Rigidity of hypersurfaces

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April, 2018

Theorem

Let $X \subset \mathbb{P}^{n+1}$ *be a smooth hypersurface of degree* $n+1$. *Assume that* $n \geq 3$. *Then every birational map* $\phi : X \dashrightarrow X'$ *to any Fano variety* X' *is an isomorphism.*

Noether-Fano method: aims to get similar results for other Fano varieties.

History

Max Noether (1870)
Fano (1908, 1915),
Segre (1942),

Iskovskikh-Manin (1971),
Pukhlikov (1987, 1998, 2002),
Corti (1995, 2000)
Cheltsov (2000, 2005),
de Fernex-Ein-Mustață (2003),
de Fernex (2013, 2016),

Z. Zhuang also with C. Stibitz, Y. Liu

Main step 1 (for any Fano variety)

Step 1. Choose m' such that $|-m'K_{X'}|$ is very ample and consider $M := \phi^{-1}|-m'K_{X'}|$ as a sub-linear system of some $|mK_X|$.

Note: ϕ is an isomorphism iff M is base point free.

Noether-Fano inequality: M must be "quite" singular at

some point of its base locus.

"Quite" singular= $(X, \frac{1}{m}M)$ is not canonical.

Main step 2 (for any hypersurface)

V: smooth hypersurface,
– $D \in |\mathcal{O}_V(m)|$ a divisor, $M \subset |\mathcal{O}_V(m)|$ a movable pencil

Lemma 1. D can be unexpectedly singular only at finitely many points.

(Simplest case: a hyperplane can be tangent only at finitely many points.)

Lemma 2. M can be unexpectedly singular only along finitely many curves.

Main step 3

Step 3. Restricting to general $W := X \cap H$ containing the "worst" point p, we get

1. $\left(W, \frac{1}{m} M|_W\right)$ is not **log** canonical at p and
2. $\left(W, \frac{1}{m} |2M_W|\right)$ is log canonical outside p.

Comments:

1. Restricting to W makes the singularity at p worse.
2. Going to $|2M|$ is a small but important trick. It controls singularties along curves.

New ingredient: Main step 4

Theorem (Zhuang)

Y: smooth projective, dimension d;
L ample and $\Delta \sim L$ a $\mathbb{Q}$-divisor. $\quad$ *($L \sim \frac{2}{m} M|_W$)*
Assume:
– Δ is log canonical outside a finite set of points and
– $\left(Y, \frac{1}{2}\Delta\right)$ is not log canonical.
Then

$$h^0\left(Y, \omega_Y \otimes L\right) \geq \tfrac{1}{2} 3^d.$$

Restate: If $\left(Y, \frac{1}{2}\Delta\right)$ is not log canonical then

$$h^0\left(Y, \omega_Y \otimes L\right) \geq \tfrac{1}{2}3^d.$$

Note: If $h^0\left(Y, L\right) > \binom{3d}{d} \sim 6.75^d$,
then there is a $D \in |L|$ such that $\operatorname{mult}_p D > 2d$, hence $\left(Y, \frac{1}{2}D\right)$ is not log canonical at p.

Informally: no accidental isolated singularities!

Hypersurface case

Steps 1–3 give

1. $W = W_{n+1}^{n-1} \subset \mathbb{P}^n$ thus $K_W \sim 0$ and
2. $\Delta \sim 2H$ such that
3. $\left(W, \frac{1}{2}\Delta\right)$ is not log canonical at p but
4. $\left(W, \Delta\right)$ is log canonical outside $p \in W$.

By Step 4

$$\binom{n+2}{2} = h^0\left(W, \omega_W(2)\right) \geq \frac{1}{2}3^{n-1}.$$

Impossible for $n \geq 5$. $\square$

Main step 1 (Noether-Fano inequality)

We have $X \overset{p}{\leftarrow} Z \overset{q}{\rightarrow} X'$ and $M \subset |-mK_X|, M_Z, M' \subset |-m'K_{X'}|$. Write

$$\begin{array}{lll} K_Z = q^*K_{X'} + E_q & M_Z = q^*M' & \text{and} \\ K_Z = p^*K_X + E_p & M_Z = p^*M - F_p. & \end{array}$$

For any c we have

$$\begin{aligned} K_Z + cM_Z &\equiv q^*(K_{X'} + cM') + E_q \\ K_Z + cM_Z &\equiv p^*(K_X + cM) + E_p - cF_p. \end{aligned}$$

$$K_Z + cM_Z \equiv q^*(K_{X'} + cM') + E_q$$
$$K_Z + cM_Z \equiv p^*(K_X + cM) + E_p - cF_p.$$

Setting $c = \frac{1}{m'}$, we see that

$$K_Z + \frac{1}{m'}M_Z \equiv q^*(K_{X'} + \frac{1}{m'}M') + E_q \equiv E_q \geq 0.$$

So $K_X + \frac{1}{m'}M \equiv p_*(E_q) \geq 0$, hence $m \geq m'$.
Setting $c = \frac{1}{m}$ gives

$$K_Z + \frac{1}{m}M_Z \equiv p^*(K_X + \frac{1}{m}M) + E_p - \frac{1}{m}F_p \equiv E_p - \frac{1}{m}F_p.$$

So $K_{X'} + \frac{1}{m}M' \equiv q_*(E_p - \frac{1}{m}F_p)$.
If $E_p - \frac{1}{m}F_p$ is effective, then $m' \geq m$. Thus

$$m = m', \quad p_*(E_q) = 0, \quad q_*(E_p - \frac{1}{m}F_p) = 0.$$

With little work: $X \cong X'$.

What is "quite" singular?

Conclusion: If $X \dashrightarrow X'$ not an isomorphism then

$$E_p - \tfrac{1}{m} F_p = \left(K_Z + \tfrac{1}{m} M_Z\right) - p^*\left(K_X + \tfrac{1}{m} M\right)$$

is **not** effective.

Question: Which p-exceptional divisor has negative coefficient?

Example. (First blow-up). If E is obtained by blowing up a codimension r center W then

$$\operatorname{coeff}(E) = (r-1) - \tfrac{1}{m} \operatorname{mult}_W M.$$

If this is negative then

$$\operatorname{mult}_W M > (r-1)m.$$

Problem. Higher blow-ups are much harder to see.

What is “quite” singular?

– variety X (smooth or normal or ...)
– $\mathbf{D}$: divisor Δ or linear system M or ideal sheaf I,
Take a log resolution $\pi : Y \to X$ and write

$$\begin{aligned} K_Y &= \pi^* K_X + \sum e_i E_i \\ \pi^* \mathbf{D} &= \sum_i a_i E_i \qquad \text{or} \\ &= \sum_i a_i E_i + (\text{free linear system}) \quad \text{or} \\ &= \mathcal{O}_Y\left(-\sum_i a_i E_i\right). \quad \text{Thus} \\ K_Y &= \pi^*\left(K_X + c\mathbf{D}\right) + \sum_i (e_i - c a_i) E_i. \end{aligned}$$

Definition:
$(X, c\mathbf{D})$ is *canonical* if $e_i - c a_i \geq 0 \quad (\forall Y, \forall i)$
$(X, c\mathbf{D})$ is *klt* if all $e_i - c a_i > -1 \quad (\forall Y, \forall i)$
$(X, c\mathbf{D})$ is *log canonical* if all $e_i - c a_i \geq -1 \quad (\forall Y, \forall i)$

Typical example

$X = \mathbb{C}^n$

$D = \left(\sum_i \lambda_i x_i^{m_i} = 0\right)$ or $M = |x_i^{m_i}|$ or $I = (x_i^{m_i})$.

$(X, \mathbf{D})$ is log canonical iff $1 \leq \sum_i \frac{1}{m_i}$.

$(X, c\mathbf{D})$ is log canonical iff $c \leq \sum_i \frac{1}{m_i}$.

Main step 2 (for any hypersurface)

Lemma (Fano, Segre, Pukhlikov, Cheltsov, Suzuki)

– V: *smooth hypersurface,*
– $D \in |\mathcal{O}_V(m)|$ *a divisor,* $M \subset |\mathcal{O}_V(m)|$ *a movable pencil*
– $\eta \in V$ *a (non-closed) point.*

1. *If* $\dim \eta \geq 1$ *then* $\operatorname{mult}_\eta D \leq m$.
2. *If* $\dim \eta \geq 2$ *then* $\operatorname{mult}_\eta (M \cdot M) \leq m^2$.

Proof. Simplest case: η generic point of a line L.
Intersect V with a general 2-plane containing L. Get $C + L$ and $C \cap L$ is $d-1$ general points on L. So

$$\begin{aligned} dm &= (C + L \cdot D) \\ &\geq (d-1)\operatorname{mult}_\eta D + (L \cdot D) \\ &\geq (d-1)\operatorname{mult}_\eta D + m. \quad \square \end{aligned}$$

Main step 3 (Corti, de Fernex, ...)

Corollary from Step 2:

– $\left(X, \frac{1}{m}M\right)$ is not canonical at a finite set of points p,
– $\left(X, \frac{1}{m}|2M|\right)$ is log canonical outside a finite set of curves.

Cut with a very general $H \ni p$ to get $W = X \cap H$.

– $\left(W, \frac{1}{m}M|_W\right)$ is not **log** canonical at p,
– $\left(W, \frac{1}{m}|2M_W|\right)$ is log canonical outside a finite set.

Main step 3 (Corti, de Fernex, ...)

Change to divisors: Set

$$\Delta = \frac{1}{m}(\text{general member of } \left|2M_W\right|).$$

Key properties

1. $\Delta \sim \mathcal{O}_W(2)$,
2. $\left(W, \frac{1}{2}\Delta\right)$ is not log canonical at p,
3. $\left(W, \Delta\right)$ is log canonical outside a finite set of points.

Multiplier ideals (prelude to Main step 4)

Take a log resolution $\pi : Y \to (X, \Delta)$. Write

$$\begin{aligned} K_Y &= \pi^* K_X + \sum e_i E_i \\ \pi^*\Delta &= \sum_i a_i E_i \end{aligned}$$

Definition: $\mathcal{J}(\Delta) = \pi_* \mathcal{O}_Y\big(\sum(e_i - [a_i])E_i$.

Note: $\mathrm{supp}\big(\mathcal{O}_X/\mathcal{J}(\Delta)\big) =$ points where (X, Δ) is not klt.

Nadel vanishing. If $L - \Delta$ is ample (or nef and big) then

$$H^i\big(X, \omega_X \otimes L \otimes \mathcal{J}(\Delta)\big) = 0 \quad \forall i > 0.$$

Step 4 in three easy lemmas

Lemma 1. If $(X, \frac{1}{2}\Delta)$ is not lc then $(X, \mathcal{J}(\Delta))$ is not lc.

Lemma 2. If (X, Δ) is lc away from finitely many points then $H^0(X, \omega_X \otimes L) \geq \mathrm{length}(\mathcal{O}_X/\mathcal{J}((1-\epsilon)\Delta))$.

Lemma 3. If (X, I) is not lc then $\mathrm{length}(\mathcal{O}_X/I) \geq \frac{1}{2}3^n$.

Proof of Lemma 2: Set $\Delta' = (1-\epsilon)\Delta$. Then $L - \Delta' \equiv \epsilon L$ is ample so

$$H^0(\omega_X \otimes L) \to H^0(\mathcal{O}_X/\mathcal{J}(\Delta')) \to H^1(\omega_X \otimes L \otimes \mathcal{J}(\Delta')).$$

Last group zero by Nadel vanishing. $\square$

Proof of Lemma 1

Since $\mathcal{J}(\Delta) = \pi_* \mathcal{O}_Y\big(\sum (e_i - [a_i]) E_i$,
$\pi^* \mathcal{J}(\Delta) \subset \mathcal{O}_Y\big(\sum (e_i - [a_i]) E_i\big)$.

So if $\pi^* \mathcal{J}(\Delta) = \mathcal{O}_Y\big(-\sum a_i' E_i\big)$ then $a_i' \geq [a_i] - e_i$.

$(X, \frac{1}{2}\Delta)$ not lc $\Rightarrow \exists i : e_i - \frac{1}{2} a_i < -1$.

$$\begin{aligned} e_i - a_i' &\leq e_i - \big([a_i] - e_i\big) = 2e_i - [a_i] < 2e_i - a_i + 1 \\ &\leq 2\big(e_i - \tfrac{1}{2} a_i\big) + 1 < -2 + 1 = -1. \end{aligned}$$

So $\big(X, \mathcal{J}(\Delta)\big)$ is not lc. $\square$

Log canonical threshold (prelude to proof of Lemma 3)

Definition.
$\operatorname{lcth}(\mathbf{D}) :=$ biggest c such that $(X, c\mathbf{D})$ is lc.

Lemma. For any smooth X and effective divisor Δ

$$\frac{\operatorname{mult}_p \Delta}{\dim X} \leq \frac{1}{\operatorname{lcth}_p(\Delta)} \leq \operatorname{mult}_p \Delta.$$

Arnol'd multiplicity: $\operatorname{lcth}_p(\Delta)^{-1}$

Informally. $\operatorname{lcth}_p(\Delta)^{-1}$ is like the multiplicity for large values, but the two are quite different for small values.

Co-length and log canonical threshold

Theorem (Corti, de Fernex-Ein-Mustață, Howald)

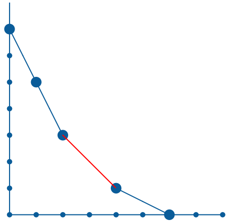
If $I \subset R = k[[x_1, \dots, x_n]]$ *then*

$$\mathrm{length}(R/I) \geq (\textit{combinatorial number from } \mathrm{lcth}(I)).$$

Proof. Using flat defomation to toric ideal ($\sim$ Gröbner basis) and lower semicontinuity of lcth, we may assume that I is monomial.

Newton polytope

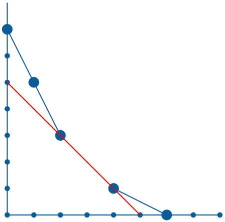
For $\prod x_i^{r_i} \in I$ we mark the point $(r_1, \ldots, r_n)$ with a big dot for generators and invisible dot for others.
The Newton polytope is the boundary of the convex hull of the marked points.



The Newton polygon of
$(y^7, y^5x, y^3x^2, yx^4, x^6)$

main face in red

- $\sum(x_i/m_i) = 1$: main face of its Newton polytope
- $I^{\mathrm{sat}} := \{\prod_i x^{r_i} : \sum(r_i/m_i) \geq 1\} \supset I$.
- $\mathrm{lcth}\left(I^{\mathrm{sat}}\right) = \sum(1/m_i)$
- Check by weighted blow up that $\mathrm{lcth}(I) = \sum(1/m_i)$.



Set $c = \mathrm{lcth}(I)$.

Corollary. (combinatorial number) $=$ the minimal number of lattice points in a simplex whose face contains $\left(\frac{1}{c}, \ldots, \frac{1}{c}\right)$. That is:

$$\min_{m_1,\ldots,m_n} \#\left\{\mathbb{N}^n \cap \left(\sum \frac{x_i}{m_i} < \frac{1}{c}\sum \frac{1}{m_i}\right)\right\}.$$

Three computations

Corollary

If $g_1, \ldots, g_n \in I$ *then* $\mathrm{length}(R/I) \geq n^n / (n! \cdot \mathrm{lcth}(I)^n)$.

Corollary

$C_1, C_2 \subset \mathbb{C}^2$ *and* $\frac{1}{m}|C_1, C_2|$ *is not lc at a point* p *then* $(C_1 \cdot C_2)_p > 4m^2$.

Corollary (= Lemma 3)

If I *is not lc then* $\mathrm{length}(R/I) \geq \frac{1}{2}3^n$.

Proof. $\left(\sum \frac{x_i}{m_i} \leq \sum \frac{1}{m_i}\right)$ contains at least half of the points in $\{0, 1, 2\}^n$.