Riemann Surfaces and Second-Order Theta Functions

by R. C. Gunning

Theta functions have played a major role in the investigation of compact Riemann surfaces ever since Riemann's own pioneering work, and remain at the core of much current research in this area. The theta functions associated to Jacobi varieties of compact Riemann surfaces have a number of quite special properties not shared by more general theta functions; that is the key to their role and the aspect of their study that will be considered here. Familiar models of such properties are Riemann's vanishing and singularity theorems; possibly less familiar models are Fay's trisecant identity and addition theorem. These properties seem inevitably to have both richer and simpler structures when expressed in terms of second-order theta functions, no doubt reflecting the fact that there are enough second-order theta functions to embody almost the full function field of the Jacobi variety. The analogues of Riemann's theorems for second-order theta functions are simple descriptions of all the subvarieties of special positive divisors in terms of these functions. Fay's theorems are linearized and hence in some ways more tractable when expressed in terms of the second-order theta functions; extensions of the addition theorem are in some ways more natural in this context, and yield a fascinating further structure that underlies the expression of solutions of some standard nonlinear partial differential equations in terms of theta functions. The goal of the present book is to discuss precisely these topics. What is known remains quite overwhelmed by what is yet unknown; this is more an introduction to an active and open area of current research than the survey of a complete and polished theory.

No previous knowledge of theta functions will be presupposed; the necessary background is provided in section A, which covers those properties of theta functions that will be needed in the subsequent discussion and establishes the notation and terminology that will be used. It is not intended as a general introduction to the whole theory of theta functions, though, so nothing is said about algebraic tori of other types, about polarizations other than the principal polarization, or about identities satisfied by more general theta functions. On the other hand at least a nodding acquaintance with Riemann surfaces will be presupposed; while a general background survey is provided in section B, it is probably too brief to serve by itself as a sufficient introduction to the subject. The primary purposes of this section, in addition to establishing the notation and terminology to be used, are to treat some topics, such as the prime functions, which are not generally covered in a first course but will be needed subsequently here, and also to introduce some new material, such as the quadratic period functions.

After these two introductory sections, the special properties of first-order theta functions on Jacobi varieties are discussed in section C. Riemann's theorem is of course the major classical result, and the first four subsections are a survey of this and other now standard material, based to a considerable extent on the treatment by John Fay in his book "Theta Functions on Riemann Surfaces" (Springer, 1973). The only novelty here is a greater reliance on the role of the prime functions than is customary. The discussion in the last three subsections of the second derivatives and second-order Gauss mapping does contain some new material though. The analogues of Riemann's theorem for second-order theta functions are discussed at the beginning of section D, and are then used to derive Fay's trisecant identity and some generalizations and limiting cases, much as in my paper in the American Journal of Mathematics (vol 108, 1986, pages 39-74). The treatment in section E of the generalized trisecant identities for hyperelliptic and trigonal surfaces, and some limiting cases of the former, are new. Most interesting here is the novel approach to demonstrating that solutions to the KDV and some more general fourth-order partial differential equations can be expressed in terms of theta functions on the Jacobi varieties of hyperelliptic surfaces. An extensive generalization of Fay's addition theorem, amounting to an expansion of the second-order theta functions on appropriate symmetric subvarieties of the Jacobi variety in terms of the canonical holomorphic and meromorphic Abelian differentials, is given in section F. This leads to a very interesting new set of invariants associated to Riemann surfaces, as discussed in sections F and G. These invariants yield a reasonably effective approach to the problem of classifying all the systems of partial differential equations satisfied by the second-order thetanullwerte. The classification of all fourth-order systems on hyperelliptic Jacobi varieties is described in detail; the new systems that appear are probably worth even further analysis. Finally the cases of Jacobi varieties of genus three and four are worked out to illustrate the general situation.

A. Theta Functions

§1. Complex tori and their period matrices.

A lattice subgroup L of a vector space $\mathbb{R}^n$ is an additive subgroup generated by n linearly independent vectors. By a suitable change of coordinates in $\mathbb{R}^n$ these generators can be taken to be the standard basis vectors, and the lattice subgroup L can thereby be identified with the subgroup $\mathbb{Z}^n \subseteq \mathbb{R}^n$; the quotient group is then just $\mathbb{R}^n/\mathbb{Z}^n = (\mathbb{R}/\mathbb{Z})^n$, the Cartesian product of n circles or an n-dimensional torus. A lattice subgroup L of a complex vector space $\mathbb{C}^g$ is just a lattice subgroup of the underlying real vector space $\mathbb{R}^{2g}$, hence is an additive subgroup generated by 2g vectors in $\mathbb{C}^g$ that are linearly independent over the real numbers; the quotient group $\mathbb{C}^g/L$ is topologically a 2g-dimensional torus, but also has the natural structure of a g-dimensional complex manifold and as such will be called a g-dimensional complex torus. These manifolds are the primary objects of interest here.

A choice of 2g generators $\lambda_1, \dots, \lambda_{2g}$ of a lattice subgroup $L \subseteq \mathbb{C}^g$ will be called a marking of the complex torus $J=\mathbb{C}^g/L$. If points in $\mathbb{C}^g$ are viewed as column vectors of length g, these 2g vectors can be viewed as forming a $g \times 2g$ complex matrix $\Lambda = \{\lambda_1, \dots, \lambda_{2g}\}$, which will be called the period matrix of the marked complex torus J. In these terms the lattice subgroup $L = \Lambda\, \mathbb{Z}^{2g} \subseteq \mathbb{C}^g$, the subgroup of $\mathbb{C}^g$ generated by the 2g columns of the matrix Λ. Note that if $\lambda'_1, \dots, \lambda'_{2g}$ is another marking of J then $\lambda'_i = \Sigma_j \lambda_j q_{ji}$ where $Q = \{q_{ji}\} \in GL(2g, \mathbb{Z})$ and $\Lambda' = \Lambda Q$; conversely any invertible $2g \times 2g$ integral matrix Q determines in this manner another marking of J, with the period matrix as indicated. Of course it is also possible to make a complex change of coordinates in the vector space $\mathbb{C}^g$, described by an invertible $g \times g$ complex matrix $C = \{c_{ij}\} \in Gl(g,\mathbb{C})$. This change of coordinates transforms the vectors λ_j in a marking of J to vectors $C\lambda_j$, and transforms the period matrix Λ to $C\Lambda$, but merely describes in another way the same complex torus with the same marking. Two period matrices Λ, Λ' will be called <u>equivalent</u> if they correspond in this way to markings of the same complex torus, that is to say, if $\Lambda' = P\,\Lambda\,Q$ for some matrices $P \in GL(g, \mathbb{C})$ and $Q \in GL(2g, \mathbb{Z})$.

<u>Lemma 1</u>. A $g \times 2g$ complex matrix Λ is the period matrix of a g-dimensional complex torus precisely when it satisfies any of the following equivalent conditions:

(i) $\Lambda x = 0$ for $x \in \mathbb{R}^{2g}$ if and only if $x = 0$;

(ii) ${}^t\Lambda z \in \mathbb{R}^{2g}$ for $z \in \mathbb{C}^g$ if and only if $z = 0$;

(iii) the $2g \times 2g$ matrix $\begin{pmatrix} \Lambda \\ \overline{\Lambda} \end{pmatrix}$ is nonsingular.

<u>Proof</u>. First note that Λ is a period matrix precisely when its columns are linearly independent over $\mathbb{R}$, which is just condition (i).

Next note that (i) and (iii) are equivalent. Indeed if Λ satisfies (i) but not (iii) there must be some nonzero vector $z \in \mathbb{C}^{2g}$ for which $\begin{Bmatrix} \Lambda \\ \overline{\Lambda} \end{Bmatrix} z = 0$, hence for which $\Lambda z = \overline{\Lambda z} = 0$; but then $\Lambda(z \pm \overline{z}) = 0$, so since either $z + \overline{z} \neq 0$ or $i(z - \overline{z}) \neq 0$ there is some nonzero vector $x \in \mathbb{R}^{2g}$ for which $\Lambda x = 0$, in contradiction to (i). On the other hand if Λ satisfies (iii) and $\Lambda x = 0$ for some vector $x \in \mathbb{R}^{2g}$ then $\overline{\Lambda} x = 0$ so that $\begin{Bmatrix} \Lambda \\ \overline{\Lambda} \end{Bmatrix} x = 0$ and hence $x = 0$. Then note that (ii) and (iii) are equivalent. Indeed if Λ satisfies (ii) but not (iii) there must be some nonzero vector $z = \begin{pmatrix} z' \\ z'' \end{pmatrix} \in \mathbb{C}^{2g}$ for which $({}^tz', {}^tz'') \begin{Bmatrix} \Lambda \\ \overline{\Lambda} \end{Bmatrix} = 0$, hence for which ${}^tz' \Lambda + {}^tz'' \overline{\Lambda} = 0$; but then

$({}^tz' \pm {}^t\bar{z}'') \Lambda = {}^tz'\Lambda \pm {}^t\bar{z}''\Lambda = {}^tz'\Lambda \mp {}^t\bar{z}' \bar{\Lambda} = ({}^tz'\Lambda) \mp (\overline{{}^tz'\Lambda})$, so that $i({}^tz' + {}^t\bar{z}'')\Lambda \in \mathbb{R}$ and $({}^tz' - {}^t\bar{z}'')\Lambda \in \mathbb{R}$, and since not both of these values are zero that leads to a contradiction to (ii). On the other hand if Λ satisfies (iii) and ${}^t\Lambda z \in \mathbb{R}^{2g}$ then $({}^tz, -{}^t\bar{z}) \begin{Bmatrix} \Lambda \\ \overline{\Lambda} \end{Bmatrix} = {}^tz\Lambda - (\overline{{}^tz\Lambda}) = 0$ so that $z = 0$. That suffices to conclude the proof.

<u>Lemma 2</u>. Any period matrix is equivalent to one of the form $\{I, \Omega\}$, where I is the $g \times g$ identity matrix and Ω is a $g \times g$ complex matrix with nonsingular imaginary part. Moreover two period matrices $\{I, \Omega\}$ and $\{I, \Omega'\}$ are equivalent precisely when

$$\Omega' = (A + \Omega C)^{-1} (B + \Omega D) \text{ where } \begin{Bmatrix} A & B \\ C & D \end{Bmatrix} \in GL(2g, \mathbb{Z}).$$

<u>Proof</u>. Any $g \times 2g$ period matrix Λ must have g linearly independent columns, and after replacing Λ by ΛQ for some permutation matrix $Q \in GL(2g, \mathbb{Z})$ it can be supposed that the first g columns are linearly independent; thus $\Lambda Q = (P, P')$ where P, P' are $g \times g$ matrix blocks and P is nonsingular, so that $P^{-1}\Lambda Q = \{I, \Omega\}$. It follows from Lemma 1 that $\begin{Bmatrix} I & \Omega \\ I & \overline{\Omega} \end{Bmatrix}$ is nonsingular, hence so is $\begin{Bmatrix} I & \Omega \\ 0 & \Omega - \overline{\Omega} \end{Bmatrix}$, and consequently $\frac{1}{2} i^{-1}(\Omega - \overline{\Omega}) = \operatorname{Im}\Omega$ is nonsingular as well.

Next, if $\{I, \Omega\}$ and $\{I, \Omega'\}$ are equivalent then $\{I, \Omega'\} = P\{I, \Omega\} Q$ where $P \in GL(g, \mathbb{C})$ and $Q = \begin{Bmatrix} A & B \\ C & D \end{Bmatrix} \in GL(2g, \mathbb{Z})$, or equivalently $I = P(A+\Omega C)$, $\Omega' = P(B+\Omega D)$; then $A+\Omega C = P^{-1}$ is nonsingular and $\Omega' = (A+\Omega C)^{-1}(B+\Omega D)$ as desired, to complete the proof.

Any period matrix is thus equivalent to one in the normal form $\Lambda = (I,\Omega)$, and the $g \times g$ matrix block Ω appearing in this normal form will also be called a period matrix; just what is meant by a period matrix will consequently depend on the size of the matrix, whether $g \times 2g$ or $g \times g$, although the former will henceforth usually be called a full period matrix for clarity. The necessary and sufficient condition that a $g \times g$ matrix Ω be a period matrix is that $\mathrm{Im}\Omega$ be nonsingular, an evident consequence of Lemma 1 as in the proof of the preceding result. For a full period matrix in normal form the marking naturally falls into two subsets: the vectors $\lambda_1,\ldots,\lambda_g$ are the columns of the identity matrix, the standard basis for the discrete subgroup $\mathbb{Z}^g \subset \mathbb{C}^g$, while the vectors $\lambda_{g+1},\ldots,\lambda_{2g}$ are the columns of the period matrix Ω. Two alternative notations will frequently and interchangeably be used in this context. First let δ_j be the Kronecker vector, the components of which are the Kronecker symbols δ_j^k; then $\lambda_j = \delta_j$ while $\lambda_{g+j} = \Omega\delta_j$ for $1 \leq j \leq g$. Alternatively it is sometimes easier or more convenient to write $\lambda_j = \alpha_j$ and $\lambda_{g+j} = \beta_j$ for $1 \leq j \leq g$, so that $\alpha_j = \delta_j$ and $\beta_j = \Omega\delta_j$.

The present discussion will for the most part be limited to a special class of complex tori, those for which the $g \times g$ period matrix Ω satisfies the further conditions (1) that Ω is a symmetric matrix, and (2) that $\mathrm{Im}\Omega$ is a positive definite matrix. Of course if Ω is symmetric then $\mathrm{Im}\Omega$ is a symmetric real matrix, so the condition that it be positive definite is meaningful, and if $\operatorname{Im}\Omega$ is positive definite it is automatically nonsingular. The set of all $g \times g$ matrices satisfying these two conditions is called the <u>Siegel upper half-space</u> of rank g, and will be denoted by $\mathfrak{H}_g$. Any $g \times g$ symmetric matrix $\Omega = \{\omega_{ij}\}$ is determined by the $\binom{g+1}{2}$ complex numbers ω_{ij} for $1 \leq i \leq j \leq g$, so that $\mathfrak{H}_g$ can be viewed as a subset of the space of $\binom{g+1}{2}$ complex variables; moreover if Ω is positive definite so are all matrices sufficiently near Ω, so that $\mathfrak{H}_g$ is actually an open subset of the space of $\binom{g+1}{2}$ complex variables.

§2. Theta Functions.

For any vector $w \in \mathbb{C}^g$ and any matrix $\Omega \in \mathfrak{H}_g$ the associated <u>theta series</u> is formally

$$\theta(w;\Omega) = \Sigma_n \exp 2\pi i [\tfrac{1}{2}\, {}^t n\Omega n + {}^t n w], \tag{1}$$

where the summation is extended over all integral vectors $n \in \mathbb{Z}^g$.

<u>Lemma 3</u>. The theta series $\theta(w;\Omega)$ is uniformly convergent on compact subsets of $\mathbb{C}^g \times \mathfrak{H}_g$, so defines a holomorphic function in $\mathbb{C}^g \times \mathfrak{H}_g$.

<u>Proof</u>. Consider any fixed point $(w_o,\Omega_o) \in \mathbb{C}^g \times \mathfrak{H}_g$. For all points $w = x + iy$, $\Omega = X+iY$ in a sufficiently small open neighborhood of (w_o,Ω_o) the coordinate y will be bounded from below by some real number a while the minimal eigenvalue of Y will be bounded from below by some positive real number $b > 0$. Then

$$\begin{aligned} &|\exp 2\pi i [\tfrac{1}{2}\, {}^t n\Omega n + {}^t n w]| \\ &\quad = \exp -2\pi [\tfrac{1}{2}\, {}^t nYn + {}^t ny] \\ &\quad \leqq \exp -2\pi [\tfrac{1}{2} b\, {}^t n\, n + an], \end{aligned}$$

so the terms of the theta series are uniformly dominated in this neighborhood by the terms of a convergent positive series and the desired result therefore follows immediately.

For many purposes it is convenient to introduce a modification of this simple theta series by introducing some auxiliary parameters called <u>characteristics</u>. Thus for any vectors $\nu, \tau, w \in \mathbb{C}^g$ and any matrix $\Omega \in \mathfrak{H}_g$ the associated theta series with characteristic $[\nu|\tau]$ is formally

$$(2) \qquad \theta[\nu|\tau](w;\Omega) = \Sigma_n \exp 2\pi i [\tfrac{1}{2}{}^t(n+\nu)\Omega(n+\nu) + {}^t(n+\nu)(w+\tau)],$$

where again the summation is extended over all integral vectors $n \in \mathbb{Z}^g$. This is related to the simple theta series (1) as follows.

<u>Lemma 4</u>. For any characteristic $[\nu|\tau]$,

$$\theta[\nu|\tau](w;\Omega) = \theta(w+\tau+\Omega\nu;\Omega)\cdot\exp 2\pi i\,{}^t\nu\,(w+\tau+\tfrac{1}{2}\Omega\nu).$$

<u>Proof</u>. This follows immediately from the simple calculation

$$\tfrac{1}{2}\,{}^t(n+\nu)\Omega(n+\nu) + {}^t(n+\nu)(w+\tau)$$

$$= \tfrac{1}{2}\,{}^tn\Omega n + {}^tn\Omega\nu + \tfrac{1}{2}\,{}^t\nu\Omega\nu + {}^tnw + {}^tn\tau + {}^t\nu w + {}^t\nu\tau$$

$$= \tfrac{1}{2}\,{}^tn\Omega n + {}^tn\,(w + \tau + \Omega\nu) + {}^t\nu(w+\tau+\tfrac{1}{2}\Omega\nu);$$

the part that is independent of n is a common factor in all the terms of the series (2), and when factored out leaves the simple theta series (1) but with w replaced by $w + \tau + \Omega\nu$, yielding the desired result.

Thus a theta series with characteristic is, aside from a holomorphic and nowhere vanishing factor, just a translate of a simple theta series. It is evident therefore that a theta series with characteristic $[\nu|\tau]$ is a holomorphic function of the parameters $\nu \in \mathbb{C}^g$, $\tau \in \mathbb{C}^g$ as well as of the variables $w \in \mathbb{C}^g$, $\Omega \in \mathfrak{H}_g$. For later reference it is worth noting here as a simple consequence of the preceding lemma that

$$(3) \qquad \theta[\nu+\nu'|\tau+\tau'](w;\Omega)$$

$$= \theta[\nu|\tau](w+\tau'+\Omega\nu';\Omega)\cdot \exp 2\pi i\,{}^t\nu'\cdot(w+\tau+\tau'+\tfrac{1}{2}\Omega\nu').$$

A particularly obvious case of this is

$$(4) \qquad \theta[\nu\,|\tau+\tau'](w;\Omega) = \theta[\nu|\tau](w+\tau';\Omega),$$

which is also an immediate consequence of the definition (2). There are further results that will follow from combining (3) with the functional equation for the theta function, as will be discussed in the next section. A special case worth mentioning here is that

$$(5) \qquad \theta[\nu+\nu'|\tau](w;\Omega) = \theta[\nu|\tau](w;\Omega) \quad \text{if } \nu' \in \mathbb{Z}^g,$$

as is quite obvious from replacing the index of summation n in (2) by $n+\nu'$.

The reader should be warned that the definition of theta functions with characteristics is not universally agreed upon, so some care is required in comparing various treatments of this topic. The definition adopted here agrees with that used by Igusa (Theta Functions, Springer-Verlag, 1972), although he writes $\theta_{\nu,\tau}(\Omega,w)$ in place of $\theta[\nu|\tau](w;\Omega)$. The definition used by Farkas and Kra (Riemann Surfaces, Springer-Verlag, 1980) differs from this just by a factor of 2 in the characteristic, so the comparison of their notation with that used here is

$$\theta\left[\begin{smallmatrix}2\nu\\2\tau\end{smallmatrix}\right](w;\Omega) = \theta[\nu|\tau]\ (w;\Omega).$$

The older literature omits the factor πi in the variables w,Ω, so the comparison of the notation of Krazer and Wirtinger (Lehrbuch der Thetafunktionen, Teubner, 1903), or Conforto (Abelsche Funktionen und Algebraische Geometrie, Springer, 1956) with that used here is

$$\theta\ \left[\begin{smallmatrix}\nu\\\tau\end{smallmatrix}\right](\pi i w;\ \pi i\Omega) = \theta\ [\nu|\tau]\ (w;\Omega).$$

Fay (Theta Functions on Riemann Surfaces, Springer Lecture Notes in Mathematics, 1956) uses the classical notation with an extra factor of 2, so the comparison of his notation with that used here is

$$\theta\ \left[\begin{smallmatrix}\nu\\\tau\end{smallmatrix}\right](2\pi i w;\ 2\pi i\Omega) = \theta[\nu|\tau](w;\Omega).$$

In the special case that $g = 1$ and the characteristics are half-integers these functions are just the Jacobi theta functions

$$\theta[0|0](\tfrac{z}{\pi};\Omega) = \theta_3(z,q), \qquad \theta[0|\tfrac{1}{2}](\tfrac{z}{\pi};\Omega) = \theta_4(z,q),$$

$$\theta[\tfrac{1}{2}|0](\tfrac{z}{\pi};\Omega) = \theta_2(z,q), \qquad \theta[\tfrac{1}{2}|\tfrac{1}{2}](\tfrac{z}{\pi};\Omega) = \theta_1(z,q),$$

where $q = e^{\pi i \Omega}$ as is customary.

§3. The functional equation for theta functions.

The basic property of a theta function $\theta[\nu|\tau](w;\Omega)$ as a function of the variable w is the functional equation it satisfies under translation by any vector in the lattice subgroup with period matrix Ω, as follows.

Theorem 1. For any vectors $p, q \in \mathbb{Z}^g$,

$$\theta[\nu|\tau](w+p+\Omega q;\ \Omega)$$

$$= \theta[\nu|\tau](w;\Omega)\cdot \exp 2\pi i[{}^t p\ \nu - {}^t q(w+\tau+\tfrac{1}{2}\Omega q)].$$

Proof. First note that replacing w by $w+p$ has the effect of multiplying each term in (2) by $\exp 2\pi i\,{}^t\nu\, p$, since $\exp 2\pi i\,{}^t n\, p = 1$, and therefore

$$\theta[\nu|\tau](w+p;\Omega) = \theta[\nu|\tau](w;\Omega)\cdot \exp 2\pi i\,{}^t p\ \nu.$$

Next note that replacing the index of summation n in (2) by $n+q$ does not change the value of the sum, since $n+q$ ranges over $\mathbb{Z}^g$ as n does, but replaces a typical term in (2) by

$$\exp 2\pi i\ [\tfrac{1}{2}\ {}^t(n+\nu+q)\ \Omega\ (n+\nu+q) + {}^t(n+\nu+q)\ (w+\tau)]$$

$$= \exp 2\pi i\ [\tfrac{1}{2}{}^t(n+\nu)\Omega(n+\nu) + {}^t(n+\nu)(w+\tau+\Omega q) + {}^t q(\tfrac{1}{2}\Omega q + w+\tau)]$$

and consequently

$$\theta[\nu|\tau](w;\Omega) = \theta[\nu|\tau](w+\Omega q;\Omega)\cdot \exp 2\pi i\,{}^t q[w+\tau+\tfrac{1}{2}\Omega q].$$

Combining these two formulas yields the desired result and thereby concludes the proof.

Both the result of the preceding theorem and the role of the characteristics can be clarified and perhaps made to appear more natural by an appropriate interpretation. In general terms, the theorem asserts that the theta function satisfies a functional equation of the form

$$(1)\qquad \theta[\nu|\tau](w+\lambda;\Omega) = \theta[\nu|\tau](w;\Omega)\cdot \mu(\lambda,w)$$

for any lattice vector $\lambda = p+\Omega q = (I,\Omega)\binom{p}{q} \in L$, where L is the lattice subgroup with period matrix Ω; here μ is a holomorphic and nowhere-vanishing function of the complex variable $w \in \mathbb{C}^g$ and depends as well on the lattice vector λ. It is a simple formal consequence of (1) that the function μ must have the property that

$$(2)\qquad \mu(\lambda_1+\lambda_2,w) = \mu(\lambda_1,w+\lambda_2)\cdot \mu(\lambda_2,w)$$

for any lattice vectors $\lambda_1,\lambda_2 \in L$. A nowhere-vanishing function μ on $L \times \mathbb{C}^g$ that satisfies (2) as a function of $\lambda \in L$ and is holomorphic as a function of $w \in \mathbb{C}^g$ is called a factor of automorphy for the action of the group L on $\mathbb{C}^g$. The simplest not totally trivial factors of automorphy are those that are actually independent of w, called the flat factors of automorphy; for these condition (2) reduces to the assertion that $\mu(\lambda_1+\lambda_2) = \mu(\lambda_1)\,\mu(\lambda_2)$, that is, that μ is a homorphism from the additive group L to the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers, which will be indicated by writing $\mu\in\mathrm{Hom}(L,\mathbb{C}^*)$. Any such factor is uniquely determined by its values on the generators of the lattice L, and these can be any values in $\mathbb{C}^*$. It is convenient to split the generators of the lattice into the two classes $\alpha_1=\lambda_1,\dots,\alpha_g=\lambda_g$ and $\beta_1=\lambda_{g+1},\dots,\beta_g=\lambda_{2g}$, and for any vector $t = {}^t(t_1,\dots,t_g)\in\mathbb{C}^g$ to introduce the two homomorphisms $\sigma_t,\rho_t\in\mathrm{Hom}\ (L,\mathbb{C}^*)$ defined by

$$
\begin{aligned}
(3)\qquad \sigma_t(\alpha_j) &= \exp 2\pi i\ t_j, &\quad \sigma_t(\beta_j) &= 1,\\
\rho_t(\alpha_j) &= 1\ , &\quad \rho_t(\beta_j) &= \exp 2\pi i\ t_j.
\end{aligned}
$$

Thus any $\phi\in\mathrm{Hom}(L,\mathbb{C}^*)$ can be written in the form $\phi = \sigma_s\rho_t$ for some vectors $s,\ t\in\mathbb{C}^g$, and these vectors are uniquely determined up to elements of the subgroup $\mathbb{Z}^g\subset\mathbb{C}^g$. Further any lattice vector $\lambda\in L$ can be written

$$\lambda = \Sigma_j(p_j\ \alpha_j + q_j\ \beta_j) = p+\Omega q$$

for some uniquely determined vectors $p,q\ \in\ \mathbb{Z}^g$, and then

$$\phi(\lambda) = \sigma_s(\lambda)\rho_t(\lambda) = \exp 2\pi i\ [{}^tp\ s + {}^tq\ t].$$

In these terms Theorem 1 can be restated as the functional equation

$$(4) \qquad \theta[\nu|\tau](w+\lambda;\Omega) = \sigma_\nu(\lambda)\rho_{-\tau}(\lambda)\ \xi(\lambda,w)\ \ \theta[\nu|\tau](w;\Omega)$$

for any lattice vector $\lambda\epsilon L$, where ξ is a factor of automorphy that is independent of the characteristic $[\nu|\tau]$; the characteristic $[\nu|\tau]$ determines the characters $\sigma_\nu, \rho_{-\tau}$. Thus the simple theta function satisfies the functional equation

$$(5) \qquad \theta(w+\lambda;\Omega) = \xi(\lambda,w)\ \ \theta(w;\Omega)$$

for any lattice vector $\lambda\epsilon L$. The factor of automorphy ξ has the explicit form

$$(6) \qquad \xi(p+\Omega q,w) = \exp -2\pi i\ {}^t q(w+\tfrac{1}{2}\Omega q),$$

or equivalently is determined by the values on the generators by

$$(7) \qquad \xi(\alpha_j,w) = 1, \qquad \xi(\beta_j,w) = \exp - 2\pi i[w_j + \tfrac{1}{2}\,\omega_{jj}]$$

where $\Omega = \{\omega_{ij}\}$.

For any given factor of automorphy μ for the action of L on $\mathbb{C}^g$, a holomorphic function f on $\mathbb{C}^g$ such that $f(w+\lambda) = \mu(\lambda,w)\ f(w)$ for all $\lambda\epsilon L$ and $w\,\epsilon\,\mathbb{C}^g$ will be called a <u>relatively automorphic function</u> for the factor of automorphy μ; the collection of all such functions form a complex vector space that will be denoted by $\Gamma(\mu)$, and the dimension of this vector space will be denoted by $\gamma(\mu)$. The theta function is essentially uniquely determined by the functional equation of the preceding theorem, a result that can be expressed in the following terms.

Theorem 2. For any vectors $\nu,\tau \in \mathbb{C}^g$ the vector space $\Gamma(\sigma_\nu\rho_{-\tau}\xi)$ is the one-dimensional vector space spanned by the theta function $\theta[\nu|\tau](w;\Omega)$.

Proof. Suppose that $f\in\Gamma(\sigma_\nu\rho_{-\tau}\xi)$ and set $f_1(w) = f(w)\cdot\exp - 2\pi i\,{}^t\nu\, w$, noting then using the explicit form for the factors of automorphy as in Theorem 1 that

$$f_1(w+p+\Omega q) = f_1(w)\cdot \exp - 2\pi i\,{}^tq(w+\tau+\tfrac{1}{2}\Omega q+\Omega\nu)$$

for any vectors $p,\ q \in \mathbb{Z}^g$. For $p = \delta_j$, $q = 0$ this means that $f_1(w+\delta_j) = f_1(w)$, so that f_1 can be viewed as a function of the g complex variables $z_j = \exp 2\pi i\, w_j$ and is then holomorphic in the region $\{(z_1,\ldots,z_g)\in\mathbb{C}^g:\ z_j \neq 0 \text{ for } j = 1,\ldots g\}$; this function has the familiar Laurent series expansion in the variables z_j, which amounts to a complex Fourier expansion in the variables w_j of the form

$$f_1(w) = \Sigma_n c_n \exp 2\pi i\,{}^tn\, w$$

for some complex constants c_n, where the summation is extended over all integral vectors $n \in \mathbb{Z}^g$. Then for $p = 0$, $q = \delta_j$ the functional equation when expressed in terms of this Fourier expansion becomes

$$\Sigma_n c_n \exp 2\pi i\ {}^t n\ [w+\Omega\delta_j]$$

$$= \Sigma_n c_n \exp 2\pi i[{}^t n\ w - {}^t\delta_j\ (w+\tau+\tfrac{1}{2}\Omega\delta_j+\Omega\nu)]$$

$$= \Sigma_n c_{n+\delta_j}\ \exp 2\pi i[{}^t n\ w - {}^t\delta_j\ (\tau+\tfrac{1}{2}\ \Omega\delta_j+\ \Omega\nu)].$$

The coefficients of $\exp 2\pi i\ {}^t n\ w$ on the two sides of this equation must be the same, so that

$$c_{n+\delta_j} = c_n \exp 2\pi i\ {}^t\delta_j[\tau+\Omega n + \tfrac{1}{2}\Omega\delta_j+\Omega\nu]$$

for every vector $n \in \mathbb{Z}^g$. This implies that all the coefficients c_n are explicitly determined by any one of them, for instance c_0 alone, and therefore the space of all such functions f_1 is one dimensional. That suffices to show the desired result.

It is perhaps worth a brief digression into generalities here, although this is not really essential to the main line of discussion. First two factors of automorphy μ,ν are called <u>equivalent</u> if there is a holomorphic and nowhere-vanishing function h on $\mathbb{C}^g$ such that $h(w+\lambda) = h(w)\ \nu(\lambda,w)/\mu(\lambda,w)$ for all $\lambda \in L$ and $w \in \mathbb{C}^g$; this is evidently an equivalence relation in the standard sense. Note that if μ,ν are equivalent then the function h can be used to give an isomorphism $\Gamma(\mu) \cong \Gamma(\nu)$ by sending a relatively automorphic function $f \in \Gamma(\mu)$ to the relatively automorphic function $h\ f \in \Gamma(\nu)$. It is easy to see that the function h is unique up to a nonzero constant factor, so this isomorphism is unique to the same extent, and it is clear that functions corresponding to one another under this isomorphism have precisely the same zeros; thus for many purposes equivalent factors of automorphy have much the same function theoretic properties, irrespective of the particular function h determining the isomorphism. The first part of the proof of Theorem 2 was really just the observation that the factor of automorphy σ_ν is equivalent to the factor of automorphy $\rho_{-\Omega\nu}$, hence that the factors of automorphy $\sigma_\nu\rho_{-\tau}\xi$ and $\rho_{-(\tau+\Omega\nu)}\,\xi$ are equivalent; with this in mind it is clear that the factors σ_ν need not play much of a role in the further discussion, and indeed they will not. Next, as a quite different point, note that a factor of automorphy μ can be used to extend the natural action of the lattice L as a group of holomorphic automorphisms of the complex manifold $\mathbb{C}^g$ to an action on the product manifold $\mathbb{C}^g\times\mathbb{C}$, by defining

$$\lambda\cdot(w,z) = (w+\lambda,\ \mu(\lambda,w)z)$$

whenever $\lambda\in L$, $w\in\mathbb{C}^g$, and $z\in\mathbb{C}^g$; the defining equation (2) for a factor of automorphy is just the condition that L does act as a group of transformations on $\mathbb{C}^g\times\mathbb{C}$ in this way. The quotient space $(\mathbb{C}^g\times\mathbb{C})/L$ has the natural structure of a complex manifold, and the obvious mapping $(\mathbb{C}^g\times\mathbb{C})/L\rightarrow\mathbb{C}^g/L$ exhibits this manifold as being locally a product $(\mathbb{C}^g/L)\times\mathbb{C}$, hence as being a holomorphic line bundle over the torus $\mathbb{C}^g/L$. The relatively automorphic functions can then be interpreted as the holomorphic sections $f\colon\mathbb{C}^g/L\rightarrow(\mathbb{C}^g\times\mathbb{C})/L$ of this line bundle, and equivalent factors of automorphy correspond precisely to biholomorphically equivalent line bundles. This more invariant approach is sometimes quite convenient and helpful.

§4. Even and odd theta functions.

The functional equation for theta functions can be expressed as a property of the characteristic rather than of the variable, by combining Theorem 1 with Lemma 4.

Theorem 3. For any vectors $p, q \in \mathbb{Z}^g$,

$$\theta[\nu+p|\tau+q](w;\Omega) = \theta[\nu|\tau](w;\Omega)\cdot \exp 2\pi i\, {}^tq\, \nu;$$

conversely whenever there are vectors $p, q \in \mathbb{C}^g$ such that

$$\theta[\nu+p|\tau+q](w;\Omega) = c\ \theta[\nu|\tau](w;\Omega)$$

for some constant $c \in \mathbb{C}$ then necessarily $p, q \in \mathbb{Z}^g$ and $c = \exp 2\pi i\, {}^tq\, \nu$.

Proof. From an application of Lemma 4 in the form of equation (2.3) and the basic functional equation of Theorem 1 it follows that for any vectors $p, q \in \mathbb{Z}^g$

$$\theta[\nu+p|\tau+q](w;\Omega) = \theta[\nu|\tau](w+q+\Omega p;\Omega)\cdot \exp 2\pi i\, {}^tp(w+\tau+q+\tfrac{1}{2}\Omega p)$$

$$= \theta[\nu|\tau](w;\Omega)\cdot \ \exp 2\pi i\ {}^tq\cdot\nu$$

as asserted, since $\exp 2\pi i\ {}^tp\ q = 1$. On the other hand if

$$\theta[\nu+p \ |\ \tau+q](w;\Omega) = c\ \theta[\nu|\tau](w;\Omega)$$

for some vectors $p,q \in \mathbb{C}^g$ and some constant $c \in \mathbb{C}$, which must be nonzero since the theta function is nontrivial, then after replacing w by $w+\lambda$ for a lattice vector $\lambda = m + \Omega n \in \mathbb{Z}^g + \Omega\,\mathbb{Z}^g = L$ and using the functional equation of Theorem 1 it follows that

$$\theta[\nu+p|\tau+q]\cdot\exp 2\pi i[{}^t m\ (\nu+p) - {}^t n\ (w+\tau+q+\tfrac{1}{2}\ {}^t\Omega n)]$$

$$= c\ \ \theta[\nu|\tau](w;\Omega)\cdot\exp 2\pi i[{}^t m\ \nu - {}^t n\ (w+\tau+\tfrac{1}{2}\ \Omega n)]$$

and hence that

$$\exp 2\pi i\,[{}^t m\ p - {}^t n\ q] = 1.$$

Since that is the case for all $m,n \in \mathbb{Z}^g$ necessarily $p,\ q \in \mathbb{Z}^g$ as well, and that suffices to conclude the proof.

The basic idea here, and one worth some emphasis, is that the functional equation of the theta function $\theta[\nu|\tau](w;\Omega)$ as a function of the variable w, as given in Theorem 1, and its functional equation as a function of the characteristic $[\nu|\tau]$, as given in Theorem 3, are really precisely equivalent to one another through the relation between the variable w and the characteristic $[\nu|\tau]$ given in Lemma 4. The functional equation in fact has a considerably simpler form in terms of the characteristic, although its basic importance is more as a property of the variable w; passing back and forth between the two forms is a very convenient tool.

There is another sort of functional equation that the theta series satisfy. It is clear that replacing the index of summation n and the parameters ν, τ, w by their negatives does not change the sum of the series (2.2), hence

(1) $\qquad \theta[-\nu|-\tau](-w;\Omega) = \theta[\nu|\tau](w;\Omega).$

In particular for the characteristic $[0|0]$ the simple theta series is an even function of the variable w,

(2) $\qquad \theta(-w;\Omega) = \theta(w;\Omega).$

It is a useful observation that the only possible theta series that can be either an even or an odd function of the variable w is that for which the characteristic $[\nu|\tau]$ is half-integral, that is to say, for which $2\nu \in \mathbb{Z}^g$ and $2\tau \in \mathbb{Z}^g$. Whether the theta function is then even or odd depends on the characteristic. To discuss this it is convenient to say that a half-integral characteristic $[\nu|\tau]$ is even or odd according as the integer $4\,{}^t\nu\,\tau$ is even or odd. In these terms the following holds.

<u>Theorem 4</u>. The function $\theta[\nu|\tau](w;\Omega)$ is an even (or odd) function of the variable w precisely when the characteristic $[\nu|\tau]$ is an even (or odd, respectively) half-integral characteristic.

<u>Proof</u>. If there is some complex constant c such that

$$c\;\theta[\nu|\tau](w;\Omega) = \theta[\nu|\tau](-w;\Omega)$$

$$= \theta[-\nu|-\tau]\;(w;\Omega)$$

$$= \theta[\nu-2\nu|\;\tau-2\tau](w;\Omega),$$

where (1) was used to pass from the first to the second line, then it follows from Theorem 3 that necessarily $2\nu \in \mathbb{Z}^g$, $2\tau \in \mathbb{Z}^g$, and moreover $c = \exp - 4\pi i\, {}^t\nu\,\tau = (-1)^{4\,{}^t\nu\cdot\tau}$ so that $c = 1$ if $[\nu|\tau]$ is an even characteristic while $c = -1$ if $[\nu|\tau]$ is an odd characteristic, just as asserted by the statement of the theorem.

Whether a half-integral characteristic is even or odd clearly only depends on the values of ν_j and τ_j modulo 1. For deciding this, and for some other purposes as well, it is convenient to view the values ν_j, τ_j as elements of $\frac{1}{2}\,\mathbb{Z}/\mathbb{Z}$, hence to view the characteristics $[\nu|\tau]$ as elements of $(\frac{1}{2}\,\mathbb{Z}/\mathbb{Z})^{2g}$. There are then 2^{2g} such characteristics altogether. Let g_+ denote the number of these that are positive, and g_- denote the number that are negative, so that $g_+ + g_- = 2^{2g}$. On the other hand

$$g_+ - g_- = \Sigma_{\nu,\tau}\,(-1)^{4\,{}^t\nu\,\tau} = \Sigma_{\nu,\tau}(-1)^{4\Sigma_j \nu_j\tau_j}$$

$$= \prod_{j=1}^{g}(\Sigma_{\nu_j,\tau_j}\;(-1)^{4\nu_j\tau_j})$$

$$= \prod_{j=1}^{g}(\;(-1)^{0\cdot 0} + (-1)^{0\cdot 1} + (-1)^{1\cdot 0} + (-1)^{1\cdot 1})$$

$$= \prod_{j=1}^{g}(2) = 2^g,$$

and solving this pair of equations for g_+, g_- leads to the values

(3) $$g_+ = 2^{g-1}\,(2^g+1), \quad g_- = 2^{g-1}\,(2^g-1).$$

There are thus approximately the same number of even as of odd half-integral characteristics for large g, although the even ones always outnumber the odd ones and do so rather markedly for small g, as in the following table:

g:	1	2	3	4	5
g_+:	3	10	36	136	528
g_-:	1	6	28	120	496
2^{2g}:	4	16	64	256	1024

For the case $g=1$ the Jacobi theta functions θ_2, θ_3, θ_4 are even while θ_1 is odd.

§5. Higher-order theta functions.

As a further extension of the theta series already considered, for any integer $r \geqq 1$ the <u>theta series of order</u> r with characteristic $[\nu|\tau]$ is formally

$$\theta_r[\nu|\tau](w;\Omega) = \theta[\tfrac{\nu}{r}|\tau](rw;r\Omega) \, . \tag{1}$$

This is evidently also a holomorphic function of all the variables and parameters ν , τ , w , Ω , and for $r = 1$ reduces to the theta function with characteristic $[\nu|\tau]$ already considered. The functional equation for this series can be deduced immediately from that of Theorem 1, and can be expressed as follows.

<u>Theorem 5</u>. For any order $r \geqq 1$, $\theta_r[\nu|\tau](w;\Omega) \in \Gamma(\sigma_\nu\rho_{-\tau}\xi^r)$.

<u>Proof</u>. For any lattice vector $\lambda = p + \Omega q$ where $p , q \in \mathbb{Z}^g$ it follows from (1) and Theorem 1 that

$$\begin{aligned}\theta_r[\nu|\tau](w + p + \Omega q;\Omega) &= \theta[\tfrac{\nu}{r}|\tau](rw + rp + r\Omega q;r\Omega)\\ &= \theta[\tfrac{\nu}{r}|\tau](rw;r\Omega) \cdot \exp 2\pi i[{}^tp\nu-{}^tq(rw + \tau + \tfrac{1}{2} r\Omega q)]\\ &= \theta_r[\nu|\tau](w;\Omega) \cdot \sigma_\nu(\lambda)\rho_{-\tau}(\lambda)\xi(\lambda,w)^r\end{aligned}$$

as desired.

A closer look at the preceding result is quite rewarding. As already observed, replacing ν by $\nu + k$ for any vector $k \in \mathbb{Z}^g$ leaves the theta functions $\theta[\nu|\tau](w;\Omega)$ unchanged. Therefore replacing ν by $\nu + rk$ for any vector $k \in \mathbb{Z}^g$ leaves the theta function $\theta_r[\nu|\tau](w;\Omega)$ unchanged but replacing ν just by $\nu + k$ may not, although the resulting new function satisfies the same functional equation since clearly $\sigma_{\nu+k} = \sigma_\nu$. It is convenient to view the parameter ν as an element of $\mathbb{C}^g/r\mathbb{Z}^g$, and then

to note that as k ranges over the set $\mathbb{Z}^g/r\mathbb{Z}^g$ there arise the r^g functions $\theta_r[\nu+k|\tau](w;\Omega) \in \Gamma(\sigma_\nu\rho_{-\tau}\xi^r)$. These functions are actually distinct, and indeed the following holds.

Theorem 6. For any order $r \geqq 1$ the vector space $\Gamma(\sigma_\nu\rho_{-\tau}\xi^r)$ has dimension $\gamma(\sigma_\nu\rho_{-\tau}\xi^r) = r^g$, and the r^g functions $\theta_r[\nu+k|\tau](w;\Omega)$ as k ranges over $\mathbb{Z}^g/r\mathbb{Z}^g$ form a basis.

Proof. The dimension $\gamma(\sigma_\nu\rho_{-\tau}\xi^r)$ can be calculated, or at least bounded, just as in the proof of Theorem 2. If $f \in \Gamma(\sigma_\nu\rho_{-\tau}\xi^r)$ then $f_1(w) = f(w) \exp - 2\pi r {}^t\nu\, w$ satisfies the functional equation

$$f_1(w + p + \Omega q) = f_1(w) \cdot \exp - 2\pi i {}^t q(rw + \tau + \tfrac{1}{2} r\Omega q + \Omega\nu)]$$

for any vectors p, $q \in \mathbb{Z}^g$. For $p = \delta_j$, $q = 0$ this means that $f_1(w + \delta_j) = f_1(w)$, so that f_1 has a complex Fourier expansion

$$f_1(w) = \sum_n c_n \exp 2\pi i {}^t n \quad w$$

for some constants c_n, where the summation is extended over all integral vectors $n \in \mathbb{Z}^g$. Then for $p = 0$, $q = \delta_j$ the functional equation when expressed in terms of this Fourier expansion becomes

$$\begin{aligned}\sum_n c_n \exp 2\pi i {}^t n \quad [w + \Omega\delta_j]\\ &= \sum_n c_n \exp 2\pi i[{}^t n\, w - {}^t\delta_j(rw + \tau + \tfrac{1}{2} r\Omega\delta_j + \Omega\nu)]\\ &= \sum_n c_{n+r\delta_j} \exp 2\pi i[{}^t n\, w - {}^t\delta_j(\tau + \tfrac{1}{2} r\Omega\delta_j + \Omega\nu)]\ .\end{aligned}$$

The coefficients of $\exp 2\pi i\,{}^t n\, w$ on the two sides of this equation must be the same, so that

$$c_{n+r\delta_j} = c_n \exp 2\pi i\,{}^t\delta_j[\tau + \Omega n + \tfrac{1}{2} r\Omega\delta_j + \Omega\nu]$$

for every vector $n \in \mathbb{Z}^g$. This means that the coefficients c_n for n varying over any congruence class modulo $r\mathbb{Z}^g$ are explicitly determined by any one of them, so that for instance all these coefficients are uniquely determined by those for which $n_j = 0,1,\ldots,r-1$. That at least implies that $\gamma(\sigma_\nu\rho_{-\tau}\xi^r) \leqq r^g$.

In order to complete the proof it is then sufficient just to show that the theta functions $\theta_r[\nu + k|\tau](w;\Omega)$ as k varies over $\mathbb{Z}^g/r\mathbb{Z}^g$ are linearly independent. For this purpose note that the series expansions of these functions have the form

$$\begin{aligned}
&\theta_r[\nu + k \mid \tau]\,|(w;\Omega) \\
&\quad = \sum_n \exp 2\pi i[\tfrac{1}{r}\,{}^t(rn + \nu + k)\Omega(rn + \nu + k) + {}^t(rn + \nu + k)(w + \tfrac{1}{r}\tau)] \\
&\quad = [\sum_n c_n \exp 2\pi i\,{}^t(rn + k)\, w] \exp 2\pi i\,{}^t\nu\, w\,.
\end{aligned}$$

Thus as k varies over $\mathbb{Z}^g/r\mathbb{Z}^g$ the separate theta series involve separate Fourier terms, so must be linearly independent as desired.

The first part of the proof of the preceding theorem can be carried out for the case $r = 0$ as well, and shows that if $f \in \Gamma(\sigma_\nu\rho_{-\tau})$ then the function f_1 defined by $f_1(w) = f(w) \cdot \exp - 2\pi i\,{}^t\nu\, w$ has a complex Fourier expansion in which the coefficients must satisfy

$$c_n = c_n \cdot \exp 2\pi i\, {}^t\delta_j(\tau + \Omega n + \Omega\nu) \; ;$$

consequently $c_n = 0$ unless ${}^t\delta_j(\tau + \Omega(n + \nu)) \in \mathbb{Z}$ for all j, that is to say, unless $\tau + \Omega(n + \nu) \in \mathbb{Z}^g$. It thus follows that

$$(2) \qquad \Gamma(\sigma_\nu \rho_{-\tau}) = 0 \quad \text{unless} \quad \tau + \Omega\nu \in \underline{L} = (I,\Omega)\, \mathbb{Z}^{2g}.$$

Moreover if $\tau + \Omega\nu = p + \Omega q$ then $\Gamma(\sigma_\nu\rho_{-\tau})$ is the one-dimensional space spanned by the function $f(w) = \exp 2\pi i\, {}^t(\nu - q)w$; this function is never zero, so the factor of automorphy $\sigma_\nu\rho_{-\tau}$ is equivalent to the trivial factor, and conversely whenever $\sigma_\nu\rho_{-\tau}$ is equivalent to the trivial factor then it has a nontrivial relatively automorphic function so by (2) necessarily $\tau + \Omega\nu \in \underline{L}$. Incidentally any representation in $\mathrm{Hom}(\underline{L},\mathbb{C}^*)$ can be written in the form $\sigma_\nu\rho_{-\tau}$ for some parameters ν, τ, and these observations show that such a representation is equivalent as a factor of automorphy to the trivial representation precisely where $\tau + \Omega\nu \in \underline{L}$. Having made these observations it is easy to see that

$$(3) \qquad \Gamma(\sigma_\nu\rho_{-\tau}\xi^r) = 0 \quad \text{whenever} \quad r < 0 \; .$$

Indeed if $f \in \Gamma(\sigma_\nu\rho_{-\tau}\xi^r)$ for some integer $r < 0$ then

$$\theta_{-r}[0|\tau_1](w;\Omega)\; f(w) \;\in\; \Gamma(\sigma_\nu\rho_{-\tau-\tau_1}) \; ,$$

and by (2) this product must vanish identically whenever τ_1 is chosen so that $\tau + \tau_1 + \Omega\nu \notin \underline{L}$, as is quite clearly possible.

Since the factor of automorphy σ_ν is equivalent to $\rho_{-\Omega\nu}$ it can often and here generally will be dropped from further consideration. The parameter ν in the characteristic $[\nu|\tau]$ is still of importance though, in describing the basis for the vector space $\Gamma(\rho_{-\tau}\xi^r)$ as in Theorem 6. It is convenient for some purposes to view the functions of this basis as the components of a vector-valued function

$$(4) \qquad \vec{\theta}_r[\tau](w;\Omega) = \{\theta_r[\nu|\tau](w;\Omega) \; : \; \nu \in \mathbb{Z}^g/r\mathbb{Z}^g\} \; .$$

As further simplifications of the notation set $\vec{\theta}_r(w;\Omega) = \vec{\theta}_r[0](w;\Omega)$, noting that

$$(5) \qquad \vec{\theta}_r[\tau](w;\Omega) = \vec{\theta}_r(w + \tau/r \; ; \; \Omega) \; .$$

In much of the further discussion the period matrix Ω will be fixed, and will simply be dropped from the notation whenever confusion is unlikely to result. Thus what is of interest is the holomorphic mapping

$$\vec{\theta}_r[\tau] \; : \; \mathbb{C}^g \to \mathbb{C}^{r^g}$$

which has the property that $\vec{\theta}_r[\tau](w + \lambda) = \rho_{-\tau}(\lambda)\xi(\lambda,w)^r \; \vec{\theta}_r[\tau](w)$ for all lattice vectors $\lambda \in \underline{L} = (I,\Omega)\,\mathbb{Z}^{2g}$. If the component functions do not have any common zeros in $\mathbb{C}^g$ then the value $\vec{\theta}_r[\tau](w)$ can be viewed as the set of homogeneous coordinates of a point in the complex projective space of dimension $r^g - 1$, and the values $\vec{\theta}_r[\tau](w)$ and $\vec{\theta}_r[\tau](w + \lambda)$ then determine the same point in projective space for any lattice vector λ . This thereby induces a holomorphic mapping

$$\vec{\theta}_r[\tau] : J \to \mathbb{P}^{r^g-1}$$

from the complex torus $J = \mathbb{C}^g/\underline{L}$ into projective space.

§6. The addition theorem.

Since $\theta[0|\tau](w;\Omega) \in \Gamma(\rho_{-\tau}\xi)$ as a function of w it is clear that

$$\theta[0|\tau_1](w;\Omega) \cdot \theta[0|\tau_2](w;\Omega) \in \Gamma(\rho_{-\tau_1-\tau_2}\xi^2)$$

and it then follows from Theorem 6 that

$$\theta[0|\tau_1](w;\Omega) \cdot \theta[0|\tau_2](w;\Omega) = \sum_\nu c_\nu(\tau_1,\tau_2) \cdot \theta_2[\nu|\tau_1+\tau_2](w;\Omega)$$

for some uniquely determined values $c_\nu(\tau_1,\tau_2)$, which being uniquely determined are easily seen to be holomorphic functions of the parameters τ_1, $\tau_2 \in \mathbb{C}^g$, where the summation is extended over all indices $\nu \in \mathbb{Z}^g/2\mathbb{Z}^g$. To examine these functions $c_\nu(\tau_1,\tau_2)$ further note that matters can be simplified somewhat by translating w so as to reduce to the special case that $\tau_2 = 0$, and it is then more convenient to set $\tau_1 = 2\tau$ so that the formula becomes

$$\theta[0|2\tau](w;\Omega)\cdot\theta[0|0](w;\Omega) = \sum_\nu c_\nu(\tau)\cdot\theta_2[\nu|2\tau](w;\Omega) .$$

Now $\theta[0|2\tau](w;\Omega) = \theta[0|w](2\tau;\Omega)$ has the property that for any vectors p, $q \in \mathbb{Z}^g$

$$\begin{aligned}\theta[0|w](2(\tau + p + \Omega q)\ ;\ \Omega) &= \theta[0|w](2\tau + 2p + \Omega 2q\ ;\ \Omega)\\ &= \theta[0|w](2\tau\ ;\ \Omega) \cdot \exp - 2\pi i\,{}^tq(4\tau + 2w + 2\Omega q) ,\end{aligned}$$

hence as a function of τ belongs to the space $\Gamma(\rho_{-2w}\xi^4)$, while on the other hand $\theta_2[0|2\tau](w;\Omega) = \theta_2[0|2w](\tau;\Omega)$ so as a function of τ belongs to the space $\Gamma(\rho_{-2w}\xi^2)$; it must consequently be the case that $c_\nu(\tau) \in \Gamma(\xi^2)$, so that $c_\nu(\tau)$ can be written as a linear combination of the second-order theta functions of τ that form the basis for $\Gamma(\xi^2)$ described in Theorem 6. There thus results an identity of the form

$$\theta[0|2\tau](w;\Omega) \cdot \theta[0|0](w;\Omega) = \sum_{\nu,\mu} c_{\nu\mu} \cdot \theta_2[\mu|0](\tau;\Omega)\ \theta_2[\nu|2\tau](w;\Omega)$$

for some constants $c_{\nu\mu}$, where the summation is extended over all indices ν , $\mu \in \mathbb{Z}^g/2\mathbb{Z}^g$. These constants can be viewed as forming a $2^g \times 2^g$ matrix C , and the above equation can be rewritten more simply as

$$\theta(w + 2\tau)\ \theta(w) = {}^t\vec{\theta}_2(\tau) \cdot C \cdot \vec{\theta}_2(w + \tau) \ ,$$

in terms of the simple first-order theta functions and the vector of second-order theta functions. This can be simplified even further by replacing w by $w - \tau$, leading to the form

$$\theta(w + \tau)\ \theta\ (w - \tau) = {}^t\vec{\theta}_2(\tau) \cdot C \cdot \vec{\theta}_2(w) \ .$$

The left-hand side is unchanged when w and τ are exchanged, since θ is an even function, so the same holds on the right-hand side; the entries in the vector ${}^t\vec{\theta}_2$ are linearly independent functions, so it is clear that the matrix C must be symmetric. It is thus possible to choose some basis for the second-order theta functions that reduces this matrix to the identity matrix, a natural basis from some point of view and one that is unique up to an orthogonal change of basis. In fact the basis chosen here already has that property, so is doubly natural.

Theorem 7. $\theta(w+\tau)\theta(w-\tau) = {}^t\vec{\theta}_2(\tau)\cdot\vec{\theta}_2(w)$.

Proof. The work lies just in calculating the coefficients $c_{\mu\nu}$ in the preceding formula, but it is as easy to rederive the whole formula in the process. Write

$$\begin{aligned}\theta(w+2\tau)\theta(w) &= \theta[0|2\tau](w;\Omega)\ \theta[0|0](w;\Omega)\\ &= \sum_{m,n} \exp 2\pi i[\ \tfrac{1}{2}{}^t m\Omega m + {}^t m(w+2\tau) + \tfrac{1}{2}\ {}^t n\Omega n + {}^t nw]\\ &= \sum_{m,n} \exp 2\pi i[\tfrac{1}{2}\ {}^t(m+n)\Omega(m+n) + {}^t(m+n)w - {}^t m\Omega n + 2\,{}^t m\tau].\end{aligned}$$

This suggests introducing $m+n$ as a new index of summation, and then possibly $m-n$ as another; but these new indices differ by an even vector so must always be congruent modulo $2\mathbb{Z}^g$, and can better be written

$$m+n = 2k+\lambda\ ,\ m-n = 2\ell+\lambda\ ,$$

where k , $\ell \in \mathbb{Z}^g$ and λ ranges over some set of coset representatives of $\mathbb{Z}^g/2\mathbb{Z}^g$. Any choice of m , n uniquely determines k , ℓ , λ and conversely since

$$m = k+\ell+\lambda\ ,\ n = k-\ell\ .$$

It is convenient to set

$$\tilde{k} = k+\lambda/2\ ,\ \tilde{\ell} = \ell+\lambda/2$$

and to note that

$$m = \tilde{k} + \tilde{\ell} \ , \ n = \tilde{k} - \tilde{\ell} \ .$$

Then the series above can be written

$$\begin{aligned}
\theta(w + 2\tau)\theta(w) &= \sum_{k,\ell,\lambda} \exp 2\pi i[2^t\tilde{k}\Omega\tilde{k} + 2^t\tilde{k}w - {}^t(\tilde{k} + \tilde{\ell})\Omega(\tilde{k} - \tilde{\ell}) + 2^t(\tilde{k} + \tilde{\ell})\tau] \\
&= \sum_{k,\ell,\lambda} \exp 2\pi i[\ \tfrac{1}{2}{}^t\tilde{k}2\Omega\tilde{k} + {}^t\tilde{k}(2w + 2\tau) + \tfrac{1}{2}\ {}^t\tilde{\ell}2\Omega\tilde{\ell} + {}^t\tilde{\ell} \quad 2\tau] \\
&= \sum_{\lambda} \theta[\lambda/2|0](2w + 2\tau \ ; \ 2\Omega) \cdot \theta[\lambda/2|0](2\tau \ ; \ 2\Omega) \\
&= \sum_{\lambda} \theta_2[\lambda|0](w + \tau \ ; \ \Omega) \cdot \theta_2[\lambda|0](\tau \ ; \ \Omega) \\
&= {}^t\vec{\theta}_2(w + \tau) \cdot \vec{\theta}_2(\tau) \ ,
\end{aligned}$$

and that is the desired result.

Sometimes it is convenient to rewrite the formula of the preceding theorem in alternative forms. For instance by setting $u = w + \tau$, $v = w - \tau$ it can be written

$$(1) \qquad \theta(u)\theta(v) = {}^t\vec{\theta}_2\left(\frac{u - v}{2}\right) \cdot \vec{\theta}_2\left(\frac{u + v}{2}\right) .$$

The general form for theta functions with characteristics follows by noting that

$$\begin{aligned}
(2) \qquad \theta[\sigma](u + v)\ \theta[\tau]\ (u - v) &= \theta(\sigma + u + v)\ \theta(\tau + u - v) \\
&= {}^t\vec{\theta}_2\left(\frac{\sigma - \tau}{2} + v\right) \cdot \vec{\theta}_2\left(\frac{\sigma + \tau}{2} + u\right) \\
&= {}^t\vec{\theta}_2[\sigma - \tau](v) \cdot \vec{\theta}_2[\sigma + \tau](u)
\end{aligned}$$

or alternatively

$$(3) \qquad \theta[\sigma](u)\ \theta[\tau](v) = {}^t\vec{\theta}_2[\sigma - \tau]\left(\frac{u - v}{2}\right) \cdot \vec{\theta}_2[\sigma + \tau]\left(\frac{u + v}{2}\right) .$$

It is also worth noting as an immediate consequence of the preceding theorem that if τ_i are any 2^g points at which the vectors $\vec{\theta}_2(\tau_i)$ are linearly independent then the 2^g products $\theta[\tau_i](w)\ \theta[-\tau_i](w) = \theta(\tau_i + w)\theta(\tau_i - w)$ are linearly independent linear combinations of second-order theta functions so also form a basis for the space $\Gamma(\xi^2)$. There always exist such points τ_i, since the component functions forming the vector $\vec{\theta}_2(w)$ are linearly independent, so it is always possible to express all elements of $\Gamma(\xi^2)$ canonically as products of first-order theta functions with suitable characteristics.

§7. The theta locus.

Since the simple theta function $\theta(w)$ is multiplied by a nonzero factor when w is replaced by $w + \lambda$ for any lattice vector $\lambda \in \underline{L} = (I,\Omega)\,\mathbb{Z}^g$, it follows that the holomorphic subvariety $\operatorname{loc}\theta = \{w \in \mathbb{C}^g : \theta(w) = 0\}$ is invariant under translation by $\underline{L}$, hence can be viewed as a holomorphic subvariety of the complex torus $J = \mathbb{C}^g/\underline{L}$; when so viewed this subvariety will be called the <u>theta locus</u> and will be denoted by $\underline{\underline{\theta}}$.

In the special case $g = 1$ this locus is just a finite set of points in the one-dimensional torus, as is no doubt quite familiar from the classical theory of elliptic functions. It may nonetheless be helpful to review some of this material here. The generators of the lattice $\underline{L}$ are the complex numbers 1 and Ω, where $\operatorname{Im}\Omega > 0$, and a fundamental domain Δ for the action of the lattice $\underline{L}$ on $\mathbb{C}$ is the rectangle sketched in Figure 1. If the sides spanned by the basis vectors 1, Ω are

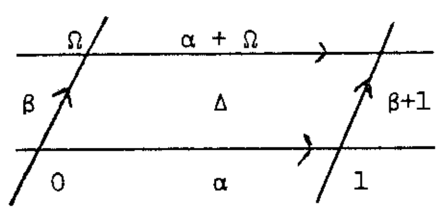

Figure 1

denoted by α, β, respectively, the boundary of the fundamental domain Δ is the oriented curve $\partial\Delta = (\alpha) + (\beta+1) - (\alpha+\Omega) - (\beta)$. Of course this configuration can be translated arbitrarily, so it can be supposed that $\underline{\underline{\theta}}$ is disjoint from $\partial\Delta$. The number of points in $\underline{\underline{\theta}}$ counting multiplicity is then just the total order of the function θ in the region Δ, so can be calculated by the familiar contour integral formula; and since the function θ satisfies the functional equations

$$\theta(w+1) = \theta(w) \ , \ \theta(w+\Omega) = \theta(w) \exp - 2\pi i(w + \tfrac{1}{2}\Omega) \ ,$$

this calculation takes the form

$$\begin{aligned} 2\pi i \cdot \text{order } \theta &= \int_{\partial\Delta} d \log \theta(w) \\ &= \int_\alpha [d \log \theta(w) - d \log \theta(w+\Omega)] + \int_\beta [d \log \theta(w+1) - d \log \theta(w)] \\ &= \int_\alpha 2\pi i \, dw = 2\pi i \int_0^1 dw = 2\pi i \ . \end{aligned}$$

Thus the locus $\underline{\underline{\theta}}$ is a single point in the torus J ; the function θ vanishes at but a single point in J , and vanishes there to the first order. To determine this point, recall that $\theta[\frac{1}{2}|\frac{1}{2}](w;\Omega) = \theta(w + \frac{1}{2} + \frac{1}{2}\Omega) \cdot \exp \pi i(w + \frac{1}{2} + \frac{1}{4}\Omega)$ is an odd function of w , since $[\frac{1}{2}|\frac{1}{2}]$ is an odd half-integral characteristic, hence this function vanishes at the origin $w = 0$. That implies that the locus defined by the function θ is just $\underline{\underline{\theta}} = \frac{1}{2} + \frac{1}{2}\Omega$, the central point of the fundamental domain Δ based at the origin.

The analogous result for $g > 1$ is rather more complicated, since it really involves properties of holomorphic or algebraic subvarieties of higher dimensions. The results can be stated and derived in such a manner as to be meaningful to those willing to accept some general properties of these subvarieties either as known or as given. The locus $\underline{\underline{\theta}}$ at which the function θ vanishes is a holomorphic subvariety of pure dimension $g - 1$ in the g-dimensional complex manifold $J = \mathbb{C}^g/\underline{L}$; at a dense open subset $\underline{\underline{\theta}}^0 \subseteq \underline{\underline{\theta}}$ called the regular part of $\underline{\underline{\theta}}$ this point set is actually a complex submanifold of dimension $g - 1$ in J. This is merely a consequence of the fact that $\underline{\underline{\theta}}$ is the zero locus of a single function. This set may or may not be reducible, that is, may or may not be expressible as a finite union of proper subsets each of which is itself a $(g-1)$-dimensional holomorphic subvariety. When $\underline{\underline{\theta}}$ is expressed as a union of its irreducible components, as a necessarily finite union $\underline{\underline{\theta}} = \cup_j \underline{\underline{\theta}}_j$, the function θ will vanish at all points of $\underline{\underline{\theta}}_j$ to a common order $n_j \in \mathbb{Z}$; the formal expression $\sum_j n_j \underline{\underline{\theta}}_j$ is then the divisor of the function θ, in analogy with the case $g = 1$ in which the irreducible components of the zero locus are just points. Each subvariety $\underline{\underline{\theta}}_j$ has topological dimension $2g - 2$ and carries a homology class $[\underline{\underline{\theta}}_j] \in H_{2g-2}(J, \mathbb{Z})$; the class $\sum_j n_j [\underline{\underline{\theta}}_j] \in H_{2g-2}(J, \mathbb{Z})$ is the analogue of the order in the case $g = 1$. This can be expressed more analytically in terms of the dual cohomology class, which can be considered as a differential form. Thus to each component $\underline{\underline{\theta}}_j$ there can be associated a differential form ϕ_j of degree 2 on the manifold J with integral periods, so that

$\int_J \phi_j \wedge \psi = \int_{\underline{\underline{\theta}}_j} \psi$ for every smooth differential form ψ of degree $2g - 2$ on J; here at a dense open subset $\underline{\underline{\theta}}_j^0 \subseteq \underline{\underline{\theta}}_j$ this locus is a differentiable submanifold, so the restriction $\psi|\underline{\underline{\theta}}_j^0$ is well defined, and the integral $\int_{\underline{\underline{\theta}}_j^0} \psi$ has a finite value that is taken to be the definition of $\int_{\underline{\underline{\theta}}} \psi$. The differential form $\phi = \sum_j n_j \phi_j$ then represents the dual cohomology class to the divisor of the function θ, and it is really this class that will be taken here as the analogue of the order of the function θ. This class is just the Chern class of the holomorphic line bundle over J represented by the factor of automorphy of the theta function.

To describe this dual cohomology class make a real linear change of coordinates in $\mathbb{C}^g = \mathbb{R}^{2g}$ by using the vectors $\delta_1,\ldots,\delta_g,\Omega\delta_1,\ldots,\Omega\delta_g$ as coordinate axes with coordinate functions $t_1,\ldots,t_g,t_{g+1},\ldots,t_{2g}$ on these respective axes; this amounts to representing the torus J as the product of the $2g$ circles represented by these coordinate axes, as observed earlier. Take the orientation of J as that for which the differential form $dt_1 \wedge dt_{g+1} \wedge dt_2 \wedge dt_{g+2} \wedge \cdots \wedge dt_g \wedge dt_{2g}$ has positive integral over J, indeed has integral equal to $+1$. The integral cohomology ring consists of exterior differential forms $\sum_{j_1<\cdots<j_k} n_{j_1\cdots j_k}\, dt_{j_1}\wedge\cdots\wedge dt_{j_k}$ with integral coefficients $n_{j_1\cdots j_k} \in \mathbb{Z}$, so the class dual to the theta divisor is represented by some differential form $\sum_{j_1<j_2} n_{j_1j_2}\, dt_{j_1}\wedge dt_{j_2}$; in these terms the desired result is the following theorem of Poincaré.

Theorem 8. The cohomology class dual to the theta divisor is $\phi = \sum_{j=1}^{g} dt_j \wedge dt_{j+g}$.

Proof. Consider first the special case that the period matrix Ω is diagonal. In that case the lattice vectors δ_k and $\Omega\delta_k$ lie in the plane of the complex coordinate function w_k and the torus J is clearly just the product $J = J_1 \times\cdots\times J_g$ of the one-dimensional complex tori $J_k = \mathbb{C}/(1,\omega_{kk})\,\mathbb{Z}^2$. The theta function is moreover clearly the product $\theta(w;\Omega) = \Pi_k\, \theta(w_k;\omega_{kk})$ of the one-dimensional theta functions, so the zero locus $\underline{\underline{\theta}}$ is the union of the zero loci of these factors. The function $\theta(w_k;\omega_{kk})$ as a function of the variable w_k alone vanishes simply at the point of J_k represented by $\frac{1}{2}+\frac{1}{2}\,\omega_{kk}$, as already noted, so as a function of g variables $\theta(w_k;\omega_{kk})$ vanishes simply on the subvariety

$$\underline{\underline{\Theta}}_k = J_1 \times \ldots \times J_{k-1} \times (\tfrac{1}{2} + \tfrac{1}{2}\,\omega_{kk}) \times J_{k+1} \times \ldots \times J_g \subseteq J\ .$$

This subvariety is clearly a connected submanifold of J , hence an irreducible subvariety; as a manifold in its own right it is biholomorphic to the (g-1)-dimensional complex torus $J_1 \times \ldots \times J_{k-1} \times J_{k+1} \times \ldots \times J_g$. Now for any differential form

$\psi = \sum_{j_1 < \ldots < j_{2g-2}} n_{j_1 \ldots j_{2g-2}}\, dt_{j_1} \wedge \cdots \wedge dt_{j_{2g-1}}$ it is evident that

$$\int_{\underline{\underline{\Theta}}_k} \psi = n_{1\ldots k-1,k+1\ldots g+k-1,g+k+1,\ldots,2g}\ .$$

On the other hand

$$dt_k \wedge dt_{k+g} \wedge \psi = n_{1\ldots k-1,k+1\ldots g+k-1,g+k+1\ldots 2g}\, dt_1 \wedge \cdots \wedge dt_{2g}$$

so that

$$\int_J dt_k \wedge dt_{k+g} \wedge \psi = \int_{\underline{\underline{\Theta}}_k} \psi \qquad .$$

Thus the differential form $dt_k \wedge dt_{k+g}$ is dual to the divisor $\underline{\underline{\Theta}}_k$, hence the differential form $\phi = \sum_k dt_k \wedge dt_{k+g}$ is dual to the divisor $\sum_k \underline{\underline{\Theta}}_k$ as desired.

To turn then to the general case, it is easy to see that the Siegel upper half-space $\mathfrak{H}_g$ is connected, so that for any period matrix $\Omega \in \mathfrak{H}_g$ there is some continuous path $\Omega_s \in \mathfrak{H}_g$ for $s \in [0,1] \subseteq \mathbb{R}$ such that $\Omega_0 = \Omega$ and Ω_1 is a diagonal period matrix. The theta function $\theta(w;\Omega_s)$ depends continuously on the parameter s , as do the homology class of its divisor in $H_{2g-2}(J, \mathbb{Z})$ and the differential form ϕ_s dual to this homology class. The coordinates $t_1, \ldots, t_{2g}$ introduced above also depend continuously on this parameter s, so that $\phi_s = \sum_{j_1<j_2} n_{j_1 j_2}(s)\, dt_{j_1} \wedge dt_{j_2}$ where $n_{j_1 j_2}(s) \varepsilon \mathbb{Z}$ is a continuous function of s and hence must be constant. That suffices to conclude the proof.

It is perhaps worth remarking that the preceding argument can be applied directly to the Chern classes of the line bundles represented by the factor of automorphy for the theta function, so that there is really less geometry involved that might appear to be the case from the version of the argument given here. It is also possible to calculate this dual cohomology class directly in general, to verify the result asserted. The calculation is more reasonably done in the context of a more general discussion and classification of factors of automorphy on complex tori, an aspect of the subject that will not be treated here.

In most of the subsequent discussion it will be supposed that the theta locus is an irreducible holomorphic subvariety of J. That is really a further restriction on the period matrix Ω, a form of nondegeneracy condition. Any thorough discussion of the significance of this condition is also most reasonably done in the context of the general classification of factors of automorphy on complex tori, so will not be covered here. Let it suffice here just to mention in passing that if the theta locus $\underline{\underline{\Theta}}$ is reducible then after a suitable change of coordinates in $\mathbb{C}^g$ and of bases for the lattice subgroup the torus J can actually be written as a product $J = J_1 \times J_2 \times \ldots \times J_n$ of tori of lower dimension, and the components of $\underline{\underline{\Theta}}$ are of the form $J_1 \times \ldots \times J_{k-1} \times \underline{\underline{\Theta}}_k \times J_{k+1} \times \ldots \times J_n$ where $\underline{\underline{\Theta}}_k \subset J_k$ is a theta locus in this factor. Thus what was observed in the proof of the preceding theorem about the theta locus of a product of one-dimensional complex tori is quite typical of the situation in general, and the diagonal period matrices are typical of the degenerate period matrices that will mostly not be considered here. The tori with reducible theta loci are described by the period matrices of the factors in this product decomposition, hence by combinations of lower dimensional subvarieties of $\mathfrak{H}_g$.

It is worth noting as an immediate consequence of the preceding theorem that the theta function $\theta(w)$ vanishes to the first order on the locus $\underline{\underline{\theta}}$, so that the divisor of $\theta(w)$ is really just the divisor $1 \cdot \underline{\underline{\theta}}$. Indeed if $\theta(w)$ vanished to some order $r > 1$ then the cohomology class dual to the theta divisor would be r times an integral form, but that is evidently not the case. The theta function actually vanishes to first order at each irreducible component of the theta locus in the reducible case as well, as could be shown as a consequence of the analysis of the reducible case mentioned in the preceding discussion or a rather more careful argument from the result of the preceding theorem; but this result will not be needed, so the verification will be omitted here. Since a theta function with characteristics is just a translate of the simple theta function, aside from a nowhere-vanishing factor, it follows that the divisor of $\theta[\nu|\tau](w;\Omega)$ is just a translate of the theta divisor, for any characteristic. For any such function the cohomology class dual to its divisor is represented by the form ϕ of Theorem 8. On the other hand $\theta(w)^r \in \Gamma(\xi^r)$ will have as its divisor $r \cdot \underline{\underline{\theta}}$, and the coholomogy class dual to that divisor is represented by the form $r\phi$. For any other section $f \in \Gamma(\xi^r)$ the quotient $\theta(w)^r/f$ is a meromorphic function on J , and any such function is known to have a divisor with trivial dual cohomology class; consequently the cohomology class dual to the divisor of any $f \in \Gamma(\xi^r)$, in particular of any r-th order theta function, is represented by the form $r\phi$, and the same holds as above for r-th order theta functions with characteristic.

It is also worth noting for later purposes some elementary properties of the theta divisor. First as a matter of notation, for any subset $X \subseteq J$ set $-X = \{-x : x \in X\}$. Then since $\theta(w)$ is an even function clearly

$$(1) \qquad -\underline{\underline{\theta}} = \underline{\underline{\theta}}\,.$$

Next as another matter of notation, the translate of a subset $X \subseteq J$ by a point $u \in J$ is defined as the set $X + u = u + X = \{u + x : x \in X\}$.

<u>Theorem 9</u>. $\underline{\underline{\theta}} + u = \underline{\underline{\theta}}$ precisely when $u \in \underline{L}$.

<u>Proof</u>. Translation by a lattice vector is a trivial operation on the torus J, so of course $\underline{\underline{\theta}} + u = \underline{\underline{\theta}}$ whenever $u \in \underline{L}$. On the other hand if $\underline{\underline{\theta}} + u = \underline{\underline{\theta}}$ for some $u \in \mathbb{C}^g$ then $\theta(w-u)$ and $\theta(w)$ both vanish to the first order on the subvariety $\underline{\underline{\theta}} + u = \underline{\underline{\theta}}$, so their quotient $f(w) = \theta(w-u)/\theta(w)$ is a holomorphic and nowhere-vanishing function on $\mathbb{C}^g$; strictly speaking this has only been demonstrated in the case that $\underline{\underline{\theta}}$ is irreducible, and actually will only be needed in that case, but as noted it does hold in general. Recall that $\theta(w) \in \Gamma(\xi)$ and $\theta(w-u) = \theta[0|-u](w) \in \Gamma(\rho_u\xi)$ so that $f(w) \in \Gamma(\rho_u)$; therefore $\Gamma(\rho_u) \neq 0$, and as in (5.2) that means that $u \in \underline{L}$ as desired.

§8 Singularities of the theta locus.

For a number of purposes it is convenient to view the theta function $\theta(w;\Omega)$ as a holomorphic function both of the variable $w \in \mathbb{C}^g$ and of the period matrix $\Omega \in \mathfrak{H}_g$, where $\mathfrak{H}_g$ is an open subset of the space of $\binom{g+1}{2}$ complex variables with coordinates ω_{ij} for $1 \leqq i \leqq j \leqq g$. As a useful preliminary observation note that this function is a solution of the following partial differential equation.

Theorem 10. $\dfrac{\partial^2\theta(w;\Omega)}{\partial w_i \partial w_j} = 2\pi i(1 + \delta^i_j)\,\dfrac{\partial\theta(w;\Omega)}{\partial\omega_{ij}}$.

Proof. It is a simple direct calculation that

$$\frac{\partial}{\partial\omega_{ij}} \exp 2\pi i[\tfrac{1}{2}\,{}^t n\Omega n + {}^t n w] = (2 - \delta^i_j)\pi i\; n_i n_j \exp 2\pi i[\tfrac{1}{2}\,{}^t n\Omega n + {}^t n w] ,$$

$$\frac{\partial^2}{\partial w_i \partial w_j} \exp 2\pi i[\tfrac{1}{2}\,{}^t n\Omega n + {}^t n w] = (2\pi i)^2 n_i n_j \exp 2\pi i[\tfrac{1}{2}\,{}^t n\Omega n + {}^t n w] .$$

Each summand in the series expansion (2.1) of the function $\theta(w;\Omega)$ thus satisfies the asserted partial differential equation, and since that series is a locally uniformly convergent series of holomorphic functions it can be differentiated term by term to any order, thus yielding the desired result.

Now associate to any pair of vectors p, $q \in \mathbb{Z}^g$ the holomorphic automorphism of the product manifold $\mathbb{C}^g \times \mathfrak{H}_g$ that sends the point $(w;\Omega) \in \mathbb{C}^g \times \mathfrak{H}_g$ to the point $(w + p + \Omega q;\Omega) \in \mathbb{C}^g \times \mathfrak{H}_g$. This is just the natural extension of the action of the lattice subgroup $\underline{L} = (I,\Omega)\,\mathbb{Z}^g$ on $\mathbb{C}^g$ considered so far, obtained by considering it as depending on the period matrix Ω as well as on the variable w; it is convenient thus to think of it as another action of the lattice subgroup $\underline{L}$, in extension of that considered earlier. The quotient space $\mathfrak{J} = (\mathbb{C}^g \times \mathfrak{H}_g)/\,\underline{L}$ is in a natural way a complex manifold of dimension $g + \binom{g+1}{2}$, sometimes called the <u>universal</u> <u>complex</u> <u>torus</u>. The projection $\mathbb{C}^g \times \mathfrak{H}_g \to \mathfrak{H}_g$ induces a holomorphic mapping $\pi : \mathfrak{J} \to \mathfrak{H}_g$ such that the inverse image or fibre $\pi^{-1}(\Omega)$ over any point $\Omega \in \mathfrak{H}_g$ is precisely the complex torus $J(\Omega)$ with period matrix Ω ; the mapping π is topologically a fibering, exhibiting $\mathfrak{J}$ locally as the product of a subset of $\mathfrak{H}_g$ and a topological torus of dimension $2g$, but is not a complex analytic fibering since the complex structures of the tori $J(\Omega)$ are not locally constant but vary with Ω . It is important to note that π is a <u>proper</u> mapping, in the sense that the inverse image of any compact subset of $\mathfrak{H}_g$ is a compact subset of $\mathfrak{J}$.

The functional equation for the theta function $\theta(w;\Omega)$ as in Theorem 1 can be viewed as a property of this extended group action of on $\mathbb{C}^g \times \mathfrak{H}_g$. The holomorphic subvariety $\mathrm{loc}\ \theta(w;\Omega) = \{(w;\Omega) \in \mathbb{C}^g \times \mathfrak{H}_g : \theta(w;\Omega) = 0\}$ is then invariant under this action of $\underline{L}$ so can be viewed as a subvariety of the universal torus $\mathfrak{J}$; it will be called the <u>universal</u> <u>theta</u> <u>locus</u> and will be denoted by $\underline{\underline{\tilde{\Theta}}}$. For any fixed point $\Omega \in \mathfrak{H}_g$ the intersection $\Theta(\Omega) = \tilde{\Theta} \cap \pi^{-1}(\Omega) \subset J(\Omega)$ is just the theta locus in the torus $J(\Omega)$. This exhibits the universal theta locus as a topological fibering over $\mathfrak{H}_g$ as well.

These rather simple observations can be coupled with general results from function theory in several complex variables to yield some useful constructions. Consider first the subset

$$\{(w;\Omega) \in \mathbb{C}^g \times \mathfrak{H}_g : \theta(w;\Omega) = \partial\theta(w;\Omega)/\partial w_j = 0 \quad \text{for} \quad j = 1,\ldots,g\} ,$$

evidently a well defined holomorphic subvariety of $\mathbb{C}^g \times \mathfrak{H}_g$. If $(w;\Omega)$ is in this locus then for any lattice vector $\lambda = p + \Omega q \in (I,\Omega)\,\mathbb{Z}^{2g}$ it is clear from the functional equation of Theorem 1 that $\theta(w+\lambda;\Omega) = 0$, and also that $\partial\theta(w+\lambda;\Omega)/\partial w_j = 0$ since it is a linear combination of $\theta(w;\Omega)$ and $\partial\theta(w;\Omega)/\partial w_j$; this locus therefore describes a subvariety $\tilde{\underline{\underline{\theta}}}^1 \subseteq \tilde{\mathfrak{J}}$, which is of course a subvariety of the universal theta locus $\tilde{\underline{\underline{\theta}}}$. For any fixed $\Omega \in \underline{\mathfrak{h}}_g$ the intersection $\tilde{\underline{\underline{\theta}}}^1 \cap \pi^{-1}(\Omega) = \underline{\underline{\theta}}^1(\Omega)$ is precisely the set of singular points of the theta locus in the torus $J(\Omega)$, those points at which the defining theta function and all its partial derivatives vanish. Those period matrices $\Omega \in \underline{\mathfrak{h}}_g$ that do not lie in the image $\pi(\tilde{\underline{\underline{\theta}}}^1) \subseteq \underline{\mathfrak{h}}_g$, those for which $\tilde{\underline{\underline{\theta}}}^1 \cap \pi^{-1}(\Omega) = \emptyset$, are precisely the period matrices $\Omega \in \underline{\mathfrak{h}}_g$ for which the theta locus is a nonsingular subvariety of the torus $J(\Omega)$. Equivalently the period matrices $\Omega \in \pi(\tilde{\underline{\underline{\theta}}}^1)$ are precisely those for which the theta locus of the torus $J(\Omega)$ has singularities. A useful result from the theory of holomorphic functions of several variables is Remmert's theorem that the image of a holomorphic subvariety under a proper holomorphic mapping is a holomorphic subvariety; thus the set of period matrices $\Omega \in \underline{\mathfrak{h}}_g$ for which the theta locus of the torus $J(\Omega)$ has singularities is a holomorphic subvariety of $\underline{\mathfrak{h}}_g$, the image of the subvariety $\tilde{\underline{\underline{\theta}}}^1 \subseteq \tilde{\mathfrak{J}}$ under the proper holomorphic mapping $\pi : \tilde{\mathfrak{J}} \to \underline{\mathfrak{h}}_g$. Of course this would be quite trivial if every theta locus had singularities; but that is not the case.

Theorem 11. The set of period matrices $\Omega \in \underline{\mathfrak{h}}_g$ for which the theta locus of the torus $J(\Omega)$ has singularities is a proper holomorphic subvariety of $\underline{\mathfrak{h}}_g$.

Proof. In view of the preceding observations, it is only necessary to show that the image $\pi(\tilde{\underline{\underline{\theta}}}^1)$ is a proper subset of $\underline{\mathfrak{h}}_g$. If that is not the case, so that the restriction $\pi|\tilde{\underline{\underline{\theta}}}^1 : \tilde{\underline{\underline{\theta}}}^1 \to \underline{\mathfrak{h}}_g$ is surjective, then it follows from rather standard properties of holomorphic mappings that the restriction admits local sections through all points outside a proper subvariety of $\tilde{\underline{\underline{\Theta}}}^1$; in particular there are an open set $U \subseteq \mathfrak{H}_g$ and a holomorphic mapping $f : U \to \mathbb{C}^g$ such that $(f(\Omega),\Omega) \in \tilde{\underline{\underline{\Theta}}}^1$ whenever $\Omega \in U$. Now in the subset $\mathbb{C}^g \times U$ the function $\theta(w;\Omega)$ is a solution of the system of complex analytic partial differential equations of Theorem 10. This system is in the form to which the Cauchy-Kowalewski initial value theorem applies: the analytic function $\theta(w;\Omega)$ is a solution of this system and satisfies the trivial initial value conditions that $\theta(f(\Omega);\Omega) = \partial\theta(f(\Omega);\Omega)/\partial w_j = 0$, hence must vanish identically. That is a contradiction, and thereby concludes the proof.

§9 Wirtinger varieties.

The zero locus of a single higher-order theta function has already been briefly considered, but of course there are a large number of such functions and their zero loci, and none particularly distinguished over the others. More interesting and intrinsic is the set of common zeros of all of these functions, although perhaps not as interesting as might have been expected.

Theorem 12. For any order $r \geqq 2$ the functions in $\Gamma(\xi^r)$ have no common zeros.

Proof. First in the special case $r = 2$, the second-order theta functions that are the components of the vector $\vec{\theta}_2(w)$ form a basis for $\Gamma(\xi^2)$, so what is to be shown is that these functions have no common zeros, or equivalently that $\vec{\theta}_2(w) \neq 0$ for any $w \in \mathbb{C}^g$. If to the contrary there is some point $w_0 \in \mathbb{C}^g$ at which $\vec{\theta}_2(w_0) = 0$ then from the formula of Theorem 7 it follows that $0 = {}^t\vec{\theta}_2(w_0) \cdot \vec{\theta}_2(w)$ $= \theta(w + w_0)\theta(w - w_0)$ for all points $w \in \mathbb{C}^g$; the product of two analytic functions cannot vanish identically unless one of the factors does, but θ is a nontrivial function and that is an immediate contradiction. That thereby shows the desired result in this special case.

Then for $r > 2$ and for any point $w_0 \in \mathbb{C}^g$ choose a value $\tau \in \mathbb{C}^g$ such that $\theta(w_0 + \tau) \neq 0$. Since $\theta(w + \tau) = \theta[0|\tau](w) \in \Gamma(\rho_{-\tau}\xi)$ as a function of w while the component functions of the vector $\vec{\theta}_2(w - (r-2)\tau/2) = \vec{\theta}_2[-(r-2)\tau](w)$ belong to $\Gamma(\rho_\tau^{r-2}\xi^2)$ it follows that the component functions of the vector $\theta(w+\tau)^{r-2}\vec{\theta}_2(w-(r-2)\tau/2)$ belong to $\Gamma(\xi^r)$. Now $\theta(w_0 + \tau) \neq 0$ by the choice of τ while $\vec{\theta}_2(w_0 - (r-2)\tau/2) \neq 0$ by the result of the first part of this proof. That shows that the functions in $\Gamma(\xi^r)$ do not all vanish at the point w_0, and since w_0 was quite arbitrary that concludes the proof.

It is worth noting explicitly that the functions that are the components of $\vec{\theta}_r(w - \tau/r) = \vec{\theta}_r[\tau](w)$ are a basis for $\Gamma(\rho_\tau \xi^r)$, so the functions in $\Gamma(\rho_\tau \xi^r)$ have no common zeros for any parameter $\tau \in \mathbb{C}^g$ and order $r \geqq 2$.

Now since $\vec{\theta}_r(w) \in \mathbb{C}^{r^g}$ is a nonzero vector it determines a point $[\vec{\theta}_r(w)] \in \mathbb{P}^{r^g-1}$ in the associated projective space, the point with homogeneous coordinates the components of the vector $\vec{\theta}_r(w)$. Thus the function $\vec{\theta}_r(w)$ can be viewed as describing a holomorphic mapping

$$[\vec{\theta}_r] : \mathbb{C}^g \to \mathbb{P}^{r^g-1} .$$

The functional equation for the r-th order theta functions implies that $\vec{\theta}_r(w)$ and $\vec{\theta}_r(w+\lambda)$ represent the same point in projective space for any lattice vector $\lambda \in \underset{\sim}{L}$, since these two vectors differ by a nonzero scalar factor; thus the above can also be viewed as describing a holomorphic mapping

$$[\vec{\theta}_r] : J \to \mathbb{P}^{r^g-1} .$$

This is a proper holomorphic mapping, since J is compact, so by Remmert's proper mapping theorem the image is a holomorphic subvariety $K_r = [\vec{\theta}_r](J)$ in that projective space; K_r is indeed an algebraic subvariety, the locus of zeros of a finite number of homogeneous polynomials, by Chow's theorem.

In the further analysis of this situation the cases $r = 2$ and $r > 2$ are rather different, and will be treated separately. The underlying reason for this difference is that $\vec{\theta}_2$ is an even mapping,

(1) $$\vec{\theta}_2(-w) = \vec{\theta}_2(w) \ ,$$

while that is not the case for $\vec{\theta}_r$ for $r > 2$. Indeed any component function of the vector $\vec{\theta}_r(w)$ is of the form $\theta_r[\nu|0](w;\Omega) = \theta[\frac{\nu}{r}|0](rw;r\Omega)$ where $\nu \in \mathbb{Z}^g/r\,\mathbb{Z}^g$, and by Theorem 4 this function can only be even when $[\frac{\nu}{r}|0]$ is an even half-integral characteristic; thus it can only be even for all indices $\nu \in \mathbb{Z}^g/r\,\mathbb{Z}^g$ when $r = 2$, and in that case it is indeed always an even half-integral characteristic so that all component functions are even. In case $r = 2$ the mapping $[\vec{\theta}_2] : J \to K$ thus cannot possibly be a one-to-one mapping, since the points w and $-w$ in $\mathbb{C}^g$ have the same image; these represent distinct points on the torus unless $2w = w - (-w) \in \underline{L}$, that is, unless w is one of the 2^{2g} half periods modulo $\underline{L}$. These exceptional points can be described equivalently as precisely the points of order two in the group J, those points $w \in J$ such that $w + w = 0 \in J$, where the origin in $\mathbb{C}^g$ is taken to represent the zero element in J. This is actually precisely the extent to which the mapping $[\vec{\theta}_2] : J \to K$ fails to be one-to-one, at least for a general period matrix.

<u>Theorem 13</u>. If $\Omega \in \underline{\mathfrak{h}}_g$ describes a complex torus J for which the theta locus $\underline{\underline{\Theta}}$ is irreducible then

$$[\vec{\theta}_2] : J \to K \subseteq \underline{\mathbb{P}}^{2^g-1}$$

exhibits J as a two-sheeted branched covering of its image subvariety

K , with 2^{2g} branch points which are precisely the points of order two in J .

Proof. If $[\vec{\theta}_2(w_1)] = [\vec{\theta}_2(w_2)]$ for some points w_1 , $w_2 \in \mathbb{C}^g$ then $\vec{\theta}_2(w_1) = c\, \vec{\theta}_2(w_2)$ for a nonzero complex number c . For any point $w \in \mathbb{C}^g$ it follows from this and the formula of Theorem 7 that

$$\begin{aligned} \theta(w + w_1)\theta(w - w_1) &= {}^t\vec{\theta}_2(w_1) \cdot \vec{\theta}_2(w) \\ &= c\ {}^t\vec{\theta}_2(w_2) \cdot \vec{\theta}_2(w) = c\theta(w + w_2)\theta(w - w_2) \ . \end{aligned}$$

This is an identity among functions of the variable w , and in terms of the zero loci of the first-order theta functions means that

$$(\underline{\underline{\theta}} - w_1) \cup (\underline{\underline{\theta}} + w_1) = (\underline{\underline{\theta}} - w_2) \cup (\underline{\underline{\theta}} + w_2) \ .$$

Since the theta locus $\underline{\underline{\theta}}$ and hence all its translates are irreducible subvarieties by hypothesis, and since the decomposition of a holomorphic subvariety of J into its irreducible components is unique up to order, it must be the case that either $\underline{\underline{\theta}} - w_1 = \underline{\underline{\theta}} - w_2$ or $\underline{\underline{\theta}} - w_1 = \underline{\underline{\theta}} + w_2$. An application of Theorem 9 shows that in the first case $w_1 - w_2 \in \underline{L}$, hence w_1 and w_2 represent the same point on the torus J , while in the second case $w_1 + w_2 = \lambda \in \underline{L}$, hence $w_1 = w_1 + w_2 - w_2 = -w_2 + \lambda$ so that w_1 and w_2 represent the same points on J as w_1 and $-w_1$. Thus the mapping $[\vec{\theta}_2] : J \to K$ is generally two-to-one, with the exceptions as discussed above, and the proof is thereby concluded.

The assumption that the theta locus $\underline{\underline{\theta}}$ is irreducible is really necessary for the preceding theorem to hold as stated. Indeed suppose that the torus J can be written as a product $J = J_1 \times J_2$ corresponding to the decomposition of the theta locus $\underline{\underline{\Theta}} = (\underline{\underline{\Theta}}_1 \times J_2) \cup (J_1 \times \underline{\underline{\Theta}}_2)$, where $\underline{\underline{\Theta}}_i$ is the theta locus in the torus J_i . The theta functions split correspondingly as products of theta functions in the variables w_1 of J_1 and those in the variables w_2 of J_2 , and the factors are also even functions. Then the four points (w_1,w_2) , $(-w_1,w_2)$, $(w_1,-w_2)$, $(-w_1,-w_2)$ have the same image under $\vec{\theta}_2$, so the mapping is at least four-to-one rather than two-to-one. It is apparent from this that something of the preceeding theorem can be salvaged in the reducible case, the map merely being some higher-order covering with further singularities possible; but this topic will not be pursued further here.

If the lattice subgroup $\underline{L}$ is enlarged by adjoining the additional transformation $T : w \to -w$, there results an extended group $\underline{\tilde{L}}$ containing $\underline{L}$ as a subgroup of index two. The extended group $\underline{\tilde{L}}$ of analytic automorphisms of $\mathbb{C}^g$ is of course no longer Abelian. The mapping $[\vec{\theta}_2] : \mathbb{C}^g \to \mathbb{P}^{2^g-1}$ then induces a one-to-one analytic mapping from the quotient space $\mathbb{C}^g/\underline{\tilde{L}}$ to the subvariety $K \subseteq \mathbb{P}^{2^g-1}$. One might expect that this would be a biholomorphic mapping between the natural quotient space $\mathbb{C}^g/\underline{\tilde{L}}$ and the projective algebraic subvariety K , but to make sense of that statement something must be said about the analytic structure of the quotient variety $\mathbb{C}^g/\underline{\tilde{L}}$. Aside from the 2^{2g} branch points, each of which will be a fixed point for some transformation in $\underline{\tilde{L}}$, the quotient space $\mathbb{C}^g/\underline{\tilde{L}}$ has the natural structure of a complex manifold, with the coordinates in $\mathbb{C}^g$ as local coordinates on $\mathbb{C}^g/\underline{\tilde{L}}$. The condition that the mapping $[\vec{\theta}_2]$ be locally biholomorphic is then just that the Jacobian matrix of this mapping from an open subset of the space of g complex variables be a maximal rank, namely g ; the image is then locally a submanifold of $\mathbb{P}^{2^g-1}$, and the mapping locally biholomorphic. At any fixed point of the group $\tilde{\underline{L}}$ the situation is rather more complicated, for the quotient space $\mathbb{C}^g/\tilde{\underline{L}}$ cannot be a complex manifold but must have an isolated singularity there whenever $g > 1$. To see that, it is enough just to consider the origin $0 \in \mathbb{C}^g$, which is a fixed point for the mapping $T : w \to -w$; it is easy to see that all fixed points are locally the same as this one. The complement of the origin in a ball about 0 is naturally mapped as a two-sheeted unbranched covering of the complement of the image of 0 in the quotient space $\mathbb{C}^g/\tilde{\underline{L}}$, and since the complement of the origin in a ball in $\mathbb{C}^g$ is simply connected for $g > 1$ it follows that the fundamental group of the complement of the image of the origin in the quotient space $\mathbb{C}^g/\tilde{\underline{L}}$ is $\mathbb{Z}/2$ whenever $g > 1$; thus the quotient space is not even topologically a manifold at any fixed point whenever $g > 1$. If the analytic structure of the quotient space $\mathbb{C}^g/\tilde{\underline{L}}$ near any fixed point is defined to be that of the image subvariety in $\mathbb{P}^{2^g-1}$ then there is a well defined complex structure on $\mathbb{C}^g/\tilde{\underline{L}}$ at all points and the mapping $[\vec{\theta}_2]$ will be locally and hence also globally biholomorphic. There are other ways to impose an analytic structure at the singularities, but then verifying that the mapping is locally biholomorphic there is a problem; that is merely avoided by the choice made here, and will not be examined further. With this understanding, the mapping $[\vec{\theta}_2] : \mathbb{C}^g/\tilde{\underline{L}} \to K$ is biholomorphic as a consequence of the following result.

Theorem 14. If $\Omega \in \mathfrak{H}_g$ describes a complex torus J for which the theta locus $\underline{\underline{\Theta}}$ is irreducible then the mapping $[\vec{\theta}_2] : J \to \mathbb{P}^{2^g-1}$ is a nonsingular holomorphic mapping at all points of J other than those of order two, so the image $K = [\vec{\theta}_2(J)]$ is a projective algebraic variety that for $g > 1$ has singularities precisely at the images of the 2^{2g} points of order two.

Proof. Consider a point $w_0 \in \mathbb{C}^g$, and for convenience relabel the component functions of the vector $\vec{\theta}_2$ as $f_0, f_1, \ldots, f_G$ where $G = 2^g - 1$ and $f_0(w_0) \neq 0$. These functions describe the mapping $[\vec{\theta}_2]$ in terms of homogeneous coordinates in $\mathbb{P}^G$, but to examine the singularities of this mapping it is really necessary to introduce suitable inhomogeneous local coordinates near $[\vec{\theta}_2(w_0)]$. Clearly it is possible to choose such coordinates so that the mapping $[\vec{\theta}_2]$ is described by the coordinate functions $h_1, \ldots, h_G$ where $h_i = f_i/f_0$. The mapping $[\vec{\theta}_2]$ is singular at w_0 precisely when the Jacobian matrix $\partial h_i/\partial w_j$ has rank $< g$ at w_0, hence precisely when there exist complex numbers $c_1, \ldots, c_g$ not all zero such that

$$\begin{aligned} 0 &= \sum_{j=1}^{g} c_j \frac{\partial h_i}{\partial w_j}(w_0) \\ &= \sum_{j=1}^{g} c_j f_0(w_0)^{-2}\Big[f_0(w_0)\frac{\partial f_i}{\partial w_j}(w_0) - f_i(w_0)\frac{\partial f_0}{\partial w_j}(w_0)\Big] \\ &= f_0(w_0)^{-1}\Big[\sum_{j=1}^{g} c_j \frac{\partial f_i}{\partial w_j}(w_0) + c_0\, f_i(w_0)\Big] \end{aligned}$$

for $i = 1, \ldots, G$, where here

$$c_0 = -f_0(w_0)^{-1} \sum_{j=1}^{g} c_j \frac{\partial f_0}{\partial w_j}(w_0) ,$$

or equivalently where c_0 is determined by the condition that

$$\sum_{j=1}^{g} c_j \frac{\partial f_0}{\partial w_j}(w_0) + c_0 f_0(w_0) = 0 .$$

Thus after introducing the new constant c_0 , the condition that $[\vartheta_2]$ be singular at w_0 is just that there exist some constants $c_0, c_1, \ldots, c_g$ not all zero such that

$$(2) \qquad c_0 f_i(w_0) + \sum_{j=1}^{g} c_j \frac{\partial f_i}{\partial w_j}(w_0) = 0 \quad \text{for} \quad i = 0,1,\ldots,G ,$$

or equivalently such that

$$(2') \qquad c_0 + \sum_{j=1}^{g} c_j \frac{\partial \log f_i}{\partial w_j}(w_0) = 0 \quad \text{for} \quad i = 0,1,\ldots,G .$$

Here f_i are simply some basis for the space $\Gamma(\xi^2)$, and it was noted earlier as a consequence of Theorem 7 that it is always possible to choose a basis of the form $f_i(w) = \theta(w + \tau_i)\,\theta(w - \tau_i)$ for some suitable points $\tau_i \in \mathbb{C}^g$; it is thus evident that this singularity condition (2') can be rewritten in the form

$$(3) \quad c_0 + \sum_{j=1}^{g} c_j \frac{\partial}{\partial w_j} \log[\theta(w + \tau)\theta(w - \tau)] = 0 \quad \text{for} \quad w = w_0 \quad \text{and all} \quad \tau .$$

If

$$(4) \qquad h(w) = \sum_{j=1}^{g} c_j \frac{\partial \log \theta(w)}{\partial w_j} = \theta(w)^{-1} \sum_{j=1}^{g} c_j \frac{\partial \theta(w)}{\partial w_j} ,$$

then condition (3) in turn is just that

(5) $$c_0 + h(w_0 + \tau) + h(w_0 - \tau) = 0 \quad \text{for all} \quad \tau \in \mathbb{C}^g .$$

Now h is clearly a meromorphic function on $\mathbb{C}^g$, with singularities at most simple poles at points of the theta locus $\underline{\underline{\theta}}$, an irreducible subvariety of J. If it really has such singularities then it follows immediately from (5) that $\underline{\underline{\theta}} + w_0 = \underline{\underline{\theta}} - w_0$, hence from Theorem 9 that $2w_0 \in \underline{L}$; thus w_0 must be a half period as desired. On the other hand if h is actually holomorphic then it follows from (4) and the functional equation of the theta function that

(6) $$h(w + p + \Omega q) = h(w) - 2\pi i \sum_{j=1}^{g} c_j q_j$$

for all vectors $p, q \in \mathbb{Z}^g$, so that the partial derivatives $\partial h/\partial w_j$ are $\underline{L}$-invariant holomorphic functions and hence constants; thus h must be a linear function, say $h(w) = a_0 + \sum_j a_j w_j = a_0 + {}^t a \cdot w$. The functional equation (6) then has the form

$$ {}^t a \cdot (p + \Omega q) = -2\pi i \, {}^t a \cdot q$$

for all $p, q \in \mathbb{Z}^g$. For $q = 0$, $p = \delta_j$ it follows that $a_j = 0$, so that actually $a = 0$; but then for $q = \delta_j$ it must be the case that $c_j = 0$ for all j, a contradiction. Therefore h cannot really be holomorphic, so the only singularities of $[\vec{\theta}_2]$ are just the half-periods, and that is enough to complete the proof.

The image $K = [\vec{\theta}_2(J)]$ is thus an algebraic subvariety of dimension g in the complex projective space of dimension $2^g - 1$, and is biholomorphically equivalent to the quotient space $\mathbb{C}^g/\underline{\tilde{L}}$ with the analytic structure as discussed. This subvariety is often called the Wirtinger variety associated with the torus J, or in the spacial case $g = 2$ the Kummer surface associated with J. The algebraic surfaces of this type were investigated by Kummer from another point of view, and have been the subject of rather extensive enquiry even since. The parameterization by theta functions was discovered by Klein. The treatment of the general case as here is due to Wirtinger. For later use it is worth stating explicitly here a simple consequence of one of the observations made in the course of the proof of the preceding theorem. As a notational convenience let $\partial_j\vec{\theta}_2(w) = \partial\vec{\theta}_2(w)/\partial w_j$.

Corollary If the theta locus $\underline{\underline{\Theta}}$ is irreducible then the $g+1$ vectors $\vec{\theta}_2(w)$, $\partial_1\vec{\theta}_2(w), \dots, \partial_g\vec{\theta}_2(w)$ are linearly dependent precisely when w is a point of order two in the torus J.

Proof The points of order two on J are precisely the singular points of the mapping $[\vec{\theta}_2]$, as a consequence of the theorem. On the other hand, in the course of the proof it was demonstrated that these singular points could also be characterized as the points $w_o \in J$ for which there are some constants c_i not all zero such that (2) holds. The desired result is an immediate consequence of these observations.

As the image of the irreducible complex manifold J, the Wirtinger variety K is an irreducible holomorphic and hence irreducible algebraic variety. It has precisely 2^{2g} isolated singularities, the images of the 2^{2g} points of order two on J, and is otherwise everywhere regular. The projective embedding being that determined by the vector space $\Gamma(\xi^2)$

spanned by the second order theta functions, it is evident that the linear hyper-surfaces on K (the intersections of K with linear hyperplanes in projective space) are the images in K of the zero loci in J of sections in $\Gamma(\xi^2)$. The interesection of the g-dimensional variety K with g generic hyper-planes will be a set of d distinct points, where d is the degree of the subvariety K. To calculate this, observe that J is a two-sheeted cover of K, so that the intersection of the zero loci of g generic sections of $\Gamma(\xi^2)$ in J will be $2d$ distinct points. This number in turn is just the topological intersection number of the divisors of g sections of $\Gamma(\xi^2)$, or equivalently the exterior product of the

dual cohomology classes evaluated on the fundamental class of J. Thus

$$\begin{aligned} 2d &= \int_J (2 \textstyle\sum_{j=1}^{g} dz_j \wedge dz_{j+g})^g \\ &= 2^g \int_J \sum dz_{j_1} \wedge dz_{j_{g+1}} \wedge \cdots \wedge dz_{j_g} \wedge dz_{j_{2g}} \\ &= 2^g \cdot g! \int_J dz_1 \wedge dz_{g+1} \wedge \cdots \wedge dz_g \wedge dz_{2g} \\ &= 2^g g! \ , \end{aligned}$$

with the orientation conventions adopted here. Thus K is an algebraic subvariety of degree

$$d = 2^{g-1} g! \tag{7}$$

The polynomials in $\mathbf{P}^{2^g-1}$ vanishing on K correspond precisely to the polynomial identities among the basic second order theta functions forming the components of the mapping $[\vartheta_2]$.

It is perhaps worth looking briefly at the simplest two cases. First for $g = 1$ the mapping $[\vec{\theta}_2] : J \to \mathbb{P}^1$ is a two-to-one mapping with branch points at the four half-periods; this is the familiar representation of an elliptic curve as a two-sheeted branched covering of the Riemann sphere $\mathbb{P}^1$ with four branch points. Next for $g = 2$ the mapping $[\vec{\theta}_2] : J \to \mathbb{P}^3$ is a two-to-one mapping from J to a quartic surface K in $\mathbb{P}^3$ with 16 branch points; the surface K is Kummer's quartic surface associated with J.