \sum_{k'=1}^{\mu_p} a^{k'}_{p,l,j}(\nu) \partial_{\theta_j} \sigma_{p,k'}(x, \theta_p) +$

(b) $\sum_{j=0}^{d-1} \sum_{k=1}^{\mu_p} \sum_{k'=1}^{\mu_p} b^{k,k'}_{p,l,j} \sigma_{p,k}(x, \theta_p) \partial_{\theta_j} \sigma_{p,k'}(x, \theta_p) - \sum_{j=0}^{d-1} \sum_{k=1}^{\mu_p} \sum_{k'=1}^{\mu_p} b^{k,k'}_{p,l,j} \frac{1}{2\pi} \int_0^{2\pi} \sigma_{p,k}(x, \theta_p) \partial_{\theta_j} \sigma_{p,k'}(x, \theta_p) d\theta_p +$

(c) $\sum_{j=0}^{d-1} \sum_{k=1}^{\mu_p} J_{p,l,j}(x, \theta_p) = \sum_{k=1}^{\mu_p} e^k_{p,l} \sigma_{p,k}(x, \theta_p)$

for $p \in \mathcal{I} \cup \mathcal{O}$, $l \in \{1, \ldots, \mu_p\}$,

where

(3.55) $J_{p,l,j}(x, \theta_p) =$

(a) $X_{\phi_p} \sigma_{p,l}(x, \theta_p) + \sum_{j=0}^{d-1} \sum_{k'=1}^{\mu_p} a^{k'}_{p,l,j}(\nu) \partial_{\theta_j} \sigma_{p,k'}(x, \theta_p) +$

(b) $\sum_{j=0}^{d-1} \sum_{k=1}^{\mu_p} \sum_{k'=1}^{\mu_p} b^{k,k'}_{p,l,j} \sigma_{p,k}(x, \theta_p) \partial_{\theta_j} \sigma_{p,k'}(x, \theta_p) +$

(c) $\sum_{j=0}^{d-1} \sum_{k=1}^{\mu_p} J_{p,l,j}(x, \theta_p) = \sum_{k=1}^{\mu_p} e^k_{p,l} \sigma_{p,k}(x, \theta_p)$

for $p \in \mathcal{P} \cup \mathcal{N}$, $l \in \{1, \ldots, \mu_p\}$,

where

(3.56) $J_{p,l,j}(x, \theta_p) =$

The constant vectors $R^{k,k'}_{3.52}$ appearing in (3.51) are given by

(3.57) $R^{k,k'}_{3.52} = \beta_{j} (\partial_{u} A_{j}(0) \cdot r_{m,k}) r_{m,k'}.$

The constant scalars $b^{k,k'}_{p,l,j}$, $e^{k,k'}_{p,l,j}$, $b^{k,k'}_{p,l,j}$, and the coefficients of the scalar linear function of $\nu$, $a^{k,k'}_{p,l,j}(\nu)$, in (3.52)-(3.55) are given by similar formulas, but now involving dot products with the vector $\epsilon_{p,l}$.

(ii) Equations (3.51), (3.52), form a hyperbolic sub-system, that is, a system in $\nu$ and $\sigma_{p,l}$ for $p \in \mathcal{I} \cup \mathcal{O}$ independent of $\sigma_{q,k}$ for all $q \in \mathcal{P} \cup \mathcal{N}$.

Remark 3.19. While the hyperbolic interior equations are independent of the elliptic profiles, the elliptic interior equations (3.54) are not in general independent of the hyperbolic profiles, since these appear in the elliptic interaction integrals if some phase paired with a hyperbolic phase forms an elliptic resonance.
Proof of Proposition 3.18. Regarding (i), (3.51) follows from application of $E_0$ to (3.31)(b), and (3.52) from application of $E_{p,l}$ to (3.31)(b). One similarly arrives at (3.54): in doing so, one finds that the second triple sum appearing in (3.52)(b) is zero for $p \in \mathcal{P} \cup \mathcal{N}$, since then spec $\sigma_{p,k}$, spec $\sigma_{p,k'} \subset Z_p \setminus 0$ which is either $Z^+ \setminus 0$ or $Z^- \setminus 0$, resulting in (3.54)(b). The differences between (3.53) and (3.55) merely account for the fact that for each nonreal $\omega$, the fact that for each nonreal $\omega$, we have that the set (3.58) is a basis for (3.60). From (3.31)(c), we get

$$r_m = \frac{\nu_m}{\nu_m} \sum_{\nu_m \in \mathcal{I} \cup \mathcal{P} \cup \mathcal{N}} \frac{\nu_m}{\nu_m} \sigma_{m,k}(x', 0, \theta_0) r_m + \frac{\nu_m}{\nu_m} \sum_{\nu_m \in \mathcal{O}} \sum_{k=1}^{\nu_m} \sigma_{m,k}(x', 0, \theta_0) r_m,$$

where $\nu_m$ denotes the mean zero part of the periodic function $\nu_m$, and we have similar for $G^*$. While (3.57)(a) gives boundary data for $r$, it remains to establish satisfactory boundary conditions on the individual profiles $\sigma_{m,k}$ from (3.57)(b). This is done with Proposition 3.21.

Lemma 3.20. Each of the sets of vectors

$$\{B(0)r_{m,k} : m \in \mathcal{I} \cup \mathcal{P}, k = 1, \ldots, \mu_m \},$$

(3.58)

$$\{B(0)r_{m,k} : m \in \mathcal{I} \cup \mathcal{N}, k = 1, \ldots, \mu_m \},$$

(3.59)

is a basis of $C^p$.

Proof. First we prove (3.58) is a basis of $C^p$. Recall our assumption of uniform stability, Assumption (2.5), which tells us that $B(0)$ maps a basis for the stable subspace $E^*(\tau, \eta)$ to a basis for $C^p$. Thus, according to Lemma 2.13, which states that

$$E^*(\tau, \eta) = \oplus_{\nu_m \in \mathcal{I} \cup \mathcal{P}} \text{Ker} L(d\phi_m),$$

(3.60)

we have that the set (3.58) is a basis of $C^p$. To see the same holds for (3.59), recall from Remark 2.9(ii) the fact that for each nonreal $\omega_m$, say, without loss of generality, $m \in \mathcal{P}$, there is another eigenvalue $\omega_m$, satisfying $\omega_m = \omega_m$, thus with $m' \in \mathcal{N}$, and that similar holds for the associated eigenvectors. It follows that

$$\{B(0)r_{m,k} : m \in \mathcal{I} \cup \mathcal{N}, k = 1, \ldots, \mu_m \} = \{B(0)r_{m,k} : m \in \mathcal{I} \cup \mathcal{P}, k = 1, \ldots, \mu_m \},$$

(3.61)

$$\{B(0)r_{m,k} : m \in \mathcal{I} \cup \mathcal{P}, k = 1, \ldots, \mu_m \} = \{B(0)r_{m,k} : m \in \mathcal{I} \cup \mathcal{P}, k = 1, \ldots, \mu_m \}. $$

(3.62)

Clearly, (3.62) spans the same subspace as (3.58), i.e. $C^p$, and so (3.59) is also a basis of $C^p$.
Proposition 3.21. The data \((\sigma_{m,k}(x',0,\theta_0); m \in \mathcal{I} \cup \mathcal{P} \cup \mathcal{N}, k \in \{1, \ldots, \mu_m\})\) are determined by the data \((\sigma_{m,k}(x',0,\theta_0); m \in \mathcal{O}, k \in \{1, \ldots, \mu_m\})\) in that there exist constant matrices \(M^\pm\) such that the zero-mean boundary condition,

(3.63) \[ B(0)\mathcal{V}_{0}^{m}(x',0,\theta_0,\ldots,\theta_0) = B(0) \left( \sum_{m=1}^{\mu_m} \sum_{k=1}^{\mu_m} \sigma_{m,k}(x',0,\theta_0)r_{m,k} \right) = G^*(x',\theta_0), \]

is equivalent to the condition

(3.64) \[ (\sigma_{m,k}^\pm(x',0,\theta_0); m \in \mathcal{I} \cup \mathcal{P} \cup \mathcal{N}, k \in \{1, \ldots, \mu_m\}) = M^\pm(G^*,\pm,\sigma_{m,k}(x',0,\theta_0); m \in \mathcal{O}, k \in \{1, \ldots, \mu_m\}), \]

(using \(+\) to denote a part with positive (negative) spectrum.)

Proof. We rewrite (3.63) as

(3.65) \[ B(0)\mathcal{V}_{0}^{m}(x',0,\theta_0,\ldots,\theta_0) = B(0) \left( \sum_{m \in \mathcal{I} \cup \mathcal{P} \cup \mathcal{N}} \sum_{k=1}^{\mu_m} \sigma_{m,k}(x',0,\theta_0)r_{m,k} + \sum_{m \in \mathcal{O}} \sum_{k=1}^{\mu_m} \sigma_{m,k}(x',0,\theta_0)r_{m,k} \right) = G^*(x',\theta_0). \]

Recall \(\mathcal{P}\)-profiles have positive spectra and \(\mathcal{N}\)-profiles have negative spectra. Thus, using the subscript \(n\) to denote the \(n\)th Fourier coefficient, we get for \(n > 0\),

(3.66) \[ B(0) \left( \sum_{m \in \mathcal{I} \cup \mathcal{P}} \sum_{k=1}^{\mu_m} \sigma_{m,k,n}(x',0)r_{m,k} \right) = G_n^*(x') - B(0) \left( \sum_{m \in \mathcal{O}} \sum_{k=1}^{\mu_m} \sigma_{m,k,n}(x',0)r_{m,k} \right), \]

and for \(n < 0\),

(3.67) \[ B(0) \left( \sum_{m \in \mathcal{I} \cup \mathcal{N}} \sum_{k=1}^{\mu_m} \sigma_{m,k,n}(x',0)r_{m,k} \right) = G_n^*(x') - B(0) \left( \sum_{m \in \mathcal{O}} \sum_{k=1}^{\mu_m} \sigma_{m,k,n}(x',0)r_{m,k} \right). \]

By Lemma 3.20, both sets \(\{B(0)r_{m,k} : k \in \{1, \ldots, \mu_m\}, m \in \mathcal{I} \cup \mathcal{P}\}, \{B(0)r_{m,k} : k \in \{1, \ldots, \mu_m\}, m \in \mathcal{I} \cup \mathcal{N}\}\) are bases for \(\mathbb{C}^n\), and the desired result follows from this.

So we now have a sub-system ((3.51), (3.52)) in just the hyperbolic profiles \(\sigma_{m,k}, m \in \mathcal{I} \cup \mathcal{O}\), and the mean \(\bar{\nu}\) with boundary data for \(\bar{\nu}\) and determination of \(\mathcal{I}\)-boundary data from \(\mathcal{O}\)-boundary data as in (3.57)(b), where (3.57)(b) is equivalent to (3.64).

Remark 3.22. We will seek \(\nu, \sigma_{p,l}\) such that, in addition to the equations above, we satisfy the initial conditions

(3.68) \[ \nu = 0 \quad \text{and} \quad \sigma_{p,l} = 0 \quad \text{in} \quad t \leq 0 \quad \text{for all} \quad p, l. \]

The large system consists of the interior equations (3.51)-(3.55) with boundary equations (3.57) and the initial conditions (3.68).

3.3 Solution of the hyperbolic sub-system of the large system

First we will obtain the ‘hyperbolic part’ of our solution, \(\mathcal{V}_{h}^{0}\), which will satisfy, in place of (3.31),

\begin{align}
& a) \quad \mathcal{E}_h \mathcal{V}_{h}^{0} = \mathcal{V}_{h}^{0} \\
& b) \quad \mathcal{E}_h \left( \mathcal{L}(\partial_x) \mathcal{V}_{h}^{0} + \mathcal{M}(\mathcal{V}_{h}^{0}) \partial_x \mathcal{V}_{h}^{0} \right) = \mathcal{E}_h (\mathcal{F}(0) \mathcal{V}_{h}^{0}) \quad \text{in} \quad x_d \geq 0 \\
& c) \quad B(0)\mathcal{G}(x') \quad \text{(}\sigma_{m,k}(x',0,\theta_0); m \in \mathcal{I}, k \in \{1, \ldots, \mu_m\}\text{)} = Lh(G^*,\sigma_{m,k}(x',0,\theta_0); m \in \mathcal{O}, k \in \{1, \ldots, \mu_m\}), \\
& d) \quad \mathcal{V}_{h}^{0} = 0 \quad \text{in} \quad t < 0.
\end{align}
Here, in the zero-mean boundary data in (3.69)(c), we denote by $L_h$ the appropriate linear function one obtains from the condition (3.64). Equation (3.69)(a) serves the same purpose as (3.31)(a), but adds the restriction that $V^0_h$ only consists of the mean and the hyperbolic profiles. Thus, recalling the form for $V^0$, we see $V^0_h$ takes the form

$$V^0_h(x, \theta_1, \ldots, \theta_M) = \psi(x) + \sum_{m \in I \cup O} \sum_{k=1}^{\mu_m} \sigma_{m,k}(x, \theta_m) r_{m,k}. \quad (3.70)$$

The condition (3.69)(b) is exactly the hyperbolic sub-system ((3.51),(3.52)) found in Proposition 3.18. This follows directly from Proposition 3.18 and the definition of $E_h$.

To solve this, we employ the same approach as that which is used in [2] to solve the full system of profile equations. This is done by solving the system with an iteration scheme such as the following

\begin{align*}
\text{a) } & E_h V^{0,n}_h = \psi^0_h \\
\text{b) } & E_h \left( \tilde{L}(\partial_v) V^{0,n}_h + \mathcal{M}(V^{0,n-1}_h) \partial_v V^{0,n}_h \right) = E_h(F(0)V^{0,n-1}_h) \text{ in } x_d \geq 0 \\
\text{c) } & B(0) \sigma^n = G(x'), \\
& (\sigma^n_{m,k}(x', 0, \theta_0); m \in I \cup O, k \in \{1, \ldots, \mu_m\}) = L_h(G^{*}, \sigma^n_{m,k}(x', 0, \theta_0); m \in I \cup O, k \in \{1, \ldots, \mu_m\}) \\
\text{d) } & V^{0,n}_h = 0 \text{ in } t < 0, \quad (3.71)
\end{align*}

where we note that (3.71)(a) means $V^{0,n}_h$ is of the form

$$V^{0,n}_h(x, \theta_1, \ldots, \theta_M) = \psi^n(x) + \sum_{m \in I \cup O} \sum_{k=1}^{\mu_m} \sigma^n_{m,k}(x, \theta_m) r_{m,k}, \quad (3.72)$$

where iterates $\psi^n(x)$, $\sigma^n_{m,k}(x, \theta_m)$ for $m \in I \cup O$ satisfy an iterated version of the hyperbolic sub-system of the profile equations, ((3.51),(3.52)) which is encapsulated by (3.71)(b).

The following proposition gives the solution to the iterated system as well as the solution to the hyperbolic sub-system itself. We omit the proofs of these as they are almost identical to the proof of Proposition 2.19 together with that of Proposition 2.21 from [2].

**Proposition 3.23.** Let $T > 0$, $m > \frac{d+3}{2} + 1$ and suppose that $G(x', \theta_0) \in H^m_T$.

(i) Setting $V^{0,0}_h = 0$, there exist unique iterates $V^{0,n}_h \in \mathbb{H}^m_{T_0}$, $n \geq 1$, solving the system (3.71).

(ii) For some $0 < T_0 \leq T$ the system (3.69) has a unique solution $V^{0}_h \in \mathbb{H}^m_{T_0}$. Furthermore,

$$\lim_{n \to \infty} V^{0,n}_h = V^{0}_h \text{ in } \mathbb{H}^{m-1}_{T_0}, \quad (3.73)$$

and the traces $V^{0,n}_{h|_{x_d=0}}, V^{0}_{h|_{x_d=0}}$, and $\sigma_{p,l|_{x_d=0}}$, $p \in I \cup O$, all lie in $H^m_{T_0}$.

### 3.4 Approximate solution of the equations for the elliptic profiles

Recall from Proposition 3.18 the elliptic interior equations:

\begin{align*}
\text{(a) } & X_{p', \mu_p} \sigma_{p,l}(x, \theta_p) + \sum_{j=0}^{d-1} \sum_{k'=1}^{\mu_p} a_{p', l, j}^{k'}(x) \partial_{\theta_{k'}} \sigma_{p, k'}(x, \theta_p) + \\
\text{(b) } & \sum_{j=0}^{d-1} \sum_{k=1}^{\mu_p} \sum_{k'=1}^{\mu_p} b_{p', l, j}^{k, k'} \sigma_{p, k}(x, \theta_p) \partial_{\theta_{k'}} \sigma_{p, k'}(x, \theta_p) + \\
\text{(c) } & \sum_{j=0}^{d-1} f_{p, l, j}(x, \theta_p) = \sum_{k=1}^{\mu_p} e_{p, l}^{k} \sigma_{p, k}(x, \theta_p) \quad (3.74)
\end{align*}

for $p \in \mathcal{P} \cup \mathcal{N}$, $l \in \{1, \ldots, \mu_p\}$.
These complex transport equations may not generally have exact solutions. Instead, our elliptic profiles $\sigma_{p,l}$ will approximately solve these equations in the sense that they will simply hold at the boundary $x_d = 0$. It is sufficient to simply evaluate the above expression at $x_d = 0$, isolate $\partial_{x_d} \sigma_{p,l}(x',0,\theta_p)$, and require that $\sigma_{p,l}$ has the appropriate $x_d$-derivative at the boundary. Of course, we will also have to adhere to the boundary conditions, so that the trace $\sigma_{p,l}(x',0,\theta_p)$ at the boundary satisfies (3.57)(b), and we must satisfy initial conditions (3.68). With Proposition 3.29, we construct such elliptic profiles $\sigma_{p,l}(x,\theta_p)$ by solving some wave equations in which $x_d$ plays the role of the time variable and with appropriate boundary data corresponding to $\sigma_{p,l}(x',0,\theta_p)$ and $\partial_{x_d} \sigma_{p,l}(x',0,\theta_p)$.

In addition to the exact elliptic interior equations, we consider semilinear equations in iterates $\sigma_{p,l}^n$ for $p \in \mathcal{P} \cup \mathcal{N}$, $l \in \{1, \ldots, \mu_p\}$, almost identical to the iteration scheme which is used to obtain the hyperbolic profiles. While we do not need elliptic iterates $\sigma_{p,l}^n$ to get the elliptic profiles $\sigma_{p,l}$, which are instead obtained by the process described above, they allow us to handle hyperbolic parts and elliptic parts uniformly in the error analysis. The semilinear equations in the elliptic iterates $\sigma_{p,l}^n$ are

\begin{align}
(3.75) & \
X_{\phi_p} \sigma_{p,l}^n(x,\theta_p) +
\end{align}

\begin{align}
(3.76) & \
\sum_{j=0}^{d-1} \sum_{\mu_p} \sum_{k'=1}^k \alpha_{p,l,j}^{k'} (\Theta_{n-1}) \partial_{\theta_j} \sigma_{p,l}^{n-1}(x,\theta_p) + \sum_{j=0}^{d-1} \sum_{\mu_p} \sum_{k'=1}^k \beta_{p,l,j}^{k',k} \sigma_{p,l}^{n-1}(x,\theta_p) \partial_{\theta_k} \sigma_{p,l}^{n-1}(x,\theta_p) +
\end{align}

\begin{align}
(3.77) & \
\sum_{j=0}^{d-1} \sum_{\mu_p} \sum_{k'=1}^k \gamma_{p,l,j}^{k',k} \sigma_{p,l}^{n-1}(x,\theta_p) + \sum_{j=0}^{d-1} \sum_{\mu_p} \sum_{k'=1}^k \delta_{p,l,j}^{k',k} \sigma_{p,l}^{n-1}(x,\theta_p)
\end{align}

\begin{align}
(3.78) & = \sum_{k=1}^{\mu_p} \epsilon_{p,l}^{k} \sigma_{p,l}^{n-1}(x,\theta_p)
\end{align}

where we define

\begin{align}
(3.79) & \nonumber
\gamma_{p,l,j}^{k',k} \sigma_{p,l}^{n-1}(x,\theta_p) = \frac{1}{2\pi} \int_0^{2\pi} (\sigma_{q,k}^{n-1})_{n_q} \left( x, \frac{n_p}{n_q} \theta_p - \frac{n_q}{n_q} \theta_r \right) \partial_{\theta_r} \sigma_{p,k'}^{n-1}(x,\theta_r) d\theta_r,
\end{align}

and similar for $\delta_{p,l,j}^{k',k} \sigma_{p,l}^{n-1}(x,\theta_p)$. Imposed on the iterates $\sigma_{p,l}^n$ are boundary conditions identical to those for the profiles $\sigma_{p,l}$, i.e. (3.64) with $\sigma_{m,k}^{n}$ in place of each $\sigma_{m,k}$, and initial conditions

\begin{align}
(3.80) & \nonumber
\sigma_{p,l}^n(x) = 0 \text{ for } t \leq 0.
\end{align}

Similar to the elliptic profiles, the elliptic iterates $\sigma_{p,l}^n$ will only approximately solve (3.75)-(3.78) in the sense that these equations will hold at $x_d = 0$. In fact, the elliptic iterates $\sigma_{p,l}^n$ are obtained with the same kind of construction used for the elliptic $\sigma_{p,l}$. In addition to the elliptic profiles, their corresponding iterates are constructed in Proposition 3.29.

For now, let us fix $p \in \{1, \ldots, M\}$, $l \in \{1, \ldots, \mu_p\}$ and an integer $n$ and begin the task of finding approximate solutions $\sigma_{p,l}$ and $\sigma_{p,l}^n$ of (3.74) and the system (3.75)-(3.78), respectively. First we rearrange (3.74), isolating the vector field applied to $\sigma_{p,l}$ in the left hand side, and denoting the resulting right hand side by $f_{p,l}$, getting something of the form

\begin{align}
(3.81) & \nonumber
X_{\phi_p} \sigma_{p,l}(x,\theta_p) = f_{p,l}(x,\theta_p).
\end{align}

Similarly, we isolate $X_{\phi_p} \sigma_{p,l}^n$ in the left hand side of (3.75)-(3.78), and define $f_{p,l}^n(x,\theta_p)$ to be what remains on the right hand side, so (3.74) becomes

\begin{align}
(3.82) & \nonumber
X_{\phi_p} \sigma_{p,l}^n(x,\theta_p) = f_{p,l}^n(x,\theta_p).
\end{align}

Now we isolate the quantities this prescribes for the traces of the $\sigma_{p,l}$ and the $\sigma_{p,l}^n$ at the boundary $x_d = 0$. 28
**Definition 3.24.** (i) The coefficient of \( \partial_{x_d} \sigma_{p,l} \) in the left hand side of (3.81) is one, so we may isolate it in this expression. We will require this equation to hold at \( x_d = 0 \), and define \( b_{p,l} \) so that doing so is represented by the condition

\[
\partial_{x_d} \sigma_{p,l} \big|_{x_d=0} = b_{p,l},
\]

noting that the right hand side depends only on the boundary data \( \sigma_{m,k} \big|_{x_d=0}, \ m = 1, \ldots, M, \ k = 1, \ldots, \mu_m \).

(ii) Observe that (3.64) determines a linear function \( L_e \) giving us the boundary data for the elliptic profiles:

\[
(3.84) \quad (\sigma_{m,k}(x', 0, \theta_0); m \in \mathcal{P} \cup \mathcal{N}, k \in \{1, \ldots, \mu_m\}) = L_e(G^*, \sigma_{m,k}(x', 0, \theta_0); m \in \mathcal{O}, k \in \{1, \ldots, \mu_m\}).
\]

Note that the right hand side has already been determined by our solution of the hyperbolic profiles. We define \( a_{p,l} \) such that the boundary condition on \( \sigma_{p,l} \) in (3.64) is equivalent to

\[
(3.85) \quad \sigma_{p,l} \big|_{x_d=0} = a_{p,l},
\]

This thus determines the right hand side of (3.83).

(iii) We isolate \( \partial_{x_d} \sigma_{p,l}^n \) on the left hand side of (3.75)-(3.78), and define \( b_{p,l}^n \) such that requiring this to hold at \( x_d = 0 \) is equivalent to

\[
(3.86) \quad \partial_{x_d} \sigma_{p,l}^n \big|_{x_d=0} = b_{p,l}^n.
\]

(iv) We impose the condition

\[
(3.87) \quad (\sigma_{m,k}^n(x', 0, \theta_0); m \in \mathcal{P} \cup \mathcal{N}, k \in \{1, \ldots, \mu_m\}) = L_e(G^*, \sigma_{m,k}^n(x', 0, \theta_0); m \in \mathcal{O}, k \in \{1, \ldots, \mu_m\}).
\]

Let us define \( a_{p,l}^n \) such that the condition on \( \sigma_{p,l}^n \) in (3.87) is equivalent to

\[
(3.88) \quad \sigma_{p,l}^n \big|_{x_d=0} = a_{p,l}^n.
\]

**Lemma 3.25.** Suppose \( G \in H^{s+1}_{T_0} \). Let \( y_0^h \in H^{s+1}_{T_0} \) be the solution of the hyperbolic sub-system constructed in Proposition 3.23. For \( a_{p,l}^n, b_{p,l}^n, a_{p,l}, \) and \( b_{p,l} \) as defined in Definition 3.24, we have \( a_{p,l}^n \in H^{s+1}_{T_0}(x', \theta_p) \), \( b_{p,l}^n \in H^s_{T_0}(x', \theta_p) \), and

\[
(3.89) \lim_{n \to \infty} a_{p,l}^n = a_{p,l} \text{ in } H^{s+1}_{T_0}(x', \theta_p),
\]

\[
(3.90) \lim_{n \to \infty} b_{p,l}^n = b_{p,l} \text{ in } H^s_{T_0}(x', \theta_p).
\]

**Proof.** Checking \( a_{p,l}^n \in H^{s+1}_{T_0}(x', \theta_p) \) is straightforward, since \( G(x', \theta_p), \sigma_{m,k}^n(x', 0, \theta_p), m \in \mathcal{O}, \) are in \( H^{s+1}_{T_0}(x', \theta_p) \). Now we show \( b_{p,l}^n \in H^s_{T_0}(x', \theta_p) \). Observe \( b_{p,l}^n \) consists of the terms evaluated at \( x_d = 0 \) (3.76), (3.77), (3.78), and

\[
(3.91) \quad (\partial_{x_d} - X_{\phi_n}) \sigma_{p,l}^n(x', 0, \theta_p).
\]

Since the \( \Sigma_{p,k}^n, \sigma_{p,k}(x', 0, \theta_p), p \in \mathcal{I} \cup \mathcal{O}, \) are in \( H^{s+1}_{T_0}(x', \theta_p) \), and \( H^s_{T_0}(x', \theta_p) \) is a Banach algebra, it follows that (3.91) and both (3.76) and (3.78) at \( x_d = 0 \) are in \( H^{s+1}_{T_0}(x', \theta_p) \). It remains to show this for (3.77) at \( x_d = 0 \). This follows from Proposition 3.16 with \( x' \) in place of \( x \). Taking into account that, for \( p \in \mathcal{I} \cup \mathcal{O}, \)

\[
(3.92) \quad \lim_{n \to \infty} \sigma_{p,k}^n \big|_{x_d=0} = \sigma_{p,k} \big|_{x_d=0} \text{ in } H^{s+1}_{T_0}(x', \theta_p),
\]

\[
(3.93) \quad \lim_{n \to \infty} \partial_{x_d} \sigma_{p,k}^n \big|_{x_d=0} = \partial_{x_d} \sigma_{p,k} \big|_{x_d=0} \text{ in } H^s_{T_0}(x', \theta_p),
\]

the proof that (3.89) and (3.90) hold is similar. \( \square \)
Remark 3.26. To simplify obtaining our approximate solutions of (3.75)-(3.78), it will be useful to extend functions \( a_{p,l}^{n}, b_{p,l}^{n} \in H^{s+1}_{T_{0}} = H^{s+1}((-\infty, T_{0}) \times \mathbb{R}^{d-1} \times \mathbb{T}) \), \( b_{p,l}^{n}, b_{p,l} \in H_{\mathcal{T}_{0}}^{s} = H^{s}((-\infty, T_{0}) \times \mathbb{R}^{d-1} \times \mathbb{T}) \) on the half-space to elements of \( H^{s+1}(\mathbb{R}^{d} \times \mathbb{T}) \) and \( H^{s}(\mathbb{R}^{d} \times \mathbb{T}) \), respectively.

Lemma 3.27. For \( s \geq 0 \) there is a continuous extension map
\[
E : H^{s}((-\infty, T_{0}) \times \mathbb{R}^{d-1} \times \mathbb{T}) \to H^{s}(\mathbb{R}^{d} \times \mathbb{T}).
\]

Proof. It is shown in 4.4 of [11] that there is a continuous extension map from \( H^{s}(\mathbb{R}^{d}) \) to \( H^{s}(\mathbb{R}^{d}) \). □

From now on, in place of \( a_{p,l}^{n}, b_{p,l}^{n}, b_{p,l} \) we refer to their extensions to the respective spaces mentioned above unless we explicitly state otherwise.

The following lemma is a standard kind of result in the theory of hyperbolic initial/boundary value problems.

Lemma 3.28. (i) For \( a \in H^{s+1}(\mathbb{R}^{d} \times \mathbb{T}) \) and \( b \in H^{s}(\mathbb{R}^{d} \times \mathbb{T}) \), there exists unique \( \varsigma \in H^{s+1}_{D} = H^{s+1}(\mathbb{R}^{d} \times [0, D] \times \mathbb{T}) \) solving the wave equation
\[
\partial^{2}_{x_{d}}\varsigma - \Delta_{x',a_{b}}\varsigma = 0,
\]
and initial-\( x_{d} \) conditions
\[
\varsigma_{|x_{d}=0} = a, \quad \partial_{x_{d}}\varsigma_{|x_{d}=0} = b.
\]

(ii) Furthermore, \( \varsigma \) satisfies
\[
|\varsigma|_{H^{s+1}_{D}} \leq C (|a|_{H^{s+1}} + |b|_{H^{s}}),
\]
for some constant \( C \) independent of the choice of initial data \( \{a, b\} \).

Justification of Lemma 3.28 can be found in Chapter 6 (see 6.18 and Remark 6.21) of [3].

Proposition 3.29. For \( a_{p,l}^{n}, b_{p,l}^{n}, a_{p,l}, b_{p,l} \) as defined in Definition 3.24, where \( p \in \mathcal{P} \cup \mathcal{N} \) there exist \( \sigma_{p,l}^{n}(x, \theta_{p}), \sigma_{p,l}(x, \theta_{p}) \in H^{s+1}_{T_{0}}(x, \theta_{p}) \) with compact \( x_{d} \)-support in \( [0, D] \) satisfying
\[
\sigma_{p,l}^{n} = \sigma_{p,l} = 0, \text{ for } t \leq 0,
\]
boundary conditions
\[
\sigma_{p,l}^{n}_{|x_{d}=0} = a_{p,l}, \quad \sigma_{p,l}^{n}_{|x_{d}=0} = a_{p,l},
\]
\[
\partial_{x_{d}}\sigma_{p,l}^{n}_{|x_{d}=0} = b_{p,l}, \quad \partial_{x_{d}}\sigma_{p,l}^{n}_{|x_{d}=0} = b_{p,l},
\]
and
\[
\lim_{n \to \infty} \sigma_{p,l}^{n} = \sigma_{p,l} \text{ in } H^{s+1}_{T_{0}}(x, \theta_{p}).
\]

Proof. We apply Lemma 3.28 to initial data \( \{a_{p,l}, b_{p,l} + \partial_{0}a_{p,l}\} \) to obtain the corresponding solution of (3.95), denoted \( \varsigma_{p,l} \in H^{s+1}_{D} \), and subsequently to each \( \{a_{p,l}^{n}, b_{p,l}^{n} + \partial_{0}a_{p,l}^{n}\} \), obtaining solutions \( \varsigma_{p,l}^{n} \in H^{s+1}_{D} \). Now we fix a smooth cutoff function \( \chi(x_{d}) \) supported in \([0, D]\), and define
\[
\sigma_{p,l}^{n}(t, x', \theta_{p}) = \chi(x_{d})\varsigma_{p,l}^{n}(t - x_{d}, x', \theta_{p}), \quad \sigma_{p,l}^{n}(t, x', \theta_{p}) = \chi(x_{d})\varsigma_{p,l}^{n}(t - x_{d}, x', \theta_{p}).
\]

It is easy to check that then \( \sigma_{p,l}^{n}, \sigma_{p,l} \) also belong to \( H^{s+1}_{D} \). Since \( \sigma_{p,l}^{n}_{|x_{d}=0}, \varsigma_{p,l}^{n}_{|x_{d}=0} \) are supported in \( t > 0 \), by finite speed of propagation for solutions to the wave equation (3.95), for any fixed \( r > 0 \), the \( (t, y, \theta_{p}) \)-supports of \( \varsigma_{p,l}^{n}|_{x_{d}=r}, \varsigma_{p,l}^{n}|_{x_{d}=r} \) are contained in the union of balls of radius \( r \) about points in \( \{(t, y, \theta_{p}) : t > 0\} \). Thus, the support is contained in \( \{(t, y, \theta_{p}) : t > -r\} \). It follows that \( \sigma_{p,l}(t, y, x_{d}, \theta_{p}), \sigma_{p,l}(t, y, x_{d}, \theta_{p}) \) are zero for all \( t \leq 0 \), since the supports of \( \sigma_{p,l}, \sigma_{p,l}^{n} \) consist only of points satisfying \( t - x_{d} > -x_{d} \), i.e. \( t > 0 \). It is easy
Applying Lemma 3.28 to the solutions (3.104) and so \( R_{\text{3.111}} \), note that Remark 3.33. Proposition 3.32. \( V_{\text{3.108}} \) and we also define \( V_{\text{3.109}} \), and we define the elliptic part of our ansatz \( V_{\text{3.106}} \). Let \( \sigma_{\text{3.112}} \) in Proposition 3.23, and let the \( V_{\text{3.108}} \), \( n \), \( V_{\text{3.109}} \), \( \sigma_{\text{3.112}} \), be the elliptic profiles and iterates obtained in Proposition 3.29.

(i) We define the elliptic part of our ansatz

\[
V^0_e(x, \theta_1, \ldots, \theta_M) := \sum_{m \in P \cup N} \sum_{k=1}^{\mu_m} \sigma_{m,k}(x, \theta_m)r_{m,k},
\]

and the corresponding nth iterate

\[
V^{0,n}_e(x, \theta_1, \ldots, \theta_M) := \sum_{m \in P \cup N} \sum_{k=1}^{\mu_m} \sigma_{m,k}^{n}(x, \theta_m)r_{m,k}.
\]

Our ansatz is defined to be

\[
V^0 := V^0_h + V^0_e,
\]

and we also define \( V^{0,n} := V^{0,n}_h + V^{0,n}_e \).

(ii) We isolate the error in the iterated interior profile equations, defining \( R^n(x, \theta) \) by

\[
R^n := E(\tilde{L}(\partial_\theta)) V^{0,n} + M(V^{0,n-1}) \partial_\theta V^{0,n} - F(0)V^{0,n-1}).
\]

**Proposition 3.32.** Suppose the hypotheses of Definition 3.31 are satisfied. Then \( V^0 \in H^{n+1}_T \), and

\[
\lim_{n \to \infty} V^{0,n} = V^0 \text{ in } H^s_T.
\]

**Proof.** The claims follow directly from Proposition 3.23 and Proposition 3.29.

**Remark 3.33.** Recalling (3.71), note that

\[
E_k(\tilde{L}(\partial_\theta)) V^{0,n} + M(V^{0,n-1}) \partial_\theta V^{0,n} - F(0)V^{0,n-1}) = 0,
\]

and so \( R^n \) of (3.109) is purely elliptic in the sense that \( E_e R^n = R^n \). Moreover,

\[
R^n = \sum_{p \in P \cup N} \sum_{l=1}^{\mu_p} (X_p \sigma_{p,l} - f_{p,l}) r_{p,l}.
\]
which is zero at the boundary $x_d = 0$, since we have satisfied (3.81) for $x_d = 0$. Summarizing the results of Proposition 3.23 and Proposition 3.29, we conclude

\[
\begin{align*}
\text{(a)} & \quad \mathcal{E}V^{0,n} = V^{0,n} \\
\text{(b)} & \quad \mathbb{E}\left(\tilde{L}(\partial_x)V^{0,n} + \mathcal{M}(V^{0,n-1}\partial_0 V^{0,n})\right) = \mathbb{E}(F(0)V^{0,n-1}) + R^n \quad \text{in } x_d \geq 0 \\
\text{(c)} & \quad B(0)V^{0,n}(x',0,\theta_0,\ldots,\theta_0) = G(x',\theta_0) \\
\text{(d)} & \quad V^{0,n} = 0 \quad \text{in } t < 0.
\end{align*}
\]

(3.113)

The sense in which an error such as $R^n$ is small is clarified by Proposition 4.6.

**Remark 3.34.** Recall the discussion in Remark 2.9(ii) which gives the bijection between the eigenvalues $\omega_m$ with $m \in \mathcal{P}$ and $\omega_m' = \omega_m$, $m' \in \mathcal{N}$ and between the corresponding eigenvectors. A careful look at the profile equations shows that the equations for the elliptic profiles come in conjugate pairs. That is, the equation for $\sigma_{m,k}$, some $m \in \mathcal{P}$, is the conjugate of the equation for $\sigma_{m',k'}$ with the corresponding $m' \in \mathcal{N}$ and $k'$. As a result the solutions we have obtained, the elliptic profiles, also come in conjugate pairs, satisfying $\sigma_{m',k'} = \sigma_{m,k}$. Meanwhile, the hyperbolic part of the solution $V^0_T$ is real. It follows from these observations and the definition of our ansatz that when we plug in $\theta = \theta(0,\xi_d)$, getting

\[
U^0(x,\theta_0,\xi_d) := V^0(x,\theta_0 + \omega_1 \xi_d,\ldots,\theta_0 + \omega_M \xi_d),
\]

we have a function $U^0(x,\theta_0,\xi_d)$ which is in fact real on $\mathbb{R}^{d+1}_x \times T \times \mathbb{R}_+$.

### 4 Convergence of the approximate solution to the exact solution

#### 4.1 The trigonometric expansion for the approximate solution

For this study, we define the spaces $E_T^s$ and $E_T^s$ as in [2].

\[
E_T^s = C(x_d,H^s_T(x',\theta_0)) \cap L^2(x_d,H^{s+1}_T(x',\theta_0))
\]

(4.1)

where by $C(x_d,H^s_T(x',\theta_0))$ we actually refer to functions in $C(x_d,H^s_T(x',\theta_0))$ with $x_d$-support in $[0,D]$ for some large enough $D$, and for $C(x_d,H^s_T(x',\theta_0))$ we use the $L^\infty(x_d,H^s_T(x',\theta_0))$ norm where the supremum is taken over $x_d \geq 0$. These spaces are algebras and are contained in $L^\infty$ for $s > \frac{d+1}{4}$. Theorem 7.1 of [13], as discussed in Section 1.1, regarding existence of solutions of the singular system, tells us that for $s$ large enough, one has existence of solutions to (1.6) in the space $E_T^s$ on a time interval $[0,T]$ independent of $\epsilon \in (0,\epsilon_0)$. For these reasons, the space $E_T^s$ and related estimates are key to our analysis, in particular for our main theorem (Theorem 4.7) which relies on Proposition 4.9, also proved in [13]. Additionally we use the spaces

\[
E_T^s = \{U(x,\theta_0,\xi_d) : \sup_{\xi_d \geq 0} |U(x,\xi_d)|_{E_T^s} < \infty\},
\]

(4.2)

which also play a role in Theorem 4.7. The proof of Theorem 4.7 uses estimates regarding functions such as $V$ in $H^{s+1}_T$ and the corresponding $U$ in $E_T^s$ which results from the substitution $\theta = \theta(0,\xi_d)$. While this moves us out of the space of periodic profiles $H^{s+1}_T(x,\theta)$, we get functions in $E_T^s(x,\theta_0,\xi_d)$ which we can approximate with trigonometric polynomials (truncated expansions\textsuperscript{24}) in $\theta = \theta(0,\xi_d)$.

**Definition 4.1.** For $k = 1, 2$ we define

\[
E_T^{s,k} := \{U(x,\theta_0,\xi_d) = V(x,\theta)|_{\theta = \theta(0,\xi_d)} : V \in H^{s+1:k}_T\},
\]

(4.3)

with the norm $|\cdot|_{E_T^s}$. (Note: The subscript $T$ has the same indication as it did for the $H^{s+1:k}_T$ used in [2], where the subscript $T$ is at first ignored.)

\textsuperscript{24}These expansions are described in Proposition 4.5.
It is verified in Proposition 4.5 that elements of $E_{T}^{s,2}$ are bounded in the $E_{T}^{s}$ norm.

**Convergence of expansions in $E_{T}^{s}$**

The following notation gives us a way to sort the spectra $\alpha \in Z^{M,2}$ of elements $V$ in $H_{T}^{s+1;2}$, which will aid in showing $U(x, \theta_{0}, \xi_{d}) = V(x, \theta)|_{\theta = \theta(\theta_{0}, \xi_{d})}$ has an expansion converging in $E_{T}^{s}$, with Proposition 4.5.

**Definition 4.2.** Let $\{z_{k}\}_{k=1}^{\infty}$ enumerate the set $\{\alpha \cdot \omega : \alpha \in Z^{M,2}\}$. For each $j, k$, define

$$C_{j,k} := \{\alpha \in Z^{M,2} : \sum_{i=1}^{M} \alpha_{i} = j, \alpha \cdot \omega = z_{k}\}$$

and pick out one element $\alpha_{(j,k)} \in C_{j,k}$.

**Remark 4.3.** A nice consequence of the fact that we work with $Z^{M,2}$ instead of general $Z^{M,k}$ is that we can easily show the $C_{j,k}$ are finite\(^{25}\). To see this, first fix $j \in \mathbb{Z}$, $k \in \{1, 2, \ldots\}$, and $p, q \in \{1, \ldots, M\}$ and take arbitrary $\alpha \in C_{j,k}$ such that all but the $p$th and $q$th components are zero (either of the $p$th and $q$th components may be zero, as well.) We claim this is the only such element of $C_{j,k}$. This is because $\omega_{p} \neq \omega_{q}$ implies

$$\begin{pmatrix} 1 \\ \omega_{p} \\ \omega_{q} \end{pmatrix} \begin{pmatrix} \alpha_{p} \\ \alpha_{q} \end{pmatrix} = \begin{pmatrix} j \\ z_{k} \end{pmatrix}$$

has a unique solution $({\alpha}_{p}, {\alpha}_{q})$. Thus, the number of such $\alpha$ in $C_{j,k}$ is bounded by the number of pairs of components, so $|C_{j,k}| \leq M(M - 1)/2$.

A function $V(x, \theta) \in H_{T}^{s+1;2}(x, \theta)$ of $(x, \theta) \in \mathbb{R}^{d+1} \times \mathbb{C}^{M}$ has a series

$$V(x, \theta) = \sum_{\alpha \in Z^{M,2}} V_{\alpha}(x) e^{i \alpha \cdot \theta}$$

and, for fixed $x_{d}$, squared $H_{T}^{s}(x', \theta)$ norm

$$|V(x, \theta)|_{H_{T}^{s}(x', \theta)} = \sum_{\alpha \in Z^{M,2} : |\beta| \leq s} |\partial_{x}^{\beta} V_{\alpha}(x)|^{2}_{L^{2}(x')} (1 + |\alpha|)^{2(s - |\beta|)}.$$

From Sobolev embedding and the fact that

$$H_{T}^{s+1}(x, \theta) \subset L^{2}(x_{d}, H_{T}^{s+1}(x', \theta)) \cap H^{1}(x_{d}, H_{T}^{s}(x', \theta)),$$

we find $V(x, \theta) \in L^{2}(x_{d}, H_{T}^{s+1}(x', \theta)) \cap C(x_{d}, H_{T}^{s}(x', \theta))$, implying the partial sums of the series (4.6) are bounded and converge in $H_{T}^{s}(x', \theta)$ uniformly with respect to $x_{d} \geq 0$. We will prove Proposition 4.5 by using these facts with the following lemma, which shows for finite truncations of expansions (4.6) that the norm $| \cdot |_{H_{T}^{s}(x', \theta)}$ dominates $| \cdot |_{H^{1}(x', \theta, \theta)}$ independent of $(x_{d}, \xi_{d})$.

**Lemma 4.4.** Suppose $V \in H_{T}^{s+1;2}(x, \theta)$ with series given by (4.6). For $\theta(\theta_{0}, \xi_{d})$ as in (1.33) and integers $M_{1} \leq M_{2}$, $0 < N_{1} \leq N_{2}$, we have the following inequality:

$$| \sum_{j=M_{1}}^{M_{2}} \sum_{k=N_{1}}^{N_{2}} \sum_{\alpha \in C_{j,k}} V_{\alpha}(x) e^{i \alpha \cdot \theta(\theta_{0}, \xi_{d})} |_{H_{T}^{s}(x', \theta, \theta_{0})}^{2} \leq | \sum_{j=M_{1}}^{M_{2}} \sum_{k=N_{1}}^{N_{2}} \sum_{\alpha \in C_{j,k}} V_{\alpha}(x) e^{i \alpha \cdot \theta} |_{H_{T}^{s}(x', \theta)}^{2}.$$

**Proof.** We estimate

$$| \sum_{j=M_{1}}^{M_{2}} \sum_{k=N_{1}}^{N_{2}} \sum_{\alpha \in C_{j,k}} V_{\alpha}(x) e^{i \alpha \cdot \theta(\theta_{0}, \xi_{d})} |_{H_{T}^{s}(x', \theta, \theta_{0})}^{2} = \sum_{j=M_{1}}^{M_{2}} \sum_{k=N_{1}}^{N_{2}} \sum_{\alpha \in C_{j,k}} \sum_{\beta} \partial_{x}^{\beta} V_{\alpha}(x) e^{i \alpha \cdot \omega_{\beta}} | \sum_{j=M_{1}}^{M_{2}} \sum_{k=N_{1}}^{N_{2}} \sum_{\alpha \in C_{j,k}} \partial_{x}^{\beta} V_{\alpha}(x) e^{i \alpha \cdot \omega_{\beta}} |_{L^{2}(x')}^{2} (1 + |j|)^{2(s - |\beta|)},$$

$$\sum_{j=M_{1}}^{M_{2}} \sum_{k=N_{1}}^{N_{2}} \sum_{\alpha \in C_{j,k}} \sum_{\beta} \partial_{x}^{\beta} V_{\alpha}(x) e^{i \alpha \cdot \omega_{\beta}} | \sum_{j=M_{1}}^{M_{2}} \sum_{k=N_{1}}^{N_{2}} \sum_{\alpha \in C_{j,k}} \partial_{x}^{\beta} V_{\alpha}(x) e^{i \alpha \cdot \omega_{\beta}} |_{L^{2}(x')}^{2} (1 + |j|)^{2(s - |\beta|)}.$$

\(^{25}\)We work with $Z^{M,2}$ due to the fact that we only have to consider quadratic interactions. Further discussion on interactions and resonances can be found in Section 3.2.
For (4.10), we used the formula which that in (4.7) generalizes. For $\alpha \in C_{j,k}$, we have $|j| = |\sum \alpha_s| \leq |\alpha|$ and $\text{Im}(\alpha \cdot \omega) \geq 0$, so for all $\xi_d \geq 0$, the sum in (4.11) is bounded by

$$
\sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} \sum_{\alpha \in C_{j,k}} V_\alpha(x) e^{i\alpha \cdot \theta(x') e^{2|H_\alpha(x')|^2}} = \sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} \sum_{\alpha \in C_{j,k}} |\beta| \leq s \sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} \sum_{\alpha \in C_{j,k}} |\beta| \leq s (1 + |\alpha|)^{2(s-|\beta|)},
$$

where the equality (4.12) follows from the formula in (4.7). \qed

With the estimate of Lemma 4.4, we are ready to prove that elements of $H^{s+1;2}_T^+$ yield elements with expansions converging in $E_{T}^\theta$ upon the substitution $\theta = \theta(x, \xi_d)$.

**Proposition 4.5.** Let $V \in H^{s+1;2}_T(x, \theta)$ with expansion (4.6) and set $U = V|_{\theta=\theta(x, \xi_d)}$ for $\theta(x, \xi_d)$ as in (1.33). Then we have $U \in E_{T}^{\theta}$:

$$
|\mathcal{U}|_{E_{T}^{\theta}} \leq C|\mathcal{V}|_{H^{s+1}_T},
$$

and we have convergence in $E_{T}^{\theta}$ to $U$ of finite partial sums, independent of arrangement, of the series

$$
U(x, \theta_0, \xi_d) = \sum_{\alpha \in \mathbb{Z}^{M/2}} V_\alpha(x) e^{i\alpha \cdot \theta(x, \xi_d)}. \tag{4.14}
$$

**Proof of Proposition 4.5.** We explicitly show the convergence of the finite partial sums

$$
\sum_{-n \leq j \leq n} \sum_{1 \leq k \leq n} V_\alpha(x) e^{i\alpha \cdot \theta(x, \xi_d)} \tag{4.15}
$$

For integers $n_1, n_2$ with $n_1 \leq n_2$, an application of Lemma 4.4 gives

$$
\sum_{-n_1 \leq j \leq n_2} \sum_{n_1 \leq k \leq n_2} V_\alpha(x) e^{i\alpha \cdot \theta(x, \xi_d)} \leq \sum_{-n_1 \leq j \leq n_2} \sum_{n_1 \leq k \leq n_2} V_\alpha(x) e^{i\alpha \cdot \theta(x, \xi_d)}. \tag{4.16}
$$

It follows from (4.16) that since the sequence of the partial sums

$$
\sum_{-n \leq j \leq n} \sum_{1 \leq k \leq n} V_\alpha(x) e^{i\alpha \cdot \theta(x, \xi_d)} \tag{4.17}
$$

is Cauchy in $H^{s}_T(x', \theta)$ uniformly with respect to $(x_d, \xi_d)$, so is the sequence (4.15) in $H^{s}_T(x', \theta_0)$, and thus we have convergence. We similarly get convergence of the (4.15) in $L^2(x_d, H^{s+1}_T(x', \theta_0))$ after integrating the inequality (4.16) (with $s+1$ in place of $s$) with respect to $x_d$ and noting the right hand side is independent of $\xi_d$. Hence, the (4.15) converge in $E_{T}^{\theta}$. It is not hard to show the limit is in fact $\mathcal{V}|_{\theta=\theta(x, \xi_d)}$, and the estimate (4.13) easily follows. The same proof works for arbitrarily ordered sums, where one instead considers finite $B_n \supset \mathbb{Z}^{M/2}$ and similarly gets that the sequence of the

$$
\sum_{\alpha \in B_n} V_\alpha(x) e^{i\alpha \cdot \theta(x, \xi_d)} \tag{4.18}
$$

is Cauchy in the desired space because the sequence of the $\sum_{\alpha \in B_n} V_\alpha(x) e^{i\alpha \cdot \theta}$ is Cauchy. \qed

### 4.2 Error analysis

The following proposition is used to make precise the notion that $\sigma_{p,l}(x, \theta_p)$ and $\sigma^n_{p,l}(x, \theta_p)$ are approximate solutions of (3.74) and (3.75)-(3.78), respectively.
Proposition 4.6. Let \( R(x, \theta) \in H_T^{s-1}(x, \theta) \) have the property that it is polarized by the elliptic projector \( \mathbb{E}_e \) defined in (3.13), i.e. that \( \mathbb{E}_e(R) = R \), and suppose

\[
R|_{x_d=0} = 0.
\]

Then

\[
\lim_{\epsilon \to 0} |R(x, \theta_0 + \omega_p \xi_d, \ldots, \theta_0 + \omega_M \xi_d)|_{\xi_d = \frac{x_d}{\epsilon}} |_{H_T^{s-1}(x, \theta_0)} = 0.
\]

Proof. It suffices to consider the case \( R(x, \theta_p) \in H_T^s(x, \theta_p), \mathbb{E}_p(R) = R \), for some \( p \in \mathcal{P} \cup \mathcal{N} \), and show

\[
\lim_{\epsilon \to 0} |R(x, \theta_0 + \omega_p \xi_d)|_{\xi_d = \frac{x_d}{\epsilon}} |_{H_T^{s-1}(x, \theta_0)} = 0.
\]

First, we show

\[
\lim \sup_{\epsilon \to 0, x_d \geq 0} |R(x, \theta_0 + \omega_p \xi_d)|_{\xi_d = \frac{x_d}{\epsilon}} |_{H_T^{s-1}(x', \theta_0)} = 0.
\]

Note \( R \) has an expansion of the form

\[
R(x, \theta_p) = \sum_{j \in \mathbb{Z}_p \setminus 0} a_j(x) e^{i j \theta_p}.
\]

Thus, noting \( R(x, \theta_0) \in H_T^s(x, \theta_0) \subset C(x_d, H_T^{s-1}(x', \theta_0)) \), fixing \( x_d \), we get the norm

\[
|R(x, \theta_p)|_{H_T^{s-1}(x', \theta_0)}^2 = \sum_{j \in \mathbb{Z}_p \setminus 0} \sum_{|\beta| \leq s-1} |\partial_x^\beta a_j(x)|_{L^2(x')}^2 (1 + |j|)^{2(s-1-|\beta|)}.
\]

We let \( \xi_d = \frac{x_d}{\epsilon} \) and find

\[
R(x, \theta_0 + \omega_p \xi_d) = \sum_{j \in \mathbb{Z}_p \setminus 0} \left( e^{-\text{Im}(j \omega_p) \xi_d} e^{i \text{Re}(j \omega_p) \xi_d} a_j(x)\right) e^{i j \theta_0}.
\]

Then it follows

\[
|R(x, \theta_0 + \omega_p \xi_d)|_{H_T^{s-1}(x', \theta_0)}^2 = \sum_{j \in \mathbb{Z}_p \setminus 0} \sum_{|\beta| \leq s-1} |\partial_x^\beta a_j(x)|_{L^2(x')}^2 (1 + |j|)^{2(s-1-|\beta|)}
\]

\[
\leq e^{2|\text{Im}(\omega_p) \xi_d|} \sum_{j \in \mathbb{Z}_p \setminus 0} \sum_{|\beta| \leq s-1} |\partial_x^\beta a_j(x)|_{L^2(x')}^2 (1 + |j|)^{2(s-1-|\beta|)}
\]

\[
eq e^{2|\text{Im}(\omega_p) \xi_d|} |R(x, \theta_0)|_{H_T^{s-1}(x', \theta_0)}^2.
\]

Now we show that, as \( \epsilon \) tends to zero,

\[
\sup_{x_d \in [0, \sqrt{\epsilon}]} |R(x, \theta_0 + \omega_p x_d)_{H_T^{s-1}(x', \theta_0)}| \to 0,
\]

and

\[
\sup_{x_d \geq \sqrt{\epsilon}} |R(x, \theta_0 + \omega_p x_d)|_{H_T^{s-1}(x', \theta_0)} \to 0.
\]

Set \( h(x_d) = R(x', x_d, \theta_0) \in H_T^{s-1}(x', \theta_0) \). To see (4.28), observe that the term is bounded by

\[
sup_{x_d \in [0, \sqrt{\epsilon}]} e^{-2|\text{Im}(\omega_p) x_d|} |h(x_d)|_{H_T^{s-1}(x', \theta_0)}^2 \leq \sup_{x_d \in [0, \sqrt{\epsilon}]} |h(x_d)|_{H_T^{s-1}(x', \theta_0)}^2.
\]
which converges to 0 as \( \epsilon \) tends to zero, since \( h \in C(x_d, H^{s-1}_T(x', \theta_0)) \) with \( h(0) = 0 \). That (4.29) holds follows from the fact that this term is bounded by

\[
\sup_{x_d \geq \sqrt{\epsilon}} e^{-2|\text{Im} \omega_d| \frac{x_d}{\epsilon}} |h(x_d)|^2_{H^{s-1}_T(x', \theta_0)} \leq \sup_{x_d \geq 0} |h(x_d)|^2_{H^{s-1}_T(x', \theta_0)} e^{-2|\text{Im} \omega_d| \frac{x_d}{\epsilon}},
\]

which also converges to 0 as \( \epsilon \) tends to zero. Now we check that

\[
\lim_{\epsilon \to 0} \left| R(x, \theta_0 + \omega_d \xi_d) \right|_{L^2(x_d, H^s_T(x', \theta_0))} = 0.
\]

We split the integral into the two pieces

\[
\int_0^{\sqrt{\epsilon}} |R(x, \theta_0 + \omega_d \frac{x_d}{\epsilon})|^2_{H^s_T(x', \theta_0)} dx_d,
\]

and

\[
\int_{\sqrt{\epsilon}}^{\infty} |R(x, \theta_0 + \omega_d \frac{x_d}{\epsilon})|^2_{H^s_T(x', \theta_0)} dx_d.
\]

Using the inequality (4.27) with \( s \) in place of \( s - 1 \), we get that the first integral is bounded above by

\[
\int_0^{\sqrt{\epsilon}} e^{-2|\text{Im} \omega_d| \frac{x_d}{\epsilon}} dx_d |R(x, \theta_0)|^2_{L^2(x_d, H^s_T(x', \theta_0))} \leq \sqrt{\epsilon} |R(x, \theta_0)|^2_{L^2(x_d, H^s_T(x', \theta_0))},
\]

and the second is bounded by

\[
e^{-2|\text{Im} \omega_d| \frac{x_d}{\epsilon}} |R(x, \theta_0)|^2_{L^2(x_d, H^s_T(x', \theta_0))},
\]

both of which tend to zero as \( \epsilon \to 0 \), and so the proof is finished.

Now we are ready to show that, as \( \epsilon \to 0 \), the approximate solution \( u^\epsilon_0 \) converges to the exact solution \( u_0 \) in \( L^\infty \). This is a corollary of the following theorem.

**Theorem 4.7.** For \( M_0 = 2(d+2) + 1 \) and \( s \geq 1 + [M_0 + \frac{d+1}{2}] \) let \( G(x', \theta_0) \in H^{s+1}_T \) have compact support in \( x' \) and vanish in \( t \leq 0 \). Let \( U_s(x, \theta_0) \in E^s_{T_0} \) be the exact solution to the singular system for \( 0 < \epsilon \leq \epsilon_0 \) given by Theorem 7.1 of [13], let \( V^0 \in H^{s+1}_T \) be the profile given by (1.29), and let \( U^0 \in E^s_T \) be defined by

\[
U^0(x, \theta_0, \xi_d) = V^0(x, \theta_0 + \omega_1 \xi_d, \ldots, \theta_0 + \omega_M \xi_d).
\]

Here \( 0 < T_0 \leq T \) is the minimum of the existence times for the quasilinear problems (1.6) and (3.31). Define

\[
U^\epsilon_0(x, \theta_0) := U^0(x, \theta_0, \frac{x_d}{\epsilon}).
\]

The family \( U^\epsilon_0 \) is uniformly bounded in \( E^s_{T_0} \) for \( 0 < \epsilon \leq \epsilon_0 \) and satisfies

\[
|U_\epsilon - U^\epsilon_0|_{E^{s-1}_{T_0}} \to 0 \text{ as } \epsilon \to 0.
\]

Upon evaluating our placeholder \( \xi_d = \frac{x_d}{\epsilon} \) in the argument of \( U^0 \in E^s_T \), we get the function \( U^\epsilon_0 \) in \( E^s_T \), the same space containing the solutions of the singular system. Indeed, the \( E^s_T \) norm is that of the estimate of Proposition 4.9, key to our proof of Theorem 4.7, and the norm in which we show our approximate solution \( U^0 \) is close to the exact solution of the singular system.

**Lemma 4.8.** For \( m \geq 0 \) suppose \( V(x, \theta) \in H^{m+1}_T \), \( E^m = V \), and set \( \mathcal{U}(x, \theta_0, \xi_d) = V(x, \theta_0 + \omega_1 \xi_d, \ldots, \theta_0 + \omega_M \xi_d) \). Then, setting \( \mathcal{U}_\epsilon(x, \theta_0) = U(x, \theta_0, \frac{x_d}{\epsilon}) \),

\[
|\mathcal{U}_\epsilon|_{E^m_T} \leq |\mathcal{U}|_{E^m_T}.
\]

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Proof. The proof is almost identical to the argument used in [2] to prove Lemma 2.25 (b) with Lemma 2.7.

Consider again the singular system, whose solution is constructed in Theorem 7.1 of [13]. This is done with the use of the following iteration scheme:

\[
\begin{align*}
\theta_{x} U_{t}^{n+1} + \sum_{j=0}^{d-1} A_{j} (\theta U_{t}^{n}) \left( \theta_{x} + \frac{\beta_{j}}{\epsilon} \theta_{x} \right) U_{t}^{n+1} &= F(\epsilon U_{t}^{n}) U_{t}^{n}, \quad (4.41) \\
B(\epsilon U_{t}^{n}) (U_{t}^{n+1})_{|x=0} &= G(x', \theta), \\
U_{t}^{n+1} &= 0 \text{ in } t < 0.
\end{align*}
\]

Once the \( U_{t}^{n} \) are obtained, \( U_{\epsilon} \) is found by taking the limit as \( n \to \infty \) of the \( U_{t}^{n} \).

In order to prove Theorem 4.7, we will use the technique of simultaneous Picard iteration. For this, we take the iterates \( \mathcal{U}^{0,n}_{\epsilon} \), which are formed by evaluation of our iterates \( \mathcal{V}^{0,n} (\theta) \) at \( \theta = (\theta_{0} + \omega_{1} \xi_{d}, \ldots, \theta_{n} + \omega_{M} \xi_{d}) \), and which converge to \( \mathcal{U}^{\epsilon}_{\epsilon} \) in the limit \( n \to \infty \) uniformly with respect to \( \epsilon \). We then show the \( \mathcal{U}^{0,n}_{\epsilon} \) tend to the iterates \( \mathcal{U}^{\epsilon}_{\epsilon} \) for the singular system as \( \epsilon \to 0 \). The proof uses the following linear estimate

**Proposition 4.9** ([13], Cor. 7.2). Let \( s \geq |M_{0} + \frac{d_{x}}{2} | \) and consider the problem (4.41), where \( G \in H^{s+1} \) has compact support and vanishes in \( t \leq 0 \) and where the right side of (4.41) is replaced by \( F \in \mathcal{E}^{s}_{x} \) with \( \text{supp } F \subset \{ t \geq 0, 0 \leq x_{d} \leq E \} \). Suppose \( U_{t}^{n} \in E^{s}_{x} \) has compact \( x \)-support and that for some \( K > 0 \), \( \epsilon_{1} > 0 \) we have

\[
|U_{t}^{n}|_{E^{s}_{x}} \leq C(\epsilon_{1}, E) \sqrt{t}(\mathcal{F})_{E^{s}_{x}} + (\mathcal{G})_{s+1,T}.
\]

Then there exist constants \( T_{0}(K) \) and \( \epsilon_{0}(K) \leq \epsilon_{1} \) such that for \( 0 < \epsilon \leq \epsilon_{0} \) and \( T \leq T_{0} \) we have

\[
|U_{t}^{n+1}|_{E^{s}_{x}} + \sqrt{T} (U_{t}^{n+1})_{s+1,T} \leq C(K,E) \sqrt{T} (\mathcal{F})_{E^{s}_{x}} + (\mathcal{G})_{s+1,T}.
\]

Our proof of Theorem 1.4 relies on an induction argument, and the above linear estimate plays a crucial role in the inductive step, allowing us to show the closeness between an iterate \( \mathcal{U}^{0,n+1}_{\epsilon} \) approximating our leading order expansion and the iterate \( U_{t}^{n+1} \).

**Proof of Theorem 4.7.** It suffices to prove boundedness of the family \( \mathcal{U}^{0}_{\epsilon} \) in \( E^{s}_{x} \) along with the following three statements:

\[
\begin{align*}
(a) \lim_{n \to \infty} U_{t}^{n} &= U_{\epsilon} \text{ in } E^{s}_{x} \text{ uniformly with respect to } \epsilon \in (0, \epsilon_{0}] \\
(b) \lim_{n \to \infty} \mathcal{U}^{0,n}_{\epsilon} &= \mathcal{U}^{0}_{\epsilon} \text{ in } E^{s}_{x} \text{ uniformly with respect to } \epsilon \in (0, \epsilon_{0}] \\
(c) \text{For each } n \lim_{\epsilon \to 0} |U_{t}^{n} - \mathcal{U}^{0,n}_{\epsilon}|_{E^{s}_{x}} = 0.
\end{align*}
\]

The uniform boundedness of \( U_{\epsilon} \) and the \( U_{t}^{n} \) in \( E^{s}_{x} \), and that (4.44)(a) holds are proved in [13], Theorem 7.1, with the use of the iteration scheme (4.41) and the linear estimate Proposition [13], Cor. 7.2.9. The desired boundedness of \( \mathcal{U}^{0}_{\epsilon} \) and the \( \mathcal{U}^{0,n}_{\epsilon} \) in \( E^{s}_{x} \) follows from the boundedness of \( \mathcal{V}^{0} \) and the uniform boundedness of the \( \mathcal{V}^{0,n} \) in \( E^{s+1}_{x} \), considered with Proposition 4.5 and Lemma 4.8. Similarly, from the fact that

\[
\lim_{n \to \infty} \mathcal{V}^{0,n} = \mathcal{V}^{0} \text{ in } E^{s}_{x},
\]

we get (4.44)(b).

Now we proceed by proving (4.44)(c). Fix \( \delta > 0 \). Noting \( E \mathcal{V}^{0,n} = \mathcal{V}^{0,n} \) and \( E \mathcal{V}^{0,n+1} = \mathcal{V}^{0,n+1} \), choose finite partial trigonometric polynomial sums \( \mathcal{V}^{0,n}_{p} \) and \( \mathcal{V}^{0,n+1}_{p} \) of (resp.) \( \mathcal{V}^{0,n} \) and \( \mathcal{V}^{0,n+1} \) such that

\[
\begin{align*}
(a) E \mathcal{V}^{0,n}_{p} = \mathcal{V}^{0,n}_{p} \text{ and } E \mathcal{V}^{0,n+1}_{p} = \mathcal{V}^{0,n+1}_{p}, \\
(b) \mathcal{V}^{0,n} - \mathcal{V}^{0,n}_{p} < \delta, \mathcal{V}^{0,n+1} - \mathcal{V}^{0,n+1}_{p} < \delta.
\end{align*}
\]
and note that, as a result of (4.46)(b), we also have

$$\|\partial_{x_{x_{i}}} U^{0,n+1}_{p} - \partial_{x_{x_{j}}} U^{0,n+1}_{p}\|_{H_{\tilde{T}_{0}}} < \delta.$$  

Observe then, as a consequence of Proposition 4.5, we get

$$\|u^{0,n} - U^{0,n}_{p}\|_{E_{\tilde{T}_{0}}} < C\delta,$$

where $u^{0,n}$ and $U^{0,n}_{p}$ are obtained from (resp.) $Y^{0,n}_{p}$ and $Y^{0,n+1}_{p}$ via the substitution $\theta = \theta(\xi_{0}, \xi_{d})$. The induction assumption

$$\lim_{\epsilon \to 0} \|U^{n}_{\epsilon} - U_{\epsilon}^{0,n}\|_{E_{\tilde{T}_{0}}^{-1}} = 0.$$  

With the boundedness of the $U^{n}_{\epsilon}$ in $E_{\tilde{T}_{0}}^{2}$, it follows

$$\lim_{\epsilon \to 0} \|F(\epsilon U^{n}_{\epsilon}) - F(0)U^{0,n}_{\epsilon}\|_{E_{\tilde{T}_{0}}^{-1}} = 0.$$

Observe that since $Y^{0,n}_{p}$ and $Y^{0,n}_{p}$ are invariant under $E$, Lemma 4.8 applies. Thus, using the estimate (4.40), from (4.50) and (4.48) we obtain

$$\|F(\epsilon U^{n}_{\epsilon}) - F(0)U^{0,n}_{\epsilon}\|_{E_{\tilde{T}_{0}}^{-1}} \leq C\delta + c(\epsilon),$$

where $c(\epsilon) \to 0$ as $\epsilon \to 0$.

Now we define

$$G_{p} = L(\partial_{x})Y^{0,n+1}_{p} + \lambda(\lambda^{0,n})\partial_{y}Y^{0,n+1}_{p}.$$  

Then

$$E(G_{p} - F(0)Y^{0,n}_{p}) = E(L(\partial_{x})(Y^{0,n+1}_{p} - Y^{0,n+1})) +$$

$$E(\lambda(\lambda^{0,n})\partial_{y}Y^{0,n+1}_{p} - \lambda(\lambda^{0,n})\partial_{y}Y^{0,n+1}) +$$

$$E(F(0)(\lambda^{0,n} - \lambda^{0,n})) + R^{n+1}.$$  

Thus using (4.46)(b) and continuity of both $E : H^{s;2} \to H^{s;1}$ and multiplication from $H^{s;1} \times H^{s;1} \to H^{s;2}$, we get

$$\|E_{p} - E(F(0)Y^{0,n}_{p}) - R^{n+1}\|_{H_{\tilde{T}_{0}}} = O(\delta).$$

We define the operator

$$L_{0} = L(\partial_{x}) + \frac{1}{\epsilon} L(d\phi_{0})\partial_{y_{0}} + \lambda'(U^{0,n}_{p})\partial_{y_{0}}.$$  

We claim

$$\|L_{0} U^{n+1}_{\epsilon} - F(\epsilon U^{n}_{\epsilon})U^{n}_{\epsilon}\|_{E_{\tilde{T}_{0}}^{-1}} \leq C\delta + c(\epsilon),$$

where $c(\epsilon) \to 0$ as $\epsilon \to 0$. This follows from (4.41)(a) and

$$\|\hat{A}_{j}(\epsilon U^{n}_{\epsilon})\partial_{x_{j}} U^{n+1}_{\epsilon} - \hat{A}_{j}(0)\partial_{x_{j}} U^{n+1}_{\epsilon}\|_{E_{\tilde{T}_{0}}^{-1}} = O(\epsilon)$$

$$\|\frac{1}{\epsilon} \hat{A}_{j}(\epsilon U^{n}_{\epsilon})\beta_{j}\partial_{y_{0}} U^{n+1}_{\epsilon} - \left(\frac{1}{\epsilon} \hat{A}_{j}(0)\beta_{j}\partial_{y_{0}} U^{n+1}_{\epsilon} + \partial_{u}\hat{A}_{j}(0)U^{n}_{\epsilon}\beta_{j}\partial_{y_{0}} U^{n+1}_{\epsilon}\right)\|_{E_{\tilde{T}_{0}}^{-1}} = O(\epsilon)$$

$$\|\partial_{u}\hat{A}_{j}(0)(U^{n}_{\epsilon} - U^{0,n}_{p})\beta_{j}\partial_{y_{0}} U^{n+1}_{\epsilon}\|_{E_{\tilde{T}_{0}}^{-1}} \leq C|U^{n}_{\epsilon} - U^{0,n}_{p}|_{E_{\tilde{T}_{0}}^{-1}} \leq c(\epsilon) + O(\delta).$$
Setting \( G'_p = \hat{L}(\partial_x)U^{0,n+1}_p + M'(U^{0,n}_p)\partial_{\theta_0}U^{0,n+1}_p \), since \( L'(\partial_{\theta_0}, \partial_{\xi_d})U^{0,n+1}_p = 0 \), we get

\[
L_0U^{0,n+1}_p = G'_p.
\]

(4.58)

From now on we use \( |(\theta_0, \xi_d) \) to denote evaluation at \( \theta = \theta(\theta_0, \xi_d) \), and \( |(\theta_0, \xi_d),\epsilon \) to indicate \( |(\theta_0, \xi_d) \) followed by evaluation at \( \xi_d = \frac{\epsilon x_d}{\epsilon} \). It is easy to check \( G'_p = G_p(\theta_0, \xi_d) \), particularly because \( V^{0,n}_p, V^{0,n+1}_p \) are finite polynomials. Thus

\[
L_0U^{0,n+1}_p - F(0)U^{0,n}_p = G'_p - F(0)U^{0,n}_p = [G_p - F(0)V^{0,n}_p]|(\theta_0, \xi_d),\epsilon
\]

(4.59)

and (iii) there exists \( V^1_p \in H^{1,2}_{\xi_d} \) such that for \( U^1_p = V^1_p|_{\theta(\theta_0, \xi_d)} \),

\[
\mathcal{L}'(\partial_{\theta_0}, \partial_{\xi_d})U^1_p = -(I - \mathcal{E})(G_p - F(0)V^{0,n}_p)|_{\theta(\theta_0, \xi_d)}.
\]

(4.63)

Noting \( \mathcal{E} \) Proposition 3.8 guarantees the existence of such \( V^1_p \), and (3.112). By requiring (3.86), and thus (3.82), to hold at \( x_d = 0 \), we have obtained \( R^{n+1} |_{x_d = 0} = 0 \). So Proposition 4.6 grants us (4.62).

Regarding (iii), note that, by Remark 3.7, (4.63) is equivalent to

\[
\mathcal{L}'(\partial_{\theta_0}, \partial_{\xi_d})U^1_p = -(I - \mathcal{E})(G_p - F(0)V^{0,n}_p)|_{\theta(\theta_0, \xi_d)},
\]

for which a sufficient condition is that, where \( U^1_p = V^1_p|_{\theta(\theta_0, \xi_d)} \),

\[
\mathcal{L}(\partial_{\theta_0})V^1_p = -(I - \mathcal{E})(G_p - F(0)V^{0,n}_p).
\]

(4.65)

Proposition 3.8 guarantees the existence of such \( V^1_p \), so proof of (iii) is complete.

Noting

\[
L_0(dU^{1,\epsilon}) = (\mathcal{L}'(\partial_{\theta_0}, \partial_{\xi_d})U^1_p)_{\epsilon} + (\hat{L}(\partial_x)dU^1_p)_{\epsilon} + M'(U^{0,n}_p)\partial_{\theta_0}(dU^{1,\epsilon})_{\epsilon},
\]

(4.66)

it follows from (4.60)-(4.63) that

\[
\|L_0(U^{0,n+1}_p + dU^{1,\epsilon}) - F(0)U^{0,n}_p|_{E^{\epsilon-1}_{\xi_d}} \leq C\delta + c(\epsilon) + K(\delta)\epsilon,
\]

(4.67)

where \( c(\epsilon) \) has been altered, but still tends to zero as \( \epsilon \to 0 \). It follows from equations (4.51), (4.56), and (4.67) that

\[
\|L_0(U^{n+1}_\epsilon - dU^{0,n+1}_p + dU^{1,\epsilon} + dU^{1,\epsilon})|_{E^{\epsilon-1}_{\xi_d}} \leq C\delta + c(\epsilon) + K(\delta)\epsilon.
\]

(4.68)

Now we claim that we have the following estimates:

\[
\begin{align*}
(a) & \quad \left| \left( \partial_{x_d} + A(dU^{0,n}_p, \partial_{x_d} + \beta \cdot \partial_{\theta_0}) \right) (U^{n+1}_\epsilon - (U^{0,n+1}_p + dU^{1,\epsilon})) \right|_{E^{\epsilon-1}_{\xi_d}} \leq C\delta + c(\epsilon) + K(\delta)\epsilon, \\
(b) & \quad \left| B(dU^{0,n}_p) (U^{n+1}_\epsilon - (U^{0,n+1}_p + dU^{1,\epsilon})) \right|_{H^{1}_{\xi_d}} \leq C\delta + c(\epsilon) + K(\delta)\epsilon.
\end{align*}
\]

(4.69)
That (4.69)(a) holds follows from (4.68) with the use of estimates similar to (4.57), and (4.69)(b) follows from the boundary conditions (1.6)(b) and (3.113)(c) together with Proposition 4.5 and Lemma 4.8. After applying the estimate of Proposition 4.9, we get that

\[(4.70) \quad |U_{\epsilon}^{n+1} - (U_{p,\epsilon}^{0,n+1} + dU_{p,\epsilon}^{1})|_{E_{T_0}^{\epsilon}} \leq C\delta + c(\epsilon) + K(\delta)\epsilon.\]

from which we conclude

\[(4.71) \quad |U_{\epsilon}^{n+1} - U_{\epsilon}^{0,n+1}|_{E_{T_0}^{\epsilon}} \leq C\delta + c(\epsilon) + K(\delta)\epsilon.\]

This finishes the induction step, completing the proof of the theorem. \qed
References


