Rigid Multiview Varieties

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Multiview Variety

What is the space of pictures of one world point?

Given n generic 3×4 camera matrices $A_1, \ldots, A_n$.

Multiview map ▶

$$\phi_A : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^2$$
$$X \mapsto (A_1X, A_2X, \ldots A_nX)$$

Multiview variety ▶ $V_A := \overline{\text{im}(\phi_A)} \subseteq (\mathbb{P}^2)^n$.
Irreducible three-fold.

Multiview ideal ▶ $I_A := I(V_A) \subseteq \mathbb{R}[u_{i0}, u_{i1}, u_{i2} : i = 1, \ldots, n]$.
$\mathbb{Z}^n$-multihomogeneous prime ideal in a polynomial ring with $3n$ variables.
Here $(u_{i0} : u_{i1} : u_{i2})$ are homogeneous coordinates on the i^{th} $\mathbb{P}^2$.

Linear system ▶ For which u_j and u_k, does:

$$\begin{cases} A_jX = \lambda_j u_j \\ A_kX = \lambda_k u_k \end{cases}$$

have a nonzero solution in X, λ_j, λ_k ? Rewrite as:

$$B^{jk}\begin{bmatrix} X \\ -\lambda_j \\ -\lambda_k \end{bmatrix} = 0 \quad \text{where} \quad B^{jk} := \begin{bmatrix} A_j & u_j & 0 \\ A_k & 0 & u_k \end{bmatrix}_{6\times 6}$$

Bilinear equations ▶ For all $1 \le j < k \le n$, $\det(B^{jk}) \in I_A$.
It equals $u_j^T F_{jk} u_k$, where F_{jk} is the fundamental matrix.

Theorem 1 (Heyden-Åström 1997)

For $n \geq 4$, the $\binom{n}{2}$ bilinear forms cut out V_A **set-theoretically**:

$$V_A = V\left(u_j^T F_{jk} u_k : \forall j, k\right).$$

Trilinear equations ▶ Maximal minors of $B^{jk\ell} := \begin{bmatrix} A_1 & u_1 & 0 & 0 \\ A_2 & 0 & u_2 & 0 \\ A_3 & 0 & 0 & u_3 \end{bmatrix}_{9\times 7}$

Quadrilinear equations ▶ Maximal minors of $B^{jk\ell m} := \begin{bmatrix} A_1 & u_1 & 0 & 0 & 0 \\ A_2 & 0 & u_2 & 0 & 0 \\ A_3 & 0 & 0 & u_3 & 0 \\ A_4 & 0 & 0 & 0 & u_4 \end{bmatrix}_{12\times 8}$

Theorem 2 (Aholt-Sturmfels-Thomas 2013)

- $\binom{n}{2}$ binilinear and $\binom{n}{3}$ trilinear forms above **minimally generate** I_A.
- The bilinear, trilinear, quadrilinear forms are a **universal Gröbner basis**.

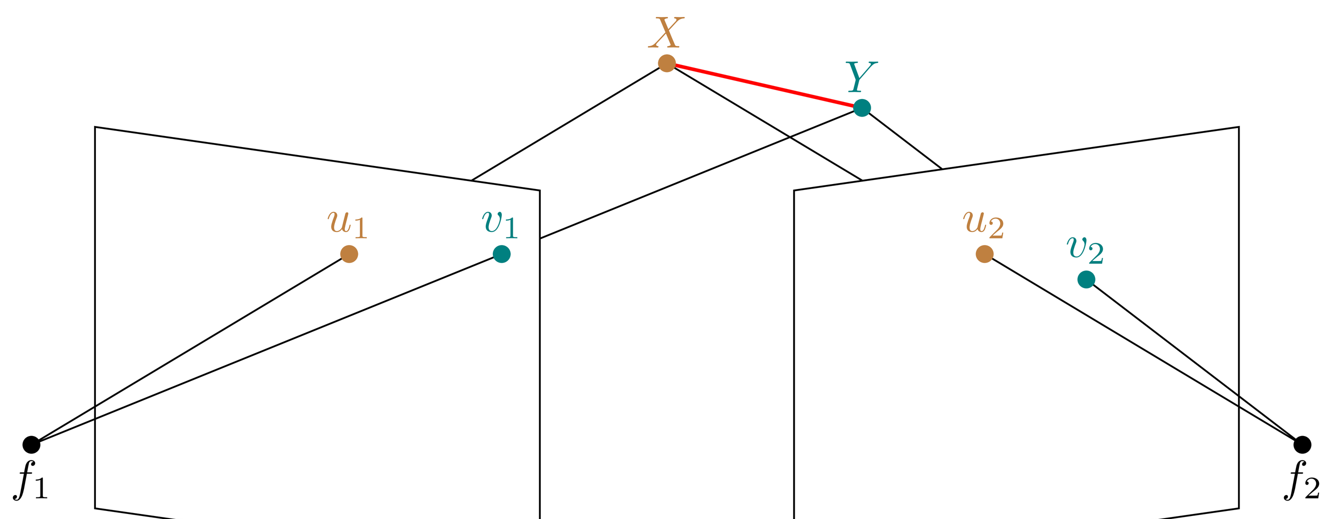
Rigid Multiview Variety

What is the space of pictures of two distance-constrained world points?

Rigid multiview map ▶

$$\psi_A : V(Q) \hookrightarrow \mathbb{P}^3 \times \mathbb{P}^3 \dashrightarrow (\mathbb{P}^2)^n \times (\mathbb{P}^2)^n,$$
$$(X, Y) \mapsto \big((A_1X, \ldots A_nX), (A_1Y, \ldots A_nY)\big).$$

where $Q(X, Y) = (X_0Y_3 - Y_0X_3)^2 + (X_1Y_3 - Y_1X_3)^2 + (X_2Y_3 - Y_2X_3)^2 - X_3^2Y_3^2$.



Rigid multiview variety ▶ $W_A := \overline{(\text{im}(\psi_A))} \subseteq \mathbb{P}^{2n}$.
Irreducible 5-fold inside $V_A \times V_A$.

Rigid multiview ideal ▶ $J_A := I(V_A) \subseteq \mathbb{R}[u_{i0}, u_{i1}, u_{i2}, v_{i0}, v_{i1}, v_{i2} : i = 1, \ldots, n]$.
$\mathbb{Z}^{2n}$-multihomogeneous prime ideal in a polynomial ring with $6n$ variables.

Triangulate with Cramer's Rule ▶ For $1 \le j < k \le n$ and $1 \le i \le 6$, let:

- $B_i^{jk}(u)$ be the 5×6 matrix that is $B^{jk}(u)$ with its i^{th} row removed
- $\widetilde{\bigwedge}_5 B_i^{jk}(u)$ be the height 6 column of signed maximal minors of $B_i^{jk}(u)$
- $C_i^{jk}(v)$ and $\widetilde{\bigwedge}_5 C_i^{jk}(v)$ be the analogs with v.

Write $Q(X, Y) = T(X, X, Y, Y)$, where $T(\bullet, \bullet, \bullet, \bullet)$ is a quadrilinear form.

Theorem 3 (J.-K.-S.-W. 2015)

The octics coming from two pairs of cameras:

$$T\left(\widetilde{\bigwedge}_5 B_{i_1}^{j_1k_1}, \widetilde{\bigwedge}_5 B_{i_2}^{j_1k_1}, \widetilde{\bigwedge}_5 C_{i_3}^{j_2k_2}, \widetilde{\bigwedge}_5 C_{i_4}^{j_2k_2}\right)$$

cut out W_A as a subvariety of $V_A \times V_A$ **set-theoretically**. For this, 16 suffice.

Ideals ▶ Above octics together with $I_A(u) + I_A(v)$ do not generate J_A.

Conjecture 4 (J.-K.-S.-W. 2015)

J_A is **minimally generated** by $\frac{4}{9}n^6 - \frac{2}{3}n^5 + \frac{1}{36}n^4 + \frac{1}{2}n^3 + \frac{1}{36}n^2 - \frac{1}{3}n$ polynomials, coming from two triples of cameras, and their number per class of degrees is:

$$(110..000..) : 1 \cdot 2\binom{n}{2} \qquad (220..111..) : 3 \cdot 2\binom{n}{2}\binom{n}{3}$$
$$(220..220..) : 9 \cdot \binom{n}{2}^2 \qquad (211..211..) : 1 \cdot n^2\binom{n-1}{2}^2$$
$$(111..000..) : 1 \cdot 2\binom{n}{3} \qquad (211..111..) : 1 \cdot 2n\binom{n-1}{2}\binom{n}{3}$$
$$(220..211..) : 3 \cdot 2n\binom{n}{2}\binom{n-1}{2} \qquad (111..111..) : 1 \cdot \binom{n}{3}^2$$

Computational proof ▶ Up to $n = 5$, when there are 4940 minimal generators.

Other Constraints, More Points, and No Labels

More points, one polynomial constraint ▶ Take n pictures of m world points constrained by a single irreducible multihomogeneous polynomial equation $Q(X^{(1)}, \ldots, X^{(m)}) = 0$. Then **Theorem 3 holds verbatim**: to cut out the image set-theoretically, equations from pairs of cameras suffice. For example if $m = 4$ and $Q = \det(X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)})$, the constraint is four points in $\mathbb{P}^3$ are coplanar, and $16\binom{n}{2}^2$ polynomials cut out set-theoretically.

More rigid points ▶ Impose distances between all pairs of m world points:

$$Q_{ij}(X, Y) = (X_0Y_3 - Y_0X_3)^2 + (X_1Y_3 - Y_1X_3)^2 + (X_2Y_3 - Y_2X_3)^2 - d_{ij}^2 X_3^2 Y_3^2$$

When $m = 3$, the image of $V(Q_{ij} : \forall i, j)$ in $(\mathbb{P}^2)^{mn}$ is six-dimensional unless:

$$(d_{12} + d_{13} + d_{23})(d_{12} + d_{13} - d_{23})(d_{12} - d_{13} + d_{23})(-d_{12} + d_{13} + d_{23}) = 0.$$

It is cut out by $27\binom{n}{2}^2$ biquadratics set-theoretically, coming from pairs of points and pairs of cameras.

No labels on world points ▶ Suppose images of m world points are **unlabeled**.
Future work: Study the *unlabeled multiview variety*, i.e. the image of:

$$(\mathbb{P}^3)^m \dashrightarrow ((\mathbb{P}^2)^m)^n \to (\text{Sym}_m(\mathbb{P}^2))^n.$$

Here $\text{Sym}_m(\mathbb{P}^2)$ is the *Chow variety* of ternary forms that are products of m linear forms. Some known equations for it inside the space $\mathbb{P}^{\binom{m+2}{2}-1}$ of all ternary forms of degree m are Brill's equations.

References

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- A. Heyden and K. Åström, *Algebraic properties of multilinear constraints*, Mathematical Methods in the Applied Sciences **20** (1997) 1135–1162.
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