§4. Examples

(a.) On a Riemann surface M of genus $g=3$ the vector of second-order theta functions takes values in the space $\mathbb{C}^8$. The subspace $L_1 \subseteq \mathbb{C}^8$ spanned by the vectors $\vec{\theta}_2(0)$ and $\partial_{jk}\vec{\theta}_2(0)$ is seven-dimensional, so the quotient space $\mathbb{C}^8/L_1 \cong \mathbb{C}$ is in this case merely a one-dimensional space. The points $w(z_1+z_2-a_1-a_2)$ cover the entire Jacobi variety, so that the values $\vec{\theta}_2(w(z_1+z_2-a_1-a_2))$ must span the entire space $\mathbb{C}^8$; thus $L_2 \cong \mathbb{C}^8$ and $L_2/L_1 \cong \mathbb{C}$. If M is not hyperelliptic the Petri space $\mathcal{P}_2$ is empty so that necessarily $L_2 = L_2^*$; the space L_2/L_1 is thus the span of the vectors $\vec{\xi}^{k_1k_2}_j$. Of course by skew-symmetry there is really just the single vector $\vec{\xi}^{23}_1$ here; it must thus be a nonzero element of L_2/L_1, actually just a nonzero complex number. The canonical curve is in this case defined by a single quartic polynomial; it is a nonsingular plane quartic. On the other hand if M is hyperelliptic then $\vec{\xi}^{k_1k_2}_j = 0$ by Theorem 6 so that $L_1 = L_2^*$ and thus L_2/L_1 is spanned by the vectors $\vec{\xi}^{k_1k_2}_{j_1j_2}$. The Petri space $\mathcal{P}_2$ in this case is one-dimensional, and if $p(x) = \sum_{jk} p_{jk} x_j x_k$ is a basis then the canonical curve is the plane quadric defined by the function $p(x)$. Here $\vec{\xi}^{k_1k_2}_{j_1j_2} = \vec{\eta}\, p_{j_1j_2}\, p_{k_1k_2}$ for some nonzero vector $\vec{\eta} \in L_2/L_1$, again just a nonzero complex number.

If M is nonhyperelliptic then $\Theta^1 = r - W_2^1$ is empty, so the theta locus $\Theta = r - W_2$ is a nonsingular surface in the Jacobi variety. Since $P_1\vec{\theta}_2(w(z-a)) = 0$ the single second-order theta function $P_1\vec{\theta}_2(t)$ vanishes on the subvariety $W_1 - W_1$ of the Jacobi variety. Since the zero locus of any second-order theta function is homologous to $2W_2$ as is the subvariety $W_1 - W_1$ (results which are discussed for instance in my paper Riemann surfaces and their associated Wirtinger varieties, Bulletin A.M.S. 11 (1984), 287-316), it follows immediately that $P_1\vec{\theta}_2(t)$ is the defining function for the surface $W_1 - W_1$. The quartic function

$$f(x) = \sum_j \partial_{j_1 j_2 j_3 j_4} P_1\vec{\theta}_2(0)\, x_{j_1} x_{j_2} x_{j_3} x_{j_4}$$

is nontrivial, since from (2.5) it follows immediately that

$$\partial_j f(w'(z)) = \frac{3}{2} \sum_{kl} \bar{\xi}_j^{kl}\, w_k'(z)\, w_l''(z)$$

is nontrivial; since $f(x)$ vanishes on the canonical curve it must therefore be the defining equation for that curve.

If M is hyperelliptic then $\Theta^1 = r - W_2^1 = r - e$ for the hyperelliptic point $e \in J$, so the theta locus $\Theta = r - W_2 \subset J$ has a single singular point. Note that $2(r-e) = k - 2e = 0 \in J$, as discussed in section E4; since $W_2^2 = \emptyset$ it must be the case that $\partial_{jk}\theta(r-e) \neq 0$ for some indices j,k by Riemann's theorem, so $r-e$ is an even half period. (It is not the origin, as is demonstrated for example in Farkas and Kra, *Riemann Surfaces*, Springer-Verlag, 1980). Now $L_1 \perp M_1 = \vec{\theta}_2(r-e)$ by Theorem 1, and since L_1 is seven-dimensional it follows that $L_1 = M_1^{\perp}$; the span of the vectors $\vec{\theta}_2(w(z-a))$ for all $z, a \in \tilde{M}$ thus consists precisely of those vectors $\vec{x} \in \mathbb{C}^8$ such that ${}^t\vec{\theta}_2(r-e) \cdot \vec{x} = 0$. Dually the second-order theta function ${}^t\vec{\theta}_2(r-e) \cdot \vec{\theta}_2(t)$ vanishes along the subvariety $W_1 - W_1 \subset J$. This function actually vanishes to the second order along that subvariety, since ${}^t\vec{\theta}_2(r-e) \cdot \vec{\theta}_2(w(z-a)) = {}^t\vec{\theta}_2(r-e) \cdot P_1 \vec{\theta}_2(w(z-a))$ and the vanishing of all the vectors $\vec{\xi}_j^{k_1 k_2}$ implies by Theorem F5 that $\partial_j P_1 \vec{\theta}_2(w(z-a)) = 0$ for all indices j. Alternatively ${}^t\vec{\theta}_2(r-e) \cdot \vec{\theta}_2(t) = \theta(t+r-e)\,\theta(t-r+e) = \mathcal{F}(2r-2e, t)\,\theta(t-r+e)^2$ since $2r-2e \in \mathcal{L}$, so this function again vanishes to second order along the subvariety $W_1 - W_1 = W_2 - e$; but moreover since $\vartheta(t-r+e)$ is the defining function for the subvariety $\Theta + r - e = W_2 - e$ the function ${}^t\vec{\vartheta}_2(r-e)\cdot\vec{\vartheta}_2(t)$ vanishes precisely along the subvariety $W_1 - W_1$ and to precisely the second order there. Next in terms of the basis $p(x) = \sum_{jk} p_{jk} x_j x_k$ of the Petri space $\mathcal{P}_2$ and the associated double differential $\omega'(z,c) = = \sum_{jk} p_{jk}\, \omega_j(z)\, \omega_k(c)$ it follows from Theorem 2 that ${}^t\vec{\vartheta}_2(r-e)\cdot\vec{\eta}\; \omega'(z,c)^2 = \omega'_{r-e}(z,c)^2$; thus after renormalizing $p(x)$ by a suitable constant factor it can be assumed that $\omega'(z,c) = \omega'_{r-e}(z,c)$, hence that

$$p_{jk} = \partial_{jk}\vartheta(r-e).$$

In this normalization ${}^t\vec{\vartheta}_2(r-e)\cdot\vec{\eta} = 1$, and $\vec{\xi}^{\,k_1k_2}_{j_1j_2} = = \vec{\eta}\,\partial_{j_1j_2}\Theta(r-e)\;\partial_{k_1k_2}\vartheta(r-e)$. It then further follows from Corollary 2 to Theorem 5 that

$$\tfrac{1}{2}\,\partial_{k_1k_2k_3k_4} P_1\vec{\vartheta}_2(0) = p_{k_1k_2}p_{k_3k_4} + p_{k_1k_3}p_{k_2k_4} + p_{k_1k_4}p_{k_2k_3}$$

so that

$$\sum_k \partial_{k_1k_2k_3k_4} P_1\vec{\vartheta}_2(0)\, x_{k_1}x_{k_2}x_{k_3}x_{k_4} = 6\,p(x)^2.$$

This quartic polynomial is thus aside from a constant factor the square of the defining function of the canonical curve; it vanishes precisely on the canonical curve, but to the second order there.

In both cases the space of all linear differential relations

$$\sum_k c_{k_1\cdots k_4}\, \partial_{k_1\cdots k_4} \vec{\theta}_2(0) + \sum_k c_{k_1 k_2}\, \partial_{k_1 k_2} \vec{\theta}_2(0) + c\, \vec{\theta}_2(0) = 0$$

is a fourteen-dimensional complex vector space. Indeed the set of all symmetric expressions $c_{k_1\cdots k_4}$ form a fifteen-dimensional complex vector space, and the constraint that

$$\sum_k c_{k_1\cdots k_4}\, \partial_{k_1\cdots k_4} P_1 \vec{\theta}_2(0) = 0$$

is a single nontrivial linear equation since the range of P_1 is one-dimensional, so determines a fourteen-dimensional space of these constants $c_{k_1\cdots k_4}$; for each such set of constants there are then uniquely determined values $c_{k_1 k_2}$, c yielding a differential relation of the desired form, since the vectors $\partial_{k_1 k_2} \vec{\theta}_2(0)$ and $\vec{\theta}_2(0)$ are linearly independent modulo symmetries. In the nonhyperelliptic case these amount precisely to the systems of equations described by the choices

$$c_{k_1\cdots k_4} = w'_{k_1}(z)\, w'_{k_2}(z)\, w'_{k_3}(z)\, w'_{k_4}(z),$$

since $\dim \Gamma(\alpha^4) = 14$. In the hyperelliptic case there are only 9 linearly independent equations of this particular form, so there must be 5 independent additional equations. On the other hand there are 14 linearly independent equations of the form described in Theorem E5, which amount to conditions of the form

$$\sum_k x_{k_1}\, w'_{k_2}(z)\, w'_{k_3}(z)\, w'_{k_4}(z)\ \partial_{k_1\cdots k_4} P_1 \vec{\theta}_2(0) = 0$$

for arbitrary values x_1, x_2, x_3; for there are 7 linearly independent elements of $\Gamma(\mathcal{O}^3)$ among the products $w'_{k_2}(z)\, w'_{k_3}(z)\, w'_{k_4}(z)$, and while the three values x_1, x_2, x_3 can be chosen arbitrarily the same equation results when all are multiplied by the same constant. Thus all the differential relations are precisely the standard sorts described before.

(b) Riemann surfaces of genus $g = 4$ are particularly interesting, being general enough to illustrate the general situation quite nicely, but at the same time being special enough both to allow the calculations involved to be performed sufficiently easily and explicitly and to exhibit some entertaining peculiarities. In this case the vector of second-order theta functions takes values in the space $\mathbb{C}^{16}$. In the subspace $L_1 \subseteq \mathbb{C}^{16}$ spanned by the vectors $\vec{\vartheta}_2(0)$ and $\partial_{jk}\vec{\vartheta}_2(0)$ it is the case that $\dim L_1 = 11$, so that $\mathbb{C}^{16}/L_1 \cong \mathbb{C}^5$. Again the points $w(z_1 + z_2 - a_1 - a_2)$ cover the entire Jacobi variety, so that $L_2 = \mathbb{C}^{16}$ and $L_2/L_1 \cong \mathbb{C}^5$. If M is not hyperelliptic then $\dim \mathcal{P}_2 = 1$; if $p(x) = \sum_j p_{j_1 j_2} x_{j_1} x_{j_2} \in \mathcal{P}_2$ is a basis then $\vec{\xi}^{\,k_1 k_2}_{j_1 j_2} = \vec{\eta}\, p_{j_1 j_2} p_{k_1 k_2}$ for some vector $\vec{\eta} \in L_2/L_2^*$, so that $\dim L_2/L_2^* \leqq 1$. The space L_2^*/L_1 is the span of the vectors $\vec{\xi}^{\,k_1 k_2}_{j}$, or what is the same in view of the skew-symmetry, the span of the four vectors $\vec{\xi}^{\,23}_{4}, \vec{\xi}^{\,13}_{4}, \vec{\xi}^{\,12}_{4}, \vec{\xi}^{\,23}_{4}$, so that $\dim L_2^*/L_1 \leqq 4$. Since $5 = \dim L_2/L_1 = \dim L_2/L_2^* + \dim L_2^*/L_1$ these inequalities must actually be equalities; thus

(1) if M is nonhyperelliptic then

$\dim L_2^*/L_1 = 4$ with basis $\vec{\xi}^{\,23}_{4}, \vec{\xi}^{\,13}_{4}, \vec{\xi}^{\,12}_{4}, \vec{\xi}^{\,23}_{4}$,

$\dim L_2/L_2^* = 1$ with basis $\vec{\eta}$, On the other hand if M is hyperelliptic $\dim \mathcal{P}_2 = 3$; if $p_i(x) = \sum_j p^i_{j_1 j_2} x_{j_1} x_{j_2} \in \mathcal{P}_2$ is a basis then $\vec{\xi}^{\,k_1 k_2}_{j_1 j_2} = \sum_i \vec{\eta}^{\,i_1}_{i_2}\, p^{i_1}_{j_1 j_2}\, p^{i_2}_{k_1 k_2}$ for some vectors $\vec{\eta}^{\,i}_j = \vec{\eta}^{\,j}_i$ with $1 \leqq i, j \leqq 3$, so that $\dim L_2/L_2^* \leqq 6$. All the vectors $\vec{\xi}^{\,k_1 k_2}_{j}$ vanish by Theorem 6, so that $L_2^* = L_1$; consequently

(2) if M is hyperelliptic then

$L_2^* = L_1$

$\dim L_2/L_2^* = 5$ and L_2/L_2^* is spanned by the six vectors $\vec{\eta}^{\,i}_j$, $1 \leqq i \leqq j \leqq 3$.

For a more detailed analysis consider first the case in which M is hyperelliptic, with hyperelliptic point $e \in \mathbb{C}^4$. Since $k - 3e = 2 \in \mathbb{Z}$ by E(1.4) the factor of automorphy $\varkappa \rho_{-2e} J^{-4} = \rho_{k-2e} J^2 = \rho_{e+2} J^2 \sim \rho_e J^2$ has $\gamma(\varkappa \rho_{-2e} J^{-4}) = 2$; if $f_1, f_2 \in \Gamma(\varkappa \rho_{-2e} J^{-4})$ is a basis then the mapping $\pi : M \longrightarrow \mathbb{P}^1$ defined by $\pi(z) = (f_1(z), f_2(z)) \in \mathbb{P}^1$ for any point $z \in \tilde{M}$ is a representation of M as a two-sheeted branched covering of the Riemann sphere as usual. By Lemma 3(b) the double differentials

$$\omega'_{j_1 j_2}(z, a) = \frac{1}{1 + \delta^{j_1}_{j_2}} \left(\frac{h_a(z)}{e(a)} \right)^2 \left(f_{j_1}(z) f_{j_2}(a) + f_{j_1}(a) f_{j_2}(z) \right)$$

for $1 \leqq j_1 \leqq j_2 \leqq 2$ form a basis for the space of Petri double differentials, and if

$$\omega'_{j_1 j_2}(z, a) = \Sigma_k \, p^{j_1 j_2}_{k_1 k_2} \, \omega'_{k_1}(z) \, \omega'_{k_2}(z)$$

then

$$\vec{\xi}^{\,k_1 k_2}_{l_1 l_2} = \Sigma_{ij} \, \vec{\eta}^{\,i_1 i_2}_{j_1 j_2} \, p^{i_1 i_2}_{k_1 k_2} \, p^{j_1 j_2}_{l_1 l_2}$$

for some uniquely determined vectors $\vec{\eta}^{\,i_1 i_2}_{j_1 j_2} \in L_2 / L_2^*$. These vectors are fully symmetric in all four indices but are otherwise linearly independent by Theorem 8 and Corollary 1 to Theorem 10; there are thus five linearly independent such vectors, and they form a basis for $L_2/L_1 = L_2/L_2^*$.

In this connection the second-order Gauss mapping can be described quite simply and explicitly. Recall from the discussion in section E1 that $\Theta^1 = r - W^1_{g-3} = r - e - W_1$ is a one-dimensional subvariety of the Jacobi variety biholomorphic to the curve itself; the points $t \in \Theta^1$ can thus be described as $t = r - e - w(x)$ where $x \in \tilde{M}$. For any such point t

$$r - t - w(a) = e + w(x) - w(a) = w(x + Ea),$$
$$r + t - w(a) = k - e - w(x) - w(a) = 2e - w(x) - w(a) = w(Ex + Ea);$$

by Theorem C10 the double differential $w_t'(z,a)$ as an abelian differential in z is characterized as vanishing at the divisor $2a + x + Ea$, and since $k = (r - t - w(a)) + (r + t - w(a)) + w(2a) = w(2a + 2Ea + x + Ex)$ is a canonical divisor it must indeed be the divisor of that abelian differential. It is thus clear that

$$w_t'(z,a) = w'_{r-e-w(x)}(z,a)$$
$$= \left(\frac{h_a(z)}{e(a)}\right)^2 \det\begin{pmatrix} f_1(z) & f_1(x) \\ f_2(z) & f_2(x) \end{pmatrix} \det\begin{pmatrix} f_1(a) & f_1(x) \\ f_2(a) & f_2(x) \end{pmatrix}$$

up to a factor that is independent of z and a, where $f_1, f_2 \in \Gamma(\mathcal{O}\rho_{-2e}\,\mathfrak{Z}^{-4})$ as before. In terms of the chosen basis for the space of Petri double differentials the second-order Gauss mapping $\mathcal{G}: \Theta^1 \longrightarrow \mathbb{P}^2$ consequently has the form

$$\mathcal{G}(r-e+w(x)) = (f_2(x)^2,\ -f_1(x)f_2(x),\ f_1(x)^2) \in \mathbb{P}^2;$$

it thus amounts essentially to the composition of the standard two-sheeted branched covering $\pi: M \longrightarrow \mathbb{P}^1$ and the imbedding $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$ of $\mathbb{P}^1$ as the rational normal plane quadric.

Next by Corollary 3 to Theorem 10 the space of fourth-order partial differential relations $\sum_j c_{j_1\cdots j_4}\,\partial_{j_1\cdots j_4} P_1\,\vec{\theta}_2(0) = 0$ has dimension 30; this is the same as the dimension of the span of all the fourth-order partial differential relations of the general form given in Corollary 5 to Theorem E5, so the latter exhaust all possible differential relations in this case. One of these 30 differential equations is the classical KDV form of Corollary 1 to Theorem E5; even if these equations corresponding to the 10 different Weierstrass points are all linearly independent, they are far from being the complete list by themselves. Since $P_1 = P_2^*$ by Theorem 6, it follows readily from the formula of Corollary 2 to Theorem 5 and what has been observed here that

$$\sum_k \partial_{k_1 k_2 k_3 k_4} P_1 \vec{\theta}_2(0)\, x_{k_1} x_{k_2} x_{k_3} x_{k_4} = 6 \sum_k \vec{\xi}^{\,k_1 k_2}_{k_3 k_4}\, x_{k_1} x_{k_2} x_{k_3} x_{k_4}$$

$$= 6 \sum_{ij} \vec{\eta}^{\,i_1 i_2}_{j_1 j_2}\, p_{i_1 i_2}(x)\, p_{j_1 j_2}(x)\,;$$

since the vectors $\vec{\eta}^{\,i_1 i_2}_{j_1 j_2}$ are fully symmetric but are otherwise linearly independent, it follows that the ideal generated by the quartic polynomials that are the components of the above vector is the same as the ideal generated by the quadrics $p_{11}(x)^2$, $p_{22}(x)^2$, $p_{11}(x)\, p_{12}(x)$, $p_{11}(x)\, p_{22}(x)$, $p_{11}(x)\, p_{22}(x) + 2\, p_{12}(x)^2$ in the second-degree Petri polynomials $p_{ij}(x)$. The point set locus of these quadrics is the same as the point set locus of the Petri polynomials $p_{ij}(x)$, which is just the canonical curve

Consider next the case in which M is not hyperelliptic. It is necessarily trigonal though, so choose a trigonal point $e \in W_3^1 \subseteq \mathbb{C}^4$ and an associated trigonal correspondence $E: z \to E'z + E''z$ for which $w(z + E'z + E''z) = e$; as pointed out earlier here, either $k - e = e \in J$ and M has a unique trigonal structure, or $k - e = e' \neq e \in J$ and M has a second trigonal structure with trigonal point e' but no further trigonal structures. These two cases can be distinguished in the following alternative manner, quite relevant to the subsequent considerations. For any point $a \in \tilde{M}$ there exists a nontrivial abelian differential vanishing at $2a + E'a + E''a$, and it is easily seen to be unique up to a constant factor; its divisor is of the form $2a + E'a + E''a + x + y$ where $x + y$ is of course uniquely determined by a. If for some point a one of the points x, y coincides with one of the points $E'a, E''a$ then this divisor is clearly of the form $2a + 2E'a + 2E''a$, and consequently $k = 2e$; conversely if $k = 2e$ then for any point $a \in \tilde{M}$ there is a nontrivial abelian differential vanishing at $2e + 2E'a + 2E''a$.

Now there is a unique Petri double differential $\omega'(z,a)$ on M, up to a constant factor, and as an abelian differential in z it must have a double zero at the point a, as observed earlier. Actually this differential form must vanish at the divisor $2a + E'a + E''a$. Indeed note that the points $a, E'a, E''a$ on M have as their images on the canonical curve three collinear points $\alpha, \alpha', \alpha''$; thus $\alpha'' = c\alpha + c'\alpha'$ for some complex constants c, c'. The Petri form $p(x) = \sum_{jk} p_{jk} x_j x_k$ vanishes at these three points, so that

$$0 = p(\alpha'') = p(c\alpha + c'\alpha')$$

$$= c^2 p(\alpha) + 2cc' p(\alpha, \alpha') + c'^2 p(\alpha')$$

$$= 2cc' p(\alpha, \alpha')$$

where $p(x,y) = \sum_{jk} p_{jk} x_j y_k$ is the polarized form of $p(x)$. Thus

$$\omega'(E'a, a) = p(\alpha', \alpha) = p(\alpha, \alpha') = 0,$$

so that $\omega'(z,a)$ vanishes at $z = E'a$ and similarly of course at $z = E''a$. From these observations and those of the preceding paragraph it follows immediately that the divisor of $w'(z,a)$ as a function of z has the form

$$\mathcal{D}_z w'(z,a) = 2a + 2E'a + 2E''a \qquad \text{if } k = 2e \in J,$$

$$\mathcal{D}_z w'(z,a) = 2a + E'a + E''a + x + y \qquad \text{if } k \neq 2e \in J,$$

(where $x \cup y$, $E'a \cup E''a$ are disjoint)

In the second case there is a second trigonal structure, which must be entirely disjoint from the first since they correspond to distinct rulings (collinearities) on the canonical curve; hence the correspondence $a \to x+y$ is precisely the trigonal correspondence for this second structure. The second-order Gauss mapping is in either case rather trivial; Θ^1 consists of either one or two points, and if $t \in \Theta^1$ then $w'_t(z,a) = c \cdot w'(z,a)$ for some nonzero constant, the same for t and $-t$, where $w'(z,a)$ is the Petri double differential

Next for the Petri space of cubics vanishing on the canonical curve $\dim \mathcal{P}_3 = 5$; the four cubics $x_j\, p(x)$ for $1 \leqq j \leqq 4$ are clearly linearly independent elements of $\mathcal{P}_3$, and in addition to these there must be another linearly independent cubic $g(x) \in \mathcal{P}_3$. The two polynomials $p(x)$, $g(x)$ generate the ideal of the canonical curve, a curve of degree 6 in $\mathbb{P}^3$. The cubic polynomials when composed with the abelian differentials on M yield cubic differentials, relatively automorphic functions in $\Gamma(\varkappa^3)$. In the considerations at hand there is another interesting set of cubic differentials, the functions

$$\sigma_{ij}(z) = \det \begin{pmatrix} w_i'(z) & w_i''(z) \\ w_j'(z) & w_j''(z) \end{pmatrix} ;$$

these are skew symmetric in the indices $1 \leqq i, j \leqq 4$, so there are really six such functions to be considered.

$\Gamma(x^3)$

Lemma ~~2~~ 4. If M is a hyperelliptic Riemann surface of genus $g=4$ then $\cancel{\Gamma(M, \mathcal{O}(\kappa^3))}$ is the direct sum of the 5-dimensional linear subspace spanned by the expressions $\sigma_{ij}(z)$ and the 10-dimensional linear subspace spanned by products of triples of the Abelian differentials $w_j'(z)$.

$\Gamma(x^3)$

Proof. Since it is quite well known that the subspace of $\cancel{\Gamma(M, \mathcal{O}(\kappa^3))}$ spanned by products of triples of Abelian differentials is 10-dimensional, it is only necessary to show that the expressions $\sigma_{ij}(z)$ span a 5-dimensional subspace of $\cancel{\Gamma(M, \mathcal{O}(\kappa^3))}$ $\Gamma(x^3)$ and that no nontrivial element of this subspace can be written as a linear combination of products of triples of Abelian differentials. These assertions are easily seen to be independent of the particular basis chosen for the space of Abelian differentials, and for the proof it is convenient to use another basis. In particular, represent M as the Riemann surface of the function

$f(z) = \left[z \prod_{j=1}^{2g+1} (z-a_j)\right]^{1/2}$, where $0, a_1, \ldots, a_{2g+1}$ are distinct complex numbers, and take the basis

$$w_1'(z)\,dz = \frac{dz}{f(z)}, \quad w_2'(z)\,dz = \frac{z\,dz}{f(z)}, \quad w_3'(z)\,dz = \frac{z^2\,dz}{f(z)}, \quad w_4'(z)\,dz = \frac{z^3\,dz}{f(z)}.$$

In terms of the local coordinate $t = cz^{1/2}$ at the point of M lying over the origin $z=0$ the functions $w_j'(t)$ evidently have local power series expansions of the form

$$w_1'(t) = 1 + c_1 t + c_2 t^2 + \cdots, \quad w_2'(t) = t^2 + c_1 t^3 + c_2 t^4 + \cdots,$$
$$w_3'(t) = t^4 + c_1 t^5 + c_2 t^6 + \cdots, \quad w_4'(t) = t^6 + c_1 t^7 + c_2 t^8 + \cdots, \tag{3}$$

(3) ~~(8)~~

if the constant $c \neq 0$ is chosen appropriately, and ~~using this and the definition (2)~~ it follows easily that the functions $\sigma_{ij}(t)$ have local power series expansions of the form the polynomials p and g alone, since the latter define the canonical curve and hence determine the Riemann surface together with the canonical Abelian differentials. To describe these polynomials it is convenient to introduce the polarized forms of the polynomials p and g, namely the multilinear functions

$$(6)\ \cancel{(11)} \qquad p(x,y) = \sum_j p_{j_1 j_2}\, x_{j_1} y_{j_2}\ , \quad g(x,y,z) = \sum_j c_{j_1 j_2 j_3}\, x_{j_1} y_{j_2} z_{j_3}\ ,$$

and to write

$$(7)\ \cancel{(12)} \qquad p(x,y) = \sum_j a_j(x) y_j\ , \quad g(x,x,y) = \sum_j b_j(x) y_j\ ,$$

where $a_j(x)$ are linear polynomials and $b_j(x)$ are quadratic polynomials.

<u>Lemma ~~3~~ 5</u>. If M is a non-hyperelliptic Riemann surface of genus $g = 4$ then the 6 expressions $\sigma_{ij}(z)$ for $i < j$ are linearly independent elements of $(\Gamma(M,\mathcal{O}(x^3)))$; the polynomials $\tilde{\sigma}_{ij}(x)$ can be taken as

Γ(x,

$$\tilde{\sigma}_{12} = c(a_3b_4 - a_4b_3)\ , \quad \tilde{\sigma}_{13} = c(-a_2b_4 + a_4b_2)\ , \quad \tilde{\sigma}_{14} = c(a_2b_3 - a_3b_2)$$
$$\tilde{\sigma}_{23} = c(a_1b_4 - a_4b_1)\ , \quad \tilde{\sigma}_{24} = c(-a_1b_3 + a_3b_1)\ , \quad \tilde{\sigma}_{34} = c(a_1b_2 - a_2b_1)$$

for some nonzero complex constant c.

<u>Proof</u>. The proof of the first assertion of this lemma is much the same as the proof of the corresponding result in the preceding lemma, by examining the orders at a Weierstrass point of an appropriately chosen basis for the Abelian differentials. As shown by T. Kato in ~~[9]~~, any non-hyperelliptic surface of genus 4 has a Weierstrass point for which the first non-gap is 4, hence for which the gap sequence is either (1,2,3,5), (1,2,3,6), or (1,2,3,7). For the natural basis for the space of Abelian differentials

insert

his paper "On Weierstrass points whose first nongaps are three", J. reine angew. Math. 316 (1980), 99-109.

(8) ~~(13)~~
$$T_x(V) = \{y \in \mathbb{C}^4 : f(x,y) = g(x,x,y) = 0\} \subseteq \mathbb{C}^4 ,$$

with the notation as in (~~11~~ 6). On the other hand, the tangent space to V at a point $x = w'(z)$ contains the vector $w'(z)$, since V is a cone, and differentiating with respect to the local coordinate z shows that it also contains the vector $w''(z)$; these 2 vectors are always linearly independent for a non-hyperelliptic Riemann surface, so this tangent space can also be described as

$$T_{w'(z)}(V) = \{y \in \mathbb{C}^4 : y, w'(z), w''(z) \text{ are linearly dependent}\} .$$

The condition that the 3 vectors $y, w'(z), w''(z)$ be linearly dependent just amounts to the vanishing of all 3×3 subdeterminants of the 4×3 matrix with these 3 vectors as columns, so this tangent space can evidently be described equivalently as

(9) ~~(14)~~
$$T_{w'(z)}(V) = \left\{ y \in \mathbb{C}^4 : \begin{array}{l} y_1\sigma_{23}(z) + y_2\sigma_{31}(z) + y_3\sigma_{12}(z) = 0 \\ y_1\sigma_{24}(z) + y_2\sigma_{41}(z) + y_4\sigma_{12}(z) = 0 \\ y_1\sigma_{34}(z) + y_3\sigma_{41}(z) + y_4\sigma_{13}(z) = 0 \\ y_2\sigma_{34}(z) + y_3\sigma_{42}(z) + y_4\sigma_{23}(z) = 0 \end{array} \right\} .$$

Since the tangent space has dimension 2 the matrix describing this system of linear equations must have rank 2; it is a straightforward matter to verify that this rank condition is equivalently that the functions $\sigma_{ij}(z)$ have no common zeros on M and satisfy the identity

(10) ~~(15)~~
$$\sigma_{12}(z)\sigma_{34}(z) - \sigma_{13}(z)\sigma_{24}(z) + \sigma_{14}(z)\sigma_{23}(z) = 0 .$$

It is also quite easy to see that aside from a common nonzero factor there is a unique system of linear equations of the form appearing in (9) ~~(14)~~

From these observations it is easy to see that $N_2 = L_2$ in this case. Since $N_2 \subseteq L_2$ by Corollary 1 to Theorem 5, it is only necessary to demonstrate the reverse inequality. For this purpose note first that (2.5) can be rewritten

$$\sum_k \partial_{jk_1k_2k_3} P_1 \vec{\theta}_2(0)\, w'_{k_1}(z)\, w'_{k_2}(z)\, w'_{k_3}(z)$$

$$= 6 \sum_{k<l} \vec{\xi}^{\,kl}_j\, \sigma_{kl}(z),$$

Since the functions $\sigma_{kl}(z)$ are linearly independent, it follows from this that $\vec{\xi}^{\,kl}_j \in N_2$ for all indices j, k, l, hence that $L_2^* \subseteq N_2$. On the other hand Corollary 2 to Theorem 5 can in this case be rewritten as

$$\frac{1}{2}\partial_{k_1k_2k_3k_4} P_2^* \vec{\theta}_2(0) = \vec{\eta}\left(p_{k_1k_2}p_{k_3k_4} + p_{k_1k_3}p_{k_2k_4} + p_{k_1k_4}p_{k_2k_3}\right),$$

from which it is clear that $\vec{\eta} \in N_2$; that is enough to show that $L_2 \subseteq N_2$ as desired. The vectors $\partial_{k_1k_2k_3k_4} P_1 \vec{\theta}_2(0)$ thus span the full five-dimensional range of the mapping P_1, so among the 35 distinct vectors there are 30 linear partial differential relations. There are thus more than such relations than the 21 coming from the KP equations of Theorem D8, but for the somewhat degenerate reason that the space $\mathbb{C}^5$ has but a very small dimension.

Insert in p 24

A

Theorem 11. If M is a nonhyperelliptic Riemann surface of genus 4 and the matrix p_{jk} is of rank 4 then the zero locus of the quartic polynomials (2.17) is precisely the canonical curve. If M is a nonhyperelliptic Riemann surface of genus 4 and the matrix p_{jk} is of rank 3, with its null vector representing a point $P \in \mathbb{P}^3$, then the zero locus of the quartic polynomials (2.17) is the union of the canonical curve and the point P.

The hypotheses of Theorem 6 do not hold in this case, nor do the conclusions of that theorem always hold. The situation is different for the two different general types of nonhyperelliptic Riemann surfaces of genus $g=4$, as follows.

replace by A

~~Theorem 11. If M is a nonhyperelliptic Riemann surface of genus $g=4$ the ideal generated by the quartic polynomials (2.17) can be described alternatively as the ideal generated by the four quartics $p(x)^2$, $g(x)\,\Sigma_j\, p_{ij}\, x_j$ for $1 \leqq i \leqq 4$.~~

Proof. Since the quartic polynomials (2.17) vanish on the canonical curve, while $p(x)$, $g(x)$ generate the ideal of that curve, there must be unique vectors $\vec{\varphi}$ and $\vec{\psi}$ of quadratic and linear polynomials respectively such that

$$(11) \qquad \Sigma_k\, \partial_{k_1 \ldots k_4} P_1 \vec{\theta}_2(0)\, x_{k_1} x_{k_2} x_{k_3} x_{k_4} = \vec{\varphi}(x)\, p(x) + \vec{\psi}(x)\, g(x).$$

Applying the projection mapping P_2^* and using Corollary 2 to Theorem 5 shows that

$$6\, \vec{\eta}\, p(x)^2 = P_2^* \vec{\varphi}(x)\, p(x) + P_2^* \vec{\psi}(x)\, g(x),$$

hence that

$$(12) \qquad P_2^* \vec{\varphi}(x) = 6\, \vec{\eta}\, p(x), \qquad P_2^* \vec{\psi}(x) = 0.$$

The vector $\vec{\Psi}(x)$ thus lies in the subspace L_2^*, so that

$$(13)\quad \vec{\Psi}(x) = \Psi_1(x)\,\vec{\xi}^{\,23}_{4} + \Psi_2(x)\,\vec{\xi}^{\,13}_{4} + \Psi_3(x)\,\vec{\xi}^{\,12}_{4} + \Psi_4(x)\,\vec{\xi}^{\,12}_{3}$$

for some uniquely determined linear polynomials $\Psi_i(x)$. To find them explicitly note that they are determined uniquely by their values at points of the canonical curve, since that curve lies in no hyperplane. Further if x lies on the canonical curve then $P_2^*\vec{\varphi}(x) = 0$ as a consequence of (12), so there is a corresponding expansion

$$(14)\quad \vec{\varphi}(x) = \varphi_1(x)\,\vec{\xi}^{\,23}_{4} + \varphi_2(x)\,\vec{\xi}^{\,13}_{4} + \varphi_3(x)\,\vec{\xi}^{\,12}_{4} + \varphi_4(x)\,\vec{\xi}^{\,12}_{3}$$

for some functions φ_i on the canonical curve. Now for any $y \in \mathbb{C}^4$ apply the differential operator $\Sigma_j\, y_j\,\partial/\partial x_j$ to (11), and consider the result at a point $x = w'(z)$ on the canonical curve; in view of (2.5) the result can be written

$$2\,\vec{\varphi}(x)\,p(x,y) + 3\,\vec{\Psi}(x)\,g(x,x,y)$$

$$= 4\,\Sigma_{jk}\,\partial_{jk_1k_2k_3}\,P_1\,\vec{\sigma}_2(0)\,y_j\,w'_{k_1}(z)\,w'_{k_2}(z)\,w'_{k_3}(z)$$

$$= 24 \sum_{\substack{j_1<j_2 \\ k}} \vec{\xi}^{\,j_1j_2}_{k}\,\tilde{\sigma}_{j_1j_2}(x)$$

where $\tilde{\sigma}_{j_1j_2}(x) = \sigma_{j_1j_2}(z)$ as in (5). Substituting this into the expansions (13), (14) and comparing coefficients of the linearly independent vectors $\vec{\xi}^{\,j_1j_2}_{1}$ readily yields the system of equations

$$(15)\quad 2\varphi_1(x)\,p(x,y) + 3\psi_1(x)\,g(x,x,y) = 24\left[y_2\,\tilde\sigma_{34}(x) - y_3\,\tilde\sigma_{24}(x) + y_4\,\tilde\sigma_{23}(x)\right]$$
$$2\varphi_2(x)\,p(x,y) + 3\psi_2(x)\,g(x,x,y) = 24\left[y_1\,\tilde\sigma_{34}(x) - y_3\,\tilde\sigma_{14}(x) + y_4\,\tilde\sigma_{13}(x)\right]$$
$$2\varphi_3(x)\,p(x,y) + 3\psi_3(x)\,g(x,x,y) = 24\left[y_1\,\tilde\sigma_{24}(x) - y_2\,\tilde\sigma_{14}(x) + y_4\,\tilde\sigma_{12}(x)\right]$$
$$2\varphi_4(x)\,p(x,y) + 3\psi_4(x)\,g(x,x,y) = 24\left[y_1\,\tilde\sigma_{23}(x) - y_2\,\tilde\sigma_{13}(x) + y_3\,\tilde\sigma_{12}(x)\right]$$

holding for all points x on the canonical curve and all points $y \in \mathbb{C}^4$. As observed earlier, $p(x,y)$ and $g(x,x,y)$ are linearly independent linear functions of y for each point x on the canonical curve, so this system of equations completely determines the values $\varphi_j(x)$ and $\psi_j(x)$ in terms of the values $\tilde\sigma_{j_1 j_2}(x)$, and the latter are determined by the values of the coefficients $a_j(x)$ and $b_j(x)$ of the forms $p(x,y)$ and $g(x,x,y)$ as in Lemma 5. For instance, comparing the coefficients of y_1 and y_2 in the first equation of (15) yields

$$2\varphi_1(x)\,a_1(x) + 3\psi_1(x)\,b_1(x) = 0$$
$$2\varphi_1(x)\,a_2(x) + 3\psi_1(x)\,b_2(x) = 24\,\tilde\sigma_{34}(x).$$

If $a_1(x)\,b_2(x) - a_2(x)\,b_1(x) \neq 0$ these equations can be solved to yield

$$\psi_1(x)\left[a_1(x)b_2(x) - a_2(x)\,b_1(x)\right] = 8\,a_1(x)\,\tilde\sigma_{34}(x),$$

and it then follows from Lemma 5 that

$$\psi_1(x) = 8\,c\,a_1(x).$$

If $a_1(x)\,b_2(x) - a_2(x)\,b_1(x) = 0$ then it is necessary to consider the coefficients of some other pair of variables, but the end result will be the same. A straightforward calculation thus yields the explicit formulas

$\Psi_1(x) = 8c\,a_1(x),\quad \Psi_2(x) = -8c\,a_2(x),\quad \Psi_3(x) = 8c\,a_3(x),\quad \Psi_4(x) = -8c\,a_4(x),$

or equivalently,

(16) $\quad \Psi_j(x) = (-1)^{j+1}\, 8c\, \Sigma_k\, p_{jk} x_k.$

Now if the quartic polynomials (2.17) all vanish it follows from (11) and (12) that $f(x) = 0$ and $g(x)\, P_2^* \Psi(x) = 0$, hence from (16) that $f(x) = 0$ and $g(x)(\Sigma_k\, p_{jk} x_k) = 0$ for all indices j. If the matrix p_{jk} is nonsingular this means just that $f(x) = g(x) = 0$; otherwise there is the additional possibility that $\Sigma_k\, p_{jk} x_k = 0$, as desired.