The Chow form of the essential variety in computer vision

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Gunnar Fløystad (Bergen)

Giorgio Ottaviani (Florence)

Algebraic vision

Multiview geometry studies 3D scene reconstruction from images. Foundations in projective geometry. *Algebraic vision* bridges to algebraic geometry (combinatorial, computational, numerical, ...).

Oct 8–9, 2015, Berlin

May 2–6, 2016, San Jose

$3D$ reconstruction

Example from *Building Rome in a Day* (2009) by S. Agarwal et al.

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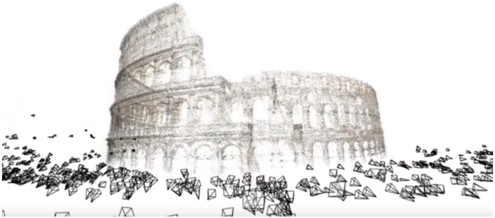
Output: configuration of cameras and 819,242 $3D$ points



How do they do it?

Over-simplification:

- Build a graph whose nodes are the images. Put an edge between two images if their views likely overlap.
- Do robust 2-view reconstruction along edges, with point pairs.
- Piece together. Refine estimate with nonlinear least squares.

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Moral: 'Tiny' reconstructions are subroutines in large-scale reconstructions. The 'tiny' reconstructions rely on super-fast, specialized polynomial equation solvers.

What is a camera?

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Determines a projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$; $\widetilde{X} \mapsto A\widetilde{X}$.

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Math	**Interpretation**
$\mathbb{P}^3$	world
$\mathbb{P}^2$	image plane
$\ker(A)$	camera center
K	internal parameters (e.g. focal length)
$[R \mid t]$	external parameters (orientation, center)

Above $A_{3\times 4} =: K_{3\times 3}\,[R_{3\times 3} \mid t_{3\times 1}]$ where K is upper triangular and R is a rotation. If $K = I$, so $A = [R \mid t]$, then A is **calibrated**.

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$$A \text{ is calibrated} \iff \begin{pmatrix} A \text{ sends the conic } \{\,(a:b:c:0) \mid a^2+b^2+c^2=0\,\} \subset \mathbb{P}^3 \\ \text{to the conic } \{\,(a:b:c) \mid a^2+b^2+c^2=0\,\} \subset \mathbb{P}^2 \end{pmatrix}$$

Our problem

Question (S. Agarwal et al. 2014)

Let $m = 6$. *Given point pairs* $\{(x^{(i)}, y^{(i)}) \in \mathbb{R}^2 \times \mathbb{R}^2 \,|\, i = 1, \ldots, m\}$. *Consider the system of equations:*

$$\begin{cases} A\widetilde{X^{(i)}} \equiv \widetilde{x^{(i)}} \\ B\widetilde{X^{(i)}} \equiv \widetilde{y^{(i)}}. \end{cases} \tag{1}$$

Here $\widetilde{x^{(i)}} = (x_1^{(i)} : x_2^{(i)} : 1)^T \in \mathbb{P}^2$ *and* $\widetilde{y^{(i)}} = (y_1^{(i)} : y_2^{(i)} : 1)^T \in \mathbb{P}^2$. *The unknowns are two* 3×4 *matrices* A, B *with rotations in their left* 3×3 *block and* $m = 6$ *points* $\widetilde{X^{(i)}} \in \mathbb{P}^3$. *When does* (1) *admit a solution?*

- Interpretation: characterize those $m = 6$ point pairs between two calibrated images that are mutually consistent.
- When $m = 5$, the system admits 10 complex solutions. Solvers for this are used in large-scale reconstructions.
- When $m = 6$, generically no exact solution. If there is a solution, then generically it is unique up to the natural symmetries.

Main result

Recall $m = 6$, RHS given, LHS unknown with A, B calibrated:

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Theorem (Fløystad, K., Ottaviani 2016)

There exists an explicit 20×20 skew-symmetric matrix $M(x, y)$ of degree $(6, 6)$ polynomials over $\mathbb{Z}$ in the coordinates of $(x^{(i)}, y^{(i)})$ such that, for generic point correspondences, (1) admits a complex solution if and only if $M(x^{(i)}, y^{(i)})$ is rank-deficient.

Here's that matrix. . .

```
R =
QQ[q_{11,12,13},q_{11,12,21},q_{11,12,22},q_{11,12,23},q_{11,12,31},q_{11,12,32},q_{11,12,33},q_{11,13,21},q_{11,13,22},q_{11,13,23},q_{11,13,31},q_{11,13,32},q_{
11,13,33},q_{11,21,22},q_{11,21,23},q_{11,21,31},q_{11,21,32},q_{11,21,33},q_{11,22,23},q_{11,22,31},q_{11,22,32},q_{11,22,33},q_{11,23,31},q_{11,23,32},q_{11,23,
33},q_{11,31,32},q_{11,31,33},q_{11,32,33},q_{12,13,21},q_{12,13,22},q_{12,13,23},q_{12,13,31},q_{12,13,32},q_{12,13,33},q_{12,21,22},q_{12,21,23},q_{12,21,31},q_
{12,21,32},q_{12,21,33},q_{12,22,23},q_{12,22,31},q_{12,22,32},q_{12,22,33},q_{12,23,31},q_{12,23,32},q_{12,23,33},q_{12,31,32},q_{12,31,33},q_{12,32,33},q_{13,21
,22},q_{13,21,23},q_{13,21,31},q_{13,21,32},q_{13,21,33},q_{13,22,23},q_{13,22,31},q_{13,22,32},q_{13,22,33},q_{13,23,31},q_{13,23,32},q_{13,23,33},q_{13,31,32},q
_{13,31,33},q_{13,32,33},q_{21,22,23},q_{21,22,31},q_{21,22,32},q_{21,22,33},q_{21,23,31},q_{21,23,32},q_{21,23,33},q_{21,31,32},q_{21,31,33},q_{21,32,33},q_{22,2
3,31},q_{22,23,32},q_{22,23,33},q_{22,31,32},q_{22,31,33},q_{22,32,33},q_{23,31,32},q_{23,31,33},q_{23,32,33},q_{31,32,33}]


M = matrix{


{0,-4*q_{11,13,22}+4*q_{11,22,31}+4*q_{13,22,33}-4*q_{22,31,33},4*q_{11,12,22}-4*q_{11,13,23}-4*q_{11,13,32}+4*q_{11,21,22}+4*q_{11,23,31}-4*q_{11,31,32}-4*q_{12,
22,33}-4*q_{13,22,23}-4*q_{13,22,32}+4*q_{13,23,33}+4*q_{13,32,33}-4*q_{21,22,33}-4*q_{22,23,31}+4*q_{22,31,32}-4*q_{23,31,33}+4*q_{31,32,33},0,0,0,-4*q_{11,12,13
}-4*q_{11,12,31}+4*q_{11,13,21}-4*q_{11,21,31}+4*q_{11,22,23}+4*q_{11,22,32}-4*q_{12,13,22}+4*q_{12,13,33}+4*q_{12,22,31}+4*q_{12,31,33}+4*q_{13,21,22}-4*q_{13,21
,33}+4*q_{21,22,31}+4*q_{21,31,33}-4*q_{22,23,33}-4*q_{22,32,33},0,4*q_{11,12,23}+4*q_{11,12,32}-4*q_{11,13,33}+4*q_{11,21,23}+4*q_{11,21,32}-4*q_{11,31,33}+4*q_{
12,22,23}+4*q_{12,22,32}-4*q_{12,23,33}-4*q_{12,32,33}-4*q_{13,22,33}+4*q_{21,22,23}+4*q_{21,22,32}-4*q_{21,23,33}-4*q_{21,32,33}+4*q_{22,31,33},4*q_{11,12,33}+4*
q_{11,21,33}+4*q_{12,22,33}+4*q_{21,22,33},0,4*q_{11,22,33}-4*q_{12,13,23}-4*q_{12,13,32}+4*q_{12,23,31}-4*q_{12,31,32}+4*q_{13,21,23}+4*q_{13,21,32}+4*q_{21,23,3
1}-4*q_{21,31,32},0,-q_{11,12,23}-q_{11,12,32}+q_{11,13,22}-q_{11,13,33}-q_{11,21,23}-q_{11,21,32}-q_{11,22,31}-
q_{11,31,33}+q_{12,22,23}+q_{12,22,32}+q_{12,23,33}+q_{12,32,33}+q_{21,22,23}+q_{21,22,32}+q_{21,23,33}+q_{21,32,33},0,-4*q_{11,12,13}-4*q_{11,12,31}+4*q_{11,13,2
1}-4*q_{11,21,31}+4*q_{11,23,33}+4*q_{11,32,33}+4*q_{12,13,22}-4*q_{12,13,33}-4*q_{12,22,31}-4*q_{12,31,33}-4*q_{13,21,22}+4*q_{13,21,33}-4*q_{21,22,31}-4*q_{21,3
1,33}-4*q_{22,23,33}-4*q_{22,32,33},0,q_{11,12,22}-q_{11,12,33}+q_{11,13,23}+q_{11,13,32}+q_{11,21,22}-q_{11,21,33}-
q_{11,23,31}+q_{11,31,32}+q_{13,22,23}+q_{13,22,32}+q_{13,23,33}+q_{13,32,33}+q_{22,23,31}-q_{22,31,32}-q_{23,31,33}+q_{31,32,33},0,-q_{11,22,23}-q_{11,22,32}-
q_{11,23,33}-q_{11,32,33}},


{4*q_{11,13,22}-4*q_{11,22,31}-4*q_{13,22,33}+4*q_{22,31,33},0,0,24*q_{11,12,13}+24*q_{11,12,31}+24*q_{11,13,21}-24*q_{11,21,31}-24*q_{11,22,23}+24*q_{11,22,32}-2
4*q_{12,13,22}-24*q_{12,13,33}+24*q_{12,22,31}-24*q_{12,31,33}-24*q_{13,21,22}-24*q_{13,21,33}-24*q_{21,22,31}+24*q_{21,31,33}+24*q_{22,23,33}-24*q_{22,32,33},0,0
,0,24*q_{11,12,22}-24*q_{11,13,23}+24*q_{11,13,32}-24*q_{11,21,22}+24*q_{11,23,31}+24*q_{11,31,32}-24*q_{12,22,33}+24*q_{13,22,23}-24*q_{13,22,32}+24*q_{13,23,33}
-24*q_{13,32,33}+24*q_{21,22,33}+24*q_{22,23,31}+24*q_{22,31,32}-24*q_{23,31,33}-24*q_{31,32,33},0,72*q_{11,22,23}-72*q_{11,22,32}-72*q_{11,23,33}+72*q_{11,32,33}
,0,0,0,0,12*q_{11,12,23}-12*q_{11,12,32}-12*q_{11,13,22}-12*q_{11,13,33}-12*q_{11,21,23}+12*q_{11,21,32}+12*q_{11,22,31}-12*q_{11,31,33}+12*q_{12,22,23}-12*q_{12,
22,32}-12*q_{12,23,33}+12*q_{12,32,33}-12*q_{21,22,23}+12*q_{21,22,32}+12*q_{21,23,33}-12*q_{21,32,33},-72*q_{11,12,22}-72*q_{11,12,33}-72*q_{11,13,23}+72*q_{11,1
3,32}+72*q_{11,21,22}+72*q_{11,21,33}+72*q_{11,23,31}+72*q_{11,31,32}+72*q_{13,22,23}-72*q_{13,22,32}-72*q_{13,23,33}+72*q_{13,32,33}+72*q_{22,23,31}+72*q_{22,31,
32}+72*q_{23,31,33}+72*q_{31,32,33},24*q_{11,22,33}-24*q_{12,13,23}+24*q_{12,13,32}+24*q_{12,23,31}+24*q_{12,31,32}-24*q_{13,21,23}+24*q_{13,21,32}-24*q_{21,23,31
}-24*q_{21,31,32},18*q_{11,12,13}+18*q_{11,12,31}+18*q_{11,13,21}-18*q_{11,21,31}+18*q_{11,23,33}-18*q_{11,32,33}+18*q_{12,13,22}+18*q_{12,13,33}-18*q_{12,22,31}+
18*q_{12,31,33}+18*q_{13,21,22}+18*q_{13,21,33}+18*q_{21,22,31}-18*q_{21,31,33}+18*q_{22,23,33}-18*q_{22,32,33},-3*q_{11,12,23}+3*q_{11,12,32}-3*q_{11,13,33}+3*q_
{11,21,23}-3*q_{11,21,32}-3*q_{11,31,33}+3*q_{12,22,23}-3*q_{12,22,32}+3*q_{12,23,33}-3*q_{12,32,33}+3*q_{13,22,33}-3*q_{21,22,23}+3*q_{21,22,32}-3*q_{21,23,33}+3
*q_{21,32,33}-3*q_{22,31,33},18*q_{11,12,33}-18*q_{11,21,33}-18*q_{12,22,33}+18*q_{21,22,33}},


{-4*q_{11,12,22}+4*q_{11,13,23}+4*q_{11,13,32}-4*q_{11,21,22}-4*q_{11,23,31}+4*q_{11,31,32}+4*q_{12,22,33}+4*q_{13,22,23}+4*q_{13,22,32}-4*q_{13,23,33}-4*q_{13,32
```

. . . after making this substitution

```
S =
QQ[x_(1,1),x_(1,2),x_(2,1),x_(2,2),x_(3,1),x_(3,2),x_(4,1),x_(4,2),x_(5,1),x_(5,2),x_(6,1),x_(6,2),y_(1,1),y_(1,2),y_(2,1),y_(2,2),y_(3,1),y_(3,2),y_(4,1),y_(4,2)
,y_(5,1),y_(5,2),y_(6,1),y_(6,2)]


q_{11,12,13} =
x_(1,2)*x_(2,1)*x_(4,2)*x_(5,1)*y_(1,2)*y_(2,2)*y_(3,2)-x_(1,1)*x_(2,2)*x_(4,2)*x_(5,1)*y_(1,2)*y_(2,2)*y_(3,2)-
x_(1,2)*x_(3,1)*x_(4,2)*x_(5,1)*y_(1,2)*y_(2,2)*y_(3,2)+x_(2,2)*x_(3,1)*x_(4,2)*x_(5,1)*y_(1,2)*y_(2,2)*y_(3,2)+x_(1,1)*x_(3,2)*x_(4,2)*x_(5,1)*y_(1,2)*y_(2,2)*y_
(3,2)-x_(2,1)*x_(3,2)*x_(4,2)*x_(5,1)*y_(1,2)*y_(2,2)*y_(3,2)-
x_(1,2)*x_(2,1)*x_(4,1)*x_(5,2)*y_(1,2)*y_(2,2)*y_(3,2)+x_(1,1)*x_(2,2)*x_(4,1)*x_(5,2)*y_(1,2)*y_(2,2)*y_(3,2)+x_(1,2)*x_(3,1)*x_(4,1)*x_(5,2)*y_(1,2)*y_(2,2)*y_
(3,2)-x_(2,2)*x_(3,1)*x_(4,1)*x_(5,2)*y_(1,2)*y_(2,2)*y_(3,2)-
x_(1,1)*x_(3,2)*x_(4,1)*x_(5,2)*y_(1,2)*y_(2,2)*y_(3,2)+x_(2,1)*x_(3,2)*x_(4,1)*x_(5,2)*y_(1,2)*y_(2,2)*y_(3,2)-
x_(1,2)*x_(2,1)*x_(4,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_(3,2)+x_(1,1)*x_(2,2)*x_(4,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_(3,2)+x_(1,2)*x_(3,1)*x_(4,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_
(3,2)-x_(2,2)*x_(3,1)*x_(4,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_(3,2)-
x_(1,1)*x_(3,2)*x_(4,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_(3,2)+x_(2,1)*x_(3,2)*x_(4,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_(3,2)+x_(1,2)*x_(2,1)*x_(5,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_
(3,2)-x_(1,1)*x_(2,2)*x_(5,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_(3,2)-
x_(1,2)*x_(3,1)*x_(5,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_(3,2)+x_(2,2)*x_(3,1)*x_(5,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_(3,2)+x_(1,1)*x_(3,2)*x_(5,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_
(3,2)-x_(2,1)*x_(3,2)*x_(5,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_(3,2)+x_(1,2)*x_(2,1)*x_(4,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_(3,2)-
x_(1,1)*x_(2,2)*x_(4,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_(3,2)-
x_(1,2)*x_(3,1)*x_(4,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_(3,2)+x_(2,2)*x_(3,1)*x_(4,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_(3,2)+x_(1,1)*x_(3,2)*x_(4,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_
(3,2)-x_(2,1)*x_(3,2)*x_(4,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_(3,2)-
x_(1,2)*x_(2,1)*x_(5,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_(3,2)+x_(1,1)*x_(2,2)*x_(5,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_(3,2)+x_(1,2)*x_(3,1)*x_(5,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_
(3,2)-x_(2,2)*x_(3,1)*x_(5,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_(3,2)-
x_(1,1)*x_(3,2)*x_(5,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_(3,2)+x_(2,1)*x_(3,2)*x_(5,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_(3,2)-
x_(1,2)*x_(2,1)*x_(3,2)*x_(5,1)*y_(1,2)*y_(2,2)*y_(4,2)+x_(1,1)*x_(2,2)*x_(3,2)*x_(5,1)*y_(1,2)*y_(2,2)*y_(4,2)+x_(1,2)*x_(3,2)*x_(4,1)*x_(5,1)*y_(1,2)*y_(2,2)*y_
(4,2)-x_(2,2)*x_(3,2)*x_(4,1)*x_(5,1)*y_(1,2)*y_(2,2)*y_(4,2)-
x_(1,1)*x_(3,2)*x_(4,2)*x_(5,1)*y_(1,2)*y_(2,2)*y_(4,2)+x_(2,1)*x_(3,2)*x_(4,2)*x_(5,1)*y_(1,2)*y_(2,2)*y_(4,2)+x_(1,2)*x_(2,1)*x_(3,1)*x_(5,2)*y_(1,2)*y_(2,2)*y_
(4,2)-x_(1,1)*x_(2,2)*x_(3,1)*x_(5,2)*y_(1,2)*y_(2,2)*y_(4,2)-
x_(1,2)*x_(3,1)*x_(4,1)*x_(5,2)*y_(1,2)*y_(2,2)*y_(4,2)+x_(2,2)*x_(3,1)*x_(4,1)*x_(5,2)*y_(1,2)*y_(2,2)*y_(4,2)+x_(1,1)*x_(3,1)*x_(4,2)*x_(5,2)*y_(1,2)*y_(2,2)*y_
(4,2)-x_(2,1)*x_(3,1)*x_(4,2)*x_(5,2)*y_(1,2)*y_(2,2)*y_(4,2)+x_(1,2)*x_(2,1)*x_(3,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_(4,2)-
x_(1,1)*x_(2,2)*x_(3,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_(4,2)-
x_(1,2)*x_(3,2)*x_(4,1)*x_(6,1)*y_(1,2)*y_(2,2)*y_(4,2)+x_(2,2)*x_(3,2)*x_(4,1)*x_(6,1)*y_(1,2)*y_(2,2)*y_(4,2)+x_(1,1)*x_(3,2)*x_(4,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_
(4,2)-x_(2,1)*x_(3,2)*x_(4,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_(4,2)-
x_(1,2)*x_(2,1)*x_(5,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_(4,2)+x_(1,1)*x_(2,2)*x_(5,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_(4,2)+x_(1,2)*x_(4,1)*x_(5,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_
(4,2)-x_(2,2)*x_(4,1)*x_(5,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_(4,2)-
x_(1,1)*x_(4,2)*x_(5,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_(4,2)+x_(2,1)*x_(4,2)*x_(5,2)*x_(6,1)*y_(1,2)*y_(2,2)*y_(4,2)-
x_(1,2)*x_(2,1)*x_(3,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_(4,2)+x_(1,1)*x_(2,2)*x_(3,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_(4,2)+x_(1,2)*x_(3,1)*x_(4,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_
(4,2)-x_(2,2)*x_(3,1)*x_(4,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_(4,2)-
x_(1,1)*x_(3,1)*x_(4,2)*x_(6,2)*y_(1,2)*y_(2,2)*y_(4,2)+x_(2,1)*x_(3,1)*x_(4,2)*x_(6,2)*y_(1,2)*y_(2,2)*y_(4,2)+x_(1,2)*x_(2,1)*x_(5,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_
(4,2)-x_(1,1)*x_(2,2)*x_(5,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_(4,2)-
x_(1,2)*x_(4,1)*x_(5,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_(4,2)+x_(2,2)*x_(4,1)*x_(5,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_(4,2)+x_(1,1)*x_(4,2)*x_(5,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_
(4,2)-x_(2,1)*x_(4,2)*x_(5,1)*x_(6,2)*y_(1,2)*y_(2,2)*y_(4,2)+x_(1,2)*x_(2,2)*x_(3,1)*x_(5,1)*y_(1,2)*y_(3,2)*y_(4,2)-
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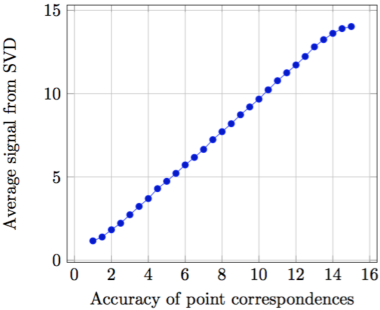
Answer

Calculate the Singular Value Decomposition of $M(x, y)$ when a noisy six-tuple of image point correspondences is plugged in. Look for a big **spectral gap** between smallest singular values.



Essential matrices

Definition

Given two calibrated cameras A, B. Consider the linear map:

$$\mathbb{P}^2 \dashrightarrow \mathsf{Gr}(\mathbb{P}^1, \mathbb{P}^3) \dashrightarrow (\mathbb{P}^2)^\vee$$

$$\widetilde{x} \mapsto A^{-1}(\widetilde{x}) \mapsto B(A^{-1}(\widetilde{x})).$$

Here the first map takes preimage. Let $E_{A,B}$ be the 3×3 real matrix representing the composite with respect to standard bases. Then $E_{A,B}$ is called an **essential matrix**. *It represents the relative pose of A and B.*

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Facts

- *Let $\widetilde{x}, \widetilde{y} \in \mathbb{P}^2$. Then there exists $\widetilde{X} \in \mathbb{P}^3$ such that $A\widetilde{X} \equiv \widetilde{x}$ and $B\widetilde{X} \equiv \widetilde{y}$ if and only if $\widetilde{y}^T E_{A,B}\, \widetilde{x} = 0$.*
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Definition/Proposition

Let $\mathcal{E} := \{E \in \mathbb{R}^{3\times 3} : \sigma_1(E) = \sigma_2(E),\, \sigma_3 = 0\}$. Then the real radical ideal is $\mathcal{E} = \{E \in \mathbb{R}^{3\times 3} : \det(E) = 0, 2(EE^T)E - \mathrm{tr}(EE^T)E = 0\}$. Let $\mathcal{E}_\mathbb{C} \subset \mathbb{P}^8_\mathbb{C}$ be the set of complex solutions, called the **essential variety**. *Dim 5, degree 10.*

Proof of main result

1. Reformulate: want the **Chow form** of $\mathcal{E}_{\mathbb{C}}$. Degree 10 equation in Plücker coordinates for the divisor $\{L \in \mathrm{Gr}(\mathbb{P}^2, \mathbb{P}^8) \mid L \cap \mathcal{E}_{\mathbb{C}} \neq \emptyset\} \subset \mathrm{Gr}(\mathbb{P}^2, \mathbb{P}^8)$.

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2. New geometric description: we prove that the singular locus of $\mathcal{E}_{\mathbb{C}}$ is the surface $\mathrm{Sing}(\mathcal{E}_{\mathbb{C}}) = \{ab^T \in \mathbb{P}^8 \mid a^T a = b^T b = 0\}$. Also, the line secant variety of the singular locus equals $\mathcal{E}_{\mathbb{C}}$, i.e. $\sigma_2(\mathrm{Sing}(\mathcal{E}_{\mathbb{C}})) = \mathcal{E}_{\mathbb{C}}$.

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3. Easier equations: we deduce that $\mathcal{E}_{\mathbb{C}}$ is a hyperplane section of the projective variety $PX^s_{4,2}$ of rank ≤ 2 symmetric 4×4 matrices.
4. Eisenbud-Schreyer theory: we construct a rank 2 **Ulrich sheaf** on $PX^s_{4,2}$. This means the corresponding graded module is Cohen-Macaulay with a linear minimal free resolution. Here it is $GL(4)$-equivariant and self-dual:

$$0 \leftarrow M \leftarrow \ydiagram{3} \leftarrow \ydiagram{3,2} \leftarrow \ydiagram{3,3,1} \leftarrow \ydiagram{3,3,3} \leftarrow 0.$$

Through the Bernstein-Gel'fand-Gel'fand correspondence, the product of the differential matrices over the **exterior algebra** gives a Pfaffian formula for the Chow form of $PX^s_{4,2}$. For $\mathcal{E}_{\mathbb{C}}$, we restrict to the hyperplane. $\square$

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Thank you!