How To Multiply Two Linear Spaces

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Abstract

We describe properties of Hadamard products of algebraic varieties, the multiplicative version of joins. We study linear spaces as an interesting case. We show any Hadamard power of a line is a linear space, and we construct star configurations from products of collinear points. Tropical geometry is used to find the degree of Hadamard products of other linear spaces.

1. Hadamard products in general

Work over $\mathbb{C}$ throughout.

Definition 1. Given points $p = [a_0 : a_1 : \ldots : a_n]$ and $q = [b_0 : b_1 : \ldots : b_n]$ in $\mathbb{P}^n$, their ***Hadamard product*** is $p \star q := [a_0b_0 : a_1b_1 : \ldots : a_nb_n]$, when defined. Given varieties $X, Y \subset \mathbb{P}^n$, their ***Hadamard product*** is $X \star Y := \overline{\{p \star q : p \in X, q \in Y, p \star q \text{ is defined}\}}$. It is a linear projection of the Segre product of X and Y. Given a positive integer r, the ***$r-$th Hadamard power*** of X is $X^{\star r} := X^{\star(r-1)} \star X$.

Three **motivations** to study Hadamard products of varieties:

- Multiplicative analog of joins and secants.
- Correspond to Minkowski sums in tropical geometry.
- Algebraic statistics: varieties of joint probability distributions arising from graphical models are Hadamard products.

Lemma 2 (Terracini's lemma). *Let $X, Y \subset \mathbb{P}^n$ be varieties, and $p \in X$, $q \in Y$ general points. Then:*

$$T_{p\star q}(X \star Y) = \langle p \star T_q(Y), q \star T_p(X)\rangle.$$

Lemma 3 (Tropical connection). *Let $X, Y \subset \mathbb{P}^n$ be irreducible varieties. The tropicalization of the Hadamard product is the Minkowski sum of the tropicalizations:*

$$\text{trop}(X \star Y) \;=\; \text{trop}(X) + \text{trop}(Y)$$

as sets. If also $X \times Y \dashrightarrow X \star Y$ is generically δ to 1, then:

$$\text{trop}(X \star Y) = \frac{1}{\delta}\big(\text{trop}(X) + \text{trop}(Y)\big)$$

as weighted balanced fans.

Example 4 (Toric varieties). *The set of varieties parameterized by monomials is closed under Hadamard product.*

2. Powers of a line

Let $L \subset \mathbb{P}^n$ be a line on which every point has at most one zero coordinate. Let $1 \leq r \leq n$.

Theorem 5. *The power $L^{\star r} \subset \mathbb{P}^n$ is a linear space of dimension r.*

Sketch. (1) $L^{\star r}$ has dimension r, by Terracini. (2) The linear span of $L^{\star r}$ has dimension r, by induction. □

Proposition 6. *Plücker coordinates for $L^{\star r}$ and L relate by:*

$$[i_0, i_1, \ldots, i_r]_{L^{\star r}} = \prod_{0 \leq j < k \leq r} [i_j, i_k]_L.$$

Corollary 7. *The hyperplane $L^{\star(n-1)} \subset \mathbb{P}^n$ is defined by:*

$$\sum_{i=0}^{n} \Big((-1)^{n+i} \prod_{\substack{0 \leq j < k \leq n \\ j,k \neq i}} [j,k]_L \Big)\, x_i \;=\; 0.$$

3. Star configurations

Let $N \geq n$.

Definition 8. A set of $\binom{N}{n}$ points $\mathbb{X} \subset \mathbb{P}^n$ is a ***star configuration*** if there exist hyperplanes $H_1, \ldots, H_N \subset \mathbb{P}^n$ such that:

- H_i are in linear general position
- $\mathbb{X} = \bigcup_{1 \leq i_1 < \ldots < i_n \leq N} H_{i_1} \cap \ldots \cap H_{i_n}$.

To construct a star configuration, one could take N random hyperplanes and solve $\binom{N}{n}$ many $n \times (n+1)$ linear systems. We found a **much** cheaper construction.

Definition 9. Let $Z \subset \mathbb{P}^n$ be a finite set of points. The ***$r-$th square-free Hadamard power*** of Z is:

$$Z^{\underline{\star} r} := \{p_1 \star \ldots \star p_r : p_i \in Z \text{ and } p_i \neq p_j \text{ for } i \neq j\}.$$

Theorem 10. *Let $L \subset \mathbb{P}^n$ be a line on which every point has at most one zero coordinate, and let $Z \subset L$ be a set of N points none of which has a zero coordinate. Then $Z^{\underline{\star} n}$ is a star configuration.*

Sketch. For hyperplanes, set $H_i = p_i \star L^{\star(n-1)}$. Then $H_i \cap H_j = p_i \star p_j \star L^{\star(n-2)}$ etc. □

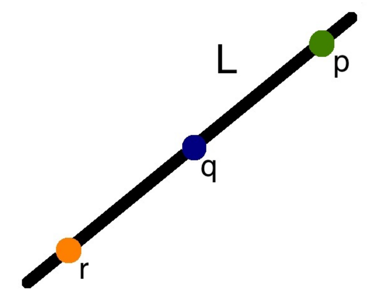
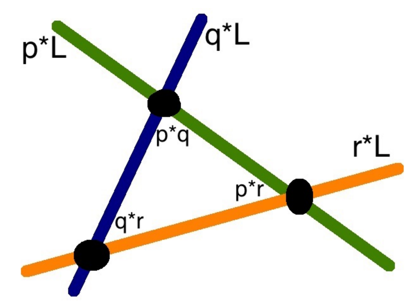


Figure 1: *Taking Hadamard square of a line in $\mathbb{P}^2$ with three marked points gives a star configuration in $\mathbb{P}^2$ of three points.*



4. Products of other linear spaces

In general, for linear spaces $L, M \subset \mathbb{P}^n$, the product $L \star M$ contains many linear spaces $\{p \star M : p \in L\} \cup \{q \star L : q \in M\}$, but is not a linear space. We quantify this by dimension of linear span and degree.

Let $L_1, \ldots, L_k \subset \mathbb{P}^n$ be generic linear spaces of dimensions $m_1, \ldots, m_k$. Let $r_1, \ldots, r_k$ be positive integers. Assume $n \gg 0$.

Proposition 11 (Linear span). *The linear span $\langle L_1^{\star r_1} \star \ldots \star L_k^{\star r_k}\rangle$ has dimension $\binom{m_1+r_1}{r_1} \ldots \binom{m_k+r_k}{r_k} - 1$.*

Corollary 12 (Identifiability). *The product $L_1^{\star r_1} \star \ldots \star L_k^{\star r_k}$ is **identifiable**, meaning $L_1^{\times r_1} \times \ldots \times L_k^{\times r_k} \dashrightarrow L_1^{\star r_1} \star \ldots \star L_k^{\star r_k}$ is generically $(r_1!) \ldots (r_k!)$ to 1.*

Theorem 13 (Degree formula). *Set $m = r_1m_1 + r_2m_2 + \ldots + r_km_k$ and $d = \binom{m}{m_1, m_1, \ldots, m_k}$ (downstairs there are r_i copies of m_i). Then $L_1^{\star r_1} \star \ldots \star L_k^{\star r_k}$ has dimension m and degree $\frac{d}{(r_1!) \ldots (r_k!)}$.*

Sketch. (1) Generically trop(L_i) equals the ***standard tropical linear space*** Λ_{m_i} of dimension m_i:

$$\text{trop}(L_i) \;=\; \Lambda_{m_i} \;:=\; \bigcup_{0 \leq j_1 < \ldots < j_{m_i} \leq n} \text{pos}(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_{m_i}}).$$

Here $\mathbf{e}_0, \ldots, \mathbf{e}_n$ are images in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ of the standard basis vectors, pos is positive span, and all facet multiplicites are 1.

(2) So, trop$(L_1)^{+r_1} + \ldots +$ trop$(L_k)^{+r_k} = d\Lambda_m$ (same support as Λ_m, all facet multiplicities d).

(3) By Lemma 3 and Corollary 12:

$$\text{trop}(L_1^{\star r_1} \star \ldots \star L_k^{\star r_k}) = \frac{d}{(r_1!) \ldots (r_k!)}\, \Lambda_m.$$

(4) For an irreducible variety, tropicalization preserves dimension and degree is recovered by stably intersecting with the standard tropical linear space of complementary dimension and then measuring the multiplicity of the origin. □

Example 14. *Let L and M be generic distinct lines in $\mathbb{P}^3$, with Plücker coordinates $[ij] := [i,j]_L$ and $\{ij\} := [i,j]_M$ respectively. Then $L \star M$ is the quadric surface in $\mathbb{P}^3$ doubly ruled by $\{p \star M : p \in L\} \cup \{q \star L : q \in M\}$ cut out by:*

$$\begin{aligned}
&[12][13][23]\{12\}\{13\}\{23\}\; x_0^2 \;+\; [02][03][23]\{02\}\{03\}\{23\}\; x_1^2 \\
&+\; [01][03][13]\{01\}\{03\}\{13\}\; x_2^2 \;+\; [01][02][12]\{01\}\{02\}\{12\}\; x_3^2 \\
&-\; [23]\{23\}\big([02][13]\{03\}\{12\} \;+\; [03][12]\{02\}\{13\}\big)\; x_0x_1 \\
&+\; [13]\{13\}\big([01][23]\{03\}\{12\} \;+\; [03][12]\{01\}\{23\}\big)\; x_0x_2 \\
&-\; [12]\{12\}\big([01][23]\{02\}\{13\} \;+\; [02][13]\{01\}\{23\}\big)\; x_0x_3 \\
&-\; [03]\{03\}\big([01][23]\{02\}\{13\} \;+\; [02][13]\{01\}\{23\}\big)\; x_1x_2 \\
&+\; [02]\{02\}\big([01][23]\{03\}\{12\} \;+\; [03][12]\{01\}\{23\}\big)\; x_1x_3 \\
&-\; [01]\{01\}\big([02][13]\{03\}\{12\} \;+\; [03][12]\{02\}\{13\}\big)\; x_2x_3.
\end{aligned}$$

Sketch. The incidence variety

$$\mathbb{X} = \overline{\{(p, L, M) \in \mathbb{P}^3 \times \text{Gr}(2,4) \times \text{Gr}(2,4) :\; p \in L \star M\}}$$

has three group actions:

- $\mathfrak{S}_2$ acts by switching L and M
- $\mathfrak{S}_4$ acts by permuting the homogeneous coordinates of $\mathbb{P}^3$
- $(\mathbb{C}^*)^4/\mathbb{C}^*$ acts by scaling the homogeneous coordinates of $\mathbb{P}^3$.

So, the defining equation of $\mathbb{X}$ is $(\mathfrak{S}_2 \times \mathfrak{S}_4)$-symmetric and $\mathbb{Z}^3$-multihomogeneous. Also, specializing $L = M$ should give the square of the linear equation in Corollary 7. □

Example 15. *Let P be a generic 2-plane in $\mathbb{P}^5$, with Plücker coordinates $[ijk] := [i,j,k]_P$. Then $P^{\star 2}$ is the cubic hypersurface in $\mathbb{P}^5$. In a defining equation, the coefficient of x_0^3 is $-(-1)^{0+0+0}$ times:*

$$[123][124][125][134][135][145][234][235][245][345].$$

The coefficient of $x_0^2x_1$ is $-(-1)^{0+0+1}$ times:

$$\big([023][045][124][125][134][135] + [024][035][123][125][134][145] + [025][034][123][124][135][145]\big)[234][235][245][345].$$

The coefficient of $x_0x_1x_2$ is $-(-1)^{0+1+2}$ times:

$$\begin{aligned}
&[013][024][134][234][125][035][235][045][145][345] + [013][124][034][234][025][135][235][045][145][345] + \\
&[023][014][134][234][125][035][135][045][245][345] + [023][124][034][134][015][135][235][045][245][345] + \\
&[123][014][034][234][025][035][135][145][245][345].
\end{aligned}$$

To get the other coefficients, act on the indices by $\mathfrak{S}_6$.

Sketch. Specialize $P = L^{\star 2}$, where L is a line. □

5. Reference

- *Hadamard products of linear spaces*, `arXiv:1504.04301`