Universally counting curves in Calabi–Yau threefolds

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Abstract

We show that curve enumeration invariants of complex threefolds with nef anti-canonical bundle are determined by their values on local curves. This statement and its proof are inspired by the proof of the Gopakumar–Vafa integrality conjecture by Ionel and Parker. The conjecture of Maulik, Nekrasov, Okounkov, and Pandharipande relating Gromov–Witten and Donaldson–Pandharipande–Thomas invariants is known for local curves by work of Bryan, Okounkov, and Pandharipande, hence holds for all complex threefolds with nef anti-canonical bundle (in particular, all Calabi–Yau threefolds).

1 Introduction

There are many ways of enumerating curves in complex threefolds [23]. These invariants turn out to satisfy some surprising relations which appear to have no straightforward explanation. In fact, the multitude of existing computations suggest that all reasonable curve enumeration theories for complex threefolds are equivalent [22].

A folk conjecture offers an explanation of this phenomenon: a complex threefold should be 'enumeratively equivalent' to a linear combination of *local curves* (rank two vector bundles over smooth proper curves). We provide a precise formulation and proof of this conjecture for complex threefolds with nef anti-canonical bundle. That is, we define a certain *Grothendieck group of 1-cycles* in complex threefolds (with nef anti-canonical bundle), and we show that this group is freely generated by local curves.

This result and its proof are inspired by the proof of the Gopakumar–Vafa integrality conjecture by Ionel–Parker [11]. They show that Gromov–Witten invariants of almost complex threefolds are integer linear combinations of Gromov–Witten invariants of local curves, which were known to satisfy Gopakumar–Vafa integrality by work of Bryan–Pandharipande [6]. Their argument may be interpreted as a proof that a certain Grothendieck group of 1-cycles in *almost* complex threefolds with nef anti-canonical bundle is freely generated by local curves (after completing by genus, later removed by Doan–Ionel–Walpuski [8]). The setting of complex threefolds is more rigid, requiring a different Grothendieck group. The proof is based on generic transversality, which explains the nef anticanonical bundle hypothesis.

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The main result opens a path to a number of conjectures relating different enumerative invariants of complex threefolds (under the assumption of nef anti-canonical bundle). We will explain here how to deduce from it the conjecture of Maulik–Nekrasov–Okounkov–Pandharipande [16, 17] relating Gromov–Witten and Donaldson–Thomas/Pandharipande–Thomas invariants (with cohomology insertions, no descendents, assuming nef anti-canonical bundle) given the calculations for local curves due to Bryan–Pandharipande [6] and Okounkov–Pandharipande [20]. The MNOP conjecture is interesting because there is no known or even proposed geometric relation between the moduli spaces giving rise to Gromov–Witten invariants and Donaldson–Thomas/Pandharipande–Thomas invariants. Of course, this is far from likely the only application (for example, see the work of Jockers–Mayr [12] and Chou–Lee [7] on quantum K-theory invariants).

The methods of this paper rely seriously on the assumption of nef anti-canonical bundle. It would be of exceptional interest to remove this assumption.

1.1 Universal enumerative invariant

There is a (very tautological) *universal* curve enumeration invariant of complex threefolds. This invariant takes values in the group $H^*_c(\mathcal{Z}(\mathrm{Cpx}_3))$, which is the homology of the double complex

$$C_*(\mathrm{Cpx}_3, C^*_c(\mathcal{Z})) = \bigoplus_{X \to \Delta^n} C^*_c(\mathcal{Z}(X/\Delta^n)) \tag{1.1}$$

in which the direct sum is over all (not necessarily proper) families $X \to \Delta^n$ of complex threefolds over a simplex, and $\mathcal{Z}(X/\Delta^n)$ denotes the space of compact (complex) 1-cycles in the fibers of $X \to \Delta^n$ (a 1-cycle $z \in \mathcal{Z}(X)$ is a formal non-negative integer linear combination $\sum_i m_i C_i$ of compact irreducible 1-dimensional subvarieties). If X is a projective threefold and $\beta \in H_2(X)$ is a homology class, then the 'universal count' of curves in X in homology class β in $H^*_c(\mathcal{Z}(\mathrm{Cpx}_3))$ is the class of the characteristic function $(\mathbf{1}_\beta : \mathcal{Z}(X) \to \mathbb{Z}) \in H^0_c(\mathcal{Z}(X))$ of the locus of 1-cycles with total homology class β (which is compact since X is projective). We call this group $H^*_c(\mathcal{Z}(\mathrm{Cpx}_3))$ the *Grothendieck group of 1-cycles in complex threefolds.*

The chain-level dual $C^*(\mathrm{Cpx}_3, C^{\mathrm{rel}\infty}_*(\mathcal{Z}))$ of the Grothendieck group of 1-cycles classifies coherent 'virtual fundamental' cycles on each relative cycle space $\mathcal{Z}(X/\Delta^n)$. A class in its homology $H^{\mathrm{rel}\infty}_*(\mathcal{Z}(\mathrm{Cpx}_3))$ is thus a 'curve enumeration theory of complex threefolds which is deformation invariant up to coherent homotopy'. Such a class determines a homomorphism out of the Grothendieck group.

The group $H^*_c(\mathcal{Z}(\mathrm{Cpx}_3))$ has a rich algebraic structure: it is a bi-algebra (product corresponds to disjoint union of cycles, while coproduct corresponds to sum of cycles). It also has bi-algebra endomorphisms corresponding to the 'multiply by d' operation on cycles.

The utility of this 'universal' discussion depends entirely on being able to make nontrivial computations. Our main result is to compute (in virtual dimension ≤ 0) the Grothendieck group $H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ whose definition is identical to $H^*_c(\mathcal{Z}(\mathrm{Cpx}_3))$ except that it considers just those 1-cycles $z = \sum_i m_i C_i$ all of whose components $C_i \subseteq X$ pair non-negatively with $c_1(TX)$.

Theorem 1.1. *In non-positive virtual dimension, the Grothendieck group $H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ is freely generated as a ring by the equivariant local curve elements $x_{g,1,k}$ with $k \geq 0$ and*

$x_{g,m,0}$.

We explain the statement. There is a natural bi-grading $H^*_c(\mathcal{Z}(\mathrm{Cpx}_3)) = \bigoplus_{i,k} H^i_c(\mathcal{Z}(\mathrm{Cpx}_3, k))$ by cohomological degree i and chern number k (pairing with $c_1(TX)$), and the 'total homological degree' (or 'virtual dimension') is $2k - i$. The equivariant local curve elements $x_{g,m,k}$ (defined in §4.3) have virtual dimension zero and correspond to the $\mathbb{C}^\times$-equivariant enumerative theory of degree m cycles on the total space of a rank two vector bundle $E \to C$ with $c_1(E) = k$ over a curve C of genus g. For example, Theorem 1.1 says that for any projective threefold X with nef anticanonical bundle and any homology class $\beta \in H_2(X)$ with zero chern number, the element $(X, \beta) \in H^0_c(\mathcal{Z}(\mathrm{Cpx}_3, 0))$ is equal to a unique polynomial in the variables $x_{g,m,0}$.

Theorem 1.1 is fundamentally a transversality statement, so the semi-Fano hypothesis appears necessary. The structure of the group $H^*_c(\mathcal{Z}(\mathrm{Cpx}_3))$ is likely to be much more complicated (likely uncomputable). An analogue of Theorem 1.1 in almost complex geometry was proven by Ionel–Parker [11]. They showed, in particular, that $H_0(\mathrm{ACpx}_3, H^0_c(\mathcal{Z}^{\mathrm{CY}}))$ (one part of the E_2 term of the spectral sequence associated to the double complex whose total homology is $H^*_c(\mathcal{Z}(\mathrm{ACpx}_3)))$ is generated by local curve elements $x_{g,m,0}$ (after completing by genus, later removed by Doan–Ionel–Walpuski [8]). Due to the rigidy of complex structures, we must work with the entire complex (1.1). The reason for this is that, while generic almost complex structures achieve transversality for all simple maps from curves, generic complex structures only achieve transversality for simple maps 'locally' on the space of cycles. Generic transversality for almost complex structures goes back to Gromov [10], while we are not aware of previous use of generic transversality in the complex setting. While we could probably prove Theorem 1.1 using a direct geometric argument, we actually only prove surjectivity geometrically and we deduce injectivity using the bi-algebra structure.

Theorem 1.1 is not the final word on the structure of enumerative invariants of complex threefolds with nef anticanonical bundle. Specifically, one could ask for the product expansion of Ionel–Parker [11] in the complex setting (perhaps deducible from their result by comparing $H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ and $H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{ACpx}_3))$ via Theorem 1.1 and an almost complex analogue thereof), namely the following.

Conjecture 1.2. *For any complex projective Calabi–Yau threefold X, the element $(X, t^{[\cdot]}) \in H^0_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))[[t^{H_2(X)}]]$ is an infinite product $\prod_\beta \prod_{g\geq 0} f_g(t^\beta)^{e_{\beta,g}(X)}$ for unique integer invariants $e_{\beta,g}(X) \in \mathbb{Z}$, where $f_g(t) = \sum_{m\geq 0} x_{g,m,0} t^m$.*

In the absence of the semi-Fano hypothesis, calculating the Grothendieck group $H^*_c(\mathcal{Z}(\mathrm{Cpx}_3))$ appears intractable. Nevertheless, we may venture the following conjecture, which might at least allow almost complex methods to be used to study invariants of complex threefolds.

Conjecture 1.3. *The map $H^*_c(\mathcal{Z}(\mathrm{Cpx}_3)) \to H^*_c(\mathcal{Z}(\mathrm{ACpx}_3))$ is an isomorphism.*

We expect the same to be true for $\mathcal{Z}^{\text{semi-Fano}}$ in place of $\mathcal{Z}$, but it is less interesting given Theorem 1.1, which presumably remains valid for $H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{ACpx}_3))$ with a similar proof.

The most interesting question is probably whether there exists a modification of the group $H^*_c(\mathcal{Z}(\mathrm{Cpx}_3))$ for which an analogue of Theorem 1.1 holds and which can be used to study enumerative invariants of complex threefolds.

1.2 MNOP correspondence

Theorem 1.1 implies that a curve enumeration invariant of complex threefolds with nef anti-canonical bundle is determined uniquely by its values on local curves. We now explain how this may be used to verify a conjecture of Maulik–Okounkov–Nekrasov–Pandharipande [16, 17] for such threefolds.

Maulik–Nekrasov–Okounkov–Pandharipande [16, 17] originally conjectured an equivalence between Gromov–Witten and Donaldson–Thomas invariants of projective threefolds. A similar conjecture relating Gromov–Witten and Pandharipande–Thomas invariants was proposed by Pandharipande–Thomas [22]. Work of Bridgeland [5] relates Donaldson–Thomas and Pandharipande–Thomas invariants, implying the two conjectures are equivalent. We will address the latter conjecture here (Pandharipande–Thomas invariants are easier to work with than Donaldson–Thomas invariants in many respects, and our work here is no exception).

We briefly recall the definition of Gromov–Witten and Pandharipande–Thomas invariants, leaving a more detailed discussion to §3.5. Given a complex projective threefold X, a homology class $\beta \in H_2(X)$, and cohomology classes $\gamma_1, \ldots, \gamma_r \in H^*(X)$, these invariants have the form

$$\mathrm{GW}(X, \beta; \gamma_1, \ldots, \gamma_r) = \int_{[\overline{\mathcal{M}}'(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{r} \pi_! \operatorname{ev}^* \gamma_i \cdot u^{-\chi} \in \mathbb{Q}((u)), \tag{1.2}$$

$$\mathrm{PT}(X, \beta; \gamma_1, \ldots, \gamma_r) = \int_{[P(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{r} \pi_!(\mathrm{ch}_2(\mathbb{F}) \cup \pi_X^* \gamma_i) \cdot q^n \in \mathbb{Z}((q)). \tag{1.3}$$

For Gromov–Witten invariants, $\overline{\mathcal{M}}'(X, \beta)$ is the moduli space of stable maps from (not necessarily connected) nodal curves to X, in homology class β, all of whose connected components are non-constant, and χ denotes the arithmetic Euler characteristic of the domain (locally constant, proper sublevel sets). For Pandharipande–Thomas invariants, $P(X, \beta)$ denotes the moduli space of stable pairs in homology class β, and n denotes the holomorphic Euler characteristic (locally constant, proper sublevel sets). The integrands are given by push/pull via the universal families over these moduli spaces.

Let us say that a pair of formal Laurent series $\mathrm{GW} \in \mathbb{Q}((u))$ and $\mathrm{PT} \in \mathbb{Z}((q))$ satisfies the *MNOP correspondence* when PT is a rational function of q whose evaluation at $q = -e^{iu}$ equals GW.

Conjecture 1.4 ([16, 17, 22])**.** *For any projective threefold X, any homology class $\beta \in H_2(X)$, and any tuple of cohomology classes $\gamma_1, \ldots, \gamma_r \in H^*(X)$, the invariants*

$$(-iu)^{\langle c_1(TX),\beta\rangle} \mathrm{GW}(X, \beta; \gamma_1, \ldots, \gamma_r) \text{ and } (-q)^{-\langle c_1(TX),\beta\rangle/2} \mathrm{PT}(X, \beta; \gamma_1, \ldots, \gamma_r) \tag{1.4}$$

satisfy the MNOP correspondence.

Conjecture 1.4 is known in many cases, essentially by computing both sides of the equality. The case of (equivariant invariants of) local curves holds by deep calcuations of Bryan–Pandharipande [6] (of Gromov–Witten invariants) and Okounkov–Pandharipande [20] (of Donaldson–Thomas invariants). Work of Maulik–Oblomkov–Okounkov–Pandharipande [18] established the conjecture for toric varieties by direct computation of both sides. Work of

Pandharipande–Pixton [21] showed the result for many threefolds (e.g. complete intersections in products of projective spaces) by degeneration to the toric case.

Combining Theorem 1.1 with the known case of equivariant local curves [6, 20], we show the following.

Theorem 1.5. *Conjecture 1.4 holds when the anticanonical bundle of X is nef (that is, when $c_1(TX)$ pairs non-negatively with every curve $C \subseteq X$).*

Indeed, Gromov–Witten invariants and Pandharipande–Thomas invariants define ring homomorphisms

$$\mathrm{GW} : H^*_c(\mathcal{Z}(\mathrm{Cpx}_3)) \to \mathbb{Q}((u)), \tag{1.5}$$

$$\mathrm{PT} : H^*_c(\mathcal{Z}(\mathrm{Cpx}_3)) \to \mathbb{Z}((q)), \tag{1.6}$$

and Conjecture 1.4 amounts to the assertion that $(-iu)^k\mathrm{GW}$ and $(-q)^{-k/2}\mathrm{PT}$ satisfy the MNOP correspondence when evaluated on the element

$$(X, \beta; \gamma_1, \ldots, \gamma_r) \in H_c^{(|\gamma_1|-2)+\cdots+(|\gamma_r|-2)}(\mathcal{Z}(\mathrm{Cpx}_3, \langle c_1(TX), \beta\rangle)). \tag{1.7}$$

represented by the product of $\mathbf{1}_\beta \in H^0_c(\mathcal{Z}(X))$ and the classes $\pi_! i^*\gamma_i \in H^{|\gamma_i|-2}(\mathcal{Z}(X))$. The results of [6, 20] imply that $(-iu)^k\mathrm{GW}$ and $(-q)^{-k/2}\mathrm{PT}$ satisfy the MNOP correspondence when evaluated on equivariant local curve elements. Thus by Theorem 1.1, they satisfy the MNOP correspondence on $H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$.

This approach is similar in spirit to [21] in that in essence we are deforming to a simpler situation where the result is already known. We obtain a stronger result from a weaker input given the strength of Theorem 1.1.

The case of descendent invariants is conspicuously missing from the discussion above. It would suffice to write down natural classes on $\mathcal{Z}(X)$ whose pullback to $\overline{\mathcal{M}}'$ and P are the respective descendent classes, but we do not know how to do this. Alternatively, it might help to consider a Grothendieck group based instead on fiber powers of the universal family $\mathcal{U}(X) \to \mathcal{Z}(X)$ (thus a Grothendieck group of multi-pointed 1-cycles).

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2 Spaces of 1-cycles

2.1 Background

A (compact holomorphic) 1-cycle z on a complex analytic manifold X is a formal non-negative integer linear combination of irreducible compact 1-dimensional subvarieties $C \subseteq X$. The set of such 1-cycles is denoted $\mathcal{Z}(X)$ (more systematically, this would be denoted $\mathcal{Z}_1(X)$, but we will not consider r-cycles $\mathcal{Z}_r(X)$ for any r other than 1 in this text, so we drop the subscript from the notation). Such a cycle will usually be written as a finite sum $z = \sum_i m_i C_i$ where it is implicitly assumed that the $C_i \subseteq X$ are distinct irreducible curves and every $m_i \neq 0$.

The total chern number of a cycle $z = \sum_i m_i C_i$ is the pairing $\langle c_1(TX), z\rangle = \sum_i m_i \langle c_1(TX), C_i\rangle$. A cycle $z = \sum_i m_i C_i$ is called semi-Fano when the pairing of *every* C_i with $c_1(TX)$ is non-negative (it bears emphasis that this is stronger than having non-negative total chern number). We denote by $\mathcal{Z}(X,k) \subseteq \mathcal{Z}(X)$ the set of cycles with total chern number k, and we denote by $\mathcal{Z}^{\text{semi-Fano}}(X) \subseteq \mathcal{Z}(X)$ the set of semi-Fano cycles. We also denote by $\mathcal{Z}(X,\beta) \subseteq \mathcal{Z}(X)$ the set of cycles in homology class $\beta \in H_2(X)$ (which, we should warn, somewhat conflicts with the previous sentence).

The set $\mathcal{Z}(X)$ has the structure of a separated reduced complex analytic space due to work of Barlet [2]. By definition, an analytic map $A \to \mathcal{Z}(X)$ from a reduced complex analytic space A is a family of 1-cycles $\{z_a \in \mathcal{Z}(X)\}_{a\in A}$ which satisfies a certain analyticity condition [2, Chapitre 1, §1, Définition fondamentale]. If the family $\{z_a \in \mathcal{Z}(X)\}_{a\in A}$ is analytic, then the union $\bigcup_{a\in A} z_a \subseteq X \times A$ is a closed analytic subset, proper over A, with fibers of pure dimension 1 and multiplicities which are constant on its irreducible components [2, Chapitre 1, §2, Théorème 1] (and the converse holds if A is normal). In particular, there is a 'universal family' $\mathcal{U}(X) \subseteq X \times \mathcal{Z}(X)$.

The homology class function $\mathcal{Z}(X) \to H_2(X)$ is locally constant; that is, each subset $\mathcal{Z}(X,\beta) \subseteq \mathcal{Z}(X)$ of cycles in homology class $\beta \in H_2(X)$ is open. In partcular, the subset $\mathcal{Z}(X,k) \subseteq \mathcal{Z}(X)$ of cycles with chern number k is open. The subset $\mathcal{Z}^{\text{semi-Fano}}(X) \subseteq \mathcal{Z}(X)$ is also open.

This discussion generalizes readily to the relative setting. Given a holomorphic submersion $X \to B$, we define $\mathcal{Z}(X/B) = \bigcup_b \mathcal{Z}(X_b)$ to be the set of cycles in fibers of $X \to B$. It is an open subset of $\mathcal{Z}(X)$, so the basic properties of $\mathcal{Z}(X)$ pass easily to $\mathcal{Z}(X/B)$.

2.2 Semi-charts

Around each point $z = \sum_i m_i C_i \in \mathcal{Z}(X)$ is a *semi-chart* defined as follows. Let $\tilde{C}_i \to C_i$ denote the normalization of C_i, so $\tilde{C}_i$ is a compact smooth curve. We consider all local deformations of $\tilde{C} = \bigsqcup_i \tilde{C}_i \to X$ (including deformations of the complex structure on the domain), and we associate to such a nearby map $\tilde{C}' = \bigsqcup_i \tilde{C}'_i \to X$ the cycle $\sum_i m_i C'_i$. We denote by $(S_z, z) \to (\mathcal{Z}(X), z)$ (a germ) the semi-chart around z. The semi-chart $S_z \to \mathcal{Z}(X)$ need not be a (germ near z of) open embedding, since it does not take into account the possibility of the topology changing (as in $y^2 = x(x-t)(x+t)$ near $t = 0$) or of curves with multiplicities breaking apart (as in $y^2 = tx$ near $t = 0$). The locus of points z for which the semi-chart around z is an open embedding is evidently open.

Lemma 2.1. *The set of points $z \in \mathcal{Z}(X)$ whose semi-chart is an open embedding is dense.*

Proof. Begin with an arbitrary cycle $z = \sum_i m_i C_i \in \mathcal{Z}(X)$, and let us produce cycles arbitrarily close to z whose associated semi-charts are open embeddings.

A nearby cycle z' determines a partition μ_i of each m_i. Partially order the set $\Pi(m)$ of partitions of m by refinement: declare $\mu \geq \mu'$ when μ' is obtained from μ by replacing each of its constituents by a partition thereof. The map from a neighborhood of $z \in \mathcal{Z}(X)$

to $\prod_i \Pi(m_i)$ has local minima arbitrarily close to z since $\prod_i \Pi(m_i)$ satisfies the descending chain condition (since it is finite). We may thus assume wlog that z is itself a local minimum of this map. This means that if we write $z = \sum_{m\geq 1} mC_m$ (C_m not necessarily irreducible) then every nearby cycle z' has the form $\sum_{m\geq 1} mC'_m$ for C'_m nearby C_m. In other words, there is a factorization $(\mathcal{Z}(X), z) = \prod_{m\geq 1}(\mathcal{Z}(X), C_m)$ of germs. This factorization reduces us to the case $z = C$ for some not necessarily irreducible curve C.

The Euler characteristic function $\chi : \mathcal{Z}(X) \to \mathbb{Z}$ near $z = C$ is bounded below since the universal family $\mathcal{U}(X) \subseteq X \times \mathcal{Z}(X)$ is finite type. We may thus assume wlog that z is a local minimum of χ. Let us argue that this implies that the semi-chart at z surjects onto a neighborhood of z (hence is an open embedding). A nearby cycle z' is simply a curve C' nearby C. Near smooth points of C, the curve C' is a nearby smooth curve, hence may be (non-canonically) identified with C (as smooth manifolds) with nearby complex structure. Near a singular point of C (necessarily isolated), choose a ball B around it so that $\tilde{C} \cap B$ is a disjoint union of disks. Now a disjoint union of disks is the unique filling of a disjoint union of circles of maximal Euler characteristic, so since $\chi(\tilde{C}') = \chi(\tilde{C})$, we conclude that $\tilde{C}' \cap B$ is also a disjoint union of disks. This shows that $\tilde{C}' \to X$ is a small perturbation of $\tilde{C} \to X$, as desired. $\square$

3 Grothendieck groups of 1-cycles

3.1 Definition

We now define the Grothendieck groups of 1-cycles which we will study.

We will consider families of complex threefolds over (real) simplices Δ^n. Such a family is, by definition, a family in the usual sense over a(n unspecified) open neighborhood of $\Delta^n \subseteq \mathbb{R}^n \subseteq \mathbb{C}^n$ (i.e. in the complexification), and an isomorphism of families is a germ of isomorphism defined in a neighborhood of Δ^n inside $\mathbb{C}^n$. In particular, if $X \to B$ is a family of complex threefolds over a smooth analytic base B and $\Delta^n \to B$ is any real analytic map, then the pullback $X \times_B \Delta^n \to \Delta^n$ is a family in the above sense, since $\Delta^n \to B$, being real analytic, extends over a neighborhood of Δ^n inside its complexification.

Definition 3.1. The group $H^*_c(\mathcal{Z}(\mathrm{Cpx}_3))$ is the homology of (the total complex associated to) the double complex $C_*(\mathrm{Cpx}_3, C^*_c(\mathcal{Z}))$, illustrated below.

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots \longrightarrow & & \bigoplus_{X\to\Delta^2} C^2_c(\mathcal{Z}(X/\Delta^2)) & \longrightarrow & \bigoplus_{X\to\Delta^1} C^2_c(\mathcal{Z}(X/\Delta^1)) & \longrightarrow & \bigoplus_{X} C^2_c(\mathcal{Z}(X)) \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots \longrightarrow & & \bigoplus_{X\to\Delta^2} C^1_c(\mathcal{Z}(X/\Delta^2)) & \longrightarrow & \bigoplus_{X\to\Delta^1} C^1_c(\mathcal{Z}(X/\Delta^1)) & \longrightarrow & \bigoplus_{X} C^1_c(\mathcal{Z}(X)) \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots \longrightarrow & & \bigoplus_{X\to\Delta^2} C^0_c(\mathcal{Z}(X/\Delta^2)) & \longrightarrow & \bigoplus_{X\to\Delta^1} C^0_c(\mathcal{Z}(X/\Delta^1)) & \longrightarrow & \bigoplus_{X} C^0_c(\mathcal{Z}(X))
\end{array} \tag{3.1}$$

The 'total complex' in this case involves direct sum over the anti-diagonals. There is a bigrading

$$H_c^*(\mathcal{Z}(\mathrm{Cpx}_3)) = \bigoplus_{i,k} H_c^i(\mathcal{Z}(\mathrm{Cpx}_3, k)) \tag{3.2}$$

by cohomological degree i (indexing the anti-diagonals of the double complex) and chern number k of the cycles. The 'total homological degree' is $2k - i$ (that is, k is 'half a homological grading'). We may also use $\mathcal{Z}^{\text{semi-Fano}} \subseteq \mathcal{Z}$ in place of $\mathcal{Z}$.

Remark 3.2. The group $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$ is the homology of the spectrum

$$\operatorname{colim}\Big(\cdots \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}} \coprod_{X\to\Delta^2} D(\mathcal{Z}(X/\Delta^2)/\infty) \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \coprod_{X\to\Delta^1} D(\mathcal{Z}(X/\Delta^1)/\infty) \rightrightarrows \coprod_X D(\mathcal{Z}(X)/\infty)\Big) \tag{3.3}$$

where by D we mean Spanier–Whitehead dual.

Since there is no 'set' of all families of complex threefolds over Δ^n, a somewhat pedantic discussion is needed to make Definition 3.1 precise. We begin by recalling the notion of a semi-simplicial object. The semi-simplex category $\mathbf{\Delta}^{\mathsf{inj}}$ consists of totally ordered sets $[n] = \{0 < \ldots < n\}$ and strictly order preserving (thus injective) maps $[n] \hookrightarrow [m]$. There is a functor from $\mathbf{\Delta}^{\mathsf{inj}}$ to topological spaces sending $[n]$ to the simplex Δ^n (with vertices labelled $0, \ldots, n$) and a morphism $[n] \hookrightarrow [m]$ to the corresponding face inclusion $\Delta^n \hookrightarrow \Delta^m$ preserving vertex order. A semi-simplicial object $X_\bullet$ in a category C is a functor $X : (\mathbf{\Delta}^{\mathsf{inj}})^{\mathsf{op}} \to \mathsf{C}$, and we write $X_n = X([n])$. A semi-simplicial set $X : (\mathbf{\Delta}^{\mathsf{inj}})^{\mathsf{op}} \to \mathsf{Set}$ may be regarded as a combinatorial specification of how to glue together standard simplices Δ^n along injective maps preserving vertex order. Associating to each $[n] \in \mathbf{\Delta}^{\mathsf{inj}}$ the groupoid of all families of complex threefolds $X \to \Delta^n$ (and to a morphism the corresponding pullback) defines a semi-simplicial groupoid which we denote by $\mathrm{Cpx}_{3,\bullet}$ (a semi-simplicial object in the 2-category of groupoids).

A coefficient system (valued in a category C) over a semi-simplicial set $X_\bullet$ is a functor $(\mathbf{\Delta}^{\mathsf{inj}} \downarrow X_\bullet) \to \mathsf{C}$ where $(\mathbf{\Delta}^{\mathsf{inj}} \downarrow X_\bullet)$ denotes the category of simplices in $X_\bullet$ (objects are maps $[n] \to X_\bullet$ and morphisms are compositions $[n] \hookrightarrow [m] \to X_\bullet$, namely maps of simplices over $X_\bullet$). The complex of chains on $X_\bullet$ with respect to a coefficient system A valued in the category $\mathsf{Ab}^{\mathsf{op}}$ is given by

$$C_*(X_\bullet; A) = \bigoplus_{n\geq 0} \bigoplus_{\sigma\in X_n} A_\sigma \otimes \mathfrak{o}_n \tag{3.4}$$

where $\mathfrak{o}_n$ denotes the orientation group of Δ^n, and the boundary operator acts on $A_\sigma \otimes \mathfrak{o}_n$ via the usual sum over faces $d_{n,i} : [n-1] \hookrightarrow [n]$ for $0 \leq i \leq n$. Dually, we may define cochains $C^*(X_\bullet, A)$ with respect to any coefficient system A valued in Ab. More generally, these constructions apply to coefficient systems valued in the category of complexes of abelian groups (and its opposite). A map of coefficient systems $A \to B$ over $X_\bullet$ induces a map $C_*(X_\bullet; A) \to C_*(X_\bullet; B)$, and a map $f : X_\bullet \to Y_\bullet$ of simplicial sets induces a map $C_*(X_\bullet, f^*A) \to C_*(Y_\bullet, A)$ for any coefficient system A over $Y_\bullet$.

Lemma 3.3. *A quasi-isomorphism of coefficient systems $A \to B$ over X induces a quasi-isomorphism $C_*(X; A) \to C_*(X; B)$.*

Proof. Let $X_{\leq k}$ denote the k-skeleton of X. The short exact sequence of complexes

$$0 \to C_*(X_{<k}; A) \to C_*(X_{\leq k}; A) \to \bigoplus_{\substack{\sigma \subseteq X \\ \dim \sigma = k}} A_\sigma \to 0 \tag{3.5}$$

induces a long exact sequence of cohomology groups. Applying the five lemma, we see that $C_*(X_{<k}; A) \to C_*(X_{<k}; B)$ a quasi-isomorphism implies $C_*(X_{\leq k}; A) \to C_*(X_{\leq k}; B)$ is a quasi-isomorphism. Finally, note that $C_*(X; A)$ is the directed colimit of $C_*(X_{\leq k}; A)$ over k and that homology commutes with directed colimits. □

Associating to each family of complex threefolds $X \to \Delta^n$ the complex of compactly supported cochains $C_c^*(\mathcal{Z}(X/\Delta^n))$ defines a coefficient system over the semi-simplicial groupoid $\mathrm{Cpx}_{3,\bullet}$. The resulting chain group $C_*(\mathrm{Cpx}_{3,\bullet}, C_c^*(\mathcal{Z}))$ is the basis of Definition 3.1, but it lacks a definition since $\mathrm{Cpx}_{3,\bullet}$ is not a semi-simplicial set, rather a semi-simplicial groupoid, rendering (3.4) meaningless. We now close this gap by defining (co)chains for semi-simplicial groupoids.

Recall that a map of semi-simplicial sets $A_\bullet \to B_\bullet$ is called a *trivial Kan fibration* when for every diagram of solid arrows

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & A_\bullet \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & B_\bullet \end{tikzcd} \tag{3.6}$$

there exists a dotted lift. If $A_\bullet \to B_\bullet$ is a trivial Kan fibration, then for any level-wise injection of semi-simplicial sets $P_\bullet \to Q_\bullet$ and every diagram of solid arrows

$$\begin{tikzcd} P_\bullet \arrow[r] \arrow[d] & A_\bullet \arrow[d] \\ Q_\bullet \arrow[r] \arrow[ur, dashed] & B_\bullet \end{tikzcd} \tag{3.7}$$

there exists a dotted lift (construct the lift one simplex at a time).

A *resolution* of a semi-simplicial groupoid $X_\bullet$ is a trivial Kan fibration $\tilde{X}_\bullet \to X_\bullet$ from a semi-simplicial set $\tilde{X}_\bullet$ (the notion of a trivial Kan fibration extends to semi-simplicial groupoids without issue). We define the (co)chain group of a semi-simplicial groupoid $X_\bullet$ with coefficients in A to be the (co)chain group of any resolution $\tilde{X}_\bullet$ of $X_\bullet$ with coefficients in the pullback of A.

Lemma 3.4. *The (co)homology of a semi-simplicial groupoid (with respect to any coefficient system) is well defined.*

Proof. Given two resolutions of $X_\bullet$, we may consider their fiber product. It thus suffices to show that for any trivial Kan fibration of semi-simplicial sets $f : X_\bullet \to Y_\bullet$ and any coefficient system A on $Y_\bullet$, the induced map $f_* : C_*(X_\bullet, f^*A) \to C_*(Y_\bullet, A)$ is a homotopy equivalence. Solving the lifting problem

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & X_\bullet \arrow[d, "f"] \\ Y_\bullet \arrow[r, "\mathbf{1}_Y"'] \arrow[ur, dashed, "s"] & Y_\bullet \end{tikzcd} \tag{3.8}$$

produces a section $s : Y_\bullet \to X_\bullet$ of f. Solving the lifting problem

$$\begin{tikzcd} X \times \partial\Delta^1 \arrow[r, "sf \sqcup \mathbf{1}_X"] \arrow[d] & X_\bullet \arrow[d, "f"] \\ X \times \Delta^1 \arrow[r, "fp_X"] \arrow[ur, dashed, "H"] & Y_\bullet \end{tikzcd} \tag{3.9}$$

produces a homotopy H between $\mathbf{1}_X$ and $sf : X_\bullet \to X_\bullet$. It then follows from functoriality of C_* that f_* is a homotopy equivalence. □

Given a semi-simplicial set B and a family of threefolds $X \to B$ (equivalently, a map $B \to \mathrm{Cpx}_{3,\bullet}$), there is a tautological map (at least on homology)

$$C_*(B, C_c^*(\mathcal{Z}(X/-))) \to C_*(\mathrm{Cpx}_3, C_c^*(\mathcal{Z})), \tag{3.10}$$

and every element of $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$ is in the image of this map for some family $X \to B$ over a finite semi-simplicial set B.

The complex $C_*(B, C_c^*(\mathcal{Z}(X/-)))$ may be described more geometrically as follows. Form $\mathcal{Z}(X/B)$ as the evident gluing of $\mathcal{Z}(X_\sigma/\sigma)$ over $\sigma \subseteq B$. To each simplex $\sigma \subseteq B$, we may associate the sheaf $\mathbb{Z}_{\mathcal{Z}(X_\sigma/\sigma)}$, namely the constant sheaf $\mathbb{Z}$ on the closed subset $\mathcal{Z}(X_\sigma/\sigma) \subseteq \mathcal{Z}(X/B)$. This defines a coefficient system on B valued in sheaves on $\mathcal{Z}(X/B)$, and its chain group will be denoted

$$\pi^*\omega_B = C_*(B; \mathbb{Z}_{\mathcal{Z}(X/-)}) = \bigoplus_{\sigma \subseteq B} \mathbb{Z}_{\mathcal{Z}(X_\sigma/\sigma)}[\dim \sigma] \tag{3.11}$$

since it is the pullback to $\mathcal{Z}(X/B)$ of the dualizing sheaf of B. Now there is a canonical identification

$$C_*(B, C_c^*(\mathcal{Z}(X/-))) = C_c^*(\mathcal{Z}(X/B), \pi^*\omega_B), \tag{3.12}$$

where the cohomology group on the right side with coefficients in a complex which is not bounded below (cohomologically) should be interpreted as the directed colimit of cohomology of truncations $C_c^*(\mathcal{Z}(X/B), \tau^{\leq -N}\pi^*\omega_B)$ as $N \to \infty$. This geometric description makes it clear that the cell decomposition of B is irrelevant; that is, there is a canonical map

$$H_c^*(\mathcal{Z}(X/B), \pi^*\omega_B) \to H_c^*(\mathcal{Z}(\mathrm{Cpx}_3)) \tag{3.13}$$

independent of the choice of triangulation of B. It would thus be more accurate to write $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3), \pi^*\omega_{\mathrm{Cpx}_3})$ in place of $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$.

In particular, we could consider a single real analytic manifold B as the base. A family of complex threefolds $X \to B$ then determines a map

$$H_c^{*+\dim B}(\mathcal{Z}(X/B)) \to H_c^*(\mathcal{Z}(\mathrm{Cpx}_3)) \tag{3.14}$$

obtained by triangulating B and summing over all top-dimensional simplices. If we consider the restriction $X' \to B'$ of this family to a submanifold $i : B' \subseteq B$, then we have a commuting diagram

$$\begin{tikzcd} H_c^{*+\dim B'}(\mathcal{Z}(X'/B')) \arrow[rr, "i_!"] \arrow[dr] & & H_c^{*+\dim B}(\mathcal{Z}(X/B)) \arrow[dl] \\ & H_c^*(\mathcal{Z}(\mathrm{Cpx}_3)) & \end{tikzcd} \tag{3.15}$$

where horizontal map $i_!$ is the 'wrong way' map defined by triangulating B so that B' is a subcomplex.

3.2 Restriction to open loci

The definition of the Grothendieck group $H^*_c(\mathcal{Z}(\mathrm{Cpx}_3))$ applies without change with the open set $\mathcal{Z}^{\text{semi-Fano}} \subseteq \mathcal{Z}$ in place of $\mathcal{Z}$, producing a group $H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$. To define the expected tautological map $H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)) \to H^*_c(\mathcal{Z}(\mathrm{Cpx}_3))$, we need to say more about our model of compactly supported cochains C^*_c (to assure that it has pushforward under open embeddings).

Let us fix a model of compactly supported cochains which is indeed functorial under open embeddings and proper maps. The singular cochains functor C^*_{sing} is a functor on topological spaces, and we consider its sheafification $C^{\#,*}_{\mathrm{sing}}$. Concretely, a section of $C^{\#,k}_{\mathrm{sing}}$ over a topological space X is a k-cochain on X modulo equivalence: two k-cochains are equivalent when there exists an open covering $X = \bigcup_i U_i$ such that their restrictions to every U_i coincide. We define $C^*_c(X)$ to be the subcomplex of $C^{\#,*}_{\mathrm{sing}}(X)$ of compactly supported sections. Now C^*_c is a contravariant functor on the category of Hausdorff topological spaces with morphisms $X \dashrightarrow Y$ given by correspondences $X \hookleftarrow U \to Y$ where $U \hookrightarrow X$ is an open embedding and $U \to Y$ is proper (composition of correspondences is via fiber product).

3.3 Algebraic structure

We now define a product and coproduct on the Grothendieck group $H^*_c(\mathcal{Z}(\mathrm{Cpx}_3))$, forming the structure of a commutative and co-commutative bi-algebra. The product corresponds to 'disjoint union of cycles', while the coproduct corresponds to 'addition of cycles'.

We begin with some generalities on products of semi-simplicial sets and coefficient systems. Given semi-simplicial sets $X_\bullet$ and $Y_\bullet$, their *product* is the product functor $X \times Y : (\mathbf{\Delta}^{\mathsf{inj}} \times \mathbf{\Delta}^{\mathsf{inj}})^{\mathsf{op}} \to \mathsf{Set}$ (such a functor is called a bi-semi-simplicial set). Just like semi-simplicial sets, a bi-semi-simplicial set may be regarded as a combinatorial specification of how to glue together products of standard simplices $\Delta^n \times \Delta^m$ along products of injective maps preserving vertex order. A product of simplices $\Delta^n \times \Delta^m$ has a standard subdivision into $\binom{n+m}{n}$ copies of Δ^{n+m}, and this subdivision is moreover compatible with products of maps. Thus a bi-semi-simplicial set $Z_{\bullet,\bullet}$ has a subdivision $s_\bullet(Z_{\bullet,\bullet})$, which is a semi-simplicial set. We denote the subdivision of a product of semi-simplicial sets $X \times Y$ by $X \boxtimes Y = s(X \times Y)$.

A coefficient system on a bi-semi-simplicial set is defined analogously with semi-simplicial sets, and we may form singular chains with coefficients in the natural way. A coefficient system A on a bi-semi-simplicial set induces a coefficient system sA on its subdivision by restriction. Given a coefficient system A on a bi-semi-simplicial set $Z_{\bullet,\bullet}$ and a coefficient system B on its subdivision $s_\bullet(Z_{\bullet,\bullet})$, a map of coefficient systems $s_*A \to B$ induces a map $C_*(Z_{\bullet,\bullet}; A) \to C_*(s_\bullet(Z_{\bullet,\bullet}), B)$. The map $C_*(Z; A) \to C_*(sZ; sA)$ is a homotopy equivalence. Given coefficient systems A and B on semi-simplicial sets X and Y, both pull back to coefficient systems on the product $X \times Y$, and there is a canonical isomorphism $C_*(X \times Y; A \otimes B) = C_*(X; A) \otimes C_*(Y; B)$.

Definition 3.5 (Product on $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$)**.** Given a pair of families $X \to \Delta^n$ and $Y \to \Delta^m$, we may consider the disjoint union family $(X \times \Delta^m) \sqcup (\Delta^n \times Y) \to \Delta^n \times \Delta^m$. That is, the product $\mathrm{Cpx}_{3,\bullet} \times \mathrm{Cpx}_{3,\bullet}$ has two families of threefolds (pulled back from the two factors), and we consider their disjoint union. The relative cycle space of the disjoint union $(X \times \Delta^m) \sqcup (\Delta^n \times Y) \to \Delta^n \times \Delta^m$ is the product of relative cycle spaces $\mathcal{Z}(X/\Delta^n) \times \mathcal{Z}(Y/\Delta^m) \to \Delta^n \times \Delta^m$. Thus cup product defines a quasi-isomorphism of coefficient systems $C_c^*(\mathcal{Z})_{(1)} \otimes C_c^*(\mathcal{Z})_{(2)} \to C_c^*(\mathcal{Z})$ on $\mathrm{Cpx}_{3,\bullet} \times \mathrm{Cpx}_{3,\bullet}$, inducing a quasi-isomorphism

$$\begin{aligned} &C_*(\mathrm{Cpx}_{3,\bullet}, C_c^*(\mathcal{Z}))^{\otimes 2} = \\ &\qquad C_*(\mathrm{Cpx}_{3,\bullet} \times \mathrm{Cpx}_{3,\bullet}, C_c^*(\mathcal{Z})_{(1)} \otimes C_c^*(\mathcal{Z})_{(2)}) \xrightarrow{\sim} C_*(\mathrm{Cpx}_{3,\bullet} \times \mathrm{Cpx}_{3,\bullet}, C_c^*(\mathcal{Z})). \end{aligned} \tag{3.16}$$

Subdividing the bi-semi-simplicial set $\mathrm{Cpx}_{3,\bullet} \times \mathrm{Cpx}_{3,\bullet}$ defines a map

$$C_*(\mathrm{Cpx}_{3,\bullet} \times \mathrm{Cpx}_{3,\bullet}, C_c^*(\mathcal{Z})) \xrightarrow{\sim} C_*(\mathrm{Cpx}_{3,\bullet} \boxtimes \mathrm{Cpx}_{3,\bullet}, C_c^*(\mathcal{Z})) \tag{3.17}$$

which is a quasi-isomorphism as remarked above. Finally, compose with the classifying map $\mathrm{Cpx}_{3,\bullet} \boxtimes \mathrm{Cpx}_{3,\bullet} \to \mathrm{Cpx}_{3,\bullet}$ to obtain a map

$$C_*(\mathrm{Cpx}_{3,\bullet} \boxtimes \mathrm{Cpx}_{3,\bullet}, C_c^*(\mathcal{Z})) \to C_*(\mathrm{Cpx}_{3,\bullet}, C_c^*(\mathcal{Z})). \tag{3.18}$$

Composing these three maps defines the product operation on $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$.

A diagram chase shows that the product is associative and unital (the unit $\eta : \mathbb{Z} \to H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$ is the constant function 1 on $\mathcal{Z}(\varnothing)$, which is indeed a cycle in $C_0(\mathrm{Cpx}_3, C_c^0(\mathcal{Z}))$). The product is also (graded) commutative: while cup product (3.16) is not commutative on the cochain level, it is commutative up to Steenrod's $\cup_1$ operation which is a chain null-homotopy of $\alpha \otimes \beta \mapsto \alpha \cup \beta - (-1)^{|\alpha||\beta|}\beta \cup \alpha$ [31, Theorem 5.1].

Definition 3.6 (Coproduct on $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$)**.** The addition map $\Sigma : \mathcal{Z}(X/\Delta^n) \times_{\Delta^n} \mathcal{Z}(X/\Delta^n) \to \mathcal{Z}(X/\Delta^n)$ is a map of (space valued) coefficient systems $\Delta^*(\mathcal{Z}) \to \mathcal{Z}$ where $\Delta : \mathrm{Cpx}_{3,\bullet} \to \mathrm{Cpx}_{3,\bullet} \boxtimes \mathrm{Cpx}_{3,\bullet}$ denotes the diagonal embedding. Applying C_c^* thus determines a map of coefficient systems $C_c^*(\mathcal{Z}) \to \Delta^* C_c^*(\mathcal{Z})$ since Σ is proper, hence a map of complexes

$$C_*(\mathrm{Cpx}_{3,\bullet}, C_c^*(\mathcal{Z})) \xrightarrow{\Sigma^*} C_*(\mathrm{Cpx}_{3,\bullet} \boxtimes \mathrm{Cpx}_{3,\bullet}, C_c^*(\mathcal{Z})). \tag{3.19}$$

Now recall from Definition 3.5 the quasi-isomorphism

$$C_*(\mathrm{Cpx}_{3,\bullet})^{\otimes 2} \xrightarrow{\sim} C_*(\mathrm{Cpx}_{3,\bullet} \boxtimes \mathrm{Cpx}_{3,\bullet}, C_c^*(\mathcal{Z})). \tag{3.20}$$

Inverting this defines a coproduct operation on $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))/\mathrm{tors}$ (note that homology does not commute with tensor product, but does modulo torsion).

A diagram chase shows that the coproduct is coassociative and counital (the counit $\varepsilon : H_c^*(\mathcal{Z}(\mathrm{Cpx}_3)) \to \mathbb{Z}$ acts on $C_*(\mathrm{Cpx}_{3,\bullet}, C_c^*(\mathcal{Z}))$ by summing the 'evaluate at the empty cycle' map over all vertices). The coproduct is trivially cocommutative.

Definition 3.7. A *bi-algebra* $(R, \eta, \mu, \varepsilon, \Delta)$ means that:

- (R, η, μ) is an algebra (satisfies unitality and associativity).

- (R, ε, Δ) is a co-algebra (satisfies co-unitality and co-associativity).
- The maps η and Δ are algebra maps (equivalently, the maps ε and μ are co-algebra maps).

A diagram chase shows that $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))/\mathrm{tors}$ is a bi-algebra.

Definition 3.8 (Division)**.** For any $d \geq 1$, the 'multiply by d map' $\mathcal{Z}(X/B) \to \mathcal{Z}(X/B)$ determines, via pullback, a map of coefficient systems $C_c^*(\mathcal{Z}) \to C_c^*(\mathcal{Z})$ over $\mathrm{Cpx}_{3,\bullet}$. We denote by ρ_d the induced map on the chain group $C_*(\mathrm{Cpx}_{3,\bullet}, C_c^*(\mathcal{Z}))$ and on its homology $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$.

The maps ρ_d are bi-algebra morphisms by inspection.

The same construction applies without change to $H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ in place of $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$. The tautological map $H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)) \to H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$ is a map of bi-algebras and commutes with ρ_d. To check compatibility with the coproduct, we should note a sum of cycles $z + z'$ is semi-Fano iff both z and z' are semi-Fano.

3.4 Virtual fundamental cycles

The chain-level dual of the Grothendieck group of 1-cycles is $C^*(\mathrm{Cpx}_3, C_*^{\mathrm{rel}\infty}(\mathcal{Z}))$, namely the total complex (product along anti-diagonals) of the following double complex.

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots \leftarrow & \prod_{X\to\Delta^2} & C_2^{\mathrm{rel}\infty}(\mathcal{Z}(X/\Delta^2)) & \leftarrow \prod_{X\to\Delta^1} & C_2^{\mathrm{rel}\infty}(\mathcal{Z}(X/\Delta^1)) & \leftarrow \prod_{X} & C_2^{\mathrm{rel}\infty}(\mathcal{Z}(X)) \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots \leftarrow & \prod_{X\to\Delta^2} & C_1^{\mathrm{rel}\infty}(\mathcal{Z}(X/\Delta^2)) & \leftarrow \prod_{X\to\Delta^1} & C_1^{\mathrm{rel}\infty}(\mathcal{Z}(X/\Delta^1)) & \leftarrow \prod_{X} & C_1^{\mathrm{rel}\infty}(\mathcal{Z}(X)) \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots \leftarrow & \prod_{X\to\Delta^2} & C_0^{\mathrm{rel}\infty}(\mathcal{Z}(X/\Delta^2)) & \leftarrow \prod_{X\to\Delta^1} & C_0^{\mathrm{rel}\infty}(\mathcal{Z}(X/\Delta^1)) & \leftarrow \prod_{X} & C_0^{\mathrm{rel}\infty}(\mathcal{Z}(X)) \end{array} \tag{3.21}$$

A cycle in this complex may reasonably be called a 'coherent collection of cycles on all cycle spaces of all complex threefolds'. A class in its homology $H_*^{\mathrm{rel}\infty}(\mathcal{Z}(\mathrm{Cpx}_3))$ will be called a *curve enumeration theory (for complex threefolds)*. Such a curve enumeration theory determines, via the tautological pairing, a homomorphism out of the Grothendieck group $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$.

The group $H_*^{\mathrm{rel}\infty}(\mathcal{Z}(\mathrm{Cpx}_3))$ has a unital product and counital coproduct by dualizing the constructions above for $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$. A curve enumeration theory $F \in H_*^{\mathrm{rel}\infty}(\mathcal{Z}(\mathrm{Cpx}_3))$ gives rise to a ring homomorphism out of $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$ when it solves the equation $\Delta(F) = F \otimes F$. We will call such a curve enumeration theory *multiplicative*.

Now let us review the practical origin of curve enumeration theories. All known curve enumeration theories arise via pushforward $H_*^{\mathrm{rel}\infty}(\mathcal{E}(\mathrm{Cpx}_3)) \to H_*^{\mathrm{rel}\infty}(\mathcal{Z}(\mathrm{Cpx}_3))$ for some $\mathcal{E}$ associating to each family of threefolds $X \to B$ over a complex analytic base B an analytic space (or Deligne–Mumford stack) $\mathcal{E}(X/B) \to B$, compatible with pullback, with a natural transformation $\mathcal{E} \to \mathcal{Z}$ which is proper (hence has pushforward on homology rel infinity). If $\mathcal{E}(X/B) \to B$ is smooth (i.e. submersive) for every $X \to B$, then the vertical/relative fundamental class is a canonical class in $H_*^{\mathrm{rel}\infty}(\mathcal{E}(\mathrm{Cpx}_3))$. Its pushforward to $H_*^{\mathrm{rel}\infty}(\mathcal{Z}(\mathrm{Cpx}_3))$ would thus be a curve enumeration theory for complex threefolds, associated canonically to $\mathcal{E}$. In practice, the map $\mathcal{E}(X/B) \to B$ will not always be smooth, rather it will carry some extra structure (either a quasi-smooth derived structure or its classical shadow, a relative perfect obstruction theory) which induces a canonical 'virtual fundamental' class in $H_*^{\mathrm{rel}\infty}(\mathcal{E}(\mathrm{Cpx}_3))$.

We now recall the theory of virtual fundamental classes and explain how it is specializes to our setting of interest. What we shall need is in fact the *relative* virtual fundamental class. Increasingly general notions of relative virtual fundamental classes were introduced by Behrend–Fantechi [3], Manolache [15], and Khan [13, 14]. We will reference the construction of Khan [13], although the framework of Manolache [15] (generalized to 'local coefficients') may also suffice. Here is what we will use:

Definition 3.9 (Virtual fundamental class)**.** Let B be a complex analytic space, and let $W \to B$ be a separated map from a complex analytic space (or Deligne–Mumford stack). Moreover, fix a derived structure on W for which the map $W \to B$ is quasi-smooth of relative virtual dimension d (the classical shadow of this structure [25], namely a relative perfect obstruction theory on $W \to B$ is probably sufficient). Such a structure induces a canonical *relative virtual fundamental class* $[W/B]^{\mathrm{vir}}$ in the degree d homology of $C^*(S_\bullet B, C_*^{\mathrm{rel}\infty}(W/-))$, where $S_\bullet B$ denotes the semi-simplicial set of real analytic maps to B and we use rational coefficients in the case W is a Deligne–Mumford stack. Namely, this is the image of the relative virtual fundamental class of Khan [13, Construction 3.6 "Fundamental class"] $[W/B]^{\mathrm{vir}} \in H_{2d}^{\mathrm{BM}}(W/B, \mathcal{F}(d))$ under the natural specialization map to the homology of $C^*(S_\bullet B, C_*^{\mathrm{rel}\infty}(W/-))$. A key property of the relative virtual fundamental class is compatibility with pullback: if $W' \to B'$ is the pullback of $W \to B$ under a map $B' \to B$, then $[W'/B']^{\mathrm{vir}}$ is the image of $[W/B]^{\mathrm{vir}}$ under the induced map $C^*(S_\bullet B, C_*^{\mathrm{rel}\infty}(W/-)) \to C^*(S_\bullet B', C_*^{\mathrm{rel}\infty}(W'/-))$ [13, Theorem 3.13 "Base change"]. It is also compatible with product in the sense that $[W \times W'/B \times B']^{\mathrm{vir}} = [W/B]^{\mathrm{vir}} \otimes [W'/B']^{\mathrm{vir}}$ [13, Theorem 3.12 "Functoriality"].

Now we will also need the relative virtual fundamental class when the base B is a semi-simplicial set. A derived complex analytic space (or Deligne–Mumford stack) W separated and quasi-smooth over B now consists of the same over open neighborhoods of $\Delta^k \subseteq \mathbb{C}^k$ for all k-simplices $\sigma \subseteq B$, with coherent restriction isomorphisms. Gluing these families together defines a complex analytic family $W_\mathbb{C} \to B_\mathbb{C}$ (the complexification of $W \to B$),[1] which thus has a virtual fundamental class $[W_\mathbb{C}/B_\mathbb{C}]^{\mathrm{vir}} \in C^*(S_\bullet B_\mathbb{C}, C_*^{\mathrm{rel}\infty}(W_\mathbb{C}/-))$ from Definition 3.9. We define $[W/B]^{\mathrm{vir}} \in C^*(B, C_*^{\mathrm{rel}\infty}(W/-))$ to be the pullback of $[W_\mathbb{C}/B_\mathbb{C}]^{\mathrm{vir}}$. The class $[W/B]^{\mathrm{vir}}$ is evidently compatible with pullback of families under maps of semi-simplicial sets $B' \to B$. It is also compatible with subdivision (recalling from §3.1 that subdivision induces a quasi-isomorphism of complexes $C^*(B, C_*^{\mathrm{rel}\infty}(W/-)))$ since subdividing B just enlarges $B_\mathbb{C}$. Finally, let us also note that the virtual fundamental class is compatible with product,

[1] Note that in practice, the family $W \to B$ has the form $\mathcal{E}(X/B) \to B$ for some family of threefolds $X \to B$, in which case we may alternatively define the complexification $\mathcal{E}(X/B)_\mathbb{C} \to B_\mathbb{C}$ as the result $\mathcal{E}(X_\mathbb{C}/B_\mathbb{C}) \to B_\mathbb{C}$ of applying the functor $\mathcal{E}$ to the complexified family of threefolds $X_\mathbb{C} \to B_\mathbb{C}$. The complexification $X_\mathbb{C} \to B_\mathbb{C}$ only involves gluing families of analytic spaces, rather than derived analytic Deligne–Mumford stacks.

in the sense that for families $W \to B$ and $W' \to B'$, we have $[W \times W'/B \times B']^{\mathrm{vir}} = [W/B]^{\mathrm{vir}} \otimes [W'/B']^{\mathrm{vir}}$ under the identification $C^*(B, C_*^{\mathrm{rel}\infty}(W/-)) \otimes C^*(B', C_*^{\mathrm{rel}\infty}(W'/-)) = C^*(B \times B', C_*^{\mathrm{rel}\infty}(W \times W'/-))$.

Now fix a functor $\mathcal{E}(X/B) \to B$ of families of threefolds $X \to B$ over complex analytic bases B, where $\mathcal{E}(X/B) \to B$ is separated and quasi-smooth. For any family of threefolds $X \to B$ over a semi-simplicial set B, we have the virtual fundamental class $[\mathcal{E}(X/B)/B]^{\mathrm{vir}}$ in the homology of $C^*(B, C_*^{\mathrm{rel}\infty}(\mathcal{E}(X/-)))$, compatible with pullback and subdivision. If $\mathcal{E}$ is multiplicative, in the sense that applying $\mathcal{E}$ to a disjoint union $(X \times B') \sqcup (B \times X') \to B \times B'$ is the product $\mathcal{E}(X/B) \times \mathcal{E}(X'/B') \to B \times B'$, then the virtual fundamental class is multiplicative $[\mathcal{E}((X \times B') \sqcup (B \times X')/(B \times B'))/(B \times B')]^{\mathrm{vir}} = [\mathcal{E}(X/B)/B]^{\mathrm{vir}} \otimes [\mathcal{E}(X'/B')/B']^{\mathrm{vir}}$. Taking the universal base $B = \mathrm{Cpx}_{3,\bullet}$, we obtain a class $[\mathcal{E}]^{\mathrm{vir}} \in H_*^{\mathrm{rel}\infty}(\mathcal{E}(\mathrm{Cpx}_3))$, which is multiplicative if $\mathcal{E}$ is.

3.5 Enumerative invariants

The Gromov–Witten and Pandharipande–Thomas ring homomorphisms

$$\mathrm{GW} : H_c^*(\mathcal{Z}(\mathrm{Cpx}_3)) \to \mathbb{Q}((u)) \tag{3.22}$$

$$\mathrm{PT} : H_c^*(\mathcal{Z}(\mathrm{Cpx}_3)) \to \mathbb{Z}((q)) \tag{3.23}$$

are defined by pairing with corresponding multiplicative virtual fundamental classes in $H_*^{\mathrm{rel}\infty}(\mathcal{Z}(\mathrm{Cpx}_3); \mathbb{Q})((u))$ and $H_*^{\mathrm{rel}\infty}(\mathcal{Z}(\mathrm{Cpx}_3))((q))$, whose definition we review here.

Underlying Gromov–Witten and Pandharipande–Thomas invariants are moduli spaces $\overline{\mathcal{M}}'(X/B)$ and $P(X/B)$ (respectively) over B associated to any family of threefolds $X \to B$, compatible with pullback. The moduli space $\overline{\mathcal{M}}'(X/B)$ is a Deligne–Mumford analytic stack representing stable maps from compact (not necessarily connected) nodal curves to fibers of $X \to B$, all of whose connected components are non-constant. The analytic space $P(X/B)$ parameterizes stable pairs on fibers of $X \to B$ (a stable pair is a coherent sheaf F of proper support of pure relative dimension one along with a section s whose cokernel has relative dimension zero [22]). There are locally constant maps

$$\chi : \overline{\mathcal{M}}'(X/B) \to \mathbb{Z} \tag{3.24}$$

$$n : \quad P(X/B) \to \mathbb{Z} \tag{3.25}$$

given by domain arithmetic Euler characteristic and holomorphic Euler characteristic, respectively.

Both $\overline{\mathcal{M}}'(X/B)$ and $P(X/B)$ carry a natural quasi-smooth derived structure. As reviewed in §3.4, there are hence induced virtual fundamental classes

$$[\overline{\mathcal{M}}'(\mathrm{Cpx}_3)]^{\mathrm{vir}} = \prod_{X \to \Delta^k} [\overline{\mathcal{M}}'_\chi(X/\Delta^k)]^{\mathrm{vir}} \in H_*^{\mathrm{rel}\infty}(\overline{\mathcal{M}}'(\mathrm{Cpx}_3); \mathbb{Q}), \tag{3.26}$$

$$[P(\mathrm{Cpx}_3)]^{\mathrm{vir}} = \prod_{X \to \Delta^k} [P_n(X/\Delta^k)]^{\mathrm{vir}} \quad \in H_*^{\mathrm{rel}\infty}(P(\mathrm{Cpx}_3)). \tag{3.27}$$

Now the maps $\overline{\mathcal{M}}' \to \mathcal{Z}$ and $P \to \mathcal{Z}$ are proper when restricted to the sets on which χ and n are bounded above by a given $N < \infty$. Pushing forward $u^\chi \cdot [\overline{\mathcal{M}}'(\mathrm{Cpx}_3)]^{\mathrm{vir}}$ and

$q^n \cdot [P(\mathrm{Cpx}_3)]^{\mathrm{vir}}$ thus defines classes

$$\mathrm{GW} \in H_*^{\mathrm{rel}\infty}(\mathcal{Z}(\mathrm{Cpx}_3); \mathbb{Q})((u)), \tag{3.28}$$
$$\mathrm{PT} \in H_*^{\mathrm{rel}\infty}(\mathcal{Z}(\mathrm{Cpx}_3))((q)), \tag{3.29}$$

which have virtual dimension zero since the virtual fundamental classes of $\overline{\mathcal{M}}'$ and P lie in relative virtual dimension $\langle c_1(T_{X/B}), \beta\rangle$ (this depends on $X \to B$ having relative dimension three). This defines the group homomorphisms (3.22)–(3.23).

The classes GW and PT are multiplicative since $\overline{\mathcal{M}}'$ and P are multplicative (see §3.4 above). That is, they satisfy $\Delta(F) = F \otimes F$, where $\Delta : H_*^{\mathrm{rel}\infty}(\mathcal{Z}(\mathrm{Cpx}_3))/\mathrm{tors} \to (H_*^{\mathrm{rel}\infty}(\mathcal{Z}(\mathrm{Cpx}_3))/\mathrm{tors})^{\otimes 2}$ is dual to product on the Grothendieck group (Definition 3.5), implying (3.22)–(3.23) are ring homomorphisms.

Classical Gromov–Witten and Pandharipande–Thomas theory is interested in evaluating GW and PT on elements of $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$ coming from projective threefolds. When X is projective, the space of cycles $\mathcal{Z}(X, \beta)$ in homology class β is compact, hence its characteristic function defines a class $(X, \beta) \in H_c^0(\mathcal{Z}(\mathrm{Cpx}_3, \langle c_1(TX), \beta\rangle))$, which has virtual dimension $2\langle c_1(TX), \beta\rangle$. Thus when $\langle c_1(TX), \beta\rangle = 0$, we may evaluate the homomorphisms GW and PT on this element to obtain invariants

$$\mathrm{GW}(X, \beta) = \int_{[\overline{\mathcal{M}}'(X,\beta)]^{\mathrm{vir}}} u^{\chi} \in \mathbb{Q}((u)), \tag{3.30}$$
$$\mathrm{PT}(X, \beta) = \int_{[P(X,\beta)]^{\mathrm{vir}}} q^n \in \mathbb{Z}((q)). \tag{3.31}$$

More generally, given cohomology classes $\gamma_1, \ldots, \gamma_r \in H^*(X)$ (called 'insertions'), we may consider the class

$$(X, \beta; \gamma_1, \ldots, \gamma_r) \in H_c^{(|\gamma_1|-2)+\cdots+(|\gamma_r|-2)}(\mathcal{Z}(\mathrm{Cpx}_3, \langle c_1(TX), \beta\rangle)) \tag{3.32}$$

given by the cohomology class $\mathbf{1}_\beta \prod_{i=1}^r \pi_! i^* \gamma_i$ on $\mathcal{Z}(X)$, namely the result of push/pull via the universal family

$$\begin{array}{ccc} \mathcal{U}(X) & \xrightarrow{i} & X \\ \big\downarrow{\scriptstyle \pi} & & \\ \mathcal{Z}(X) & & \end{array} \tag{3.33}$$

Evaluating GW and PT on this class produces Gromov–Witten invariants and Pandharipande–Thomas invariants of X in homology class β with insertions $\gamma_1, \ldots, \gamma_r$

$$\mathrm{GW}(X, \beta; \gamma_1, \ldots, \gamma_r) = \int_{[\overline{\mathcal{M}}'(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^r \pi_! \operatorname{ev}^* \gamma_i \cdot u^{-\chi} \in \mathbb{Q}((u)) \tag{3.34}$$
$$\mathrm{PT}(X, \beta; \gamma_1, \ldots, \gamma_r) = \int_{[P(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^r \pi_! (\mathrm{ch}_2(\mathbb{F}) \cup \pi_X^* \gamma_i) \cdot q^n \in \mathbb{Z}((q)) \tag{3.35}$$

where the integrand involves push/pull for the universal families

$$\begin{array}{ccc} \overline{\mathcal{U}}'(X) & \xrightarrow{\text{ev}} & X \\ {\scriptstyle\pi}\big\downarrow & & \\ \overline{\mathcal{M}}'(X) & & \end{array} \qquad\qquad \begin{array}{ccc} P(X)\times X & \xrightarrow{\pi_X} & X \\ {\scriptstyle\pi}\big\downarrow & & \\ P(X) & & \end{array} \tag{3.36}$$

and $\mathbb{F}$ denotes the universal stable pair on $P(X) \times X$ (note that the second chern character $\mathrm{ch}_2(\mathbb{F})$ is simply the fundamental cycle of the support of $\mathbb{F}$, a codimension four cohomology class on $P(X, \beta) \times X$). These invariants are nonzero when the virtual dimension $2\langle c_1(TX), \beta\rangle - \sum_i(|\gamma_i| - 2)$ is zero.

4 Local curves

In the study of enumerative invariants of complex threefolds, the term *local curve* refers to (the total space of) a rank two vector bundle E over a smooth proper (usually connected) curve C. Given a local curve $E \to C$, one is then interested in enumerating curves *supported on the zero section* $C \subseteq E$ (though this *a priori* has no meaning since $\mathbb{Z}_{\geq 0} \cdot [C] \subseteq \mathcal{Z}(E)$ is usually not open). The goal of this section is to recall the various ways to make sense of the enumerative theory of local curves by working equivariantly, and to show how this enumerative theory may be realized within the framework of the Grothendieck group $H^*_c(\mathcal{Z}(\mathrm{Cpx}_3))$.

Remark 4.1. It is not hard to show that local curves are classified up to deformation by the pair of integers $g = g(C) \geq 0$ and $c = c_1(E) \in \mathbb{Z}$. The chern number of the zero section is given by $k = c_1(TE) = c_1(E) + c_1(TC) = 2 - 2g + c$ and is a more convenient index than c. We write $E_{g,k}$ for the (unique up to deformation) local curve of genus $g \geq 0$ and chern number k.

4.1 Equivariant homology

The flavor of equivariant homology relevant for our present discussion is called *co-Borel equivariant homology*, which measures 'homotopically S^1-invariant cycles' on an S^1-space. We will employ the following concrete definition of this homology theory.

Definition 4.2 (co-Borel equivariant homology)**.** Let X be an S^1-space with reasonable topology (say paracompact Hausdorff and locally homeomorphic to a finite CW-complex of uniformly bounded dimension). The co-Borel S^1-equivariant homology of X is the inverse limit

$$H_*^{cS^1}(X) = \varprojlim_n H_{*+2n}\left(\frac{X \times S^{2n+1}}{S^1}\right) \tag{4.1}$$

where $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ is the unit sphere acted on by the unit circle $S^1 \subseteq \mathbb{C}$ by multiplication. The quotient $(X \times S^{2n+1})/S^1$ is a locally trivial fibration over $S^{2n+1}/S^1 = \mathbb{C}P^n$ with fiber

X. The diagram

$$\begin{array}{ccc} \dfrac{X \times S^{2n+1}}{S^1} & \longrightarrow & \dfrac{X \times S^{2n+3}}{S^1} \\ \big\downarrow & & \big\downarrow \\ \mathbb{C}P^n & \longrightarrow & \mathbb{C}P^{n+1} \end{array} \tag{4.2}$$

thus determines maps $H_{*+2n+2}((X \times S^{2n+3})/S^1) \to H_{*+2n}((X \times S^{2n+1})/S^1)$ ('intersect with a hyperplane'), which are the structure maps of the inverse system in (4.1). These structure maps fit into a long exact sequence with third term $H_{*+2n+1}(X)$, so the inverse system is constant in degree d once $2n+1 > \dim X - d$. This eventual constancy is why the inverse limit of homology (4.1) yields the 'correct' homology groups. It is evident that $H_*^{cS^1}(X)$ is supported in degrees $\leq \dim X$, and it is typically nontrivial in arbitrarily negative degrees.

For example, $H_*^{cS^1}(\mathrm{pt}) = \varprojlim_n H_{*+2n}(\mathbb{C}P^n) = \varprojlim_n H^{-*}(\mathbb{C}P^n) = \mathbb{Z}[t]$ (free polynomial algebra) where t is the class of a hyperplane and lies in homological degree -2 (cohomological degree 2). Intersection of cycles gives $H_*^{cS^1}(\mathrm{pt})$ the structure of a ring and gives each $H_*^{cS^1}(X)$ the structure of a module over it.

Definition 4.3 (Tate equivariant homology)**.** The Tate S^1-equivariant homology is the localization of co-Borel equivariant homology at $t \in H_{-2}^{cS^1}(\mathrm{pt})$, namely it is the direct limit

$$H_*^{tS^1}(X) = \varinjlim_i H_{*-2i}^{cS^1}(X) \tag{4.3}$$

where the transition maps are multiplication by t (compare Greenlees–May [9, Corollary 16.3]).

The key property of Tate equivariant homology is that it vanishes for (almost) free S^1-spaces (with rational coefficients), hence by the long exact sequence and excision, depends rationally only on the fixed set. This is known as the *equivariant localization theorem*, which originates in the work of Smith [28, 29, 30], reformulated cohomologically by Borel [4], and then formalized in its present form by Atiyah–Segal [1, 26] and Quillen [24].

Proposition 4.4. *The map $H_*^{tS^1}(X^{S^1}) \to H_*^{tS^1}(X)$ is an isomorphism over $\mathbb{Q}$.*

Proof. We assume that our spaces have a reasonable S^1-equivariant cell decomposition (which holds in the cases we care about by real analyticity). Precisely speaking, this means that X is glued out of cells of the form $S^1/\Gamma \times (D^k, \partial D^k)$ for subgroups $\Gamma \subseteq S^1$, where S^1 acts by multiplication on the first factor (and trivially on the second factor). Given such a cell decomposition of X, to show that $H_*^{tS^1}(X, X^{S^1}) = 0$ it suffices (by the long exact sequence and excision) to show that $H_*^{tS^1}(S^1/\Gamma \times (D^k, \partial D^k)) = 0$ for $\Gamma \subsetneqq S^1$ a *proper* subgroup. We have $H_*^{tS^1}(S^1/\Gamma \times (D^k, \partial D^k)) = H_{*-k}^{tS^1}(S^1/\Gamma)$, so we are reduced to showing that $H_*^{tS^1}(S^1/\Gamma) = 0$ for $\Gamma \subsetneqq S^1$. Since Γ is finite, there is a 'transfer' map $H_*^{tS^1}(S^1/\Gamma) \to H_*^{tS^1}(S^1)$ whose composition with the pushforward map $H_*^{tS^1}(S^1) \to H_*^{tS^1}(S^1/\Gamma)$ is multiplication by $\#\Gamma$ on $H_*^{tS^1}(S^1/\Gamma)$. It thus suffices to show that $H_*^{tS^1}(S^1) = 0$, which follows from calulating $H_*^{cS^1}(S^1) = \mathbb{Z}$. $\square$

The significance of equivariant localization is the following. Given a class in $H_*^{cS^1,\mathrm{rel}\infty}(X)$, we may push forward to $H_*^{cS^1}(\mathrm{pt})$ provided X is compact. However, if we are satisfied with pushing forward to the Tate group $H_*^{tS^1}(\mathrm{pt})$, then equivariant localization provides such a pushforward map when just the fixed set X^{S^1} is compact.

4.2 Equivariant enumerative invariants

Curve enumeration theories, namely classes in $H_*^{\mathrm{rel}\infty}(\mathcal{Z}(\mathrm{Cpx}_3))$, specialize to virtual fundamental classes in $H_*^{\mathrm{rel}\infty}(\mathcal{Z}(X))$ for complex threefolds X. It turns out that a curve enumeration theory also determines S^1-equivariant virtual fundamental classes, namely classes in $H_*^{\mathrm{rel}\infty,cS^1}(\mathcal{Z}(X))$ for X with an $\mathbb{C}^\times$-action. Indeed, we have

$$H_*^{cS^1,\mathrm{rel}\infty}(\mathcal{Z}(X)) = \varprojlim_n H_{*+2n}^{\mathrm{rel}\infty}\left(\frac{\mathcal{Z}(X)\times(\mathbb{C}^{N+1}-0)}{\mathbb{C}-0}\right) \tag{4.4}$$

and $(\mathcal{Z}(X)\times(\mathbb{C}^{N+1}-0))/\mathbb{C}^\times$ is the relative cycle space of the family $(X\times(\mathbb{C}^{n+1}-0))/\mathbb{C}^\times \to \mathbb{C}P^N$, so we have

$$H_*^{cS^1,\mathrm{rel}\infty}(\mathcal{Z}(X)) = \varprojlim_n H_{*+2n}^{\mathrm{rel}\infty}\left(\mathcal{Z}\left(\frac{X\times(\mathbb{C}^{n+1}-0)}{\mathbb{C}-0}\;\Big/\;\mathbb{C}P^N\right)\right). \tag{4.5}$$

A curve enumeration theory gives rise to a coherent system of classes in this inverse system, hence to a class in $H_*^{cS^1,\mathrm{rel}\infty}(\mathcal{Z}(X))$. Such a class determines numerical invariants by capping with something in $H_c^*(\mathcal{Z}(X))$ and pushing forward to $H_*^{cS^1}(\mathrm{pt}) = \mathbb{Z}[t]$. In particular, this defines equivariant Gromov–Witten and Pandharipande–Thomas invariants.

Let us now consider the case of a local curve $E = E_{g,k}$ equipped with the fiberwise scaling action of $\mathbb{C}^\times$. An equivariant virtual fundamental class thus lies in $H_*^{cS^1,\mathrm{rel}\infty}(\mathcal{Z}(E))$. Restricting to cycles $\mathcal{Z}(E,m)\subseteq\mathcal{Z}(E)$ of degree m (homology class $m[C]$ for $C\subseteq E$ the zero section), this class lies in degree $2km$. The fixed locus $\mathcal{Z}(E)^{S^1}$ is just $\mathbb{Z}_{\geq 0}\times[C]$ (multiples of the zero section), so we may appeal to equivariant localization (Proposition 4.4) to push forward this class to Tate equivariant homology $H^{tS^1}(\mathrm{pt}) = \mathbb{Z}[t,t^{-1}]$. This defines equivariant enumerative invariants in $H_{2mk}^{tS^1}(\mathrm{pt}) = \mathbb{Z}\cdot t^{-mk}$ which roughly speaking 'S^1-equivariantly count curves representing the cycle $m[C]$ in $E_{g,k}$'. In particular, we obtain invariants

$$\mathrm{GW}_{S^1}(E,m) \in \mathbb{Q}((u))\cdot t^{-mk} \tag{4.6}$$

$$\mathrm{PT}_{S^1}(E,m) \in \mathbb{Z}((q))\cdot t^{-mk} \tag{4.7}$$

for any local curve $E_{g,k}\to C$ and integer multiplicity $m\geq 0$.

Theorem 4.5 ([6, 20])**.** *The pair of power series $(-iu)^k\mathrm{GW}_{S^1}(E_{g,k},m)$ and $(-q)^{-k/2}\mathrm{PT}_{S^1}(E_{g,k},m)$ satisfy the MNOP correspondence.*

4.3 Elements of the Grothendieck group

Now we would like to realize the equivariant enumerative invariants of local curves $E_{g,m,k}\to C_g$ as the ordinary non-equivariant enumerative invariants of certain elements $x_{g,m,k}\in H_c^{2km}(\mathcal{Z}(\mathrm{Cpx}_3,km))$ (virtual dimension zero) called *equivariant local curve elements.*

Consider a map

$$\mathcal{Z}(E,m) \xrightarrow{\cap E_p} \operatorname{Sym}^m E_p \xrightarrow{\operatorname{Sym}^m \lambda} \operatorname{Sym}^m \mathbb{C} \xrightarrow{\beta_r} \mathbb{C} \tag{4.8}$$

associated to a point $p \in C$, a linear map $\lambda : E_p \to \mathbb{C}$, and a homogeneous symmetric polynomial $\beta_r : \operatorname{Sym}^m \mathbb{C} \to \mathbb{C}$ of degree r. This map is $\mathbb{C}^\times$-equivariant for the weight r action on the target $\mathbb{C}$. It thus determines a section f of $\mathcal{L}^{\otimes r}$ over $(\mathcal{Z}(E,m)\times(\mathbb{C}^{N+1}-0))/\mathbb{C}^\times$, where $\mathcal{L}$ denotes the tautological line bundle on $\mathbb{C}P^N$. We may choose such functions $f_1,\ldots,f_n$ so that their joint zero set $f_1^{-1}(0)\cap\cdots\cap f_n^{-1}(0)$ will be just the point $m[C] \in \mathcal{Z}(E,m)$.

We now define the equivariant local curve element $x_{g,m,k} \in H_c^{2km}(\mathcal{Z}(\mathrm{Cpx}_3, km))$ to be the image of

$$\mathbf{1}_m \prod_{i=1}^n r_i^{-1} f_i^* \tau_{\mathcal{L}^{\otimes r_i}} \in H_c^{2n}\Big(\mathcal{Z}\Big(\frac{E\times(\mathbb{C}^{N+1}-0)}{\mathbb{C}-0} \Big/ \mathbb{C}P^N\Big)\Big) \to H_c^{2n-2N}(\mathcal{Z}(\mathrm{Cpx}_3, km)) \tag{4.9}$$

for any collection of sections f_i of $\mathcal{L}^{\otimes r_i}$ as above for which $f_1^{-1}(0)\cap\cdots\cap f_n^{-1}(0)$ is compact (so the cocycle above is indeed compactly supported) and $n = N + km$ (where $\mathbf{1}_m$ denotes the characteristic function of the cycles of degree m).

Lemma 4.6. *The expression* (4.9) *defining* $x_{g,m,k} \in H_c^{2km}(\mathcal{Z}(\mathrm{Cpx}_3, km))$ *is independent of the choice of* n*-tuple of sections* $f_1,\ldots,f_n$.

Proof. It is enough to show that the class is unchanged by removing f_n and decrementing N, provided the joint zero set of $f_1^{-1}(0)\cap\cdots\cap f_{n-1}^{-1}(0)$ is compact. The cocycle $f_n^*\tau_{\mathcal{L}^{\otimes r_n}}$ is cohomologous to r_n times the hyperplane $\mathbb{C}P^{N-1} \subseteq \mathbb{C}P^N$, which means that the 'wrong way map'

$$H_c^{*+2N-2}\Big(\mathcal{Z}\Big(\frac{E\times(\mathbb{C}^{N}-0)}{\mathbb{C}-0} \Big/ \mathbb{C}P^{N-1}\Big)\Big) \to H_c^{*+2N}\Big(\mathcal{Z}\Big(\frac{E\times(\mathbb{C}^{N+1}-0)}{\mathbb{C}-0} \Big/ \mathbb{C}P^{N}\Big)\Big) \tag{4.10}$$

sends $r_1^{-1}f_1^*\tau_{\mathcal{L}^{\otimes r_1}}\cup\cdots\cup r_{n-1}^{-1}f_{n-1}^*\tau_{\mathcal{L}^{\otimes r_{n-1}}}$ to $r_1^{-1}f_1^*\tau_{\mathcal{L}^{\otimes r_1}}\cup\cdots\cup r_n^{-1}f_n^*\tau_{\mathcal{L}^{\otimes r_n}}$, so we are done by commutativity of (3.15). □

For $k \ge 0$, the elements $x_{g,m,k} \in H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ evidently lift canonically to $H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ and are well defined there as well. We have $x_{g,0,k} = 1$ (take $n = N = 0$).

Proposition 4.7. *Given any curve enumeration theory (class in* $H_*^{\mathrm{rel}\infty}(\mathcal{Z}(\mathrm{Cpx}_3))$*), the resulting equivariant count of a local curve* E *in degree* m *is given by the pairing with* $x_{g,m,k} \in H_c^{2km}(\mathcal{Z}(\mathrm{Cpx}_3, km))$ *times* t^{-km}.

Proof. Every class in $H_{-2d}^{cS^1}(\mathrm{pt})$ is of the form $a\cdot t^d$ for some integer a. The coefficient a may be recovered by realizing the class inside some $H_{2N-2d}(S^{2N+1}/S^1)$ and pairing with H^{N-d} (power of the hyperplane class), provided $N \ge d$ so that this makes sense. We can apply the same recipe to find the image in $H_*^{cS^1}(\mathrm{pt})$ of a class in $H_*^{cS^1}(X)$ for any S^1-space X. Namely the image of a class in $H_{-2d}^{cS^1}(X) = \varprojlim_N H_{2N-2d}((X \times S^{2N+1})/S^1)$ in $H_*(\mathrm{pt}) = \mathbb{Z}[t]$ is given by t^d times its pairing with H^{N-d} for any $N \geq d$.

Now the equivariant enumerative invariants of $(E_{g,k}, m)$ are defined by pushing forward (after localizing at t) the virtual fundamental class in (the inverse limit wrt N of) $H_{2mk+2N}((\mathcal{Z}(E_{g,k}, m) \times S^{2N+1})/S^1)$. This pushforward is only defined in Tate homology, that is we must multiply by t^n and lift to $H_{2km+2N-2n}((\mathcal{Z}(E_{g,k}, m)^{S^1} \times S^{2N+1})/S^1)$ (which is guaranteed to be possible for n sufficiently large by Proposition 4.4) before pushing forward. This multiplication by t^n is precisely realized by $(r_1 \cdot \cdots \cdot r_n)^{-1} f_1^* \tau_{\mathcal{L}^{\otimes r_1}} \cup \cdots \cup f_n^* \tau_{\mathcal{L}^{\otimes r_n}}$. After multiplying by t^n, the pushforward to a point lies in degree $2km - 2n$, so following the above procedure we should cap with $H^{N-(n-km)}$ for $N \geq n - km$ to determine its image in $H_{2km-2n}^{cS^1}(\mathrm{pt})$. We can simply take $N = n - km$, so there is no cap with a power of H, and we conclude that the coefficient in front of t^{n-km} is the evaluation of our curve enumeration theory on $x_{g,m,k}$ as desired. □

Corollary 4.8. *The invariants $(-iu)^k \mathrm{GW}(x_{g,m,k})$ and $(-q)^{-k/2} \mathrm{PT}(x_{g,m,k})$ satisfy the MNOP correspondence.*

Proof. Combine Theorem 4.5 with Proposition 4.7. □

Lemma 4.9. *We have $\rho_d(x_{g,m,0}) = x_{g,m/d,0}$ if $d|m$.*

Proof. Inspection: the pullback of f_i of degree r_i under multiplication by d map is another such map of degree r_i. □

We now calculate the value of the coproduct Δ applied to $x_{g,m,k}$. First note that in the definition of $x_{g,m,k}$, we could in fact take $n = N+km+\ell$ for any integer ℓ. Denote the resulting elements by ${}_\ell x_{g,m,k} \in H_c^{2mk+2\ell}(\mathcal{Z}(\mathrm{Cpx}_3, mk))$, which have virtual dimension -2ℓ. Since the relative cycle space $\mathcal{Z}((E \times (\mathbb{C}^{N+1} - 0))/\mathbb{C}^\times, m)$ has finite type, its compactly supported cohomology vanishes in sufficiently large degree, so we have ${}_\ell x_{g,m,k} = 0$ for sufficiently large ℓ (depending on (g, m, k)). In fact, we do not need to know the dimension of the cycle space to estimate the vanishing of ${}_\ell x_{g,m,k}$: if the simultaneous zero set of some n functions f_i is compact, then we can take $N = \max(0, n - km)$, which means multiplication by H^ℓ is zero once $\ell > N$, so we see that ${}_\ell x_{g,m,k} = 0$ for $\ell > \max(0, n - km)$.

Lemma 4.10. *We have $\Delta(x_{g,m,k}) = \sum_{a+b=m} \sum_\ell {}_\ell x_{g,a,k} \otimes {}_{-\ell} x_{g,b,k}$ where $x_{g,0,k} = 1$ by convention.*

Proof. Realize $x_{g,m,k}$ by the expression (4.9) for a local curve $E = E_{g,k}$ and some sections f_i of $\mathcal{L}^{\otimes r_i}$ as in (4.8) whose joint zero set $f_1^{-1}(0) \cap \cdots \cap f_n^{-1}(0)$ is compact, and $n = N + km$. To define the coproduct $\Delta(x_{g,m,k})$ (Definition 3.6), consider the disjoint union family

$$\left(\frac{E \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0} \times \frac{\mathbb{C}^{N+1} - 0}{\mathbb{C} - 0}\right) \sqcup \left(\frac{(\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0} \times \frac{E \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0}\right) \to \mathbb{C}P^N \times \mathbb{C}P^N, \quad (4.11)$$

whose relative cycle space is the product of relative cycle spaces

$$\mathcal{Z}\Big(\frac{E\times(\mathbb{C}^{N+1}-0)}{\mathbb{C}-0}\Big/\mathbb{C}P^N\Big)\times\mathcal{Z}\Big(\frac{E\times(\mathbb{C}^{N+1}-0)}{\mathbb{C}-0}\Big/\mathbb{C}P^N\Big)\to\mathbb{C}P^N\times\mathbb{C}P^N. \tag{4.12}$$

Over the diagonal $\Delta(\mathbb{C}P^N)\subseteq\mathbb{C}P^N\times\mathbb{C}P^N$, there is an addition map Σ from this relative cycle space to the relative cycle space of $(E\times(\mathbb{C}^{N+1}-0))/\mathbb{C}^\times\to\mathbb{C}P^N$. The coproduct $\Delta(x_{g,m,k})$ is represented by the disjoint union family (4.11) equipped with the cocycle $\Delta_!\Sigma^*\mathbf{1}_m\prod_{i=1}^n r_i^{-1}f_i^*\tau_{\mathcal{L}^{\otimes r_i}}$.

Now we may consider the product of this cocycle with some number of cocycles $r^{-1}g^*\tau_{\mathcal{L}_1^{\otimes r}}$ or $r^{-1}h^*\tau_{\mathcal{L}_2^{\otimes r}}$ for sections g or h as in (4.8) coming from either copy of E (thus pulled back from the corresponding factor). If we increase N by the number of such cocycles, the argument of Lemma 4.6 shows that the result still calculates $\Delta(x_{g,m,k})$. Now if the sections g and h we choose have compact joint zero set (which is certainly attainable), then we can apply the argument of Lemma 4.6 to remove each $r_i^{-1}f_i^*\tau_{\mathcal{L}^{\otimes r_i}}$ factor. We thus conclude that $\Delta(x_{g,m,k})$ is represented by

$$\Delta_!\mathbf{1}_m\prod_{i=1}^p r_i^{-1}g_i^*\tau_{\mathcal{L}_1^{\otimes r_i}}\prod_{j=1}^q r_j^{-1}h_j^*\tau_{\mathcal{L}_2^{\otimes r_j}}\in H_c^{2N+2p+2q}\Big(\mathcal{Z}\Big(\frac{E\times(\mathbb{C}^{N+1}-0)}{\mathbb{C}-0}\Big/\mathbb{C}P^N\Big)^2\Big) \tag{4.13}$$

where $p+q=N+km$. Now expand $\mathbf{1}_m=\sum_{a+b=m}\mathbf{1}_{a,b}$ and $\Delta_!=\sum_{c+d=N}H^c\otimes H^d$ to see that this cocycle represents

$$\sum_{a+b=m}\sum_{c+d=N}{}_{c+p-ak-N}x_{g,a,k}\otimes{}_{d+q-bk-N}x_{g,b,k}. \tag{4.14}$$

Note that the quantities $c+p-ak-N$ and $d+q-bk-N$ sum to zero. Their maximum values are $p-ak$ and $q-bk$, respectively. Since p and q are arbitrary, we can simply sum over all integers $\ell=c+p-ak-N$ (note that one factor of the tensor product will be zero for all but finitely many ℓ), thus obtaining the desired result. □

5 Transversality

We prove a 'generic transversality' result, which says that simple (not multiply covered) maps from smooth curves to complex manifolds with generic (in a certain precise sense) complex structures are unobstructed (transverse). We derive from this that $H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ is generated by certain equivariant local curve elements.

5.1 Regularity

The deformation theory of a map $u:C\to X$ from a smooth proper curve C to a smooth complex analytic manifold X is controlled by $H^*(C,u^*TX)$. The deformation theory of C itself is controlled by $H^*(C,TC[1])$. The deformation theory of the pair (C,u) is controlled by $H^*(C,[TC[1]\to u^*TX])$. A deformation problem (in any of the above flavors) is said to be *unobstructed* when $H^{\geq 1}=0$.

Given a complex analytic submersion $X \to B$, we may also consider the deformation theory of pairs $(b, u : C \to X_b)$. Note that this differs from the deformation theory of maps from C to the total space X (which we will not ever consider). If $X \to B$ is a pullback of a submersion $X' \to B'$, then a pair $(b, u : C \to X_b)$ in $X \to B$ which is unobstructed remains unobstructed when pushed forward to $X' \to B'$.

Definition 5.1 (Regular)**.** Let $u : C \to X$ be a holomorphic map from a compact smooth curve C. A point $x \in C$ will be called *special* (for u) when $du(x) = 0$ or $\#u^{-1}(u(x)) > 1$. The set $S \subseteq C$ of special points is finite provided $\dim_{\mathbb{C}} X \geq 2$, which we now assume. We now consider the deformation theory of the triple (C, S, u) subject to the constraint that the points S remain special with the same discrete data, meaning that all conditions $u(x) = u(x')$ and $(D^r u)(x) = 0$ which hold for u are preserved. We say that the map u is *regular* when this deformation problem is unobstructed.

To clarify the meaning of the point constraints, we note that they add to the (complex) index the quantity

$$|S| - \dim_{\mathbb{C}} X \cdot \Big(|S| - |u(S)| + \sum_{p \in S} \operatorname{ord}_p(du)\Big). \tag{5.1}$$

When $\dim_{\mathbb{C}} X \geq 3$, this quantity is < 0 unless $S = \varnothing$.

Regularity is also defined for curves in fibers of a family $X \to B$, meaning the deformation problem includes variation in the base parameter. If $X' \to B'$ is a pullback of $X \to B$, then regularity in $X' \to B'$ implies regularity of the pushforward to $X \to B$.

In contrast to curves and maps from curves, it is not so clear whether 1-cycles have a reasonable deformation theory. We will call a (possibly relative) 1-cycle $z = \sum_i m_i C_i$ *semi-regular* when the map $\bigsqcup_i \tilde{C}_i \to X$ is regular in the sense of Definition 5.1. Semi-regularity evidently measures properties of the semi-chart from §2.2. In particular, if $z \in \mathcal{Z}(X/B)$ is semi-regular, then the semi-chart through z is a smooth subvariety of $\mathcal{Z}(X/B)$ of dimension $\dim B + \sum_i \langle c_1(TX), C_i \rangle$.

We denote by $\mathcal{Z}_{\text{semi-reg}} \subseteq \mathcal{Z}$ the locus of semi-regular cycles, and we call points in its interior $\mathcal{Z}^{\circ}_{\text{semi-reg}} \subseteq \mathcal{Z}_{\text{semi-reg}}$ *interior semi-regular*.

If $z \in \mathcal{Z}(X/B)$ is semi-regular, then it evidently remains semi-regular upon pushing forward to a family $X' \to B'$ of which $X \to B$ is a pullback. In contrast, interior semi-regularity need not be so preserved, which is a significant technical trip hazard!

Lemma 5.2. *$\mathcal{Z}^{\text{semi-Fano}}(X/B)^{\circ}_{\text{semi-reg}}$ has dimension $\leq \dim B + 2\langle c_1(TX), z \rangle$.*

Proof. The set of points of $\mathcal{Z}(X/B)^{\circ}_{\text{semi-reg}}$ whose associated semi-chart is an open embedding is dense by Lemma 2.1. At such a point, the dimension of $\mathcal{Z}(X/B)^{\circ}_{\text{semi-reg}}$ equals the dimension of the semi-chart. At semi-regular points $z = \sum_i m_i C_i$, this dimension is $\dim B + 2\sum_i \langle c_1(TX), C_i \rangle \leq \dim B + 2\sum_i m_i \langle c_1(TX), C_i \rangle = \dim B + 2\langle c_1(TX), z \rangle$ (where we have crucially used the semi-Fano condition $\langle c_1(TX), C_i \rangle \geq 0$ for all i). $\square$

5.2 Semi-regular Grothendieck group

Define a 1-cycle $z \in \mathcal{Z}(X/\Delta^n)$ to be semi-regular when it is semi-regular inside the minimal stratum of Δ^n containing it (i.e. consider variations in the base which are tangent to the stratum containing z). Thus for any injection $[i] \hookrightarrow [n]$ and any family $X \to \Delta^n$, we have $\mathcal{Z}(X \times_{\Delta^n} \Delta^i/\Delta^i)_{\text{semi-reg}} = \mathcal{Z}(X/\Delta^n)_{\text{semi-reg}} \times_{\Delta^n} \Delta^i$. There is thus a correspondence

$$\mathcal{Z}(X \times_{\Delta^n} \Delta^i/\Delta^i)^\circ_{\text{semi-reg}} \hookleftarrow \mathcal{Z}(X/\Delta^n)^\circ_{\text{semi-reg}} \times_{\Delta^n} \Delta^i \to \mathcal{Z}(X/\Delta^n)^\circ_{\text{semi-reg}} \tag{5.2}$$

in which the left arrow is an open embedding and the right arrow is a pullback of $\Delta^i \to \Delta^n$ (hence a closed embedding, but in particular proper). Now given that C^*_c is functorial for open embeddings and proper maps (see §3.2), we obtain a coefficient system $C^*_c(\mathcal{Z}^\circ_{\text{semi-reg}})$ on $\mathrm{Cpx}_{3,\bullet}$ and hence chain groups $C_*(\mathrm{Cpx}_{3,\bullet}, C^*_c(\mathcal{Z})^\circ_{\text{semi-reg}})$, whose homology we denote by $H^*_c(\mathcal{Z}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg}})$ and call the *Grothendieck group of interior semi-regular 1-cycles*. There is an evident map of coefficient systems $C^*_c(\mathcal{Z}^\circ_{\text{semi-reg}}) \to C^*_c(\mathcal{Z})$ on $\mathrm{Cpx}_{3,\bullet}$ (functoriality under open embeddings), inducing a map on Grothendieck groups $H^*_c(\mathcal{Z}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg}}) \to H^*_c(\mathcal{Z}(\mathrm{Cpx}_3))$.

5.3 Generation by local curves

Define a *geometric local curve element* in $H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg}})$ to be the Poincaré dual of a point of $\mathcal{Z}^{\text{semi-Fano}}(X/B,k)^\circ_{\text{semi-reg}}$ whose semi-chart is an open embedding of dimension $2k + \dim B$. Since the set of points whose semi-chart is an open embedding is dense (Lemma 2.1), the Poincaré dual of *any* dimension $2k + \dim B$ smooth point of $\mathcal{Z}^{\text{semi-Fano}}(X/B,k)^\circ_{\text{semi-reg}}$ is a geometric local curve element. The *topological type* of a point $z = \sum_i m_i C_i \in \mathcal{Z}^{\text{semi-Fano}}(X/B,k)^\circ_{\text{semi-reg}}$ is the collection of tuples (g_i, m_i, k_i) consisting of the genus g_i of C_i, the multiplicity m_i, and the chern number $k_i = \langle c_1(T_{X/B}), C_i\rangle \geq 0$ (since we are working with $\mathcal{Z}^{\text{semi-Fano}}$); this is constant over any semi-chart. The dimension of the semi-chart at $z = \sum_i m_i C_i$ is $\dim B + 2\sum_i k_i \leq \dim B + 2\sum_i m_i k_i = \dim B + 2k$, so equality only occurs when $(m_i - 1)k_i = 0$ for all i. Every geometric local curve element has virtual dimension zero (it has chern number $\sum_i m_i k_i$ and cohomological degree $2\sum_i m_i k_i$).

Proposition 5.3. *The group $\bigoplus_{2k-i\leq 0} H^i_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3, k)^\circ_{\text{semi-reg}})$ is generated by geometric local curve elements.*

Proof. Represent a class in $H^i_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3, k)^\circ_{\text{semi-reg}})$ by a finite semi-simplicial set $B_\bullet$, a map $B_\bullet \to \mathrm{Cpx}_{3,\bullet}$ (i.e. a family of complex threefolds $X \to B$), and a cycle $\lambda \in C_*(B_\bullet, C^*_c(\mathcal{Z}^{\text{semi-Fano}}(X/-,k)^\circ_{\text{semi-reg}}))$. The components $\lambda_\sigma \in C^{i+\dim\sigma}_c(\mathcal{Z}^{\text{semi-Fano}}(X_\sigma/\sigma,k)^\circ_{\text{semi-reg}})$ associated to top-dimensional simplices $\sigma \in B_\bullet$ are cocycles. Since $i + \dim\sigma \geq 2k + \dim\sigma = \dim \mathcal{Z}^{\text{semi-Fano}}(X_\sigma/\sigma, k)$ (Lemma 5.2), the cohomology class $[\lambda_\sigma] \in H^{i+\dim\sigma}_c(\mathcal{Z}^{\text{semi-Fano}}(X_\sigma/\sigma,k))$ is a linear combination of Poincaré duals of smooth points of dimension $2k + \dim\sigma$, namely geometric local curve elements (such smooth points lie over the interior of σ, hence their Poincaré duals are cycles in $C_*(B_\bullet, C^*_c(\mathcal{Z}^{\text{semi-Fano}}(X/-,k)^\circ_{\text{semi-reg}})))$. By subtracting these, we may reduce to the case that $[\lambda_\sigma] = 0$ in cohomology for top-dimensional simplices σ. Thus by adding a boundary to our cycle, we may reduce the dimension of $B_\bullet$. Iterating, we have reduced our class to zero by adding geometric local curve elements. $\square$

Conjecture 5.4. *Every geometric local curve element coincides with the equivariant local curve element of the same topological type.*

Note that Conjecture 5.4 is false in the almost complex setting by the analysis of Ionel–Parker [11, §7]. In the analytic setting, we would expect any 'walls' to be complex codimension one, hence real codimension two, so there is no wall crossing. This is nontrivial to make precise over real analytic bases B given the delicate nature of interior semi-regularity.

5.4 Generic transversality

It is a standard result that for generic *almost* complex structures, all simple maps are unobstructed (see [10, 19, 32] for precise statements). We now derive analogous results for complex structures. Since complex structures are much more rigid (for example, they have no local perturbations supported inside a small ball), these results are weaker than those in the almost complex setting.

For complex manifolds U and V, denote by $\operatorname{An}(U, V)$ the space of analytic maps $U \to V$ with relatively compact image. If V admits an open embedding into some $\mathbb{C}^n$, then $\operatorname{An}(U, V)$ is a complex analytic Banach manifold, locally modelled on the space of n-tuples of bounded holomorphic functions on U.

Given a complex manifold U, let $\mathcal{R}(U) = \operatorname{An}(U^-, U)$ (the space of 'regluings') for $U^- \subseteq U$ a(n unspecified) large relatively compact open subset. More formally, let us regard $\mathcal{R}(U)$ as the inverse system of all neighborhoods of the identity $\mathbf{1}_U \in \operatorname{An}(U^-, U)$ over all relatively compact open sets $U^- \subseteq U$. Composition makes $\mathcal{R}(U)$ into a group object. In all cases of interest to us, U will admit an open embedding into some $\mathbb{C}^n$, implying that $\mathcal{R}(U)$ is a (pro) Banach analytic manifold.

Definition 5.5. Given a complex manifold X with an open cover $X = A \cup B$, we may deform X by modifying the identification between open sets $A \supseteq A \cap B \subseteq B$. More formally, we consider the family $\tilde{X} \to \mathcal{R}(A \cap B)$ by taking the trivial families A and B over $\mathcal{R}(A \cap B)$ and gluing via the base parameter $A \times \mathcal{R}(A \cap B) \ni (a, \gamma) \sim (\gamma(a), \gamma) \in B \times \mathcal{R}(A \cap B)$. To make this construction precise, and to ensure the result is Hausdorff, we may fix compact sets $A^- \subseteq A$ and $B- \subseteq B$ and glue $(A^- \sqcup B^-) \times \mathcal{R}(A \cap B)$ to obtain a proper map $\tilde{X}^- \to \mathcal{R}(A \cap B)$ so that $\tilde{X}$ is well defined as a germ containing $\tilde{X}^-$.

We will in fact only need a special case of the above construction, namely when the regluing takes place in a small neighborhood of a *divisor* (a closed complex submanifold of codimension one).

Definition 5.6 (Deforming complex structure near a divisor)**.** Let X be a complex manifold, and let $D \subseteq X$ be a smooth divisor. Regarding X as the gluing of $X \setminus D$ and $\operatorname{Nbd} D$ over their common intersection, Definition 5.5 provides a family $\tilde{X} \to \mathcal{R}(\operatorname{Nbd} D \setminus D)$. This family is smoothly trivial (analytic perturbations of the identity map on $\operatorname{Nbd} D \setminus D$ extend smoothly to $\operatorname{Nbd} D$), so a choice of trivialization determines a family of complex structures on X parameterized by $\mathcal{R}(\operatorname{Nbd} D \setminus D)$. We will also denote this base space by $\mathcal{J}_D(X)$ (complex structures on X obtained by regluing near D). Of course, this isn't really a space but rather a family of spaces depending on choices of neighborhoods, etc. Sometimes we will need to fix a specific one, but we will do this at the relevant time.

The same construction applies to families $X \to B$ of complex manifolds. Given a *relative divisor* $D \subseteq X \to B$, meaning a divisor which is submersive over B, we may consider the set $\mathcal{J}_D(X/B) = \mathcal{R}_B(\operatorname{Nbd} D \setminus D) = \bigcup_{b\in B} \mathcal{R}(\operatorname{Nbd} D_b \setminus D_b) \to B$, a holomorphic section α of which determines a 'vertical' (i.e. over B) regluing $X_\alpha \to B$ of X.

The tangent space to $\mathcal{R}(\operatorname{Nbd} D \setminus D)$ at the identity is the space of germs of holomorphic vector fields on $\operatorname{Nbd} D$ possibly singular along D. We denote this space by $H^0(D, TX(-\infty D))$ (implicitly restricting the sheaf of holomorphic sections of TX over X to the divisor D). Such a vector field thus gives a first order deformation of the complex structure on X modulo gauge, that is an element of $H^1(X, TX)$. Concretely, this map $H^0(D, TX(-\infty D)) \to H^1(X, TX)$ sends a holomorphic vector field v to (the Dolbeaut cohomology class represented by) $\bar\partial((1-\varphi)\cdot v)$ for a smooth function $\varphi : X \to [0, 1]$ supported inside an open set $U \subseteq X$ containing D such that v is defined on $U \setminus D$ and $\varphi \equiv 1$ in a neighborhood of D. The choice of φ does not matter since $\bar\partial((1-\varphi)\cdot v) - \bar\partial((1-\varphi')\cdot v) = \bar\partial((\varphi'-\varphi)\cdot v)$ is exact in the Dolbeat complex since $(\varphi'-\varphi)\cdot v$ is a smooth vector field on X (in contrast to $\varphi \cdot v$, which has singularities along D, or $(1-\varphi)\cdot v$, which is defined only on U).

Given a divisor $D \subseteq X$, a map $u : C \to X$ from a smooth curve C will be called D-*controlled* when $u^{-1}(D) \subseteq C$ is discrete and intersects every component of C. It is elementary to observe that being D-controlled is an open and condition on u. A cycle $z = \sum_i m_i C_i$ in X will be called D-controlled when $\tilde C_i \to X$ is D-controlled for every i.

Lemma 5.7. *The set of D-controlled cycles in $\mathcal{Z}(X/B)$ is open for any relative divisor $D \subseteq X \to B$.*

Proof. Suppose $z = \sum_i m_i C_i \in \mathcal{Z}(X/B)$ is D-controlled. Since C_i intersects D geometrically, the algebraic intersection number $C_i \cdot D$ is positive by positivity of intersection. If $z' = \sum_i m_i' C_i'$ is close to z, then every C_i' is homologous to a positive linear combination of some C_i's, hence also has positive algebraic intersection with D, thus *a fortiori* intersects it geometrically. $\square$

We now come to the key technical result underlying generic transversality, which says that the restrictions of D-deformations to D-controlled curves are sufficiently rich. For any family $X \to B$, the deformation complex of a map $u : C \to X_b$ maps to the deformation complex of the pair $(b, u : C \to X_b)$, with cokernel T_bB. This induces a map from T_bB to the obstruction space of the map u, whose cokernel is the obstruction space of the pair (b, u). Explicitly, this map is simply restriction (e.g. of Dolbeaut representatives) from $H^1(X, TX)$ to $H^1(C, TX)$. More generally, we could consider constrainted deformation problems.

Lemma 5.8 (Enough first order deformations)**.** *Let $u : C \to X$ be a simple map from a smooth proper curve C to a complex manifold X, and let $D \subseteq X$ be a divisor. If u is D-controlled, then the map*

$$H^0(D \cap u(C), TX(-\infty D)) \to H^1(C, TX) \tag{5.3}$$

is surjective. In fact, it is surjective onto the obstruction space for deforming the map $u : C \to X$ subject to any finite number of point constraints (such as those appearing in the notion of 'regularity' Definition 5.1).

Proof. Recall from above that the map in question sends a vector field v to $\bar{\partial}((1-\varphi)\cdot v)$ (which we note is not exact since the 'primitive' $(1-\varphi)\cdot v$ is not defined globally on X). Let us fix a rather singular cutoff function φ so that $d\varphi$, hence also $\bar{\partial}((1-\varphi)\cdot v)$, is supported along the boundary of a fixed small tubular neighborhood $\operatorname{Nbd}(D)$ of D. In fact, fix a local projection $\pi : X \to \mathbb{C}$ with $D = \pi^{-1}(0)$, and take $\operatorname{Nbd}(D) = \pi^{-1}(D^2_\varepsilon)$ (near a given point of intersection $u(C) \cap D$). Now $\pi \circ u : C \to \mathbb{C}$ is a ramified cover near the origin. In particular, the inverse image $u^{-1}(\partial \operatorname{Nbd}(D))$ is a finite number of circles near $u^{-1}(D)$ which together meet every component of C. We will show that by appropriate choice of v, we can make $\bar{\partial}((1-\varphi)\cdot v)$ approximate the delta function at any point of this union of circles K. Every nonzero element of $H^0(C, K_C \otimes T^*X) = H^1(C, TX)^*$ has nonzero restriction to $u^{-1}(\partial \operatorname{Nbd}(D))$ by holomorphicity and unique continuation (recall they meet all components of C), so we obtain the desired surjectivity. Finitely many point constraints added to TX do not affect this argument: it is enough to approximate delta functions away from these special points.

It thus remains to prove that we can make $\bar{\partial}((1-\varphi)\cdot v)$ approximate a delta function at any point of $u^{-1}(\partial \operatorname{Nbd}(D))$. Fix local coordinates $X = \mathbb{C}_z \times \mathbb{C}^2_{x,y}$ near an (isolated, by hypothesis) intersection point $u(C) \cap D$ in which π is the z-coordinate so $D = \{z = 0\} = 0 \times \mathbb{C}^2_{x,y}$. Choose φ to be a smoothing of the characteristic function of the unit disk in $\mathbb{C}_z$, so that $\bar{\partial}(1-\varphi)$ is a smoothing of the δ-mass along the unit circle $\delta(z\bar{z}-1)zd\bar{z}$. Writing $v = \sum_k f_k(x,y)z^k\partial_z$, we calculate $\bar{\partial}((1-\varphi)\cdot v) = \bar{\partial}(1-\varphi)\cdot v$ is a smoothing of $\delta(z\bar{z}-1)\sum_k f_k(x,y)z^{k+1}d\bar{z}\partial_z$. Now the factor $\sum_k f_k(x,y)z^{k+1}d\bar{z}\partial_z$ can approximate any continuous function on $\partial D^2 \times \mathbb{C}^2_{x,y}$ which is holomorphic on fibers $e^{i\theta} \times \mathbb{C}^2_{x,y}$ (use approximation by Fourier polynomials in the ∂D^2 direction). It is thus enough to show that the restrictions of such functions on $\partial D^2 \times \mathbb{C}^2_{x,y} = \partial \operatorname{Nbd}(D)$ are dense in continuous functions on $u^{-1}(\partial \operatorname{Nbd}(D))$, which is evident from the fact that $\pi \circ u : C \to \mathbb{C}$ is a ramified covering. $\square$

Let us now explain how the existence of enough infinitesimal deformations (Lemma 5.8) implies various flavors of generic transversality. We say that (the complex structure on) X is *D-regular* when every D-controlled simple map is regular (Definition 5.1) and hence every D-controlled cycle is semi-regular.

Lemma 5.9 (Generic transversality)**.** *For any complex manifold X, any divisor $D \subseteq X$, and any finite set $A \subseteq D$, after possibly removing a closed subset of X disjoint from A, generic elements of $\mathcal{J}_D(X)$ are D-regular.*

Proof. This is a typical argument based on Smale's infinite-dimensional Sard theorem [27].

We begin by fixing a precise space $\mathcal{J}_D(X)$ to consider. Let $\mathbb{C}_{|\cdot|\leq 1} \subseteq \mathbb{C}$ denote the unit disk. Fix coordinates $\mathbb{C}_{|\cdot|\leq 1} \times \mathbb{C}^{n-1}_{|\cdot|\leq 1}$ near each point $a \in A$ with $a = (0,0)$ and $D = 0 \times \mathbb{C}^{n-1}_{|\cdot|\leq 1}$. We let $\mathcal{J}_D(X)$ consist of holomorphic maps $f : \mathbb{C}_{|\cdot|\leq 1} \times \mathbb{C}^{n-1}_{|\cdot|\leq 1} \to \mathbb{C}^n$ with $\|f - \mathbf{1}\|_2 < \varepsilon$ for some $\varepsilon > 0$. By smearing the Cauchy Integral Formula and appealing to Cauchy–Schwarz, we see that $\|f - \mathbf{1}\|_\infty$ over any compact subset of the interior of $\mathbb{C}^n_{|\cdot|\leq 1}$ is bounded linearly in ε. Thus for sufficiently small $\varepsilon > 0$, the reglued family (Definition 5.6) is defined (we remove from X a small neighborhood of $\partial\mathbb{C}_{|\cdot|\leq 1} \times \mathbb{C}^{n-1}_{|\cdot|\leq 1}$). Using the L^2-norm here guarantees that the space $\mathcal{J}_D(X)$ is separable.

Now consider a compact smooth (not necessarily connected!) surface C and a smooth family of almost complex structures on C parameterized by a finite-dimensional smooth manifold $\mathcal{J}(C)$. Now $W^{k,2}(C,X)$ is a smooth Banach manifold for any integer $k \geq 2$ (which guarantees $W^{k,2} \subseteq C^0$), whose product with $\mathcal{J}(C) \times \mathcal{J}_D(X)$ carries the smooth Banach bundle

$$\begin{array}{c} \mathcal{J}_C \times W^{k,2}(C,X) \times_{W^{k-1,2}(C,X)} W^{k-1,2}(C, \overline{TC} \otimes_{\mathbb{C}} TX) \times \mathcal{J}_D(X) \\ {\scriptstyle\bar\partial} \big\uparrow \big\downarrow \\ \mathcal{J}_C \times W^{k,2}(C,X) \times \mathcal{J}_D(X) \end{array} \tag{5.4}$$

with a section $\bar\partial$ measuring failure of the map $C \to X$ to be holomorphic. Now the linearization (derivative) of $\bar\partial$ at a triple $(u : C \to X, j, J)$ is a map

$$W^{k,2}(C, u^*TX) \oplus T_j\mathcal{J}(C) \oplus T_J\mathcal{J}_D(X) \to W^{k-1,2}(C, \overline{TC} \otimes_{\mathbb{C}} u^*TX) \tag{5.5}$$

whose restriction to the first direct summand is the deformation complex of the map u. Lemma 5.8 guarantees that the restriction to $T_J\mathcal{J}_D(X)$ surjects onto the obstruction space $H^1(C, TX)$ if u is D-controlled.

Now restrict to the clopen subdset of $W^{k,2}(C,X)$ consisting of those maps whose restriction to every component of C has positive algebraic intersection with D (thus a holomorphic map is D-controlled iff it lies in this set). Over this clopen set, the zero set $\bar\partial^{-1}(0)$ is thus a smooth Banach manifold, and the projection map

$$\bar\partial^{-1}(0) \to \mathcal{J}_D(X) \tag{5.6}$$

is Fredholm by ellipticity of the linearized operator. Now Sard–Smale [27] implies that generic elements of $\mathcal{J}_D(X)$ will have regular fibers. We can cover all curves using countably many pairs $(C, \mathcal{J}_C)$ with smoothly varying point constraints, so we conclude that for generic elements of $\mathcal{J}_D(X)$, all D-controlled simple maps are unobstructed.

Regularity is stronger than unobstructedness, since it involves a deformation problem with point constraints. To prove regularity of D-controlled simple maps with respect to generic elements of $\mathcal{J}_D(X)$, we consider triples $(C, \mathcal{J}(C), \gamma)$ where γ is a finite set of point constraints (again, countably many such triples suffice to cover all possible situations). $\square$

Lemma 5.10 (Generic transversality in a family)**.** *Fix a family of complex threefolds $X \to B$ over a semi-simplicial set B, a relative divisor $D \subseteq X \to B$, and a set $A \subseteq D$ whose map to B is proper with finite fibers. After subdividing B and removing from X a closed subset contained in $D \setminus A$, there exist (simplex-wise) analytic sections of $\mathcal{J}_D(X/B) \to B$ which are D-regular. Moreover, given in addition a semi-simplicial subset $B' \subseteq B$, there exist sections vanishing on B' which are D-regular over $B \setminus B'$.*

Proof. By subdividing B, we can ensure that over each simplex $\sigma \subseteq B$, there exist finitely many disjoint charts

$$(\mathbb{C}_{|\cdot|\leq 1} \times \mathbb{C}^{n-1}_{|\cdot|\leq 1} \times \sigma, 0 \times \mathbb{C}^{n-1}_{|\cdot|\leq 1} \times \sigma) \to (X_\sigma, D_\sigma) \tag{5.7}$$

which together cover A. By induction, it suffices to consider the case $(B, B') = (\sigma, \partial\sigma)$ for a single simplex $\sigma = \Delta^k$. We now consider the Banach space of analytic sections $B \to \mathcal{J}_D(X/B)$ given by those analytic maps $f : \mathbb{C}_{|\cdot|\leq 1} \times \mathbb{C}^{n-1}_{|\cdot|\leq 1} \times \sigma \to \mathbb{C}^n$ with small L^2-norm over $\mathbb{C}_{|\cdot|\leq 1} \times \mathbb{C}^{n-1}_{|\cdot|\leq 1}$ times a fixed neighborhood of $\sigma \subseteq \mathbb{C}^{\dim\sigma}$ and vanishing on $\Delta \times \Delta^{n-1} \times \partial\sigma$. We may now proceed as in Lemma 5.9. $\square$

To get any mileage out of the above generic transversality results, we need a sufficiently rich collection of divisors. Producing a divisor which controls a single map or cycle is elementary. We now generalize this assertion to families, where a more subtle argument is required.

Proposition 5.11 (Enough divisors)**.** *Let $X \to B$ be a family of complex threefolds over a finite semi-simplicial set, and let $K \subseteq \mathcal{Z}(X/B)$ be a compact analytic set whose projection map $K \to |B|$ is injective. After possibly removing a closed subset from X disjoint from K, there exists a finite collection of disjoint relative divisors $D_i \subseteq X \times_B U_i \to U_i$ ($U_i \subseteq |B|$ open) such that every $z \in K$ is D_i-controlled for some i.*

Proof. First, let us discuss how to construct (germs of) relative divisors $D \subseteq X \to B$ locally near a given point of x. Suppose x lies over the interior of a simplex $\sigma \in B_\bullet$. Begin with a holomorphic map $\pi : X_\sigma \to \mathbb{C}$ defined near x with transverse zero set $\pi^{-1}(0) = D_\sigma$. To explain the term 'holomorphic' for π, recall that $X_\sigma \to \sigma$ is the restriction of a given family $X_\sigma^{\mathbb{C}} \to \sigma^{\mathbb{C}} \cong \mathbb{C}^{\dim \sigma}$ over the complexification, so it makes sense to require that π be the restriction to X_σ of a (necessarily unique) holomorphic function on $X_\sigma^{\mathbb{C}}$. For D_σ to be a *relative* divisor, we need it to be submersive over σ, which in terms of π is the condition that $d\pi|T_{X/B}$ is surjective. To extend D_σ to a neighborhood of x in the total space X, it suffices to extend the holomorphic map π (note that surjectivity of $d\pi|T_{X/B}$ is an open condition). Proceeding by induction on simplices, we are reduced to the question of extending (near the origin) an analytic function from $\partial\mathbb{R}^n_{\geq 0} \times \mathbb{C}^n$ to $\mathbb{R}^n_{\geq 0} \times \mathbb{C}^n$. This extension problem is solved by the standard formula $f(y_1, \ldots, y_n, z) = \sum_{\varnothing \neq S \subseteq \{1,\ldots,n\}} (-1)^{|S|-1} f(\{y_i\}_{i \notin S}, \{0\}_{i \in S}, z)$. There is extra freedom to add any analytic function times $y_1 \cdots y_n$, which will be important below. We note that the resulting germ of relative divisor $D \subseteq X \to B$ can be promoted to a true (not germ) relative divisor by removing a suitable closed subset of X.

Given the local existence of relative divisors, compactness of K immediately produces a finite collection of relative divisors $D_i \subseteq X \times_B U_i \to U_i$ ($U_i \subseteq |B|$ open) such that every $z \in K$ is D_i-controlled for some i. These divisors, however, need not be disjoint. Note that it suffices to ensure that $D_i \cap D_j \cap z = \varnothing$ for all $z \in K$, as then $\bigcup_{i<j} D_i \cap D_j \subseteq X$ is closed and disjoint from K so we can simply remove it. To produce divisors with this property, we use an inductive argument, the key being that intersections $D \cap D' \cap z$ generically happen over a codimension two ($\dim(\mathcal{U}/\mathcal{Z}) - \operatorname{codim} D - \operatorname{codim} D' = -2$) subset of K.

Consider the following more general problem. In addition to the data of $X \to B$ and $K \subseteq \mathcal{Z}(X/B)$, fix a relative singular divisor $D^{\mathrm{prev}} \subseteq X \to B$. We then ask for a finite collection of divisors D_i which together control all $z \in K$ and which are disjoint from each other and from D^{prev}. Our original problem is the special case $D^{\mathrm{prev}} = \varnothing$.

We now show how to reduce the problem for a given (D^{prev}, K) to that of another pair $(D^{\mathrm{prev}\prime}, K')$. Consider any choice of relative divisors D_i controlling all $z \in K$. We claim that if the problem associated to

$$(D^{\mathrm{prev}\prime}, K') = \Biggl(D^{\mathrm{prev}} \cup \bigcup_i D_i,$$
$$\pi_B\biggl(\Bigl(\mathcal{U}(X/B) \times_{\mathcal{Z}(X/B)} K\Bigr) \cap \bigcup_i \Bigl(D_i \cap \bigl(D^{\mathrm{prev}} \cup \bigcup_{j \neq i} D_j\bigr)\Bigr)\biggr)\Biggr) \tag{5.8}$$

has a solution, then our original problem (D^{prev}, K) has a solution. Consider divisors D_i' solving the modified problem; they are disjoint from D^{prev} and from every D_i, and they control all $z \in K'$, hence all z in a neighborhood of this set (Lemma 5.7). For $z \in K \setminus \mathrm{Nbd}\, K'$, we use some D_i to control z. The intersections of these divisors with each other and with D^{prev} will be disjoint from $\mathcal{U}(X/B) \times_{\mathcal{Z}(X/B)} K$ by definition of K' hence can simply be removed from X.

We now claim that the more general problem has a solution provided $D^{\mathrm{prev}} \cap \mathcal{U}(X/B) \times_{\mathcal{Z}(X/B)} K \to K$ has relative dimension zero. We argue by induction on $\dim K$, the case $K = \varnothing$ being trivial. For the inductive step, we simply note that in the construction above, the set K' will have at most complex codimension one inside K provided the D_i are chosen generically (this uses the fact that $D^{\mathrm{prev}} \cap \mathcal{U}(X/B) \times_{\mathcal{Z}(X/B)} K \to K$ has relative dimension zero), which also ensures that $D^{\mathrm{prev}\prime} \cap \mathcal{U}(X/B) \times_{\mathcal{Z}(X/B)} K \to K$ also has relative dimension zero. ☐

We now use generic transversality to argue that the Grothendieck group of interior semi-regular 1-cycles coincides with that of all 1-cycles.

Proposition 5.12. *The map $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg}}) \to H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$ is surjective (and the same for $\mathcal{Z}^{\text{semi-Fano}} \subseteq \mathcal{Z}$ in place of $\mathcal{Z}$).*

Proof. Fix a class in $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$, and let us show it is in the image of $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg}})$. Represent our class by a finite semi-simplicial set $B_\bullet$, a family of threefolds $X \to B$ (i.e. a map $B_\bullet \to \mathrm{Cpx}_{3,\bullet}$ in the sense of §3.1), and a cycle $\lambda \in C_*(B_\bullet, C_c^*(\mathcal{Z}(X/-))$ consisting of cochains $\lambda_\sigma \in C_c^*(\mathcal{Z}(X_\sigma/\sigma))$ indexed by the simplices $\sigma \in B_\bullet$.

The pair (B, λ) is equivalent in $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$ to its stabilization $(B \times \mathbb{R}^N, \lambda \cup \pi_{\mathbb{R}^N}^*[0])$ (we will leave the choice of triangulation of $B \times \mathbb{R}^N$ implicit). It is also equivalent to the modified stabilization $(B \times \mathbb{R}^N, \lambda \cup (\pi_{\mathbb{R}^N} - i)^*[0])$ for any map $i : \mathcal{Z}(X/B) \to \mathbb{R}^N$. Now the product $\lambda \cup (\pi_{\mathbb{R}^N} - i)^*[0]$ is supported along the graph of i denoted $\Gamma_i \subseteq \mathcal{Z}(X/B) \times \mathbb{R}^N = \mathcal{Z}((X \times \mathbb{R}^N)/(B \times \mathbb{R}^N))$. Taking i to be an analytic embedding, we may ensure that Γ_i is analytic and the projection $\Gamma_i \to \mathbb{R}^N$ (hence *a fortiori* the projection $\Gamma_i \to B \times \mathbb{R}^N$) is injective.

We have thus shown that (B, λ) is equivalent in $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$ to another pair, which we now rename as (B, λ), which comes with a compact analytic set $K \subseteq \mathcal{Z}(X/B)$ for which $K \to B$ is injective and with a lift of λ to a cycle

$$\tau \in C_*(B_\bullet, C_K^*(\mathcal{Z}(X/-)) \tag{5.9}$$

where $C_Z^*(A) = C^*(A, A \setminus Z)$.

Now the fact that $K \to B$ is a injective allows us to appeal to Proposition 5.11 to fix disjoint relative divisors $D_i \subseteq X \times_B U_i \to U_i$ for constructible closed sets $U_i \subseteq B$ (i.e. unions of closed simplices) which together control K (this requires deleting a closed subset of X disjoint from K and subdividing B, neither of which change the class of (B, λ) in $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$).

Now finally we are in a situation which can be deformed to a semi-regular situation using generic transversality. Consider the family $\tilde{X} = X \times \mathbb{R} \to B \times \mathbb{R} = \tilde{B}$, and consider a collection Φ of analytic functions $\varphi_i : \tilde{B} \to \mathcal{J}_{\tilde{D}_i}(\tilde{X}/\tilde{B}) = \mathcal{J}_{D_i}(X/B) \times \mathbb{R}$ supported inside $U_i \times \mathbb{R}$ and vanishing on $B \times 0$. By generic transversality Lemma 5.10, the resulting reglued family $\tilde{X}_\Phi \to \tilde{B}$ (Definition 5.6) is $(\bigsqcup_i D_i)$-regular over $B \times (\mathbb{R} \setminus 0)$ for some Φ.

Choose a local retraction $\rho : \mathcal{Z}(\tilde{X}/\tilde{B}) \to \mathcal{Z}(X/B)$ near K. Let $\tilde{B}_t$ denote the fiber of $\tilde{B}$ over $t \in \mathbb{R}$. For sufficiently small $t > 0$, the restriction $\rho_t : \mathcal{Z}(\tilde{X}_t, \tilde{B}_t) \to \mathcal{Z}(X/B)$ may be used to define a pullback cycle

$$\rho_t^* \tau \in C_*(\tilde{B}_t, C^*_{\rho_t^{-1}(K)}(\mathcal{Z}(\tilde{X}_{\Phi,t}/-))) \tag{5.10}$$

which, by regularity of Φ, determines a class in $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg}})$. Now $\rho_t^*\tau$ is homologous to τ by consideration of the pullback $\rho^*\tau$ paired with the chain $[0,t]$, which is a chain in $C_*(\tilde{B}, C^*_{\rho^{-1}(K)}(\mathcal{Z}(\tilde{X}_\Phi/-)))$ with boundary $\rho_t^*\tau - \tau$. This shows surjectitivity of the map $H_c^*(\mathcal{Z}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg}}) \to H_c^*(\mathcal{Z}(\mathrm{Cpx}_3))$. The same argument applies to $\mathcal{Z}^{\text{semi-Fano}} \subseteq \mathcal{Z}$. $\square$

It would appear that a relative version of the same argument would show that the map in Proposition 5.12 is also injective. We do not need this fact, so we will not pursue it here.

Lemma 5.13. *$\rho_d(x_{g,m,k}) = 0$ in $H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ for $k > 0$, $m > 0$, and $d > 1$.*

Proof. Combining Proposition 5.12 with Proposition 5.3, we see that $H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ vanishes in negative virtual dimension. The virtual dimension of

$$\rho_d(x_{g,m,k}) \in H_c^{2mk}(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3, mk/d)) \tag{5.11}$$

is $2mk/d - 2mk$, which is negative for $k > 0$, $m > 0$, and $d > 1$. $\square$

5.5 Filtration

We now compute generators for the virtual dimension ≤ 0 part of $H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg}})$ using a filtration argument.

The first step is to argue that to understand the virtual dimension ≤ 0 part of the group $H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg}})$, we can replace $\mathcal{Z}^{\text{semi-Fano}}(-)^\circ_{\text{semi-reg}}$ with the interior of the subset consisting of cycles with *smooth support* (say $z = \sum_i m_i C_i$ has smooth support when $\bigcup_i C_i \subseteq X$ is smooth, equivalently when $\bigsqcup_i \tilde{C}_i \to X$ is a smooth embedding). The point will be that having non-smooth support is a codimension two phenomenon.

We indicate cycles with smooth support using the subscript $\mathcal{Z}_{\text{smooth}}$.

Lemma 5.14. *$\mathcal{Z}_{\text{smooth}} \subseteq \mathcal{Z}$ is an analytic constructible subset.*

Proof. The set of points $p \in \mathcal{U}$ in the universal family at which the fiber of $\mathcal{U} \to \mathcal{Z}$ is smooth is an analytic constructible subset. Since $\mathcal{U} \to \mathcal{Z}$ is proper, the image of a constructible subset is constructible. $\square$

Lemma 5.15. *The map*

$$H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}}) \to H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg}}) \tag{5.12}$$

is an isomorphism in virtual dimension ≤ 0 and surjective in virtual dimension 1.

Proof. While $\mathcal{Z}^{\text{semi-Fano}}(X/B,k)^\circ_{\text{semi-reg}}$ has dimension $\leq 2k+\dim B$ by Lemma 5.2, the same argument shows that the complement of its subset $\mathcal{Z}^{\text{semi-Fano}}(X/B,k)^\circ_{\text{semi-reg,smooth}}$ has dimension at most this quantity minus two, since singularities impose codimension two constraints. Now the map in question fits into a long exact sequence whose third term is

$$H^*_c(((\mathcal{Z}^{\text{semi-Fano}})^\circ_{\text{semi-reg}} \setminus (\mathcal{Z}^{\text{semi-Fano}})^\circ_{\text{semi-reg,smooth}})(\mathrm{Cpx}_3)) \tag{5.13}$$

which for the aforementioned dimension reasons is supported in virtual dimension ≥ 2. $\square$

Having restricted to smooth cycles, we may consider the following filtration.

Definition 5.16 (Multiplicity filtration)**.** Let $\mathbf{M} = \bigsqcup_{n\geq 0}\mathbb{Z}^n_{\geq 1}/S_n$, so that there is a map $\mathcal{Z} \to \mathbf{M}$ associating to each cycle $z = \sum_i m_iC_i$ the multi-set $\mathbf{m}$ of multiplicities m_i. Partially order $\mathbf{M}$ by declaring that $\mathbf{m}' \leq \mathbf{m}$ whenever $\mathbf{m}$ may be obtained from $\mathbf{m}'$ by grouping together the multiplicities and replacing each group with some positive integer linear combination thereof. The map $\mathcal{Z} \to \mathbf{M}$ is not in general upper semi-continuous, however it is so at every point $\sum_i m_iC_i$ with all C_i disjoint. In particular, it is upper semi-continuous on $\mathcal{Z}_{\text{smooth}}$. Thus the loci

$$(\mathcal{Z}(-)_{\text{smooth}})_{\leq \mathbf{m}} \subseteq \mathcal{Z}(-)_{\text{smooth}} \tag{5.14}$$

are open.

We thus have groups $H^*_c((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{\leq \mathbf{m}})$ (and similarly with $< \mathbf{m}$ in place of $\leq \mathbf{m}$). The tautological map

$$H^*_c((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{<\mathbf{m}}) \to H^*_c((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{\leq \mathbf{m}}) \tag{5.15}$$

fits into a long exact sequence with third term $H^*_c((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{=\mathbf{m}})$.

Proposition 5.17. *The virtual dimension ≤ 0 part of $H^*_c((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{=\mathbf{m}})$ has a canonical generating set indexed by combinatorial types of smooth semi-Fano cycles (multi-sets of triples (g,m,k) of genus, multiplicity, and non-negative chern number) of multiplicity* $\mathbf{m}$ *which are* non-deficient *(meaning each triple (g,m,k) satisfies $(m-1)k=0$).*

Proof. The space $(\mathcal{Z}^{\text{semi-Fano}}(X/B)^\circ_{\text{semi-reg,smooth}})_{=\mathbf{m}}$ consists of cycles $\sum_i m_iC_i$ for $(m_1,\ldots,m_n) = \mathbf{m}$ and $\bigsqcup_i C_i \to X$ a smooth embedded curve unobstructed relative B. It is thus a manifold of dimension $\dim B + 2\sum_i k_i$, hence has cohomology up to this degree. The map to the Grothendieck group reduces cohomological degree by $\dim B$, and the chern number is $\sum_i k_im_i$. Hence it contributes cohomology in virtual dimension $\geq 2\sum_i(m_i-1)k_i \geq 0$. Thus classes of virtual dimension ≤ 0 only exist when $(m_i-1)k_i = 0$ for all i (i.e. non-deficient), and they are generated by Poincaré duals of points. We declare such Poincaré duals to be our canonical generators. It thus suffices to show that the Poincaré dual of a non-deficient point of $(\mathcal{Z}^{\text{semi-Fano}}(X/B)^\circ_{\text{semi-reg,smooth}})_{=\mathbf{m}}$, regarded as a class in $H^*_c((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{=\mathbf{m}})$, depends only on its combinatorial type.

Let us show how to associate a class in $H^*_c((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{=\mathbf{m}})$ to any non-deficient smooth semi-Fano cycle z of multiplicity $\mathbf{m}$ in a threefold X_0 (regarded as a germ around z). Consider any family $X \to B$ with fiber X_0 over a specified point $0 \in B$, with the property that z is semi-regular in $\mathcal{Z}(X/B)$. Now choose a relative divisor $D \subseteq X \to B$ controlling z, and consider analytic sections $\Phi : B \times \mathbb{R} \to \mathcal{J}_D(X/B)$ vanishing on $B \times 0$. By generic transversality Lemma 5.10, there exists such a section which is D-regular over $B \times (\mathbb{R} \setminus 0)$. Since $z \in \mathcal{Z}(X/B)$ is semi-regular, its semi-chart in $\mathcal{Z}((X \times \mathbb{R})_\Phi/B \times \mathbb{R})$ is smooth and submersive over $\mathbb{R}$. Now this semi-chart has dimension $\dim(B \times \mathbb{R}) + 2\sum_i k_i = \dim(B \times \mathbb{R}) + 2\sum_i m_i k_i$, and (inside the interior semi-regular locus) the closure of the non-smooth locus has dimension at most two less (as in Lemma 5.15). Thus the Poincaré dual of an interior smooth point of this semi-chart lying over $\mathbb{R}_{>0}$ nearby z is a well defined element of

$$H_c^{\dim(B\times\mathbb{R})+2\sum_i m_i k_i}((\mathcal{Z}((X \times \mathbb{R})_\Phi/B \times \mathbb{R})^\circ_{\text{semi-reg,smooth}})_{=\mathbf{m}}), \tag{5.16}$$

hence its image in $H_c^{2\sum_i m_i k_i}((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{=\mathbf{m}})$ is also well defined.

Now let us argue that this class in $H_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{=\mathbf{m}})$ depends only on z and X_0 (i.e. is independent of the choice of family $X \to B$ with specified fiber X_0, relative divisor $D \subseteq X \to B$ controlling z, and perturbation Φ). To show independence of Φ, consider the product $X \times \mathbb{R} \times \mathbb{R}$ with Φ on $X \times \mathbb{R} \times 0$ and Φ' on $X \times 0 \times \mathbb{R}$, and choose $\bar{\Phi}$ on $X \times \mathbb{R} \times \mathbb{R}$ extending Φ and Φ' using Lemma 5.10 making it D-regular on $X \times (\mathbb{R} \setminus 0) \times (\mathbb{R} \setminus 0)$. To show dependence on just $D_0 = D \cap X_0$, consider a finite-dimensional vector space V mapping to $H^0(D, T_{X/B}(-\infty D))$. Associated to such a map is a family $X_V \to B \times V$ obtained by regluing near D via the exponential of V (Definition 5.6). Choose V so that z is semi-regular in $(X_0)_V \to V$, and choose Φ over $B \times V \times \mathbb{R}$ vanishing on $B \times V \times 0$ so that its restrictions to $0 \times 0 \times (\mathbb{R} \setminus 0)$, $B \times 0 \times (\mathbb{R} \setminus 0)$, $0 \times V \times (\mathbb{R} \setminus 0)$, and $(B \setminus 0) \times (V \setminus 0) \times (\mathbb{R} \setminus 0)$ are all D-regular (apply Lemma 5.10 inductively four times). It follows that the element associated to $X \to B$ and D coincides with that associated to $(X_0)_V \to V$ and $D_0 \times V$. Independence of V follows from the same argument comparing V, V', and $V \oplus V'$. Finally, to show independence of D_0, note that given D_0 and D_0', there exists D_0'' disjoint from both, and the invariants associated to D_0 and $D_0 \sqcup D_0''$ are evidently the same (a V for D_0 induces one for $D_0 \sqcup D_0''$ by extension by zero).

Now we claim that the element of $H_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{=\mathbf{m}})$ associated to a non-deficient smooth semi-Fano cycle z of multiplicity $\mathbf{m}$ in some threefold is invariant under deformation. This is immediate, since given a family $X_{[0,1]} \to [0,1]$ and non-deficient smooth semi-Fano cycles $z_t \in \mathcal{Z}(X_t)$ of multiplicity $\mathbf{m}$, we can (at least after dividing into subintervals) find $X \to B$ with our family pulled back under a map $[0,1] \to B$, and every z_t semi-regular inside $\mathcal{Z}(X/B)$. We are now done since smooth cycles in threefolds are classified up to deformation by their combinatorial type (deform to the normal cone and appeal to Remark 4.1). $\square$

We now lift the elements $x_{g,m,k} \in H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ for $k \geq 0$ and $(m-1)k = 0$ to elements $\tilde{x}_{g,m,k} \in H_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{\leq(m)})$. Lemma 5.15 and Proposition 5.12 show that the maps

$$\begin{aligned} &H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}}) \\ &\qquad\qquad \to H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg}}) \\ &\qquad\qquad\qquad\qquad \to H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)) \end{aligned} \tag{5.17}$$

are bijective in virtual dimension ≤ 0 and surjective, respectively. We will inspect their proofs to obtain a cycle in $C_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{\leq(m)})$ lifting $x_{g,m,k}$. Represent $x_{g,m,k}$ via its definition (4.9), and use (the trace of) a fiber of $E \to C$ as relative divisor D (which controls all cycles in E). Use generic transversality Lemma 5.10 to produce a family of D-deformations $\Phi : \mathbb{C}P^N \times \mathbb{R} \to \mathcal{J}_D(((E \times (\mathbb{C}^{N+1} - 0))/\mathbb{C}^\times)/\mathbb{C}P^N)$ (piecewise real analytic) which is trivial over $\mathbb{C}P^N \times 0$ and D-regular over $\mathbb{C}P^N \times (\mathbb{R} \setminus 0)$. Pulling back the cocycle (4.9) under a local retraction from the relative cycle space over $\mathbb{C}P^N \times \mathbb{R}$ to that over $\mathbb{C}P^N \times 0$ and restricting to that over $\mathbb{C}P^n \times t$ for some generic small $t > 0$ defines a class in

$$H_c^{2n}\Bigg(\mathcal{Z}^{\text{semi-Fano}}\Big(\Big(\frac{E \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0}\Big)_{\Phi_t} \Big/ \mathbb{C}P^N\Big)^\circ_{\text{semi-reg}}\Bigg) \\ \to H_c^{2km}(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3, km)^\circ_{\text{semi-reg}}) \quad (5.18)$$

lifting $x_{g,m,k}$. The resulting element of $H_c^{2km}(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3, km)^\circ_{\text{semi-reg}})$ is well defined by considering families of deformations over $\mathbb{C}P^N \times \mathbb{R}^2$. Now by the dimension count in Lemma 5.15, the class in the domain of (5.18) lifts uniquely to

$$H_c^{2n}\Bigg(\mathcal{Z}^{\text{semi-Fano}}\Big(\Big(\frac{E \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0}\Big)_{\Phi_t} \Big/ \mathbb{C}P^N\Big)^\circ_{\text{semi-reg,smooth}}\Bigg). \quad (5.19)$$

Finally, note that this lift lies in the $\leq (m)$ part since cycles on E of total degree m all have multiplicity tuple $\leq (m)$. This defines the lift $\tilde{x}_{g,m,k} \in H_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{\leq(m)})$.

Lemma 5.18. *For $k \geq 0$ and $(m-1)k = 0$, the lift $\tilde{x}_{g,m,k} \in H_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{\leq(m)})$ maps to the generator of $H_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{=m})$ from Proposition 5.17 associated to the topological type (g, m, k).*

Proof. Recall that $x_{g,m,k} \in H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ (4.9) is given by

$$\mathbf{1}_m \prod_{i=1}^n r_i^{-1} f_i^* \tau_{\mathcal{L}^{\otimes r_i}} \in H_c^{2n}\Big(\mathcal{Z}\Big(\frac{E \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0} \Big/ \mathbb{C}P^N\Big)\Big) \\ \to H_c^*(\mathcal{Z}(\mathrm{Cpx}_3)) \quad (5.20)$$

where $n = N + km$. The lift $\tilde{x}_{g,m,k} \in H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})$ is defined by perturbation as detailed just above.

Now the cycles on E of degree m all have multiplicity $\leq (m)$, and the multiplicity $= (m)$ locus inside $\mathcal{Z}(E, m)$ is canonically identified with $\mathcal{Z}(E, 1) = H^0(C, E)$. Let us take $E = L \oplus L'$ for generic line bundles L and L' of degrees $g - 1$ and $g - 1 + k$, respectively, which ensures that $h^0(C, L) = 0$, $h^0(C, L') = k$, and $h^1(C, L) = h^1(C, L') = 0$. In particular, the multiplicity m inside $\mathcal{Z}(E, m)$ is semi-regular and smooth.

Now let us consider the restriction of the cocycle (5.20) to the multiplicity $= (m)$ locus, which is thus a class in

$$H_c^{2n}\Big(\frac{H^0(C, E) \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0}\Big). \quad (5.21)$$

Note that $(m-1)k = 0$ implies $mk = k$, so we have $n = N + k$. The restriction of a degree r_i function $f_i : \mathcal{Z}(E, m) \to \mathbb{C}$ of the form (4.8) to the multiplicity m locus identified as $\mathcal{Z}(E, 1)$

has the same form (4.8) of the same degree r_i. The restriction of (5.20) to the multiplicity $= (m)$ locus is thus a cocycle of precisely the same form, just with m replaced with 1. By Lemma 4.6, the resulting class is independent of the choice of functions f_i. Taking k linear functions f_i which together define an isomorphism $\mathcal{Z}(E,1) = H^0(C,E) \xrightarrow{\sim} \mathbb{C}^k$, we see that this restricted class in (5.21) is the Poincaré dual of a point.

Now we are interested in the image of the lift $\tilde{x}_{g,m,k}$ in $H_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{=m})$. The lift $\tilde{x}_{g,m,k}$ is defined by perturbing the cocycle representing $x_{g,m,k}$. Since the multiplicity m locus is semi-regular and smooth, it remains so after the perturbation, and the point class remains the point class. It may not be *interior* smooth or *interior* semi-regular, but the loci where these fail are codimension two, and the point class lifts uniquely to the point class. ☐

A product of lifts $\tilde{x}_{g_i,m_i,k_i}$ is an element of $H_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{\le (m_i)_i})$. Lemma 5.18 implies that its image in $H_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{=(m_i)_i})$ is the generator associated to the topological type of the set of triples $(g_i, m_i, k_i)_i$ (it is evident from the definition that products of canonical generators in the sense of Proposition 5.17 are themselves canonical generators, with topological type the disjoint union of the topological types of the factors).

Corollary 5.19. *The group $H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})$ is generated in virtual dimension ≤ 0 by monomials in the lifted equivariant local curve elements $\tilde{x}_{g,m,k}$ with $k \ge 0$ and $(m-1)k = 0$.*

Proof. By a direct limit argument, it suffices to show that $H_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{\le \mathfrak{m}})$ is generated in virtual dimension ≤ 0 by monomials in the $\tilde{x}_{g,m,k}$ with $k \ge 0$ and $(m-1)k = 0$ with multiplicity tuple $\le \mathfrak{m}$. Now we prove this statement by induction.

Every element of $H_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{<\mathfrak{m}})$ is in the image of $H_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smoo}}$ for some $\mathfrak{m}' < \mathfrak{m}$, so the induction hypothesis implies that $H_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{<\mathfrak{m}})$ is generated in virtual dimension ≤ 0 by monomials in the $\tilde{x}_{g,m,k}$ for $k \ge 0$ and $(m-1)k = 0$ with multiplicity tuple $< \mathfrak{m}$. The monomials with multiplicity tuple $= \mathfrak{m}$ generate $H_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{=\mathfrak{m}})$ by Lemma 5.18. Now appealing to the long exact sequence (5.15), we conclude the desired generation statement for $H_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{\le \mathfrak{m}})$. ☐

Theorem 5.20. *The group $H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ is generated as a ring in virtual dimension ≤ 0 by the equivariant local curve elements $x_{g,m,k}$ with $k \ge 0$ and $(m-1)k = 0$.*

Proof. The map $H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg}}) \to H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ is surjective by Proposition 5.12. The map $H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}}) \to H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg}})$ is an isomorphism in virtual dimension ≤ 0 by Lemma 5.15. By Corollary 5.19, the group $H_c^*(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})$ is generated by monomials in equivariant local curve elements $x_{g,m,k}$ with $k \ge 0$ and $(m-1)k = 0$. ☐

The above argument is very close to giving a full proof of Theorem 1.1 (free generation by local curve elements) rather than just generation (Theorem 5.20). The missing ingredient is a proof that the connecting map

$$H_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{=\mathbf{m}}) \xrightarrow{+1} H_c^*((\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3)^\circ_{\text{semi-reg,smooth}})_{<\mathbf{m}}) \quad (5.22)$$

vanishes in virtual dimension 1 mapping to virtual dimension 0. This assertion appears reasonable over complex analytic bases (a generic path in $(\mathcal{Z}^{\text{semi-Fano}}(X/B)^{\circ}_{\text{semi-reg,smooth}})_{=\mathbf{m}}$ would avoid the closure of $(\mathcal{Z}^{\text{semi-Fano}}(X/B)^{\circ}_{\text{semi-reg,smooth}})_{<\mathbf{m}}$ since this subset has codimension two), but becomes less clear once we consider real simplices. Instead, we will prove the injectivity part of Theorem 1.1 using an algebraic argument in the next section.

Proof of Theorem 1.5. By Corollary 4.8, the ring homomorphisms $(-iu)^k\mathrm{GW}$ and $(-q)^{-k/2}\mathrm{PT}$ satisfy the MNOP correspondence when evaluated on all local curve elements $x_{g,m,k}$. These local curve elements generate $H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ in virtual dimension zero by Theorem 5.20, so they satisfy the MNOP correspondence on all of $H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$. We may thus evaluate on the element $(X, \beta; \gamma_1, \ldots, \gamma_r)$ (see §3.5) to obtained the desired result. $\square$

6 Algebraic constraints

We use the bi-algebra structure on $H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ and the nontriviality of certain Gromov–Witten invariants to show that the sub-algebra generated by equivariant local curve elements $x_{g,m,k}$ for $k \geq 0$ and $(m-1)k = 0$ is free.

Consider the free polynomial ring $R = \mathbb{Z}[x_{g,m,k}]$ on formal variables $x_{g,m,k}$ indexed by integers $g \geq 0$, $m \geq 0$, and $k \geq 0$, satisfying $(m-1)k = 0$, modulo the relation that $x_{g,0,k} = 1$. Equip R with the co-unit and co-multiplication maps given by

$$\eta : R \to \mathbb{Z} \qquad\qquad \Delta : R \to R \otimes R \tag{6.1}$$

$$x_{g,m,k} \mapsto 0 \quad \text{for } m > 0 \qquad\qquad x_{g,m,k} \mapsto \sum_{a+b=m} x_{g,a,k} \otimes x_{g,b,k} \tag{6.2}$$

on generators and extended to be algebra maps. This makes R into a commutative and co-commutative bi-algebra. Sending $x_{g,m,k} \in R$ to the equivariant local curve element $x_{g,m,k} \in H^{2mk}_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3, mk))$ defines a ring homomorphism $R \to H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ and a bi-algebra homomorphism $R \to H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))/\mathrm{tors}$ by Lemma 4.10 (and the fact that $H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ vanishes in negative virtual dimension).

Let $\rho_d : R \to R$ $(d \geq 1)$ be given on generators by

$$\rho_d(x_{g,m,k}) = \begin{cases} x_{g,m/d,k} & m \text{ divisible by } d \text{ and } k = 0 \text{ or } d = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

and extended multiplicatively. This ρ_d is a map of bi-algebras (commutes with Δ and η) by inspection. The map $R \to H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ is compatible with the operations ρ_d by Lemma 4.9 and Lemma 5.13.

We now wish to analyze the kernel $A \subseteq R$ of the map $R \to H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))/\mathrm{tors}$. Compatibility of this map with ρ_d implies that $\rho_d(A) \subseteq A$. Compatibility with Δ implies that $\Delta(A) \subseteq (A \otimes R) + (R \otimes A)$ (at least modulo torsion issues which we will address later). Our goal is to prove that these constraints, along with a simple Gromov–Witten invariant calculation, forces $A = 0$.

The *weight* of a monomial in the variables $x_{g,m,k}$ is a function $w : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined by $w(ab) = w(a) + w(b)$ and $w(x_{g,m,k}) = m \cdot \mathbf{1}_{g,k}$. Given an arbitrary element $a \in R$, we denote by a_w its weight w part. A tensor product of monomials $a \otimes b$ has a bi-weight $(w(a), w(b))$ and a total weight $w(a) + w(b)$. The coproduct Δ preserves (total) weight.

Lemma 6.1 (Weight splitting). *Let $A \subseteq R$ be a subgroup with the property that $\Delta(A) \subseteq (A \otimes R) + (R \otimes A)$. Let w be any nonzero weight. If A has an element with nonzero weight w part, then A has an element with nonzero weight $m\mathbf{1}_{g,k}$ part for some $(g, k) \in \operatorname{supp} w$.*

Proof. The idea is to use Δ to 'split' the weight w until its support becomes a singleton. We have $\Delta(a)_w = \Delta(a_w)$. If the support of w is not a singleton, then write $w = w_1 + w_2$ for nonzero w_1 and w_2 of disjoint support. Since the supports of w_1 and w_2 are disjoint, the result of applying Δ to a monomial of weight w will have a *unique* monomial of bi-weight (w_1, w_2). In particular, $a_w \neq 0$ implies that $\Delta(a)_{w_1,w_2} \neq 0$. Since $\Delta(a) \in (A \otimes R) + (R \otimes A)$, we conclude that A must have an element with nonzero weight w_1 part or weight w_2 part. Now we replace w with whichever of w_1 or w_2 it is and repeat until the support of w is a singleton. $\square$

Lemma 6.2 (Weight purifying). *Let $A \subseteq R$ be a subgroup with the property that $\Delta(A) \subseteq (A \otimes R) + (R \otimes A)$. If A has an element with nonzero weight $m\mathbf{1}_{g,k}$ part, then A has an element containing the single variable monomial $x_{g,m',k}$ for some $m' \leq m$.*

Proof. The argument is similar to 'weight splitting' Lemma 6.1. Let $w = m\mathbf{1}_{g,k}$. Let $a \in A$ have nonzero weight w part. Among the weight w monomials appearing in a, consider the factor $x_{g,m',k}$ with m' the largest possible. Now consider the monomials in $\Delta(a)_w$ of the form $x_{g,m',k} \otimes -$. How can a given monomial in a_w contribute such a monomial to $\Delta(a)_w$? The factors $x_{g,m'',k}$ with $m'' < m'$ must go completely on the right. Of the factors of $x_{g,m',k}$, exactly one must go completely on the left, and the rest must go completely on the right. Thus the monomials in $\Delta(a)_w$ of the form $x_{g,m',k} \otimes -$ are in bijection with the monomials in a_w with at least one $x_{g,m',k}$ factor, and the effect of Δ is to multiply their coefficient by the number of such factors. In particular, $\Delta(a)_w$ contains monomials of the form $x_{g,m',k} \otimes -$. Appealing to $\Delta(a) \in (A \otimes R) + (R \otimes A)$, we have 'split' w unless $m' = m$, in which case we have proven the desired result. $\square$

Lemma 6.3 (Weight dividing). *Let $A \subseteq R$ be a subgroup with the property that $\rho_d(A) \subseteq A$. If A has an element containing the single variable monomial $x_{g,m,0}$, then A has an element containing the single variable monomial $x_{g,1,0}$ and no single variable monomials $x_{g,m',0}$ for $m' > 1$.*

Proof. Take an element of A containing single variable monomials $x_{g,m,0}$ for various m, and apply ρ_d to it where d is the maximum m among them. $\square$

Order the pairs $(g, k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ lexicographically, namely $(g, k) < (g', k')$ when either $g < g'$ or $g = g'$ and $k < k'$. This is evidently a well-ordering.

Proposition 6.4. *Let $A \subseteq R$ be a subgroup with the property that $\Delta(A) \subseteq (A \otimes R) + (R \otimes A)$ and $\rho_d(A) \subseteq A$. If $A \neq 0$, then there exists an element of A containing the single variable monomial $x_{g,1,k}$ and no single variable monomials $x_{g,m,k}$ with $m > 1$ or $x_{g',m',k'}$ with $(g', k') < (g, k)$.*

Proof. Consider the set of all weights of all monomials appearing in elements of A. Each such weight has a maximum pair (g,k) in its support. Fix (g,k) to be the minimum such pair. Let $a \in A$ have a monomial whose weight has (g,k) as the maximum element in its support. By Lemma 6.1, there exists $a \in A$ with a monomial of weight $m\mathbf{1}_{g,k}$. By Lemma 6.2, there exists $a \in A$ with a single variable monomial $x_{g,m,k}$ (possibly different m). By Lemma 6.3, there exists $a \in A$ with a single variable monomial $x_{g,1,k}$ and no single variable monomials $x_{g,m,k}$ with $m > 1$. Finally, a has no single variable monomials $x_{g',m',k'}$ with $(g',k') < (g,k)$ by choice of (g,k). □

Lemma 6.5. *Let $k \geq 0$. There exists a group homomorphism $\mathrm{GW}_{g,k} : H^*_c(\mathcal{Z}(\mathrm{Cpx}_3)) \to \mathbb{Q}$ such that $\mathrm{GW}_{g,k}(x_{g,1,k}) = 1$, $\mathrm{GW}_{g,k}(x_{g',m',k'}) = 0$ for $(g',k') > (g,k)$, and $\mathrm{GW}_{g,k}$ evaluates to zero on any monomial of degree > 1.*

Proof. Let $\mathrm{GW}_{g,k}$ integrate over the virtual fundamental cycle of the moduli space $(\mathcal{M}'_g)_{c_1=k}$ of non-constant stable maps from connected nodal curves of arithmetic genus g representing a homology class with chern number k. Since the image of a connected space is connected, $\mathrm{GW}_{g,k}$ annihilates monomials of degree > 1.

We have $\mathrm{GW}_{g,k}(x_{g',m',k'}) = 0$ if $g' > g$, since there are no non-constant maps from a nodal curve of arithmetic genus g to a curve of genus $g' > g$. In the case $g = g'$, the map would have to have degree $d = k/k' < 1$, hence cannot exist.

Finally, let us calculate $\mathrm{GW}_{g,k}(x_{g,1,k}) = 1$. We can represent $x_{g,1,k}$ by a curve C of genus g and $E = L \oplus L'$ for $c_1(L) = g-1$ and $c_1(L') = g-1+k$. Generically we have $h^1(L) = h^1(L') = 0$ and $h^0(L) = 0$ and $h^0(L') = k$. That is, C is part of a transversely cut out k-dimensional moduli space of sections (which locally coincides with $\mathcal{M}_g$), and the equivariant local curve element is by definition the Poincaré dual of a point in this space. □

Lemma 6.6. *The kernel A of any morphism $F : X \to Y$ of co-algebras over a field satisfies $\Delta(A) \subseteq (A \otimes X) + (X \otimes A)$.*

Proof. Compatibility of F with Δ implies that $\Delta(A) \subseteq \ker(F \otimes F)$. Now $\ker(F \otimes F) = (\ker F) \otimes X + X \otimes (\ker F)$ since indeed for any pair of morphisms of vector spaces $f : V \to W$ and $f' : V' \to W'$ we have $\ker(f \otimes f') = (\ker f) \otimes V' + V \otimes (\ker f')$. □

Proposition 6.7. *The map $R \to H^*_c(\mathcal{Z}^{\text{semi-Fano}}(\mathrm{Cpx}_3))$ is injective.*

Proof. Since R is torsion free, it suffices to show that the map is injective after rationalizing. The kernel A of this map on rationalizations satisfies $\Delta(A) \subseteq (A \otimes R) + (R \otimes A)$ by Lemma 6.6 and $\rho_d(A) \subseteq A$. If this kernel is nonzero, then Proposition 6.4 produces an element of it on which $\mathrm{GW}_{g,k}$ is nonzero by Lemma 6.5, a contradiction (the preceding lemmas work just the same over $\mathbb{Q}$ as over $\mathbb{Z}$). □

Proof of Theorem 1.1. Combine Theorem 5.20 and Proposition 6.7. □

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