E. Special Riemann Surfaces

1. The hyperelliptic multisecant identities.

The identities for second-order theta functions considered in the preceding part were initially derived from the description of the subvarieties W_r in terms of these theta functions. There are further identities that arise from the description of the subvarieties W_r^ν for $\nu > 0$, for those special Riemann surfaces for which these subvarieties are nonempty. The simplest case is that involving the subvariety W_2^1, and will be the first case to be considered here. If $W_2^1 \neq \emptyset$ and $g > 1$ the Riemann surface M is hyperelliptic, as discussed in section B10. As noted there W_2^1 consists of a single point $e \in J$, called the hyperelliptic point, so there is essentially a unique representation of M as a two-sheeted branched holomorphic covering of $\mathbb{P}^1$. For points $z_i \in M$ it is the case that $w(z_1+z_2) \in W_2^1$ precisely when $\pi^{-1}(p) = \{z_1,z_2\}$ for some point $p \in \mathbb{P}^1$; in particular $2w(z_1) \in W_2^1$ precisely when z_1 is one of the 2g+2 branch points of the mapping $\pi: M \to \mathbb{P}^1$, the Weierstrass points of M. The interchange of the sheets of this branched covering is a well defined biholomorphic mapping of M to itself, called the <u>hyperelliptic involution</u>; the fixed points of this involution are again the branch points of the mapping π, and the quotient space of M under the cyclic group of order two generated by this involution is just the Riemann sphere $\mathbb{P}^1$. Note that if $z' \in M$ is the image of the point $z \in M$ under this involution then $w(z+z') = e$; clearly then $W_1 = e-W_1$, and from B(9.5) it follows that $W_r = W_1 + \ldots + W_1 = r\,e-W_1 - \ldots - W_1 = r\,e-W_r$ for any index $r \geq 1$. From this and B(9.12) it follows in turn that

$$W_r^\nu = W_{r-\nu} \ominus (-W_\nu) = W_{r-\nu} \ominus (-\nu e' + W_\nu)$$

(1)

$$= \nu e + W_{r-2\nu} \qquad \text{whenever } r-\nu \leq g-1 \ ;$$

thus for hyperelliptic surfaces the subvarieties W_r^ν can be described quite simply in terms of the hyperelliptic point and the elementary subvarieties W_r, and in particular $W_{2\nu}^\nu = \nu e$ whenever $0 \leq \nu \leq g-1$.

For present purposes it is convenient to describe the special properties of hyperelliptic surfaces in terms of the universal covering space $\tilde{M}$ of M. Choose a base point $z_o \in \tilde{M}$ lying over the base point $p_o \in M$, and let p_o' be the image of p_o under the hyperelliptic involution. For any choice of a point $E\, z_o \in \tilde{M}$ lying over p_o' there is a lifting of the hyperelliptic involution to a biholomorphic mapping $E: \tilde{M} \to \tilde{M}$ such that $E\, z_o$ is the image of the base point z_o. The most general such lifting is then of the form $T\,E$ for any element T of the covering translation group Γ. The mapping E and the group Γ generate a group $\tilde{\Gamma}$ of automorphisms of $\tilde{M}$, where $\tilde{\Gamma}$ is clearly independent of the lifting E; the quotient space $\tilde{M}/\tilde{\Gamma}$ is just the Riemann sphere. There are various ways to construct such a lifting E, but perhaps the simplest is as follows. Recall that the universal covering space $\tilde{M}$ can be described as the space of homotopy classes of paths in M beginning at the base point $p_o \in M$; the other end point is arbitrary, but the homotopy must leave both end points fixed. If z is some representative path beginning at p_o then its image under the hyperelliptic involution is a path z' beginning at p_o'. If α

is any path from p_o to p'_o in M then the path obtained by traversing first α and then z' is another path $Ez = \alpha + z'$ at p_o, and the mapping $z \to Ez$ is the desired lifting of the hyperelliptic involution. The choice of lifting is thus determined by the choice of the connecting path α, up to homotopy. A particularly nice choice of connecting path α consists of a path α_1 from p_o to some fixed point p_1 of the hyperelliptic involution, followed by the image α'_1 of α_1 under the hyperelliptic involution but traversed in the opposite direction; this path can be written $\alpha = \alpha_1 - \alpha'_1$. With this choice $Ez = \alpha_1 - \alpha'_1 + z'$ and $E(Ez) = \alpha_1 - \alpha'_1 + \alpha'_1 - \alpha_1 + z = z$, or equivalently, $E^2 = I$. Thus it is always possible to choose some lifting E that is also an involution of $\tilde{M}$. Some such choice will be supposed made, so that E is henceforth an analytic automorphism $E : \tilde{M} \to \tilde{M}$ such that E induces the hyperelliptic involution on M and $E^2 = I$. Any other choice must be of the form TE for some element $T \in \Gamma$ for which $TETE = I$. The universal covering space $\tilde{M}$ is biholomorphically equivalent to the unit disc, so E will correspond to an elliptic linear fractional transformation of the disc, and consequently E has a single fixed point in $\tilde{M}$. It is worth noting explicitly that the condition that E be a lifting of the hyperelliptic involution implies that $E\Gamma E^{-1} = \Gamma$; for whenever $T \in \Gamma$ then $ETz = T'Ez$ for some element $T' \in \Gamma$. It is often conveient also to use E to denote the hyperelliptic involution itself, the mapping of M induced by $E :$ $\tilde{M} \to \tilde{M}$.

Since $w(z) + w(Ez) \in W_2^1 \subseteq J$ for all points $z \in \tilde{M}$ and since W_2^1 is a single point of the torus it is clear that there is a uniquely determined point $e \in \mathbb{C}^g$ such that

(2) $w(z) = w(Ez) = e$ for all $z \in \tilde{M}$;

this point will also be called the hyperelliptic point. It should be noted that this point does not depend on the choice of the lifting E, and that the choice of another lifting TE leads to the point $e + \omega(T)$, where $T \in \Gamma$ is any element for which TETE = I. Differentiating (2) with respect to the canonical coordinates on $\tilde{M}$ yields the relation

(3) $w'(Ez) \cdot E'(z) = -w'(z)$,

where $E'(z)$ denotes the derivative of the function representing the mapping E when both the domain and range of that mapping are described by the canonical coordinates on $\tilde{M}$. Since the Abelian differentials $w_j'(z)$ have no common zeros it is clear from (3) that $E'(a) = -1$ where a is the unique fixed point of the mapping E. It is also clear from (3) that if $c \in \mathbb{C}^g$ and $z_1 \in \tilde{M}$ is a point at which ${}^t c \cdot w'(z_1) = 0$ then ${}^t c \cdot w'(Ez_1) = 0$ as well; thus the divisor on M of any Abelian differential is necessarily of the form $\underset{\sim}{\mathfrak{z}} = \sum_{i=1}^{g-1}(z_i + Ez_i)$ when none of the zeros is a fixed point of the hyperelliptic involution, and the evident limiting argument shows that the same is true even when some of the points $z_i \in M$ are fixed points. Consequently $k = \sum_{i=1}^{g-1} w\,(z_i + Ez_i) = (g-1)\,e$ in J, or equivalently

(4) $k - (g-1) \cdot e \in \underset{\approx}{L} \subseteq \mathbb{C}^g$,

where as usual $k \in \mathbb{C}^g$ is the canonical point. For the fixed point a of the mapping E it is clear from (2) that $w(a) = e/2$. This point $a \in \tilde{M}$ represents one of the Weierstrass points of M, selected from among the others merely by the choice of the lifting E of the hyperelliptic involution. In general

$z_1 \in \tilde{M}$ represents a Weierstrass point of M precisely when $Ez_1 = Tz_1$ for some $T \in \Gamma$; in that case $e = w(z_1) + w(Ez_1) = w(z_1) + w(Tz_1) = 2w(z_1) + \omega(T)$ so that $\frac{e}{2} - w(z_1) = \frac{1}{2}\omega(T)$ is a half-period, and conversely whenever $\frac{e}{2} - w(z_1)$ is a half-period then z_1 represents a Weierstrass point of M. There are thus precisely $2g+2$ distinct half-periods on the translate $\frac{e}{2} - W_1 \subseteq J$, and they correspond to the Weierstrass points of M. Actually it is only for hyperelliptic curves that any translate either of W_1 or of $-W_1$ can contain even two distinct half-periods of J. Indeed if $\lambda_1, \lambda_2 \in \underline{L}$ are such that $\frac{1}{2}\lambda_1$, $\frac{1}{2}\lambda_2$ represent distinct points of J and $\frac{1}{2}\lambda_i \in W_1 + t$ for some $t \in \mathbb{C}^g$ then $w(a_i) = \frac{1}{2}\lambda_i - t$ where a_1, a_2 represent distinct points of M, and $w(2a_1 - 2a_2) = 2w(a_1) - 2w(a_2) = \lambda_1 - \lambda_2 \in \underline{L}$ so that there must be a meromorphic function f on M with divisor $\underline{\vartheta}(f) = 2a_1 - 2a_2$; this function exhibits M as a two-sheeted branched covering of $\mathbb{P}^1$, so shows that M is hyperelliptic.

With these background properties of hyperelliptic Riemann surfaces established, the derivation of the hyperelliptic trisecant identity follows the pattern of the earlier derivations quite readily. If $z_1, z_2, z_3, z_4 \in \tilde{M}$ represent distinct points of M and if $\underline{\vartheta} = w(z_1 + z_2 + z_3 + z_4)$ then by Theorem D2 the four vectors $\vec{\theta}_2[e - w(\underline{\vartheta})]\,(w(z_i))$ for $1 \leq i \leq 4$ span a subspace of $\mathbb{C}^{2^g}$ of dimension at most two; the same of course continues to hold by continuity even if there are coincidences among these points, with the proper interpretation as usual. On the other hand it follows easily from the same theorem that if z_1, z_2 represent distinct points of M then the first two of these vectors are linearly dependent precisely when $e - w(z_3 + z_4) = 0$ in J, hence precisely when $z_4 = Ez_3$ on M. Thus so long as $z_1 \neq z_2$ on M and $z_4 \neq Ez_3$ on M there are uniquely determined values f_1, f_2 such that

$$(5)\quad \vec{\theta}_2[e - w(\underline{\vartheta})]\,(w(z_3)) = f_1\,\vec{\theta}_2[e - w(\underline{\vartheta})]\,(w(z_1)) + f_2\,\vec{\theta}_2[e - w(\underline{\vartheta})]\,(w(z_2)).$$

These values are holomorphic funtions of the variables $z_i \in \tilde{M}$ when the variables are restricted as above, but extend to meromorphic functions on $\tilde{M}^4$ with singularities at most along the subvarieties $z_1 = Tz_2$ and $z_3 = TEz_4$ for arbitrary $T \in \Gamma$. With this pattern in mind though, the coefficient functions can be read directly from the ordinary trisecant identity as follows.

Theorem 1. If M is a hyperelliptic Riemann surface and $z_1,z_2,z_3,z_4 \in \tilde{M}$ are arbitrary points of its universal covering space then

$$\begin{aligned} 0 = {} & q(z_1,Ez_4)\, q(z_2,z_3)\, \overset{+}{\theta}_2\left(\frac{e}{2} + \tfrac{1}{2}w(z_1 - z_2 - z_3 - z_4)\right) \\ & - q(z_2,\, Ez_4)\, q(z_1,z_3)\, \overset{+}{\theta}_2\left(\frac{e}{2} + \tfrac{1}{2}w(z_2 - z_1 - z_3 - z_4)\right) \\ & + q(z_3,\, Ez_4)\, q(z_1,z_2)\, \overset{+}{\theta}_2\left(\frac{e}{2} + \tfrac{1}{2}w(z_3 - z_1 - z_2 - z_4)\right). \end{aligned}$$

Proof. This follows immediately from the ordinary trisecant identity, Theorem D5, merely replacing z in the formula of that theorem by Ez_4; for $w(Ez_4) = \frac{e}{2} - w(z_4)$.

Although the hyperelliptic trisecant identity is thus really just a trivial reformulation of the ordinary trisecant identity, there is some point in considering it explicitly. On the one hand it is naturally of some interest to see the analogues of the multisecant identities for the subvarieties W_r^ν for $\nu > 0$; that nothing new arises, at least in the case of the subvariety W_2^1, is of itself interesting, and turns out to be a model of what happens in general, as will be shown. On the other hand the reformulation is quite suggestive; for it is but natural in the context of this reformulation to look at limiting cases, not only as z_1 tends to z_2 for instance, but also as z_1 tends to Ez_2. These extra limiting cases that arise for hyperelliptic and other special classes of Riemann surfaces are of considerable interest; but once having had the idea, the limiting processes can be applied to the ordinary trisecant or multisecant identities.

Corollary. If M is a hyperelliptic Riemann surface then for all points $z_1, z_2 \in \tilde{M}$

$$0 = q(z_1, Ez_1)\, q(z_2, Ez_2)\, \vec{\theta}_2(0)$$
$$+ q(z_1, Ez_2)\, q(Ez_1, z_2)\, \vec{\theta}_2(w(z_1 - z_2))$$
$$- q(z_1, z_2)\, q(Ez_1, Ez_2)\, \vec{\theta}_2(w(z_1 - Ez_2)).$$

Proof Setting $z_4 = z_1$ in the formula of the theorem yields the identity

$$(6) \quad 0 = q(z_1, Ez_1)\, q(z_2, z_3)\, \vec{\theta}_2\left(\frac{e}{2} - \tfrac{1}{2}w(z_2 + z_3)\right)$$
$$+ q(Ez_1, z_2)\, q(z_1, z_3)\, \vec{\theta}_2\left(\frac{e}{2} + \tfrac{1}{2}w(z_2 - z_3) - w(z_1)\right)$$
$$- q(z_1, z_2)\, q(Ez_1, z_3)\, \vec{\theta}_2\left(\frac{e}{2} + \tfrac{1}{2}w(z_3 - z_2) - w(z_1)\right).$$

Setting $z_3 = Ez_2$ in this gives the desired result almost immediately since then $w(z_2 + z_3) = e$, $w(z_2 - z_3) = 2w(z_2) - e$, and $w(z_3 - z_2) = 2w(z_3) - e$; the theta function is an even function, so the sign of its argument is immaterial.

This corollary has a rather interesting geometric interpretation. For any given point $a \in \tilde{M}$ the mapping that sends a point $z \in \tilde{M}$ to the point $\vec{\theta}_2(w(z-a)) \in \mathbb{C}^{2^g}$ induces as usual a holomorphic mapping from M into the projective space $\mathbb{P}^{2^g-1}$; the image is an algebraic curve M_a lying in the Wirtinger variety $K \subseteq \mathbb{P}^{2^g-1}$. Two points $z_1, z_2 \in \tilde{M}$ have the same image in M_a precisely when $w(z_1 - a) = \pm\, w(z_2 - a)$, as is evident from Theorem A13, and by Abel's Theorem that means that the divisor $z_1 - a$ on M is linearly equivalent to either $z_2 - a$ or $a - z_2$; the first case can only occur when $z_1 = z_2$ on M, while the second case amounts to the linear equivalence $z_1 + z_2 \sim 2a$ and can only occur when $z_1 = Ez_2$ and $a = Ea$ on M. Thus if a is not a Weierstrass point this mapping is one-to-one from M to M_a; the curve M_a is then a birational model of M in the Wirtinger variety K. On the other hand if a is a Weierstrass point this mapping factors through the hyperelliptic covering $M \to \mathbb{P}^1$; the curve M_a is a rational curve in the Wirtinger variety K, and the mapping $M \to M_a$ is another realization of the hyperelliptic covering. In the first case, that in which a is not a Weierstrass point, the corollary shows that the points a, z and Ez have images in M_a that are collinear. This provides a geometric interpretation of the hyperelliptic involution E on the birational model M_a of the curve M : <u>for any point</u> $z \in M_a$, <u>the line in</u> $\mathbb{P}^{2^g-1}$ <u>joining</u> $a \in M_a$ <u>to</u> $z \in M_a$ <u>intersects the curve</u> M_a <u>again in the point</u> Ez. Note that this gives a number of trisecants of the Wirtinger variety passing through the point $\vec{\theta}_2(0)$.

If a is a Weierstrass point then $w(a) = \frac{e}{2} + \frac{\lambda}{2}$ for some period $\lambda \in \underset{\sim}{L}$; the mapping just considered is that taking the point $z \in M$ to the point in projective space represented by the vector $\chi(\lambda)\ \vec{\theta}_2(\frac{e}{2} - w(z))$, where $\chi(\lambda)$ is the matrix described in Theorem D3. To discuss this mapping it is generally sufficient to ignore the linear mapping $\chi(\lambda)$ and to consider just the mapping described by the function $\vec{\theta}_2(\frac{e}{2} - w(z))$. As noted earlier, the value of this mapping is unchanged when z is replaced by Ez, a result even more obvious upon noting that $\vec{\theta}_2(\frac{e}{2} - w(z)) = \vec{\theta}_2(\frac{1}{2} w(Ez-z))$ and recalling that $\vec{\theta}_2$ is an even function; the mapping thus factors through the hyperelliptic covering $M \to \mathbb{P}^1$, and again as noted earlier provides a birational mapping from $\mathbb{P}^1$ to the image curve M_a. An interesting identity in this connection arises by applying the differential operator $\partial/\partial z_3$ in canonical local coordinates on $\tilde{M}$ to (6) and then setting $z_3 = z_2$; the result is

$$(7)\quad 0 = -\, q(z_1,Ez_1)\ \vec{\theta}_2\left(\tfrac{e}{2} - w(z_2)\right)$$
$$+ \left[q(Ez_1,z_2)\ \partial_2 q(z_1,z_2) - q(z_1,z_2)\ \partial_2 q(Ez_1,z_2)\right]\ \vec{\theta}_2\left(\tfrac{e}{2} - w(z_1)\right)$$
$$- q(z_1,z_2)\ q(Ez_1,z_2)\ \sum_j \partial_j \vec{\theta}_2\left(\tfrac{e}{2} - w(z_1)\right)\, w'_j(z_2).$$

If z_1 is not a Weierstrass point then $q(z_1, Ez_1) \neq 0$ and $\frac{e}{2} - w(z_1)$ is not a half-period. In that case as in the corollary to Theorem A14 the g+1 vectors $\vec{\theta}_2\left(\frac{e}{2} - w(z_1)\right)$, $\partial_1\vec{\theta}_2\left(\frac{e}{2} - w(z_1)\right), \ldots, \partial_g\vec{\theta}_2\left(\frac{e}{2} - w(z_1)\right)$ are linearly independent so span a $(g+1)$ - dimensional linear subspace $L_{z_1} \subseteq \mathbb{C}^{2^g}$, and this determines a g-dimensional linear subspace $[L_{z_1}] \subseteq \mathbb{P}^{2^g-1}$. For any $z_2 \in \tilde{M}$ the identity (7) shows that $\left[\vec{\theta}_2\left(\frac{e}{2} - w(z_2)\right)\right] \in [L_{z_1}]$; the entire rational curve M_a thus lies in $[L_{z_1}]$, where $a \in \tilde{M}$ is the Weierstrass point for which $w(a) = \frac{e}{2}$. There are g+1 linearly independent functions among the 2g components of the vector $\vec{\theta}_2\left(\frac{e}{2} - w(z)\right)$, so the image curve M cannot be contained within any proper linear subspace of $[L_{z_1}]$; but $M_a \subseteq [L_{z_1}]$ for every point z_1 that is not a Weierstrass point, and hence $[L_{z_1}] = [L]$ <u>must be independent of the point</u> z_1.

To turn from the geometric to the analytic aspects of the identity (7), since the first line of that equation is unchanged when z_2 is replaced by Ez_2 while the g+1 vectors on the second and third lines are linearly independent whenever z_1 is not a Weierstrass point, it is evident that each coefficient of these g+1 vectors is also unchanged when z_2 is replaced by Ez_2. For the coefficient of $\partial_j \overset{+}{\theta}_2(\frac{e}{2} - w(z_1))$ it thus follows that

$$q(z_1,z_2)\ q(Ez_1,z_2)\ w'_j(z_2) = q(z_1,Ez_2)\ q(Ez_1,Ez_2)\ w'_j(Ez_2),$$

and since $w'_j(Ez_2)\ E'(z_2) = -w'_j(z_2)$ and not all of the values $w'_j(z_2)$ are zero necessarily

$$(8)\quad E'(z_2) = -\frac{q(z_1,Ez_2)\ q(Ez_1,Ez_2)}{q(z_1,z_2)\ \ q(Ez_1,z_2)}\ .$$

The right-hand side of this formula is almost symmetric in the variables z_1 and z_2, but of course not really so since it must actually be a function just of the variable z_2; indeed from this formula it is clear that

$$\frac{E'(z_2)}{E'(z_1)} = \frac{q(z_1,Ez_2)^2}{q(Ez_1,z_2)^2}\ .$$

This suggests introducing the function

$$(9)\quad e(z) = E'(z)^{1/2},$$

for some choice of a branch of the square root, a well defined holomorphic function on $\widetilde{M}$ since $E'(z)$ is nowhere vanishing. In terms of this function then

$$\frac{e(z_2)}{e(z_1)} = \pm\frac{q(z_1,Ez_2)}{q(Ez_1,z_2)}$$

where the sign is really well determined since the same branch of the square root is involved in both numerator and denominator on the left-hand side; setting $z_1 = z_2$ shows immediately that it must be the negative sign, so that

$$(10)\quad q(z_1,Ez_2)\ e(z_1) = q(z_2,Ez_1)\ e(z_2),$$

a fully symmetric function of the variables z_1 and z_2. It should be noted incidentally that since $E^2 = I$ then $E'(Ez)\ E'(z) = 1$ and hence $e(Ez)\ e(z) = \pm 1$, where again the sign is really well determined. At the fixed point $a \in \widetilde{M}$ of the mapping E it was observed that $E'(a) = -1$, hence $e(a) \pm i$ and consequently $e(Ea)\ e(a) = e(a)^2 = -1$; therefore

(11) $e(Ez)\ e(z) = -1$

for all points $z \in \tilde{M}$. It should also be noted that taking the logarithmic derivative of (10) with respect to the variable z_2 leads to the result that

$$w_{Ez_2}(z_1)\ E'(z_2) = w_{z_2}(Ez_1) + \frac{d}{dz_2} \log e(z_2)$$

or equivalently

(12) $w_{Ez_2}(z_1)\ E'(z_2) = w_{z_2}(Ez_1) + \frac{1}{2}\ \frac{E''(z_2)}{E'(z_2)}$,

since $\partial \log q(z,a) / \partial a = w_a(z)$. Replacing z_1 by Ez_1 here and subtracting the result from (12) yields the interesting further identity

(13) $w'_{z_1,Ez_1}(Ez_2)\ E'(z_2) = -w'_{z_1,Ez_1}(z_2)$,

since $w_a(z_1) - w_a(z_2) = w'_{z_1,z_2}(a)$. This is easily seen also to be just the identity that arises from the observation that the coefficient of $\vec{\theta}_2\left(\frac{e}{2} - w(z_1)\right)$ in (7) is unchanged when z_2 is replaced by Ez_2, to complete the line of argument from the beginning of this paragraph.

It was noted earlier that $W^{\nu}_{2\nu} = \nu e$ whenever $0 \leq \nu \leq g-1$, so the preceding arguments with the trisecant identity for $\nu = 1$ can be extended to the analogous arguments with the multisecant identity for general values of ν. Thus if $z_1,\ldots,z_{2\nu+2} \in \tilde{M}$ represent distinct points of M and $\mathfrak{z} = w(z_1 + \ldots + z_{2\nu+2})$ then by Theorem D2 the $2\nu+2$ vectors $\vec{\theta}_2[\nu . e - w(\mathfrak{z})]\ (w(z_i))$ for $1 \leq i \leq 2\nu+2$

span a linear subspace of $\mathbb{C}^{2^g}$ of dimension at most $\nu+1$; by continuity the same assertion holds even if there are coincidences among these points, with the appropriate interpretation. On the other hand it follows easily from the same theorem that if $z_1, \ldots, z_{\nu+1}$ represent distinct points of M then the first $\nu+1$ of these vectors are linearly dependent precisely when $w(z_{\nu+2} + \ldots + z_{2\nu+2}) \in \nu \cdot e - W_{\nu-1}$; this last condition means that the point $z_{\nu+2} + \ldots + z_{2\nu+2} \in \tilde{M}^{(\nu+1)}$ is restricted to lie in a holomorphic subvariety of codimension two in the complex manifold $\tilde{M}^{(\nu+1)}$. Thus so long as $z_1, \ldots, z_{\nu+1}$ represent distinct points of M and $w(z_{\nu+2} + \ldots + z_{2\nu+2}) \notin \nu.e - W_{\nu-1}$ there are uniquely determined values $f_1, \ldots, f_{\nu+1}$ such that

$$(14) \quad \vec{\theta}_2[\nu.e-w(\underline{\vartheta})]\,(w(z_{\nu+2})) = \sum_{i=1}^{\nu+1} f_i\, \vec{\theta}_2\,[\nu.e-w(\underline{\vartheta})]\,(w(z_i)).$$

These values f_i are holomorphic functions of the variables $z_j \in \tilde{M}$ when restricted as noted, but extend holomorphically across the subvariety $\nu.e-W_{\nu-1} \subseteq M^{\nu+1}$ since it has codimension at least two, and by Cramer's rule extend meromorphically to all points $z_j \in \tilde{M}$; thus (14) holds identically in the values $z_j \in \tilde{M}$, where f_i are uniquely determined meromorphic functions with poles at most along the subvarieties $z_j = Tz_k$ for $T \in \Gamma$ and $1 \leq j, k \leq \nu+1$. Again these functions can be read directly from the ordinary multisecant identity as follows, in a more symmetric form that does not isolate one of the terms as in (14).

<u>Theorem 2</u>. If M is a hyperelliptic Riemann surface of genus g then for any index $\nu \geqq 1$ and any points $z_1, \ldots z_{2\nu+2} \in \tilde{M}$

$$0 = \sum_{i=1}^{\nu+2} \frac{\prod_{j=\nu+3}^{2\nu+2} q(z_i, Ez_j)}{\prod_{\substack{k=1\\k\neq i}}^{\nu+2} q(z_i, z_k)} \, \vec{\theta}_2\left(\frac{\nu \cdot e}{2} + \frac{1}{2}w(z_i - z_1 - \cdots - z_{i-1} - z_{i+1} - \cdots - z_{2\nu+2})\right)$$

<u>Proof</u>. This follows immediately from the formula of Theorem D9 for $n = \nu+2$, upon replacing x_j by $Ez_{\nu+2+j}$ for $1 \leqq j \leqq \nu$ and noting that then $w(x_j) = w(Ez_{\nu+2+j}) = e - w(z_{\nu+2+j})$; by Corollary 2 to Theorem D9 this formula holds whenever $n = \nu+2 \geq 3$.

For the case $\nu=1$ this yields the formula of Theorem 1 upon multiplying by $q(z_1,z_2)\ q(z_1,z_3)\ q(z_2,z_3)$ to make all the coefficients holomorphic. As another illustrative example, for the case $\nu=2$ after multiplying by $\prod_{1 \leqq j < k \leqq 4} q(z_j, z_k)$ to make all the coefficients holomorphic the formula of the theorem takes the form

$$\begin{aligned}
(15)\quad 0 = {} & q(z_2,z_3)q(z_2,z_4)q(z_3,z_4)q(z_1,Ez_5)q(z_1,Ez_6)\ \vec{\theta}_2(e+\tfrac{1}{2}w(z_1-z_2-z_3-z_4-z_5-z_6)) \\
& - q(z_1,z_3)q(z_1,z_4)q(z_3,z_4)q(z_2,Ez_5)q(z_2,Ez_6)\ \vec{\theta}_2(e+\tfrac{1}{2}w(z_2-z_1-z_3-z_4-z_5-z_6)) \\
& + q(z_1,z_2)q(z_1,z_4)q(z_2,z_4)q(z_3,Ez_5)q(z_3,Ez_6)\ \vec{\theta}_2(e+\tfrac{1}{2}w(z_3-z_1-z_2-z_4-z_5-z_6)) \\
& - q(z_1,z_2)q(z_1,z_3)q(z_2,z_3)q(z_4,Ez_5)q(z_4,Ez_6)\ \vec{\theta}_2(e+\tfrac{1}{2}w(z_4-z_1-z_2-z_3-z_5-z_6))
\end{aligned}$$

2. Limiting forms of the hyperelliptic multisecant identities.

It is convenient to discuss briefly some further general properties of the hyperelliptic involution $E: \tilde{M} \to \tilde{M}$ before continuing the examination of the multisecant identities. It may be recalled that $ETE = ETE^{-1} \in \Gamma$ whenever $T \in \Gamma$. Then replacing z by Tz in (1.2) leads to the result that

$$\begin{aligned} e &= w(Tz)+w(ETz) = w(Tz) + w(ETE\cdot Ez) \\ &= w(z) + \omega(T) + w(Ez) + \omega(ETE) \\ &= e + \omega(T) + \omega(ETE), \end{aligned}$$

hence that

(1) $\omega(ETE) = -\omega(T)$ for all $T \in \Gamma$.

As noted earlier, the period homomorphism $\omega: \Gamma \to \underline{L}$ identifies the lattice subgroup $\underline{L} = \omega(\Gamma) \subset \mathbb{C}^g$ with the Abelianization of Γ, so its kernel is the commutator subgroup $[\Gamma,\Gamma] \subset \Gamma$; thus (1) is equivalent to

(2) $ETET \in [\Gamma,\Gamma]$ for all $T \in \Gamma$.

This is an interesting result by itself, with various interpretations and alternative derivations.

Next the notation can often be simplified by introducing the auxiliary function

(3) $h_a(z) = q(z,a)\, q(z,Ea)$,

a holomorphic function on $\tilde{M} \times \tilde{M}$ that for each fixed $a \in \tilde{M}$ is a relatively automorphic function $h_a \in \Gamma(\zeta_a \zeta_{Ea}) = \Gamma(\rho_e \zeta^2)$, determined uniquely up to a constant factor by the condition that its divisor is a+Ea on M. The projective space $\mathbb{P}\Gamma(\rho_e \zeta^2) = \mathbb{P}^1$ is just the Riemann sphere, and the mapping that associates to any point $a \in \tilde{M}$ the class $[h_a] \in \mathbb{P}^1$ depends only on the point of M represented by a and amounts to the standard representation of M as a two-sheeted branched covering of M. The condition that $h_a \in \Gamma(\rho_e \zeta^2)$ is just that

(4) $\quad h_a(Tz) = \rho_e(T)\ \zeta(T,z)^2\ h_a(z)$

for all $T \in \Gamma$, while from (1.8) it follows readily that

(5) $\quad h_a(Ez) = -E'(z)\ h_a(z) = -e(z)^2\ h_a(z);$

thus h_a is actually a relatively automorphic function for the extended group $\tilde{\Gamma}$ acting on $\tilde{M}$ with quotient $\mathbb{P}^1$, with respect to the factor of automorphy $\zeta_e(\tilde{T},z)$ for $\tilde{\Gamma}$ defined by

(6) $\quad \zeta_e(T,z) = \rho_e(T)\zeta(T,z)^2$ for $T \in \Gamma$, $\zeta_e(E,z) = -e(z)^2$.

To consider the behavior of this function in the other variable, note that from the definition (3) and (1.10) it follows readily that

(7) $\quad h_z(a) = -h_a(z)\ \frac{e(z)}{e(a)}.$

From this and (5) it is evident that

$$(8) \qquad h_{Tz}(a)\ e(Tz)^{-1} = \rho_e(T)\ \zeta(T,z)^2\ h_z(a)e(z)^{-1},$$

while from the definition (3) it is clear that

$$(9) \qquad h_{Ez}(a) = h_z(a).$$

On the other hand from (3)

$$\begin{aligned} h_{Tz}(a) &= q(a,\ Tz)\ q(a,ETz) = q(a,\ Tz)\ q(a,ETE\cdot Ez) \\ &= \rho_{w(a)}(T)\ \zeta(T,z)\ q(a,z)\cdot\rho_{w(a)}(ETE)\ \zeta(ETE,Ez)\ q(a,Ez) \\ &= \zeta(T,z)\ \zeta(ETE,Ez)\ h_z(a), \end{aligned}$$

where $\rho_{w(a)}(ETET) = 1$ in view of (2). A comparison of this with (8) shows that

$$(10) \qquad e(Tz) = \frac{\zeta(ETE,Ez)}{\rho_e(T)\ \zeta(T,z)}\quad e(z).$$

This implies that the factor of automorphy $\zeta(ETE,\ Ez)$ for Γ is analytically equivalent to the factor of automorphy $\rho_e(T)\ \zeta(T,z)$.

Next consider the holomorphic function $q(Ez,z)$ on $\tilde{M}$, and note from what has just been demonstrated and the explicit form for the canonical factor of automorphy found in the proof of Theorem B9 that

$$\begin{aligned} q(ETz,\ Tz) &= q(ETE\cdot Ez,\ Tz) \\ &= \rho_{w(Tz)}(ETE)\ \zeta(ETE,Ez)\cdot\rho_{w(Ez)}(T)\ \zeta(T,z)\cdot q(Ez,z) \\ &= \rho_{e-2w(z)-\omega(T)}(T)\ \zeta(ETE,Ez)\ \zeta(T,z)\ q(Ez,z) \\ &= \rho_e(T)\ \kappa(T,z)\ \zeta(T,z)^3\ \zeta(ETE,Ez)\ q(Ez,z) \end{aligned}$$

hence in view of (10) that

$$\text{(11)} \qquad q(ETz,Tz)\, e(Tz)^{-1} = [\rho_e(T)\zeta(T,z)^2]^2\, \kappa(T,z)\, q(Ez,z)\, e(z)^{-1}.$$

The function $q(Ez,z)$ vanishes precisely at those points of $\tilde{M}$ representing the $2g+2$ Weierstrass points of M, since only at such a point $z \in \tilde{M}$ do Ez and z represent the same point of M. On the other hand by (11) this function is a relatively automorphic function for a factor of automorphy analytically equivalent to $\rho_{2e}(T)\ \zeta(T,z)^4\ \kappa(T,z) = \rho_{\kappa+2e}(T)\ \zeta(T,z)^{2g+2}$; it must therefore have $2g+2$ zeros altogether on M, so actually has a simple zero at each Weierstrass point.

With these auxiliary results established, it is now possible to consider limiting cases of the quadrisecant identity in the form (1.15). Perhaps the most natural limit is that for which $z_1=z_4, z_2=z_5$, and $z_3=z_6$; although the formula itself reduces to a triviality in this case, an interesting result does arise if the differential operation $\partial/\partial z_1$ is applied first.

Theorem 3. If M is a hyperelliptic Riemann surface then for any points $x,z,a\in\tilde{M}$

$$q(Ez,a)q(z,a)^{-1}\sum_j \partial_j\vec{\theta}_2(w(z-a))\, w_j'(x)$$

$$= \vec{\theta}_2(0)\left\{ -q(Ez,z)e(z)e(a)\,\frac{h_a(x)}{h_z(x)}\, w_{Ea}'(x)+q(Ea,a)\,\frac{h_z(x)}{h_a(x)}\, w_z'(x)\right.$$

$$\left. + h_z(a)w_a'(z)\ [w_{Ez,z}'(x)-w_{Ea,a}'(x)]\right\}$$

$$+ \tfrac{1}{2}\sum_{jk} \partial_{jk}\,\vec{\theta}_2(0)\ \left\{q(Ez,z)\frac{e(z)}{e(a)}\,\frac{h_a(x)}{h_z(x)}\, w_j'(x)w_k'(a)+q(Ea,a)\,\frac{h_z(x)}{h_a(x)}\, w_j'(x)w_k'(z)\right.$$

$$\left. + h_z(a)[w_{Ez,z}'(x)-w_{Ea,a}'(x)]\ w_j'(z)w_k'(a)\right\}.$$

Proof. Upon setting $z_5=z_2$ and $z_6=z_3$, equation (1.15) takes the form

$$\begin{aligned}(12)\quad 0 &= q(z_2,z_3)q(z_2,z_4)q(z_3,z_4)q(z_1,Ez_2)q(z_1,Ez_3)\vec{\theta}_2(e-\tfrac{1}{2}w(-z_1+z_4+2z_2+2z_3))\\ &\quad - q(z_1,z_3)q(z_1,z_4)q(z_3,z_4)q(z_2,Ez_2)q(z_2,Ez_3)\vec{\theta}_2(e-\tfrac{1}{2}w(z_1+z_4+2z_3))\\ &\quad + q(z_1,z_2)q(z_1,z_4)q(z_2,z_4)q(z_3,Ez_2)q(z_3,Ez_3)\vec{\theta}_2(e-\tfrac{1}{2}w(z_1+z_4+2z_2))\\ &\quad - q(z_1,z_2)q(z_1,z_3)q(z_2,z_3)q(z_4,Ez_2)q(z_4,Ez_3)\vec{\theta}_2(e-\tfrac{1}{2}w(z_1-z_4+2z_2+2z_3)).\end{aligned}$$

To this apply the differential operator $\partial/\partial z$, and then set $z_4=z_1$. The second and third lines contain the factor $q(z_1,z_4)$ that vanishes when $z_4=z_1$, so their only nontrivial contribution is that arising from applying the differential operator to that factor. The first and fourth lines are the same except for sign when $z_4=z_1$; but $w(z_1)$ appears with different signs in the argument of the theta functions on those two lines, so differentiation there leads to a nontrivial contribution (the same for both lines), while z_1 and z_4 appear in different places in the prime function factors, so that too leads to a nontrivial contribution. To calculate these last terms note that

$$\frac{\partial}{\partial z_1}\left[q(z_1,Ez_2)q(z_1,Ez_3\right]=q(z_1,Ez_2)q(z_1,Ez_3)\frac{\partial}{\partial z_1}\log\left[q(z_1,Ez_2)q(z_1,Ez_3)\right]$$

$$= q(z_1,Ez_2)q(z_1,Ez_3)\ \left[w_{z_1}(Ez_2) + w_{z_1}(Ez_3)\right],$$

and similarly in the fourth line. Altogether there results the formula

$$0 = q(z_1,z_3)^2 q(z_2,Ez_2)q(z_2,Ez_3)\ \vec{\theta}_2(e-w(z_1+z_3))$$

$$- q(z_1,z_2)^2 q(z_3,Ez_2)q(z_3,Ez_3)\ \vec{\theta}_2(e-w(z_1+z_2))$$

$$+ q(z_2,z_3)h_{z_2}(z_1)h_{z_3}(z_1)\cdot\left\{\sum_j \partial_j\vec{\theta}_2(e-w(z_2+z_3))\ w_j'(z_1)\right.$$

$$\left. + \left[w_{z_1}(Ez_2)+w_{z_1}(Ez_3)-w_{z_1}(z_2)-w_{z_1}(z_3)\ \right]\ \vec{\theta}_2(e-w(z_2+z_3))\right\}.$$

By the Corollary to Theorem B11 the expression in brackets above can be rewritten $w'_{Ez_2,z_2}(z_1) + w'_{Ez_3,z_3}(z_1)$, while Theorem D6 together with (1.2) and (1.3) imply that

$$\vec{\theta}_2(e-w(z+a) = \vec{\theta}_2(w(Ez-a)$$

$$= q(Ez,a)^2 w_a'(Ez)\vec{\theta}_2(0) - \tfrac{1}{2} q(Ez,a)^2 E'(z)^{-1} \sum_{jk} \partial jk\ \vec{\theta}_2(0) w_j'(z) w_k'(a);$$

this last identity is symmetric in the variables z and a, which amounts to some further identities that are just consequences of (1.10) and (1.12) and means that there is some formal variety in the results obtained by applying it. With these observations the preceding formula can be rewritten

$$q(z_2,z_3)\ h_{z_2}(z_1)\ h_{z_3}(z_1) \sum_j \partial_j\ \vec{\theta}_2(w(Ez_2-z_3))\ w'_j(z_1)$$

$$= \vec{\theta}_2(0)\Big\{ -q(z_2,Ez_2)q(z_2,Ez_3)\ h_{z_3}(z_1)^2\ w'_{Ez_3}(z_1)$$

$$+q(z_3,Ez_2)q(z_3,Ez_3)\ h_{z_2}(z_1)^2\ w'_{Ez_2}(z_1)$$

$$-q(Ez_2,z_3)h_{z_2}(z_1)h_{z_3}(z_1)h_{z_2}(z_3)w'_{z_3}(Ez_2)\left[w'_{Ez_2,z_2}(z_1)+w'_{Ez_3,z_3}(z_1)\right]\Big\}$$

$$+ \tfrac{1}{2}\sum_{jk} \partial_{jk}\vec{\theta}_2(0)\ \Big\{ q(z_2,Ez_2)q(z_2,Ez_3)h_{z_3}(z_1)^2 E'(z_3)^{-1} w_j'(z_1) w_k'(z_3)$$

$$-q(z_3,Ez_2)q(z_3,Ez_3)h_{z_2}(z_1)^2\ E'(z_2)^{-1} w_j'(z_1) w_k'(z_2)$$

$$+q(z_2,Ez_3)h_{z_2}(z_1)h_{z_3}(z_1)h_{z_3}(z_2)E'(z_3)^{-1}\left[w'_{Ez_2,z_2}(z_1)+w'_{Ez_3,z_3}(z_1)\right]$$

$$w_j'(z_2)w_k'(z_3)\Big\}\ .$$

Upon setting $z_1=x$, $z_2=Ez$, $z_3=a$ and using the various identities that have been established heretofore to simplify the result the desired formula follows fairly directly, to conclude the proof.

For any fixed points $z, a \in \tilde{M}$ the various functions of x appearing in the formula of the preceding theorem can be viewed as meromorphic differential forms on M with singularities at most at the divisor $z+Ez+a+Ea$; that is quite obvious for such functions as $w'_j(x)$ and $w'_{Ez,z}(x)$, but is also clear for such functions as $w'_z(x)\ h_z(x)/h_a(x)$ since $h_z(x)$ and $h_a(x)$ are relatively automorphic functions for the same factor of automorphy so their quotient is a meromorphic function on M leading altogether to the singularities as asserted. The space of all such differential forms has dimension $g+3$, and if z,Ez,a,Ea represent distinct points of M a convenient basis consists of the functions

$$(13) \qquad w'_1(x)\ ,\ldots,\ w'_g(x),\ w'_{z,a}(x),\ w'_{Ez,z}(x),\ w'_{Ea,a}(x).$$

The differential forms appearing in the formula of the theorem are either listed here or uniquely expressible as a linear combination of these forms; a particularly interesting case is as follows.

Lemma 1. In terms of the quadratic period functions,

$$\frac{q(Ez,z)}{h_a(z)}\,\frac{h_a(x)}{h_z(x)}\,w'_j(x) = w'_j(z)w'_{Ez,z}(x) + \sum_k \left[\varphi^k_j(z;a)+\varphi^k_j(Ez;a)E'(z)\right]w'_k(x).$$

Proof. Since the differential form of interest here has singularities at most at the divisor $z+Ez$ on M it can be written

$$(14)\qquad \frac{h_a(x)}{h_z(x)}\, w_j'(x) = f_j(z,a)\, w_{Ez,z}'(x) + \sum_k f_j^k\,(z,a)\, w_k'(x)$$

for some coefficients $f_j(z,a)$, $f_j^k\,(z,a)$ which are uniquely determined so long as Ez and z represent distinct points of M and hence which are meromorphic functions of $(z,a) \in \tilde{M}^2$ with singularities at most along the subvarieties $z=z_i$, where $z_i \in \tilde{M}$ are the representatives of the 2g+2 Weierstrass points of M. Clearly

$$f_j(z,a) = -\lim_{x\to z} q(x,z)\, \frac{h_a(x)}{h_z(x)}\, w_j'(x) = \frac{h_a(z)}{q(Ez,z)}\, w_j'(z),$$

so the functions $f_j(z,a)$ have at most simple poles along the subvarieties $z=z_i$ since the function $q(Ez,z)$ has simple zeros there. With these functions thus determined (14) can be rewritten

$$(15)\qquad \sum_k f_j^k(z,a)\, w_k'(x) = \frac{h_a(x)}{h_z(x)}\, w_j'(x) - \frac{h_a(z)}{q(Ez,z)}\, w_j'(z)\, w_{Ez,z}'(x);$$

the functions $w_{Ez,z}'(x)$ also vanish whenever $z=z_i$, so the right-hand side has singularities at most along the subvarieties $z=Tx$, $z=TEx$ for $T\in\Gamma$, and it is apparent from this that the functions $f_j^k(z,a)$ must necessarily be holomorphic.

Now it follows immediately from (15) and the functional equation (8) that

$$(16)\qquad f_j^k\,(z,Ta)\; e(Ta)^{-1} = \rho_e(T)\; \zeta(T,a)^2\; f_j^k\,(z,a)\; e(a)^{-1}$$

for all $T \in \Gamma$. On the other hand by Theorem B11

$$w'_{ETz,Tz}(x) = w_x(ETz) - w_x(Tz) = w_x(ETE \cdot EZ) - w_x(Tz)$$

$$= w_x(Ez) + 2\pi i \sum_j \beta_j(ETE) w'_j(x) - w_x(z) - 2\pi i \sum_j \beta_j(T) w'_j(x)$$

$$= w'_{Ez,z}(x) - 4\pi i \sum_j \beta_j(T) w'_j(x),$$

since $\beta_j(ETE) = -\beta_j(T)$ by (2); and using this together with (4), (8), and (11) shows from (15) after a straightforward calculation that

$$(17) \qquad f^k_j(Tz,a)\, e(Tz)$$

$$= \rho_e(T)^{-1}\, \zeta(T,z)^{-2} \left\{ f^k_j(z,a) e(z) + 4\pi i \frac{e(z)\, h_a(z)}{q(Ez,z)} w'_j(z) \beta_k(T) \right\}.$$

These functional equations are somewhat simpler for the modified functions

$$(18) \qquad F^k_j(z,a) = \frac{q(Ez,z)}{h_a(z)} f^k_j(z,a) = -\frac{q(Ez,z)\, e(z)}{h_z(a) e(a)} f^k_j(z,a),$$

for which it is readily verified that they become

$$(16') \quad F^k_j(z, Ta) = F^k_j(z,a)$$

$$(17') \quad F^k_j(Tz,a)\, \underline{k}(T,z)^{-1} = F^k_j(z,a) + 4\pi i\, \beta_k(T)\, w'_j(z).$$

It is evident from (18) and what is known about $f_{jk}(z,a)$ that the functions $F^k_j(z,a)$ are meromorphic with singularities at most simple poles along the subvarieties $z=Ta$ and $z=TEa$ for $T\in\Gamma$. On the other hand it is evident from (15) that

$$f_j^k(a,a) = f_j^k(Ea,a) = \delta_j^k ,$$

hence

$$(19)\quad \lim_{z\to a} q(z,a)\, F_j^k(z,a) = \lim_{z\to Ea} q(z,Ea)\, F_j^k(z,a) = -\delta_j^k \ ;$$

thus $F_j^k(z,a)$ is actually holomorphic if $j\neq k$, but has nontrivial simple poles at a and Ea if $j=k$. Finally it is clear from the definitions that

$$(20)\qquad f_j^k(z,Ea) = f_j^k(z,a),\ F_j^k(z,Ea) = F_j^k(z,a),$$

while it is a simple calculation to verify from (15) that

$$(21)\qquad f_j^k(Ez,a)=f_j^k(z,a),\quad F_j^k(Ez,a)\ E'(z)=F_j^k(z,a).$$

Now consider the linear combination

$$\begin{aligned}(22)\qquad G_j^k(z,a) &= F_j^k(z,a) - \varphi_j^k(z;a) - \varphi_j^k(z;Ea)\\ &= F_j^k(z,a) - 2\varphi_j^k(z) + \delta_j^k\, w_z(a) + \delta_j^k\, w_z(Ea)\end{aligned}$$

of the function $F_j^k(z,a)$ and the quadratic period functions. This is a meromorphic function on $\tilde{M}^2$, and from (16') and Theorem B11 it is readily verified that it satisfies

$$G_j^k(z,Ta) = G_j^k(z,a)$$

for all $T\in\Gamma$ while from (17′) and the corollary to Theorem B12 it is as readily verified that it satisfies

$$G_j^k(Tz,a)\ \underline{k}(T,z)^{-1} = G_j^k(z,a)$$

for all $T\in\Gamma$; thus it is actually a meromorphic differential form on M in the variable z and a meromorphic function on M in the variable a. From the definition it is clear that its singularities are at most simple poles along the subvarieties $z=Ta$ and $z=TEa$ for all $T\in\Gamma$; but from (19)

$$\lim_{z\to a} q(z,a)\ G_j^k(z,a) = -\delta_j^k+\delta_j^k \lim_{z\to a} q(z,a)w_z(a) = 0$$

and similarly at Ea, so that $G_j^k(z,a)$ is actually holomorphic. Consequently it is of the form

$$G_j^k(z,a) = \sum_{\ell} c_{j\ell}^k\ w_\ell'(z)$$

for some constants $c_{j\ell}^k$, and thus

$$(23) \qquad F_j^k(z,a) = \varphi_j^k(z;a)+ \varphi_j^k(z;Ea) + \sum_{\ell} c_{j\ell}^k\ w_\ell'(z)$$

$$= 2\varphi_j^k(z) + \sum_{\ell} c_{j\ell}^k w_\ell'(z) - \delta_j^k w_z(a)-\delta_j^k w_z(Ea)$$

for some constants $c_{j\ell}^k$. Now it follows readily from (1.3) and (1.12) that

$$F^k_j(Ez,a)E'(Z) = 2\varphi^k_j(Ez)E'(z) - \sum_{\ell} c_{j\ell}^{k} w'_{\ell}(z)$$
$$- \delta^k_j \left[w_z(Ea) + w_z(a) + 2\, e'(z)/e(z)\right],$$

so in view of (21)

$$(24) \qquad \varphi^k_j(Ez)E'(z) = \varphi^k_j(z) + \sum_{\ell} c_{j\ell}^{k} w'_{\ell}(z) + \delta^k_j\, e'(z)/e(z);$$

thus the constants $c_{j\ell}^{k}$ can be viewed as determined by the way in which the quadratic period functions transform under the action of the hyperelliptic involution. On the other hand eliminating these constants from (23) and (24) and using (1.12) again shows that

$$F^k_j(z,a) = \varphi^k_j(z) + \varphi^k_j(Ez)E'(z) - \delta^k_j\left[w_z(a)+w_z(Ea)+e'(z)e(z)\right]$$
$$= \varphi^k_j(z) + \varphi^k_j(Ez)E'(z) - \delta^k_j\left[w_z(a) + w_{Ez}(a)\, E'(z)\right]$$

hence that

$$(25) \qquad F^k_j(z,a) = \varphi^k_j(z;a) + \varphi^k_j(Ez;a)\, E'(z).$$

The desired result follows immediately from (15), (18), and (25), thereby concluding the proof.

This lemma can be used to simplify the result of Theorem 3 as follows.

<u>Corollary</u>. If M is a hyperelliptic Riemann suface then for any points $z,a \in \tilde{M}$ and any index j

$$q(z,a)^{-2}\,\partial_j\vec{\theta}_2(w(z-a)) - \vec{\theta}_2(0)\,h_j(z,a)$$

$$= \tfrac{1}{2}\sum_{k,\ell}\partial_{k\ell}\,\vec{\theta}_2(0)\,\Big\{\Big[\varphi^{j}_{\ell}(a;z)+\varphi^{j}_{\ell}(Ea;z)E'(a)\Big]\,w_k'(z)$$

$$- \Big[\varphi^{j}_{\ell}(z;a) + \varphi^{j}_{\ell}(Ez;a)\,E'(z)\Big]w_k'(a)\Big\}$$

where $h_j(z,a)$ are meromorphic functions on $\tilde{M}^2$, with singularities at most double poles along the subvarieties $z=Ta$ and $z=TEa$ for $T\in\Gamma$, and are symmetric in the variables z and a.

<u>Proof</u>. Substituting the formula of the lemma, and the same with the variables z and a interchanged, into the result of Theorem 3 yields the result that

$$q(Ez,a)q(z,a)^{-1}\sum_j \partial_j\,\vec{\theta}_2\,(w(z-a))\,w_j'(x) - \vec{\theta}_2(0)\,h(z,a,x)$$

$$= \tfrac{1}{2}\,h_z(a)\sum_{jk\ell}\partial_{jk}\,\vec{\theta}_2(0)w_\ell'(x)\,\Big\{\Big[\varphi^{\ell}_{j}(a;z)+\varphi^{\ell}_{j}(Ea;z)E'(a)\Big]w_k'(z)$$

$$-\Big[\varphi^{\ell}_{j}(z;a)+\varphi^{\ell}_{j}(Ez;a)E'(z)\Big]w_k'(a)\Big\}$$

for some function $h(z,a,x)$, which can be written out explicitly as in the theorem; the only property of this function needed here is just that it is some linear combination of the differentials (13), actually only involving the ordinary Abelian differentials $w_j'(x)$ since none of the others appear in the rest of the formula. Comparing the coefficients of the linearly independent functions $w_j'(x)$ gives a formula of the desired form, for some functions $h_j(z,a)$. Since the latter are uniquely determined from this formula it is evident that they have the asserted properties, to conclude the proof.

Thus far limiting cases of both the trisecant and quadrisecant formula have been considered; the former involve theta functions translated by $e/2$, as in (1.7), whereas in the latter the hyperelliptic point can be eliminated altogether, as in Theorem 3 and its Corollary. There are analogous limiting cases for general multisecant formulas, the even cases again leading to simpler results as follows.

<u>Theorem 4</u>. If M is a hyperelliptic Riemann surface then for any index $n \geq 1$ and any points $x, z_1, \dots, z_{2n} \in \tilde{M}$

$$\left\{ \prod_{j=1}^{2n} \frac{q(x,Ez_j)}{q(x,z_j)} \right\} \left\{ \sum_{k=1}^{g} \partial_k \vec{\theta}_2(w(Ez_1 + \dots + Ez_n - z_{n+1} - \dots - z_{2n}))\, w'_k(x) \right.$$

$$\left. + \vec{\theta}_2(w(Ez_1 + \dots + Ez_n - z_{n+1} - \dots - z_{2n})) \sum_{k=1}^{2n} w'_{Ez_k, z_k}(x) \right\}$$

$$= -\sum_{i=1}^{2n} \frac{\prod_{j=1}^{2n} q(z_i,Ez_j)}{\prod_{\substack{k=1\\k\neq i}}^{2n} q(z_i,z_k)}\, q(x,z_i)^{-2}\, \vec{\theta}_2(w(z_i - x + Ez_1 + \dots + Ez_n - z_{n+1} - \dots - z_{2n})).$$

<u>Proof</u>. In the formula of Theorem 2 for $\nu=2n$ note that the $2n+2$ variables $z_1, \dots, z_{2n+2}$ and the $2n$ variables $z_{2n+3}, \dots, z_{4n+2}$ play somewhat different roles. In particular there are no factors $q(z_i,z_j)^{-1}$ where $i \leq 2n+2$ and $j \geq 2n+3$, so that it is possible to set $z_{2n+j} = z_j$ for $j=3,\dots,2n+2$ in that formula without any undue complications; the result can be written

$$0 = \frac{\prod_{j=3}^{2n+2} q(z_1,Ez_j)}{\prod_{\substack{k=1\\k\neq 1}}^{2n+2} q(z_1,z_k)} \vec{\theta}_2(ne+\tfrac{1}{2}w(z_1-z_2)-w(z_3+\dots+z_{2n+2}))$$

$$+ \frac{\prod_{j=3}^{2n+2} q(z_2,Ez_j)}{\prod_{\substack{k=1\\k\neq 2}}^{2n+2} q(z_2,z_k)} \vec{\theta}_2(ne+\tfrac{1}{2}w(z_1-z_2)-w(z_3+\dots+z_{2n+2}))$$

$$+ \sum_{i=3}^{2n+2} \frac{\prod_{j=3}^{2n+2} q(z_i,Ez_j)}{\prod_{\substack{k=1\\k\neq i}}^{2n+2} q(z_i,z_k)} \vec{\theta}_2(ne+\tfrac{1}{2}w(z_1+z_2)-w(z_3+\dots+z_{i-1}+z_{i+1}+\dots+z_{2n+2})),$$

in which now the variables z_1 and z_2 play special roles. Multiply this formula by $q(z_1,z_2)$, then apply the differential operation $\partial/\partial z_1$, and finally take the limit as $z_2 \to z_1$. In the first line the factor $q(z_1,z_2)$ in the denominator is thus cancelled; the differential operator can be applied first to the theta function, yielding in the limit

$$(26) \quad \sum_{\ell=1}^{g} \partial_\ell \vec{\theta}_2 \,(ne-w(z_3+\dots+z_{2n+2}))\, w_\ell'(z_1),$$

and then to the prime factors, where by logarithmic differentiation it yields the same product of prime factors multiplied by

$$\sum_{\ell=3}^{2n+2} w'_{Ez_\ell, z_\ell}(z_1)$$

since as observed many times before

$$\frac{\partial}{\partial z_1} \log \frac{q(z_1, Ez_j)}{q(z_1, z_j)} = w'_{Ez_j, z_j}(z_1).$$

In the second line the factor $q(z_2, z_1)$ in the denominator is cancelled but with the introduction of a negative sign; the differential operator is only applied to the theta function, yielding in the limit (26) but with another negative sign. In the remaining terms there is only a nontrivial result when the factor $q(z_1, z_2)$ is differentiated, and it tends to 1 in the limit. Combining these observations gives the result that

$$0 = \frac{\prod_{j=3}^{2n+2} q(z_1, Ez_j)}{\prod_{k=3}^{2n+2} q(z_1, z_k)} \left\{ \sum_{\ell=1}^{g} \partial_\ell \vec{\theta}_2 (ne-w(z_3+\ldots+z_{2n+2}))\, w'_j(z_1) \right.$$

$$\left. + \vec{\theta}_2(ne-w(z_3+\ldots+z_{2n+2})) \sum_{\ell=3}^{2n+2} w'_{Ez_\ell, z_\ell}(z_1) \right\}$$

$$+ \sum_{i=3}^{2n+2} \frac{\prod_{j=3}^{2n+2} q(z_i, Ez_j)}{\prod_{\substack{k=3 \\ k\neq i}}^{2n+2} q(z_i, z_k)} q(z_i, z_1)^{-2}\, \vec{\theta}_2(ne-w(z_1+z_3+\ldots+z_{i-1}+z_{i+1}+\ldots+z_{2n+2})).$$

Noting that $ne-w(z_3+\ldots+z_{2n+2}) = w(Ez_3+\ldots+Ez_{n+2} - z_{n+3}-\ldots-z_{2n+2})$ and $ne-w(z_1+z_3+\ldots+z_{i-1}+z_{i+1}+\ldots+z_{2n+2}) = w(z_i-z_1+Ez_3+\ldots+Ez_{n+2} - z_{n+3} -\ldots- z_{2n+2})$ and making the obvious change of notation then yields the desired result, to complete the proof.

For the case n=1 the result of the preceding theorem reduces to that of Theorem 3 upon using the formula of Theorem D6. For the general case it seems easier to use the result in the form given rather than to seek a further reduction, using the analogues of Theorem D6 to be developed in the next chapter. A slight change of notation is sometimes convenient though, as follows.

<u>Corollary</u>. If M is a hyperelliptic Riemann surface then for any index $n \geq 1$ and any points $x, z_1, \ldots, z_n, a_1, \ldots, a_n \in \tilde{M}$

$$\left\{ \prod_{j=1}^{n} \frac{q(x,z_j)\, q(x,Ea_j)}{q(x,Ez_j)\, q(x,a_j)} \right\} \left\{ \sum_{k=1}^{g} \partial_k \vec{\theta}_2 \,(w(z_1+\ldots+z_n-a_1-\ldots-a_n))\, w_k'(x) \right.$$

$$\left. + \vec{\theta}_2(w(z_1+\ldots+z_n-a_1-\ldots-a_n)) \sum_{k=1}^{n} (w'_{z_k,Ez_k}(x) - w'_{a_k,Ea_k}(x) \right\}$$

$$= - \sum_{i=1}^{n} \left\{ \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{q(Ez_i,z_j)\, q(Ez_i,Ez_j)}{q(Ez_i,Ez_j)\, q(Ez_i,a_j)} \right\} \cdot$$

$$\cdot \frac{q(Ez_i,\, z_i)\, q(Ez_i,Ea_i)}{q(Ez_i,a_i)\, q(Ez_i,x)^2} \vec{\theta}_2 \,(w(Ex-z_i+z_1+\ldots+z_n-a_1-\ldots-a_n))$$

$$- \sum_{i=1}^{n} \left\{ \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{q(a_i,z_j)\, q(a_i,Ea_j)}{q(a_i,Ez_j)\, q(a_i,a_j)} \right\} \cdot$$

$$\cdot \frac{q(a_i,z_i)\, q(a_i,Ea_i)}{q(a_i,Ez_i)\, q(a_i,x)^2} \vec{\theta}_2 \,(w(a_i-x+z_1+\ldots+z_n-a_1-\ldots-a_n)).$$

<u>Proof</u>. This follows immediately from the theorem upon replacing z_j by Ez_j and z_{j+n} by a_j for $1 \leq j \leq n$ and noting that $w(Ez_i-x)=w(Ex-z_i)$.

3. A special limiting form: the KDV equation.

There are interesting further limiting forms of the hyperelliptic quadrisecant identity, obtained by setting $z=a$ in the Corollary to Theorem 3. The direct substitution leads as usual to a trivial identity, but by differentiating first there results the following analogue of Theorem D8.

<u>Theorem 5</u>. If M is a hyperelliptic Riemann surface then for any index j and any point $z \in \tilde{M}$

$$\sum_{k_1 k_2 k_3} \partial_{j k_1 k_2 k_3} \vec{\theta}_2(0) w'_{k_1}(z) w'_{k_2}(z) w'_{k_3}(z)$$

$$+ \sum_k \partial_{jk} \vec{\theta}_2(0) \left\{ - \tfrac{1}{2} w'''_k(z) + 3 w''_k(z) q(z,Ez)^{-1} \partial_2 q(z,Ez) E'(z) - 3 w'_k(z) f(z) \right\}$$

$$+ \vec{\theta}_2(0) h_j(z)$$

$$= -3 \sum_{k,\ell} \partial_{k\ell} \vec{\theta}_2(0) w'_k(z)^2 \frac{\partial}{\partial z} \left\{ \frac{\varphi^j_\ell(z) + \varphi^j_\ell(Ez) E'(z)}{w'_k(z)} \right\}$$

where $\varphi^j_\ell(z)$ are the quadratic period functions, $h_j(z)$ are holomorphic functions, and

$$f(z) = \lim_{a \to z} \left\{ 8\, q_3(z,z) + \frac{\partial^2}{\partial z^2} \log q\,(Ez,a) - \frac{\partial^2}{\partial z \partial a} \log q\,(Ez,a) \right\}.$$

<u>Proof</u>. Multiply the formula of the corollary to Theorem 3 by $q(z,a)^2$ apply the differential operator $\partial^3/\partial z^3$, and then set $z=a$. For the left-hand side note from the chain rule for differentiation that

$$\partial^3 \partial_j \vec{\theta}_2(w(z-a)/\partial z^3$$

$$= \sum_k \partial_{jk_1k_2k_3} \vec{\theta}_2(w(z-a))w'_{k_1}(z)w'_{k_2}(z)w'_{k_3}(z)$$

$$+ 3 \sum_k \partial_{jk_1k_2} \vec{\theta}_2 (w(z-a)) w''_{k_1}(z)w'_{k_2}(z)$$

$$+ \sum_k \partial_{jk} \vec{\theta}_2 (w(z-a))w'''_k (z),$$

and $\partial_{jk_1k_2}\vec{\theta}_2(0)=0$ since $\vec{\theta}_2(w)$ is an even function of w. For the right-hand side recall that $\varphi^j_\ell(z;a)=\varphi^j_\ell(z)-\delta^j_\ell w_z(a)$ and consider first those terms involving the holomorphic functions φ^j_ℓ. For a nontrivial result the factor $q(z,a)^2$ must be differentiated exactly twice, so these terms contribute altogether

$$\lim_{z\to a} 3\sum_{k,\ell} \partial_{k\ell} \vec{\theta}_2(0) \frac{\partial}{\partial z} \left\{ \left[\varphi^j_\ell(a)+\varphi^j_\ell(Ea)E'(a)\right]w'_k(z)-\left[\varphi^j_\ell(z)+\varphi^j_\ell(Ez)E'(z)\right]w'_k(a)\right\}$$

$$= -3\sum_{k,\ell} \partial_{k\ell}\vec{\theta}_2(0)\ w'_k(a)^2 \frac{\partial}{\partial a} \left[\frac{\varphi^j_\ell(a) + \varphi^j_\ell(Ea)E'(a)}{w'_k(a)} \right].$$

The remaining contribution from the right-hand side is

$$\lim_{z\to a} \frac{\partial^3}{\partial z^3} \tfrac{1}{2} q(z,a)^2 \sum_k \partial_{jk}\vec{\theta}_2(0)\Big\{ w_k'(a) \Big[w_z(a)+w_{Ez}(a)E'(z)\Big]$$

$$- w_k'(z) \Big[w_a(z)+w_{Ea}(z)\ E'(a)\Big]\Big\}$$

$$= \lim_{z\to a} \tfrac{1}{2} \sum_k \partial_{jk}\vec{\theta}_2(0) \frac{\partial^3}{\partial z^3} \Big\{ q(z,a) \Big[w_k'(a) \frac{\partial}{\partial z} q(z,a) - w_k'(z) \frac{\partial}{\partial a} q(z,a)\Big]$$

$$+ q(z,a)^2 \Big[w_k'(a) \frac{\partial}{\partial z} \log q(Ez,a) - w_k'(z)\frac{\partial}{\partial a} \log q(z,Ea)\Big]\Big\}$$

$$= \tfrac{1}{2} \sum_k \partial_{jk}\ \vec{\theta}_2(0)\Big\{ 48\ q_3(a,a)w_k'(a) + 3w_k'''\ (a)$$

$$+ 6w_k'(a)\Big[\frac{\partial^2}{\partial z^2} \log q(Ez,a) - \frac{\partial^2}{\partial z\partial a} \log q\ (Ez,a)\Big]_{z=a}$$

$$- 6\ w_k''(a)\ \Big[\frac{\partial}{\partial a} \log q\ (z,Ea)\Big]_{z=a} \Big\}$$

$$= \sum_k \partial_{jk}\ \vec{\theta}_2(0)\Big\{ \frac{3}{2}\ w_k'''\ (a) - 3w_k''\ (a)\ q(a,Ea)^{-1}\ \partial_2 q(a,Ea)E'(a)$$

$$+ 3\ w_k'(a)\ \Big[8q_3(a,a) + \frac{\partial^2}{\partial z^2} \log q\ (Ez,a) - \frac{\partial^2}{\partial z\partial a} \log q\ (Ez,a)\Big]_{z=a}\Big\}.$$

Combining these observations and changing variables by replacing a by z lead rather easily to the asserted result and thereby conclude the proof.

There are 3g-2 linearly independent functions among the products $w'_{k_1}(z)w'_{k_2}(z)\,w'_{k_3}(z)$ on a hyperelliptic Riemann surface of genus g, and the index j is arbitrary, so the preceding theorem amounts to at most g(3g-2) independent linear combinations of the fourth-order theta derivatives at the origin expressible as linear combinations of lower-order derivatives; there are more such relations than those described by the formula of Theorem D8. Alternatively the preceding theorem can be viewed as expressing some particular third-degree homogeneous polynomials in the Abelian differentials in terms of other explicitly given functions in $\Gamma(\kappa^3)$. To discuss this in a bit more detail, note first that the function $\varphi_j^i(Ez)E'(z)$ satisfies the same functional equation under the action of an element $T\in\Gamma$ as does the function $\varphi_j^i(z)$, the equation described in Theorem B12; this is a simple verification, using (1.3) and (2.2). Consequently the function

$$(1) \quad \varphi^{*i}_j(z) = \tfrac{1}{2}(\varphi_j^i(z) + \varphi_j^i(Ez)E'(z))$$

must satisfy the same functional equation as well, so differs from $\varphi_j^i(z)$ by an Abelian differential and can therefore be viewed as another normalization of the quadratic period function. It is indeed evident that the function (1) is uniquely characterized by the functional equation of Theorem B12 for all transformations $T\in\Gamma$ together with the condition that

$$(2) \quad \varphi^{*i}_j\,(Ez)\;E'(z) = \varphi^{*i}_j\,(z),$$

since no nontrivial Abelian differential is invariant under E as a consequence of (1.3). This normalization is particularly appropriate for hyperelliptic Riemann surfaces of course, and is much easier to handle than the period normalization for $\varphi_{\ell}(z)$ as described in Theorem B12. The right-hand side of the formula of the preceding theorem can be rewritten even more simply in terms of this normalization as

$$(3) \qquad -6 \sum_{k\ell} \partial_{k\ell}\vec{\theta}_2(0)\; w_k'(z)^2 \frac{\partial}{\partial z} \left\{ \varphi^*_{\ell}(z)/w_k'(z) \right\} .$$

¶ Multiplying the formula of the preceding theorem by arbitrary constants c_j and summing the result over j leads to a system of linear differential equations in the second-order theta functions of the form

$$0 = \sum_{jk} \partial_{jk_1k_2k_3}\vec{\theta}_2(0)\; c_j w_{k_1}'(z)\; w_{k_2}'(z)\; w_{k_3}'(z) + \text{lower-order terms};$$

the lower-order terms are actually fully determined by the fourth-order terms, since the vectors $\partial_{jk}\,\vec{\theta}_2(0)$, $\vec{\theta}_2(0)$ are linearly independent modulo symmetries. On a hyperelliptic Riemann surface of genus g the products $w_{k_1}'(z)\; w_{k_2}'(z)\; w_{k_3}'(z)$ span a linear subspace of dimension 3g-2 in the space $\Gamma(\kappa^3)$ of cubic differentials; the g constants c_j can be quite arbitrary, but multiplying all by the same constant leads to essentially the same differential equation, so that this really amounts to a linear system of (g-1)(3g-2) fourth order-differential equations in the second-order theta functions at the origin. Among these differential equations are some particularly interesting special cases, arising when z is taken to be the fixed point of the hyperelliptic involution and the constants c_j are chosen appropriately.

Corollary 1. If $a \in \tilde{M}$ is the fixed point of the hyperelliptic involution $E : \vec{M} \to \vec{M}$ then

$$\sum_{k_1k_2k_3k_4} \partial_{k_1k_2k_3k_4} \vec{\theta}_2(0) w'_{k_1}(a) w'_{k_2}(a) w'_{k_3}(a)\; w'_{k_4}(a)$$

$$= \sum_{jk} \partial_{jk}\vec{\theta}_2(0)\; b_j\; w'_k(a) + \vec{\theta}_2(0)\; 2\; d$$

for some constants b_j, d.

Proof. Set $z=a$ in the formula of Theorem 5, multiply the result by $w'_j(a)$, and then sum over the index j. The left-hand side of the result is clearly

$$\sum_{jk} \partial_{jk_1k_2k_3} \vec{\theta}_2(0) w'_j(a)\; w'_{k_1}(a) w'_{k_2}(a)\; w'_{k_3}(a)$$

$$+ \sum_{jk} \partial_{jk}\; \vec{\theta}_2(0)\; w'_j(a)\; b'_k - \vec{\theta}_2(0)\; 2d$$

where b'_k is the expression in braces in the formula of Theorem 5 at $z=a$ and $-2d=\Sigma_j h_j(a) w'_j(a)$. In view of (3) and the observation that $\varphi^{*j}_{\ell}(a)=0$ for all indices j,ℓ as an obvious consequence of (2), the right-hand side can be written

$$- 6 \sum_{jk\ell} \partial_{k\ell}\; \vec{\theta}_2(0) w'_k(a) \varphi^{*j}_{\ell}{}'(a)\; w'_j(a)$$

$$= -\sum_{k\ell} \partial_{k\ell}\; \vec{\theta}_2(0)\; w'_k(a)\; b''_{\ell}$$

where $\varphi^{*}_{j}{}'(a) = d\,\varphi^{*}_{j}(a)/da$ and $b_j'' = 6\sum_j \varphi^{*}_{j}{}'(a) w_j'(a)$. That yields the desired result with $b_j = -b_j' - b_j''$.

This too can be rewritten in terms of first-order theta functions, in analogy with the discussion in section D5. Introduce the vectors

$$(4) \qquad U_j = w_j'(a), \; W_j = \tfrac{1}{4} b_j$$

in $\mathbb{C}^g$, together with an entirely arbitrary vector $Z \in \mathbb{C}^g$, and in terms of auxiliary variables $x, t \in \mathbb{C}$ introduce the holomorphic function f on $\mathbb{C}^2$ defined by

$$f(x,t) = \theta(Ux+Wt+Z).$$

Note that $f_x = \partial f/\partial x = \Sigma_j \partial_j \theta \; w_j'(a)$, $f_t = \partial f/\partial t = \tfrac{1}{4} \Sigma_j \partial_j \theta \; b_j$, and similarly for the higher-order derivatives.

<u>Corollary 2</u>. With the notation as above, the function f satisfies the partial differential equation

$$0 = f\, f_{xxxx} - 4 f_x f_{xxx} + 3\, f_{xx}^2 + 4 f_x f_t - 4 f f_{xt} - d f^2.$$

<u>Proof</u>. Multiply the formula of Corollary 1 on the left by the matrix $\overset{t\rightarrow}{\theta_2}(t)$ for an arbitrary point $t \in \mathbb{C}^g$. It follows readily from Lemmas D3 and D4 that the result can be written

$$2\sum_{k}\left\{\theta(t)\partial_{k_1k_2k_3k_4}\theta(t)-4\partial_{k_1}\theta(t)\partial_{k_2k_3k_4}\theta(t)+3\partial_{k_1k_2}\theta(t)\ \partial_{k_3k_4}\theta(t)\right\}$$
$$w'_{k_1}(a)\dots w'_{k_4}(a)$$

$$= 2\sum_{jk}\left\{\theta(t)\ \partial_{jk}\theta(t)-\partial_j\theta(t)\ \partial_k\theta(t)\right\} b_j w'_k(a) + 2\ d\ \theta(t)^2.$$

Upon taking the argument $t\in\mathbb{C}^g$ to be of the form $Ux+Wt+Z$, dividing by 2, and rewriting this in terms of the auxiliary function f, the formula of the corollary follows immediately.

The auxiliary function f for a hyperelliptic Riemann surface thus satisfies a somewhat simpler partial differential equation than that satisfied by the analogous functions for a general Riemann surface, as is evident upon comparing the preceding result with Corollary 2 to Theorem D8. This amounts to the same thing as asserting that the function $u(x,t)=2\partial^2\log f(x,t)/\partial x^2$ satisfies the classical Korteweg-deVries (KdV) equation, whereas the analogous function in the general case satisfies the KP equation of Corollary 3 to Theorem D8. These matters are nicely discussed in the survey article Theta functions and nonlinear equations by B.A. Dubrovin (Russian Math. Surveys 36 (1981), 11-92). In the traditional approach the Abel-Jacobi mapping is normalized quite explicitly for hyperelliptic Riemann surfaces, among other things by taking the base point to be a Weierstrass point; that amounts here to taking a as the fixed point of the hyperelliptic involution. Another special case of interest is the following.

Corollary 3. Let $a \in \tilde{M}$ again be the fixed point of the hyperelliptic involution $E : \tilde{M} \to \tilde{M}$, and let c_j be an eigenvector of the matrix $\varphi^{*j}_{k}{}'(a) = d\varphi^{*j}_{k}(z)/dz|_{z=a}$ with eigenvalue c, so that $\Sigma_j c_j \varphi^{*j}_{k}{}'(a) = c\, c_k$. Then

$$\sum_{jk} \partial_{jk_1k_2k_3} \vec{\theta}_2(0)\ c_j\ w'_{k_1}(a)\ w'_{k_2}(a)\ w'_{k_3}(a)$$

$$= \sum_{jk} \partial_{jk}\ \vec{\theta}_2(0)\ c_j\ b_k + \vec{\theta}_2(0)\ 2d$$

for some constants b_j, d depending on the choice of eigenvector.

Proof. Set $z=a$ in the formula of Theorem 5, multiply the result by c_j, and then sum over the index j. The left-hand side of the result is clearly

$$\sum_{jk} \partial_{jk_1k_2k_3} \vec{\theta}_2(0)\ c_j\ w'_{k_1}(a)\ w'_{k_2}(a)\ w'_{k_3}(a)$$

$$+ \sum_{jk} \partial_{jk}\ \vec{\theta}_2(0)\ c_j\ b'_k - \vec{\theta}_2(0)\ 2d$$

where b'_k is the expression in braces in the formula of Theorem 5 at $z=a$ and $-2d = \Sigma_j\ h_j(a)\ c_j$. In view of (3) and the observation that $\varphi^{*j}_{\ell}(a) = 0$ for all indices j, ℓ as a consequence of (2), the right-hand side can be rewritten as

$$- 6 \sum_{jk\ell} \partial_{k\ell} \vec{\theta}_2(0)\, w_k'(a)\, \varphi^{*j}_{\ell}{}'(a)\, c_j$$

$$= - 6 \sum_{k\ell} \partial_{k\ell} \vec{\theta}_2(0) w_k'(a)\, c\, c_\ell\ .$$

That yields the desired result with $b_k = -b_k' - 6\, c\, w_k'(a)$.

In this case introduce the auxiliary vectors

$$(5) \qquad U_j = w_j'(a),\ V_j = c_j,\ W_j = \tfrac{1}{4}\, b_j$$

in $\mathbb{C}^g$, together with an entirely arbitrary vector $Z \in \mathbb{C}^g$, and in terms of auxiliary variables x, y, $t \in \mathbb{C}^g$ introduce the holomorphic function f on $\mathbb{C}^3$ defined by

$$f(x,y,t) = \theta(Ux+Vy+Wt+Z).$$

In these terms the preceding corollary can be rewritten as follows.

Corollary 4. With the notation as above, the function f satisfies the partial differential equation

$$0 = f\, f_{xxxy} - f_y f_{xxx} - 3 f_x f_{xxy} + 3\, f_{xy}\, f_{xx}$$

$$+ 4\, f_y f_t - 4\, f\, f_{yt} - d f^2$$

Proof. Multiply the formula of Corollary 3 on the left by the matrix ${}^t\vec{\theta}_2(t)$ for an arbitrary point $t \in \mathbb{C}^g$. It follows readily from Lemmas D3 and D4 that the result can be written

$$2\sum_{jk}\Big\{\theta(t)\,\partial_{jk_1k_2k_3}\theta(t)-\partial_j\theta(t)\,\partial_{k_1k_2k_3}\theta(t)-3\,\partial_{k_1}\theta(t)\,\partial_{jk_2k_3}\theta(t)$$

$$+3\,\partial_{jk_1}\,\theta(t)\,\partial_{k_2k_3}\theta(t)\Big\}\,c_j\,w'_{k_1}(a)\,w'_{k_2}(a)\,w'_{k_3}(a)$$

$$=2\sum_{jk}\Big\{\theta(t)\,\partial_{jk}\,\theta(t)-\partial_j\theta(t)\,\partial_k\theta(t)\Big\}\,c_jb_k+\theta(t)^2\,2d.$$

Upon taking the argument $t \in \mathbb{C}^g$ to be of the form $Ux+Vy+Wt+Z$, dividing by 2, and rewriting this in terms of the auxiliary function f, the formula of the colollary follows immediately.

In the general case the differential equation of Theorem 5 can be rewritten as follows.

Corollary 5. For any vector $x_j \in \mathbb{C}^g$ and any point $z \in \tilde{M}$ there are constants b_j, c_j, c'_j, d depending linearly on x and analytically on z such that

$$\sum_{jk}\partial_{jk_1k_2k_3}\vec{\theta}_2(0)\;x_jw'_{k_1}(z)\;w'_{k_2}(z)\;w'_{k_3}(z)$$

$$=\sum_{jk}\partial_{jk}\,\vec{\theta}_2(0)\,\left[x_jb_k+c_jw'_k(z)+c'_jw''_k(z)\right]+2d\,\vec{\theta}_2(0).$$

Proof. This follows immediately from the formula of Theorem 5 upon multiplying by x_j and summing over j; here

$$b_k = \tfrac{1}{2}w'''_k(z) - 3w''_k(z)\ q(z,Ez)^{-1}\partial_2 q(z,Ez)\ E'(z)+3w'_k(z)f(z)$$

$$c_j = 3\sum_{\ell} x_{\ell}\ d\varphi^{*\ell}_{j}(z)/dz$$

(6)
$$c'_j = -3\sum_{\ell} x_{\ell}\ \varphi^{*\ell}_{j}(z)$$

$$d\ = \tfrac{1}{2}\sum_{\ell} x_{\ell} h_{\ell}(z)$$

where $f(z)$, $h_j(z)$ are as in Theorem 5.

It is worth noting explicitly that the formula of the corollary is a homogeneous linear function of the variable $x \in \mathbb{C}^g$. To express this general formula in terms of first-order theta functions introduce the auxiliary vector

$$(7)\qquad U_j=w'_j(z),\ U'_j=w''_j(z),\ V_j=x_j,\ V'_j=c_j, V''_j=c'_j,\ W_j=b_j$$

together with an arbitrary vector $Z \in \mathbb{C}^g$, and in terms of auxiliary variables $x, x', t, t', t'', y \in \mathbb{C}$ consider the holomorphic function f on $\mathbb{C}^6$ defined by

$$f(x,x',y,t,t',t'') = \Theta(Ux+U'_x+Vt+V't'+V''t''+Wy+Z).$$

With these conventions there is the following result.

Corollary 6. With the notation as above, the function f satisfies the partial differential equation

$$0 = ff_{xxxt} - f_t\ f_{xxx} - 3f_x f_{xxt} + 3f_{xt}\ f_{xx}$$

$$+ f\ f_{yt} - f_y f_t + ff_{xt'} - f_x\ f_{t'} + f\ f_{x't''} - f_{x'} f_{t''} + df^2.$$

Proof. Multiply the formula of Corollary 5 on the left by the matrix $\overset{t\rightarrow}{\theta}_2(t)$ for an arbitrary point $t \in \mathbb{C}^g$. It follows readily from Lemmas D3 and D4 that the result can be written

$$2\sum_{jk} \Big[\theta(t)\ \partial_{jk_1k_2k_3}\theta(t) - \partial_j\theta(t)\ \partial_{k_1k_2k_3}\theta(t) - 3\ \partial_{k_1}\theta(t)\ \partial_{jk_2k_3}\theta(t)$$

$$+ 3\ \partial_{jk_1}\theta(t)\ \partial_{k_2k_3}\theta(t)\ \Big]\ x_j\ w'_{k_1}(z)\ w'_{k_2}(z)\ w'_{k_3}(z)$$

$$= 2\sum_{jk} \big[\theta(t)\ \partial_{jk}\theta(t) - \partial_j\theta(t)\partial_k\theta(t)\big]\ \big[x_jb_k + c_jw'_k(z) + c'_jw''_k(z)\big]$$

$$+ 2\ \theta(t)^2\ d.$$

Upon setting $t = Ux+U'x'+ Vt+V't'+V''t''+Wy+Z$, dividing by 2, and rewriting this in terms of the auxiliary function f, the formula of the corollary ensues.

4. The trigonal multisecant identities.

The next simplest class of Riemann surfaces beyond the hyperelliptic ones are those for which $W_3^1 \neq \emptyset$. Any such surface M has a nontrivial meromorphic function with poles of total order at most 3, and if $g > 1$ and M is not hyperelliptic this function has poles of total order 3 so describes a three-to-one holomorphic mapping $\pi : M \to \mathbb{P}^1$; thus M can be represented as a three-sheeted branched holomorphic covering of the Riemann sphere. The surfaces M of this form, that are of genus $g > 1$ and are not hyperelliptic, are called trigonal Riemann surfaces.

Since any Riemann surface of genus g=2 is hyperelliptic the trigonal surfaces must have genus $g \geq 3$. It follows from the Riemann-Roch theorem in the geometric form B(9.8) that $W_3^1 = k - W_1$ when g=3, so every nonhyperelliptic Riemann surface of genus g=3 is trigonal; moreover since W_3^1 is one-dimensional in this case there are a number of quite distinct representations of such a surface as a branched three-sheeted covering of the Riemann sphere. Any nonhyperelliptic Riemann surface of genus g=4 is trigonal, and as observed in the discussion in section B9 in this case W_3^1 consists of either one or two points. A general Riemann surface of genus $g \geq 5$ is neither hyperelliptic nor trigonal, but if it is trigonal then W_3^1 consists of a single point so the surface has an essentially unique representation as a branched three-sheeted covering of the Riemann sphere; this sitatuion is discussed in the book by H. M. Farkas and I. Kra (Riemann Surfaces, Springer-Verlag, 1980).

Suppose then that M is a trigonal Riemann surface and that $\pi : M \to \mathbb{P}^1$ is a representation of M as a three-sheeted branched covering of M associated to a point $e \in W_3^1$. For any point $p \in M$ the set $\pi^{-1}(\pi(p))$ can be viewed as a divisor of degree 3 on M, say

$$(1) \qquad \pi^{-1}(\pi(p)) = p + p'+p'' \ ;$$

in general this divisor will consist of three distinct points, but if p lies over a branch point of the mapping π then it will consist of just one or two points. For this divisor it is of course the case that

$$(2) \qquad w(p+p'+p'') = e \in W_3^1 \subset J,$$

and conversely whenever (2) holds these three points are related as in (1).

Now in general a divisor of degree r on M can be viewed as an unordered set of r points of M, so the set of all such divisors can naturally be identified with the quotient space $M^{(r)} = M^r/\mathfrak{S}_r$ where the symmetric group $\mathfrak{S}_r$ acts as a group of complex analytic automorphisms of the product manifold M^r by permuting the factors; the quotient space $M^{(r)}$ is itself an r-dimensional complex manifold, a standard result proved among other places in my book (Lectures on Riemann Surfaces: Jacobi Varieties, Princeton University Press, 1972). The mapping that associates to any point $p \in M$ the divisor $p'+p'' \in M^{(2)}$ is a well defined holomorphic mapping $M \to M^{(2)}$; its image is a one-dimensional analytic subvariety of $M^{(2)}$ by Remmert's proper mapping theorem, and is birationally equivalent to M since this mapping between M and its image is evidently one-to-one. If $\tilde{M}$ is the universal covering space of M then it is quite easy to see that $\tilde{M}^{(2)}$ is the universal covering space of $M^{(2)}$. Indeed $\tilde{M}^{(2)}$ can be identified with the set of divisors of degree 2 on $\tilde{M}$, so is evidently at least a covering space of $M^{(2)}$. On

? branched covering space

the other hand $\tilde{M}$ can be identified with the unit disc in the complex plane, and the mapping that associates to any point $(z_1,z_2) \in \tilde{M}^2 \subset \mathbb{C}^2$ the point $(w_1,w_2) \in \mathbb{C}^2$ where $w_1 = z_1+z_2$, $w_2=z_1z_2$ clearly identifies the quotient space $\tilde{M}^{(2)}$ with an open subset of $\mathbb{C}^2$; this open subset is contractible along the parabolic paths $\{(tw_1,t^2w_2) : 0 \le t \le 1\}$, so is simply connected. The mapping $M \to M^{(2)}$ can be lifted to a holomorphic mapping $E : \tilde{M} \to \tilde{M}^{(2)}$ from the universal covering space of $\tilde{M}$ into the universal covering space of $M^{(2)}$; this mapping associates to any point $z \in \tilde{M}$ a divisor $Ez = E'z+E''z \in \tilde{M}^{(2)}$, where if $z \in \tilde{M}$ covers a point $p \in M$, then $E'z+E''z$ covers the divisor $p'+p''$. Of course E' and E'' can be defined separately by making a choice for each point $z \in M$, but are not then necessarily even continuous let alone analytic; however the divisor $E'_z+E''z$ does depend analytically on z. There any many possible liftings, each being determined uniquely by the choice of a divisor $E'z_0+E''z_0$ covering $p_0'+p_0''$ where $z_0 \in \tilde{M}$ is the base point of the marking of M and covers the point $p_0 \in M$; any other lifting will correspond to another choice $T'E'z_0+T''E''z_0$ for some arbitrary elements

$T', T'' \in \Gamma$. It will be supposed that some choice has been made, so that the mapping E is given for the remainder of the discussion.

With this choice it follows from (1) that $w(z+Ez) = e \in J$, and since the divisor Ez depends continuously on z there is a unique point of $\mathbb{C}^g$ which will also be denoted by e such that

(3) $w(z+E'z+E''z) = e \in \mathbb{C}^g$

for all points $z \in \tilde{M}$. This point will be called a trigonal point; it depends on the choice of a point in W_3^1 if $g \leq 4$ and also on the choice of the lifting E. If $E'a \neq E''a$ for some point a then it is possible to choose the points $E'z$, $E''z$ so that each will be a holomorphic function of z near a; in that case it follows from (3) that

(4) $w'(z) + w'(E'z) \dfrac{d}{dz} E'(z) + w'(E''z) \dfrac{d}{dz} E''(z) = 0$

in this neighborhood, in terms of canonical local coordinates on $\tilde{M}$ as usual. This is just the observation that any holomorphic differential form on M when symmetrized over the covering mapping $\pi : M \to \mathbb{P}^1$ will yield a holomorphic differential form on $\mathbb{P}^1$, which of course can only be identically zero since there are no nontrivial such forms on $\mathbb{P}^1$.

The derivation of the first trigonal multisecant identity follows almost precisely the pattern of the earlier discussion of the hyperelliptic trisecant identity. If $z_1,\ldots,z_5 \in \tilde{M}$ represent distinct points of M and $\underset{\sim}{\theta} = w(z_1+\ldots+z_5)$ then by Theorem D2 the five vectors

$\vec{\theta}_2\ [e-w(\underset{\sim}{\theta})]\ (w(z_i))$ for $1 \le i \le 5$ span a subspace of $\mathbb{C}^{2^g}$ of dimension at most three, while the first three of these vectors are linearly dependent precisely when $e-w(z_4+z_5) \in W_1$; this last condition is just that $e = w(z_4+z_5+z_6)$ for some point $z_6 \in \tilde{M}$, so is equivalent to the condition that $z_5 \in \Gamma\ E'z_4 \cup \Gamma\ E''z_4$. Thus there must be an identity of the form

$$(5)\quad \vec{\theta}_2\ [e-w(\underset{\sim}{\theta})]\ (w(z_4) = \sum_{i=1}^{3} f_i(z_1,\ldots,z_5)\ \vec{\theta}_2[e-w(\underset{\sim}{\theta})]\ (w(z_i))$$

for some uniquely determined meromorphic functions f_i with singularities at most along the subvarieties $z_j \in \Gamma\ z_k$ for $1 \le j < k \le 5$ or $z_5 \in \Gamma\ E'z_4 \cup \Gamma\ E''z_4$. These functions in a symmetrized form can be obtained directly from the ordinary quadrisecant identity as follows.

<u>Theorem 6</u>. If M is a trigonal Riemann surface with trigonal point e then for any points $z_1,\ldots,z_6 \in \tilde{M}$

$$0 = \sum_{i=1}^{4} \frac{q(z_i,E'z_5)q(z_i,E''z_5)}{\prod_{\substack{j=1\\ j\neq i}} q(z_i,z_j)}\ \vec{\theta}_2\ [e-w(z_1+\ldots+z_5)]\ (w(z_i)).$$

<u>Proof</u>. This follows immediately from the case n=4 of Theorem D9 upon setting $x_1 = E'z_5$, $x_2 = E''z_5$, and noting that $w(E'z_5+E''z_5-z_1-\ldots-z_4) = e-w(z_1+\ldots+z_5)$ in view of (3).

Here too there is a particularly interesting limiting case of the identity, that for which $z_5 = z_4$. To simplify the notation introduce in analogy to the corresponding situation for hyperelliptic surfaces the auxiliary function

(6) $h_a(z) = q(z,a)\ q(z,E'a)\ q(z,E''a)$;

this expression is symmetric in $E'a$ and $E''a$, so is a holomorphic function on $\tilde{M} \times \tilde{M}$. For any fixed point $a \in \tilde{M}$ it is a relatively automorphic function $h_a \in \Gamma(\zeta_a\zeta_{E'a}\zeta_{E''a}) = \Gamma(\rho_e\zeta^3)$, and is clearly determined uniquely up to a constant factor as that function in $\Gamma(\rho_e\zeta^3)$ that vanishes at the point a. The projective space $\mathbb{P}\Gamma(\rho_e\zeta^3)=\mathbb{P}^1$ is just the Riemann sphere, and the mapping that associates to any point $a\in\tilde{M}$ the class $[h_a] \in \mathbb{P}^1$ depends only on the point of M represented by a and amounts to the representation of M as a three-sheeted branched covering of $\mathbb{P}^1$ associated to the point $e\in W_3^1$.

Corollary 1. If M is a trigonal Riemann surface with trigonal point e and z,z_1,z_2,z_3 are any points of $\tilde{M}$ then with $f=$
$= \frac{1}{2}e-\frac{1}{2}\ w(z_1+z_2+z_3)$

$$\frac{q(z_1,z_2)q(z_1,z_3)q(z_2,z_3)}{q(z_1,z)q(z_2,z)q(z_3,z)}\ q(z,E'z)q(z,E''z)\ \vec{\theta}_2(f)$$

$$= h_z(z_1)q(z_2,z_3)q(z,z_1)^{-2}\ \vec{\theta}_2(f-w(z-z_1))$$

$$+ h_z(z_2)q(z_3,z_1)q(z,z_2)^{-2}\ \vec{\theta}_2(f-w(z-z_2))$$

$$+ h_z(z_3)q(z_1,z_2)q(z,z_3)^{-2}\ \vec{\theta}_2(f-w(z-z_3))$$

Proof. This follows immediately from the formula of the preceding theorem upon setting $z_4 = z_5 = z$ and multiplying through by $q(z_1,z_2)q(z_1,z_3)q(z_2,z_3)$.

Corollary 2. If M is a trigonal Riemann surface then for any points $z, a \in \tilde{M}$

$$\begin{aligned} 0 = {} & q(z,E'z)q(z,E''z)q(a,E'a)\, q(a,E''a)\, \vec{\theta}_2(0) \\ & + h_a(z)\, h_z(a)\, q(z,a)^{-2}\, \vec{\theta}_2\,(w(z-a)) \\ & - h_a(z)\, h_z(E'a)\, q(a,E''a)\, q(E'a,E''a)^{-1}\, q(z,E'a)^{-2}\, \vec{\theta}_2\,(w(z-E'a)) \\ & + h_a(z)\, h_z(E''a)\, q(a,E'a)\, q(E'a,E''a)^{-1}\, q(z,E''a)^{-2}\, \vec{\theta}_2\,(w(z-E''a)). \end{aligned}$$

Proof. This in turn follows immediately from the preceding corollary upon setting z_1=a, z_2=E'a, Z_3=E"a and noting that with these choices f=0.

These corollaries have interpretations rather analogous to those of the corresponding result for hyperelliptic Riemann surfaces. First for any fixed point $a \in \tilde{M}$ the mapping that sends a point $z \in \tilde{M}$ to the point $\vec{\theta}_2(w(z-a)) \in \mathbb{C}^{2^g}$ induces as before a holomorphic mapping from M into the projective space $\mathbb{P}^{2^g-1}$, the image being an algebraic curve M_a lying in the Wirtinger variety $K \subseteq \mathbb{P}^{2^g-1}$; since M is not hyperelliptic this mapping is always one-to-one, so that M_a is a birational model of M in the Wirtinger variety. It is evident from the corollary that the point $a \in M$ together with any three points $z,z',z'' \in M$ related as in (1) have as images four coplanar points of $M_a \subseteq K$. This is true for any representation of M as a three-sheeted branched covering of $\mathbb{P}^1$, even in those cases in which W^1_3 has more than a single point so that there is more than one such representation. The result of the first corollary can be interpreted geometrically as a somewhat more general relation of coplanarity.

In a slightly different direction, the result of Corollary 2 can be rewritten in a rather interesting alternative form by using the formula of Theorem D6; a simple calculation shows that it has the form

$$0 = \vec{\theta}_2(0)\Big\{ q(z,E'z)q(z,E''z)q(a,E'a)q(a,E''a) + h_a(z)h_z(a)w'_z(a)$$
$$- h_a(z)\ h_z(E'a)q(a,E''a)q(E'a,E''a)^{-1}\ w'_z(E'a)$$
$$+ h_a(z)\ h_z(E''a)q(a,E'a)q(E'a,E''a)^{-1}\ w'_z(E''a)\Big\}$$
$$+ \tfrac{1}{2}\sum_{j,k} \partial_{jk}\ \vec{\theta}_2(0)\Big\{ h_a(z)h_z(a)w'_j(z)w'_k(a)$$
$$- h_a(z)h_z(E'a)q(a,E''a)q(E'a,E''a)^{-1}\ w'_j(z)w'_k(E'a)$$
$$+ h_a(z)h_z(E''a)q(a,E'a)q(E'a,E''a)^{-1}\ w'_j(z)\ w'_k(E''a)\ \Big\}.$$

Here $\partial_{jk}\ \vec{\theta}_2(0) = \partial_{kj}\ \vec{\theta}_2(0)$, but aside from this obvious symmetry the vectors $\vec{\theta}_2(0)$ and $\partial_{jk}\ \vec{\theta}_2(0)$ are linearly independent; this result will only be demonstrated in Theorem F4, but the proof does not rely on anything beyond result already established in Section C so the theorem can be used freely here. It follows that the preceding identity is really equivalent to a few simpler identities. A consideration of the coefficient of $\tfrac{1}{2}\partial_{jj}\vec{\theta}_2(0)$ shows after dividing by the nontrivial function $h_a(z)\ h_z(a)\ w'_j(z)$ that

$$(7) \qquad 0 = w_k'(a) - \frac{h_z(E'a)\ q(a,E''a)}{h_z(a)\ q(E'a,E''a)}\ w_k'(E'a)$$

$$+ \frac{h_z(E''a)\ q(a,E'a)}{h_z(a)\ q(E'a,E''a)}\ w_k'(E''a).$$

This is a formula of the form (3), and must actually amount precisely to the same thing as (3). Indeed if there were two formulas of the same general form as (3) then there would clearly be an identity of the form $w_k'(E'z) = f(z)\ w_k'(E''z)$ for some function $f(z)$, and that would mean that $E'z$ and $E''z$ would have the same image in the canonical curve, an impossibility since M is not hyperelliptic. It follows from this that the expressions $h_z(E'a)/h_z(a)$ and $h_z(E''a)/h_z(a)$ must be independent of z. It was noted earlier that the functions h_z lie in the two-dimensional space $\Gamma(\rho_e \zeta^3)$, and that the functions associated to two parameters z_1, z_2 representing distinct points of M are linearly independent; two such functions form a basis for $\Gamma(\rho_e \zeta^3)$, and since $h_z(E'a) = c\ h_z(a)$ for some value c independent of z and for all points $z \in \tilde{M}$ it is evident that $h(E'a) = c\ h(a)$ for the same value c and for any $h \in \Gamma(\rho_e \zeta^3)$. Consequently (7) can be rewritten

$$(8) \qquad 0 = w_k'(z) - \frac{h(E'z)\ q(z,E''z)}{h(z)\ q(E'z,E''z)}\ w_k'\ (E'z)$$

$$+ \frac{h(E''z)\ q(z,E'z)}{h(z)\ q(E'z,E''z)}\ w_k'\ (E''z)$$

for any nontrivial function $h \in \Gamma(\rho_e\zeta^3)$, where the quotients $h(E'z)/h(z)$ and $h(E''z)/h(z)$ are independent of the choice of the function h. It should be noted that this formula is symmetric in $E'z$ and $E''z$. This provides an extension of (3) to arbitrary points z of $\tilde{M}$, for which $z, E'z, E''z$ need not represent distinct points of M; but if there are coincidences the coefficients need not be well defined. The coefficients are only meromorphic functions at points for which $E'z$ and $E''z$ can separately be defined as holomorphic mappings. A consideration of the coefficient of $\vec{\theta}_2(0)$ in the preceding identity shows similarly that

$$(9) \qquad \frac{q(z,E'z)\; q(z,E''z)\; q(a,E'a)\; q(a,E''a)}{h_z(a)\; h_a(z)}$$

$$= w_z'(a) - \frac{h(E'a)\; q(a,E''a)}{h(a)\; q(E'a,E''a)}\, w_z'(E'a) + \frac{h(E''a)\; q(a,E'a)}{h(a)\; q(E'a,E''a)}\, w_z'(E''a)$$

for any nontrivial function $h \in \Gamma(\rho_e\zeta^3)$. This is somewhat analogous to (8) but for the Abelian differential of the second kind.

Just as in the case of hyperelliptic surfaces so also in the case of trigonal surfaces are multiples νe of any point $e \in W_3^1$ also rather special points, associated to even more interesting multisecant identities than is the point e itself. The situation is rather more complicated for trigonal surfaces than it is for hyperelliptic surfaces though, since the properties of the points νe depend on the particular surface under consideration. The discussion of these properties is perhaps most conveniently based on the trigonal invariants that were apparently introduced by A. Maroni (Le serie lineari speciali sulle curve trigonali, Ann. di Mat. 25 (1946), 341-354); these are instances of analogous invariants that can be defined for arbitrary Riemann surfaces, associated to the minimal representations of that surface as a branched covering of the Riemann sphere. There is a thorough treatment of these invarients in the paper of A. Andreotti and A. Mayer (On period relations for Abelian integrals on algebraic curves, Ann. Scuola Norm. Sup. Pisa 21 (1967), 189-238), but a summary of the properties that will be needed here in a convenient form for the uses that will be made of them will be included for the sake of completeness.

A point $e \in W_3^1$ describes a line bundle or factor of automorphy $\tau=\rho_e\zeta^3$ for which $\gamma(\tau)=2$. If $f_0, f_1 \in \Gamma(\tau)$ is any basis for the space of relatively automorphic functions associated to this factor of automorphy then for any index $\nu\geq 1$ the $\nu+1$ functions f_1^{ν}, $f_1^{\nu-1} f_0$, $f_1^{\nu-2} f_0^2$, ..., f_0^{ν} are linearly independent relatively automorphic functions for the factor of automorphy τ^{ν}, so that $\gamma(\tau^{\nu})\geq\nu+1$. There is an index ν_2 such that $\gamma(\tau^{\nu})=\nu+1$ for $1\leq\nu<\nu_2$ but $\gamma(\tau^{\nu_2}) = \nu_2+2$; thus in addition to homogeneous polynomials of degree ν_2 in f_0 and f_1 there is a further linearly independent relatively automorphic function $f_2 \in \Gamma(\tau^{\nu_2})$. For any index $\nu\geq\nu_2$ the functions f_1^{ν}, $f_1^{\nu-1} f_0$, ..., f_0^{ν}, $f_2f_1^{\nu-\nu_2}$, $f_2f_1^{\nu-\nu_2-1}f_0$, ..., $f_2f_0^{\nu-\nu_2}$ are linearly independent relatively automorphic functions for the factor of automorphy τ^{ν}, so that $\gamma(\tau^{\nu}) \geq 2\nu-\nu_2+2$. There is another index ν_3 such that $\gamma(\tau^{\nu})=2\nu-\nu_2+2$ for

$\nu_2 \le \nu < \nu_3$ but $\gamma(\tau^{\nu_3}) = 2\nu_3 - \nu_2 + 3$, hence an additional relatively automorphic function $f_3 \in \Gamma(\tau^{\nu_3})$. For any index $\nu \ge \nu_3$ the functions $f_1^{\nu}, f_1^{\nu-1}f_o, \ldots, f_o^{\nu}, f_2\, f_1^{\nu-\nu_2}, f_2\, f_1^{\nu-\nu_2-1} f_o, \ldots, f_2 f_o^{\nu-\nu_2}$, $f_3\, f_1^{\nu-\nu_3}, f_3 f_1^{\nu-\nu_3-1}\, f_o, \ldots, f_3\, f_o^{\nu_3}$ are a basis for $\Gamma(\tau^{\nu})$ so that $\gamma(\tau^{\nu}) = 3\nu - \nu_2 - \nu_3 + 3 = 3\nu - g + 1$. These indices ν_2, ν_3 are the invariants of Maroni. Note that $\gamma(\tau^{\nu}) = 3\nu + 1 - g$ for all sufficiently large values of ν by the Riemann-Roch theorem, so that actually $\nu_2 + \nu_3 = g+2$; thus ν_2 alone specifies the Maroni invariant. This index varies over the interval $\frac{1}{3}(g+2) \le \nu_2 \le \frac{1}{2}(g+2)$.

As a simple alternative way of thinking of the indices of Maroni, note from the preceding discussion that

$$(10) \qquad \gamma(\tau^{\nu}) - \gamma(\tau^{\nu-1}) = \begin{cases} 1 \text{ if } 1 \le \nu < \nu_2, \\ 2 \text{ if } \nu_2 \le \nu < \nu_3, \\ 3 \text{ if } \nu_3 \le \nu. \end{cases}$$

For g=3 it follows from the inequality $\nu_2 \ge \frac{1}{3}(g+2)$ that $\nu_2 \ge 2$, and from the equality $\nu_2 + \nu_3 = 5$ that the only possibility is $\nu_2 = 2$, $\nu_3 = 3$; thus the values $\gamma(\tau^{\nu})$ are as follows:

(1)

		:	1	2	3	4	5	...
g=3	$\gamma(\tau^{\nu})$	:	2	4	7	10	13	...

On the other hand for g=4 there are the two possibilities $\nu_2 = 2$, $\nu_3 = 4$ and $\nu_2 = 3$, $\nu_3 = 3$; the values of $\gamma(\tau^{\nu})$ in these cases are as follows:

(12)

	ν :	1	2	3	4	5	...
$g=4, \nu_2=2, \nu_3=4$	$\gamma(\tau^\nu)$:	2	4	6	9	12	...
$g=4, \nu_2=3, \nu_3=3$	$\gamma(\tau^\nu)$:	2	3	6	9	12	...

These two cases are distinguished by the value of $\gamma(\tau^2)$. Note from the Riemann-Roch theorem that $\gamma(\underline{\kappa}\tau^{-2})=\gamma(\tau^2)-3$ so that $\gamma(\underline{k}\tau^{-2}) = 1$ or 0 according as $\nu_2=2$ or 3; but $c(\underline{\kappa}\tau^{-2}) = 0$, so in the first case $\underline{\kappa} = \tau^2$ while in the second case $\kappa \neq \tau^2$. It also follows from the Riemann-Roch theorem that $\kappa\, \tau^{-1} \in W_3^1$, so if $\nu_2=3$ then $\kappa\tau^{-1}\neq\tau$ and M is of the class of Riemann surfaces of genus 4 for which W_3^1 consists of two points. On the other hand if τ_1, $\tau_2 \in W_3^1$ are distinct points and f_0, $f_1 \in \Gamma(\tau_1)$ and $g_0, g_1 \in \Gamma(\tau_2)$ are bases then f_0g_0, f_0g_1, f_1g_0 f_1g_1 are linearly independent functions, so that $\gamma(\tau_1\tau_2) \geq 4=g$ while $c(\tau_1\tau_2) =$ $= 6 = 2g-2$; in that case $\kappa = \tau_1\tau_2$, and for both τ_1 and τ_2 the Maroni invariant is $\nu_2=3$. The two possible cases for the Maroni invariant thus correspond to the two distinct types of nonhyperelliptic Riemann surfaces of genus 4 discussed earlier. It is characteristic that the smaller values of the Maroni invariant ν_2 correspond to special Riemann surfaces of a fixed genus, as in this case. For other small values of the genus the invariants are as follows:

(13) $g=5$: $\nu_2=3,\ \nu_3=4$;

$g=6$: $\nu_2=3,\ \nu_3=5$ if $\kappa=\tau^3\zeta_a$ for some $a\in\tilde{M}$,
$\nu_2=4,\ \nu_3=4$ otherwise;

$g=7$: $\nu_2=3,\ \nu_3=6$ if $\kappa=\tau^4$,
$\nu_2=4,\ \nu_3=5$ otherwise;

$g=8$: $\nu_2=4,\ \nu_3=6$ if $\kappa=\tau^4\zeta_{a_1}\zeta_{a_2}$ for some $a_i\in\tilde{M}$,
$\nu_2=5,\ \nu_3=5$ otherwise;

$g=9$: $\nu_2=4,\ \nu_3=7$ if $\kappa=\tau^5\zeta_a$ for some $a\in\tilde{M}$,
$\nu_2=5,\ \nu_3=6$ otherwise;

$g=10$: $\nu_2=4,\ \nu_3=8$ if $\kappa=\tau^6$ hence $\gamma(\kappa\tau^{-5})=2$,
$\nu_2=5,\ \nu_3=7$ if $\gamma(\kappa\tau^{-5})=1$,
$\nu_2=6,\ \nu_3=6$ if $\gamma(\kappa\tau^{-5})=0$.

From these and the other observations already made it then follows that

$$(14)\qquad \gamma(\rho_{2e}\zeta^6)=\gamma(\tau^2)=\begin{cases}4 \text{ if } g=3 \text{ or } g=4,\ \underset{\sim}{\kappa}=\tau^2,\\ 3 \text{ otherwise;}\end{cases}$$

$$(15)\qquad \gamma(\rho_{4e}\zeta^{12})=\gamma(\tau^4)=\begin{cases}10 \text{ if } g=3,\\ 9 \text{ if } g=4,\\ 8 \text{ if } g=5,\\ 7 \text{ if } g=6 \text{ or } g=7,\ \nu_2=3,\\ 6 \text{ if } g=7,\ \nu_2=4 \text{ or } g=8,\ \nu_2=4\\ \qquad \text{or } g=9,\ \nu_2=4, \text{ or } g=10, \nu_2=4\\ 5 \text{ otherwise.}\end{cases}$$

If $\nu_2\geq5$ clearly $\gamma(\tau^4)=5$; it is only possible that $\gamma(\tau^4)>5$ when $\nu_2\leq4$, and from the inequality $\frac{1}{3}(g+2)\leq\nu_2$ that can only happen for a Riemann surface of genus $g\leq10$. The specific instances in which $\gamma(\tau^4)$ exceeds the minimal value are those tabulated in (15). In general $\gamma(\rho_{\nu e}\zeta^{3\nu}) = \gamma(r^{\nu}) = \nu+1$, as is always the case whenever $\nu_2>\nu$ and in particular for any Riemann surface of genus $g > 3\nu-2$.

To return then to the discussion of further trigonal multisecant identities, note as a consequence of the preceding discussion that $2e\in W_6^2$ for any trigonal Riemann surface. If $z_1,\ldots,z_8\in\tilde{M}$ represent distinct points of M and $\underset{\sim}{\theta} = z_1 +\ldots+ z_8$ then by theorem D2 the eight vectors $\vec{\theta}_2[2e-w(\underset{\sim}{\theta})](w(z_i))$ for $1\le i\le 8$ span a linear subspace of $\mathbb{C}^{2^g}$ of dimension at most five, and the first five of these vectors are linearly dependent precisely when $2e-w(z_6+z_7+z_8)\in W_3$; this last condition only holds for all points $z_6,z_7,z_8\in\tilde{M}$ when $2e\in W_3\ominus(-W_3)=W_6^3$, in view of B(9.12), and $2e\in W_6^3$ only for the special Riemann surfaces for which $\nu_2=2$ as described in (14). Thus so long as M is not one of these special surfaces there must be an identity of the form

$$\vec{\theta}_2\ [2e-w(\underset{\sim}{\theta})]\ (w(z_6)) = \sum_{i=1}^{5} f_i(z_1,\ldots,z_8)\ \vec{\theta}_2[2e-w(\underset{\sim}{\theta})]\ (w(z_i))$$

for some uniquely determined meromorphic functions f_i. On the other hand for these special surfaces the first five vectors are always linearly dependent, but by Theorem D2 again the first four are generally linearly independent; thus in these cases there must be an identity of the form

$$\vec{\theta}_2\ [2e-w(\underset{\sim}{\theta})]\ (w(z_5)) = \sum_{i=1}^{4} f_i(z_1,\ldots,z_8)\ \vec{\theta}_2[2e-w(\underset{\sim}{\theta})]\ (w(z_i))$$

for some uniquely determined meromorphic functions f_i. These functions can be obtained directly from the ordinary multisecant identities, but a few further preliminary observations are helpful for the second case.

If $2e \in W_6^3$ then for any divisor $z_1+z_2+z_3$ on M there is another divisor $z_1^*+z_2^*+z_3^*$ on M such that

$$(16) \qquad 2e = w(z_1+z_2+z_3+z_1^*+z_2^*+z_3^*),$$

and this second divisor is unique so long as $2e-w(z_1+z_2+z_3) \notin W_3^1$. Thus upon identifying divisors on M with points of the symmetric product of M and setting

$$(17) \qquad X = \left\{ z_1+z_2+z_3 \in M^{(3)} : 2e-w(z_1+z_2+z_3) \in W_3^1 \right\}$$

the mapping that associates to any divisor $z_1+z_2+z_3 \in M^{(3)} \sim X$ the divisor $z_1^*+z_2^*+z_3^*$ is a well defined mapping

$$(18) \qquad F : M^{(3)} \sim X \to M^{(3)};$$

this mapping is evidently holomorphic, since the unique element $h \in \mathbb{P}\,\Gamma(\rho_{2e}\zeta^6)$ vanishing at a divisor $\underset{\sim}{\theta} = z_1+z_2+z_3 \in M^{(3)} \sim X$ depends holomorphically on $\underset{\sim}{\theta}$ and $F(\underset{\sim}{\theta})$ is the divisor of h. If M is a trigonal Riemann surface of genus $g=4$ for which $W_3^1 \subset J$ consists of the single point e then X is clearly the one-dimensional subvariety consisting of all divisors of the form $z+E'z+E''z$ as z varies over M; the natural mapping $z \to z+E'z+E''z$ factors through the trigonal projection $M \to \mathbb{P}^1$, and exhibits X as being biholomorphic to $\mathbb{P}^1$. If M is a trigonal

Riemann surface of genus $g=3$ then $W_3^1=k-W_1$, so that $e=k-w(a)$ for some fixed point $a \in M$, and $2e-w(z_1+z_2+z_3)=2k-w(z_1+z_2+z_3+2a)\in W_3^1$ precisely when $k=w(z_1+z_2+z_3+2a-z)$ for some point $z \in M$; but any meromorphic Abelian differential with at most a simple pole must actually be holomorphic, since it has total residue zero, so for a canonical divisor of this form necessarily either $z=z_i$ for some index i or $z=a$. Thus in this case X consists of all divisors $z_1+z_2+z_3$ for which $z_1+z_2+z_3+a$ is a canonical divisor. If ω',ω'' are any two linearly independent Abelian differentials vanishing at a then the canonical divisors of this form are precisely the divisors of the differentials $c'\omega'+c''\omega''$ for all points $(c',c'') \in \mathbb{P}^1$, so again X is a one-dimensional subvariety of $M^{(3)}$ biholomorphic to $\mathbb{P}^1$. The symmetric product $\tilde{M}^{(3)}$ is the universal covering space of $M^{(3)}$, by essentially the same argument as for the two-fold symmetric product, and the simply-connected subvariety $X=\mathbb{P}^1 \subset M^{(3)}$ lifts to a subset $\tilde{X} \subset \tilde{M}^{(3)}$ that consists of a number of disjoint homeomorphic copies of X. Here $\tilde{M}$ is an open disc, and the homogeneous polynomials of degree three exhibit $\tilde{M}^{(3)}$ as an open subset of $\mathbb{C}^3$; the complement of a set of disjoint two-spheres in an open subset of $\mathbb{R}^6$ is simply-connected, so that $\tilde{M}^{(3)} \sim \tilde{X}$ is the universal covering space of $M^{(3)} \sim X$. At any rate F can always be lifted to a holomorphic mapping

$$(18') \qquad F : \tilde{M}^{(3)} \sim \tilde{X} \to M^{(3)},$$

and since this is a mapping between subsets of $\mathbb{C}^3$ and $\tilde{X}$ is a one-dimensional holomorphic subvariety it follows from known removable singularity theorems that F extends further to a holomorphic mapping

$$(18'') \qquad F : \tilde{M}^{(3)} \to \tilde{M}^{(3)}.$$

Thus to any divisor $\theta = z_1+z_2+z_3$ on $\tilde{M}$ there is associated a unique divisor $F(\theta) = z_1^*+z_2^*+z_3^*$ on $\tilde{M}$, depending holomorphically on θ, such that (16) holds for these divisors and the hyperelliptic point $e \in \mathbb{C}^g$. There are clearly a number of choices for this extension (18″) of the mapping (18), even for a particular choice of $e \in \mathbb{C}^g$.

Theorem 7. If M is a trigonal Riemann surface with trigonal point e then for any points $z_1,\ldots,z_8 \in \tilde{M}$

$$0 = \sum_{i=1}^{6} \frac{q(z_i,E'z_7)q(z_i,E''z_7)q(z_i,E'z_8)q(z_i,E''z_8)}{\prod\limits_{\substack{1\le j\le 6\\ j\ne i}} q(z_i,z_j)} \vec{\theta}_2[2e-w(z_1+\ldots+z_8)]\,(w(z_i)).$$

If M is one of the special trigonal surfaces with Maroni invariant $\nu_2=2$ and $F(z_6+z_7+z_8) = z_6^*+z_7^*+z_8^*$ for the mapping (18″) then in addition

$$0 = \sum_{i=1}^{5} \frac{q(z_i,z_6^*)\, q(z_i,z_7^*)\, q(z_i,z_8^*)}{\prod\limits_{\substack{1\le j\le 5\\ j\ne i}} q(z_i,z_j)} \vec{\theta}_2\,[2e-w(z_1+\ldots+z_8)]\,(w(z_i)).$$

Proof. The first assertion of the theorem follows immediately from the case n=6 of Theorem D9 upon setting $x_1=E'z_7$, $x_2=E''z_7$,

$x_3=E'z_8$, $x_4=E'z_8$ and noting from (3) that $w(x_1+\ldots+x_4-z_1-\ldots-z_6) = 2e-w(z_1+\ldots+z_8)$. The second assertion follows from the case n=5 of that theorem upon setting $x_1=z_6^*$, $x_2=z_7^*$, $x_3=z_8^*$ for the divisor $F(z_6+z_7+z_8)=z_6^*+z_7^*+z_8^*$ and noting from (16) that $w(x_1+\ldots+x_8-z_1-\ldots-z_5) = 2e-w(z_1+\ldots+z_8)$.

The general situation is much like the special one just considered, but with a greater variety of supplemental trigonal multisecant identities for surfaces of small genus. Rather than attempting here to describe the general situation it is enough just to examine one further special case, that involving the point 4e. For any trigonal surface of genus g>10 it was noted in (15) that $\gamma(\rho_{4e}\zeta^{12})=5$, so that $4e \in W^4_{12}\sim W^5_{12}$; however for surfaces of genus $g\le 10$ the value $\gamma(\rho_{4e}\zeta^{12})=\nu+1$ may exceed 5, as tabulated in (15), so that $4e \in W^{\nu}_{12} \sim W^{\nu+1}_{12}$ for the value of ν indicated. In any case if $z_1,\ldots,z_{14} \in \tilde{M}$ represent distinct points of M and $\underset{\sim}{\theta}=z_1+\ldots+z_{14}$ then by Theorem D2 the fourteen vectors $\vec{\theta}_2[4e-w(\underset{\sim}{\theta})]\,(w(z_i))$ for $1\le i\le 14$ span a linear subspace of $\mathbb{C}^{2^g}$ of dimension $13-\nu$. There must thus be an identity of the form

$$\vec{\theta}_2[4e-w(\underset{\sim}{\theta})]\;(w(z_{14-\nu}) = \sum_{i=1}^{13-\nu} f_i(z_1,\ldots,z_{14})\;\vec{\theta}_2[4e-w(\underset{\sim}{\theta})](w(z_i)$$

for some uniquely determined meromorphic functions f_i. The general case is that in which $\nu=4$, as observed above, and yields results precisely paralleling the general case for the point 2e.

If $\gamma(\rho_{4e}\zeta^{12})=\nu+1$ then for any divisor $z_1+\ldots+z_\nu$ on M there is another divisor $z_1^*+\ldots+z_{12-\nu}^*$ on M such that

$$(19) \qquad 4e = w(z_1+\ldots+z_\nu+z_1^*+\ldots+z_{12-\nu}^*),$$

since there will be some nontrivial function in $\Gamma(\rho_{4e}\zeta^{12})$ vanishing at the divisor $z_1+\ldots+z_\nu$ and the full divisor of that function has image $4e$ in the Jacobi variety. If

$$X_\nu = \left\{ z_1 +\ldots+z_\nu \in M^{(\nu)} \ : \ 4e-w(z_1+\ldots+z_\nu) \in W_{12-\nu}^1 \right\}$$

then whenever $z_1+\ldots+z_\nu \notin X_\nu$ the divisor $z_1^*+\ldots+z_{12-\nu}^*$ satisfying (19) is uniquely determined, so must as usual depend holomorphically on $z_1+\ldots+z_\nu$; there is thus a well defined holomorphic mapping

$$F_\nu \ : M^{(\nu)} \sim X_\nu \to M^{(12-\nu)}$$

such that the divisors $z_1+\ldots+z_\nu$ and $F(z_1+\ldots+z_\nu)=z_1^*+\ldots+z_{12-\nu}^*$ satisfy (19). Note that in terms of a basis $f_i \in \Gamma(\rho_{4e}\zeta^{12})$ the set X_ν can be characterized as the set of those divisors $z_1+\ldots+z_\nu$ such that rank $\{f_i(z_j)\}<\nu$ when the points z_i are distinct, with the customary modification for coincidences; thus X_ν is necessarily a proper holomorphic subvariety of $M^{(\nu)}$. Now the universal covering space of $M^{(\nu)}$ is $\tilde{M}^{(\nu)}$, just as for the case $\nu=2$; so if $\pi: \tilde{M}^{(\nu)}\to M^{(\nu)}$ is the covering mapping and $\tilde{X}_\nu = \pi^{-1}(X_\nu)$ the induced mapping $F_\nu\pi: \tilde{M}^{(\nu)} \sim \tilde{X}_\nu \to M^{(12-\nu)}$ can be lifted to a holomorphic mapping

$$F_\nu \ : \tilde{M}^{(\nu)} \sim \tilde{X}_\nu \to \tilde{M}^{(12-\nu)},$$

which is unique up to a covering translation on $\tilde{M}^{(12-\nu)}$. Here $\tilde{M}^{(\nu)}$ and $\tilde{M}^{(12-\nu)}$ can be identified with bounded open subsets of $\mathbb{C}^{\nu}$ and $\mathbb{C}^{12-\nu}$ respectively; the component functions of the mapping F_{ν} are then bounded holomorphic functions in the complement of the holomorphic subvariety $\tilde{X}_{\nu}$, and by familar theorems on removable singularities they extend to holomorphic functions on all of $\tilde{M}^{(\nu)}$. Thus there is actually a holomorphic mapping

$$(20) \qquad F_{\nu} : \tilde{M}^{(\nu)} \to \tilde{M}^{(12-\nu)}$$

that associates to any divisor $z_1+\ldots+z_{\nu}$ on $\tilde{M}$ a divisor $F_{\nu}(z_1+\ldots+z_{\nu})=z_1^*+\ldots+z_{12-\nu}^*$ on $\tilde{M}$ such that (19) holds; the mapping can be chosen so that $e \in \mathbb{C}^g$ in this formula is the trigonal point (3), but is still not uniquely determined. In the general case, that for which $\nu=4$, all the functions in $\Gamma(\rho_{4e}\zeta^{12})$ can be expressed in terms of products of functions in $\Gamma(\rho_e\zeta^3)$ and this construction can be expressed in terms of the trigonal correspondence; indeed

$$F_4(z_1+z_2+z_3+z_4) = E'z_1+E''z_1+\ldots+ E'z_4+E''z_4 \in \tilde{M}^{(8)}$$

in this case, but there is nothing similar otherwise.

Theorem 8. If M is a trigonal Riemann surface with trigonal point e then for any points $z_1,\ldots,z_{14} \in \tilde{M}$

$$0 = \sum_{i=1}^{10} \frac{\prod_{j=11}^{14} q(z_i, E'z_j)q(z_i, E''z_j)}{\prod_{\substack{k=1\\k\neq i}}^{10} q(z_i, z_k)} \vec{\theta}_2[4e-w(z_1+\ldots+z_{14})]\ (x(z_i)).$$

If in particular $e \in W_{12}^{\nu} \sim W_{12}^{\nu+1}$ for $4 \le \nu \le 9$ and $F_\nu(z_{15-\nu}+\ldots+z_{14}) = x_1+\ldots+x_{12-\nu}$ then in addition

$$0 = \sum_{i=1}^{14-\nu} \frac{\prod_{j=1}^{12-\nu} q(z_i, x_j)}{\prod_{\substack{k=1\\k\neq i}}^{14-\nu} q(z_i, z_k)} \vec{\theta}_2[4e-w(z_1+\ldots+z_{14})]\ (w(z_i)).$$

<u>Proof</u>. The first assertion of the theorem follows immediately from the case n=10 of Theorem D2 upon setting $x_1=E'z_{11}$, $x_2=E''z_{11}, \ldots,$ $x_7=E'z_{14}$, $x_8=E''z_{14}$ and noting from (3) that $w(x_1+\ldots+x_8-z_1-\ldots-z_{10})=$ $4e-w(z_1+\ldots+z_{14})$. The second assertion follows from the case $n=14-\nu$ of Theorem D2 upon setting $x_1+\ldots+x_{12-\nu}=F_\nu^*(z_{15-\nu}+\ldots+z_{14})$ and noting from (19) that $w(x_1+\ldots+x_{12-\nu} - z_1-\ldots-z_{14-\nu}) = 4e-w(z_1+\ldots+z_{14})$.