DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **00**, Number **0**, Xxxx **XXXX**

Website: http://AIMsciences.org

pp. **000–000**

REMARKS ON A SMOLUCHOWSKI EQUATION

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Abstract.
We study the long time dynamics of a Smoluchowski equation arising in the modeling of nematic liquid crystalline polymers. We prove uniform bounds for the long time average of gradients of the distribution function in terms of the nondimensional parameter characterizing the intensity of the potential. In the two dimensional case we obtain lower and upper bounds for the number of steady states. We prove that the system is dissipative and that the potential serves as unique determining mode of the system.

1. **Introduction.** Certain descriptions of the rheology of non-Newtonian complex fluids containing liquid crystalline polymers combine macroscopic partial differential equations with microscopic stochastic differential equations ([9], [11], [14], [12], [7]). A simple model of nematic liquid crystalline polymers - the rigid rod model - using the Maier-Saupe potential, gives rise to a Smoluchowski equation for the single particle distribution function on the unit sphere ([2], [10], [4],[13], [5]). In spite of its simplicity, this equation exhibits nontrivial nonlinear dynamical features, in contrast with classical Fokker-Planck equations for noninteracting particles ([8]).

We study long time properties of this equation in function of one parameter $b > 0$ representing the nondimensional potential intensity. We formulate the problem in n dimensions. In the general high dimensional case we obtain bounds for the long time average of gradients of the distribution function which are independent of initial data. In the $n = 3$ case we obtain long time bounds for the mean square average of the distribution function. The bounds are independent of initial data. In this case we also obtain bounds for the long time average of mean square gradients of the distribution, which again are independent of initial data. We study steady states, and parametrize them. In general, the steady state equations reduce to finitely

1991 *Mathematics Subject Classification.* 35K35, 76A05, 76A15, 82D60.
Key words and phrases. Rheology, non-Newtonian fluids, nematic liquid crystalline polymers, Smoluchowski equation, long time dynamics, number of steady solutions, gradient bounds, determining modes.
* Research partially supported by NSF DMS-0202531.
† Research partially supported by AFOSR (Dynamics and Control) and NSF (ITR) .
‡ Research partially supported by NSF DMS-0204794, and by the US Civilian Research and Development Foundation (CRDF) under grant No. RM1-2343-MO-02. This work was completed while E.S.T. was the Stanislaw M. Ulam Visiting Scholar at the Center for Nonlinear Studies in the Los Alamos National Laboratory supported by the Department of Energy under contract W–7405–ENG–36.

many transcendental equations for a trace-free matrix and a normalization factor. The steady states and corresponding matrix equations have a natural covariance with respect to conjugacy by rotation. This symmetry has to be taken into account when counting distinct steady states. In the $n = 2$ case we prove that the uniform (constant) distribution is the unique steady state when $0 < b \leq 4$. We prove that, for $b > 4$ there are at least two distinct steady states. We also prove that, for arbitrary $b > 4$ there are at most $2\left[\frac{b}{4}\right]$ distinct steady states ($[x]$ denotes the largest integer not exceeding x). We study in more detail the $n = 2$ dynamics. A cancellation is used to prove that the system is dissipative in $H^{-\frac{1}{2}}(S^1)$: all solutions enter an absorbing ball in finite time, never to leave it again. The system has a finite dimensional global attractor. The long time behavior of solutions can be observed by monitoring one determining mode - the potential itself - and that can be done by measuring the potential at one location alone. Thus, two solutions will approach each other globally if their potentials approach each other at one fixed angle.

2. **A Smoluchowski equation.** Let S^{n-1} be the unit sphere in $\mathbf{R}^n$. We consider the Smoluchowski equation written in local coordinates ($\phi = (\phi_1, \dots \phi_{n-1})$) as:

$$\partial_t \psi = \frac{1}{\sqrt{g}} \partial_i \left(e^{-V} \sqrt{g} g^{ij} \partial_j (e^V \psi)\right) \tag{1}$$

The potential V is given by

$$V(x,t) = -b x_i x_j S^{ij} \tag{2}$$

where x_i are Cartesian coordinates in $\mathbf{R}^n$, $i, j = 1, 2, \dots, n$, and b is a positive constant. The matrix S is determined by

$$S^{ij}(t) = \int_{S^{n-1}} x_i(\phi) x_j(\phi) \psi(\phi, t) \sigma(d\phi) - \frac{1}{n}\delta_{ij} \tag{3}$$

with $\sigma(d\phi) = \sqrt{g} d\phi$ the surface area. Thus, $V(x,t)$ is a homogeneous polynomial of second degree, restricted to the sphere. We will focus on the examples $n = 3$ and $n = 2$. When $n = 2$, the unit circle has local coordinate $\phi \in [0, 2\pi]$, and one has $x_1(\phi) = \cos\phi$, $x_2(\phi) = \sin\phi$, $g^{11} = g = 1$, $\partial_1 = \partial_\phi$. When $n = 3$, the coordinates on the two dimensional unit sphere are $\phi = (\theta, \varphi)$, $x_1(\theta, \varphi) = \sin\theta\cos\varphi$, $x_2(\theta, \varphi) = \sin\theta \sin\varphi$, $x_3(\theta, \varphi) = \cos\theta$. Recall also that

$$g^{11} = 1, \;\; g^{22} = (\sin\theta)^{-2}, \;\; g^{ij} = 0, i \neq j$$

with $\partial_\theta = \partial_1$ and $\partial_\varphi = \partial_2$ and that $\sqrt{g} = \sin\theta$.

The equation keeps ψ positive and normalized, if it starts so.

Theorem 2.1. *Let ψ_0 be a nonnegative, continuous function on S^{n-1}. The solutions of (1) with initial data $\psi(\cdot, 0) = \psi_0$ exist for all nonnegative time, are smooth, nonnegative and normalized*

$$\int_{S^{n-1}} \psi(\phi, t)\sigma(d\phi) = \int_{S^{n-1}} \psi_0(\phi)\sigma(d\phi) \tag{4}$$

The solutions are real analytic for positive time.

The proof is elementary and will not be given here. From now on we will choose the normalization

$$\int_{S^{n-1}} \psi(\phi, t)\sigma(d\phi) = 1. \tag{5}$$

Because of this normalization it follows from the definition (3) that the matrix S is trace-free

$$Tr(S(t)) = 0 \tag{6}$$

and, consequently, from (2), that the quadratic polynomial $V(x,t)$ is harmonic

$$\Delta_x V(x,t) = 0. \tag{7}$$

This implies that, when restricted to the unit sphere, V is an eigenfunction of the Laplace-Beltrami operator

$$\Delta_g = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right)$$

with eigenvalue equal to $-2n$

$$\Delta_g V = -2nV. \tag{8}$$

Because of the way (2) V depends on ψ, the evolution of the quantity

$$P(t) = -\int_{S^{n-1}} V(x(\phi),t)\psi(\phi,t)\sigma(d\phi) \tag{9}$$

is given by

$$\frac{d}{2dt}P = -\int_{S^{n-1}} V(x(\phi),t)\psi_t(\phi,t)\sigma(d\phi)$$

and, using the evolution (1) and one integration by parts, one obtains

$$\frac{d}{2dt}P(t) = \int_{S^{n-1}} g^{ij}\partial_i V(\psi\partial_j V + \partial_j\psi)\sigma(d\phi).$$

Integrating by parts and using (8) we deduce

$$\frac{d}{2dt}P(t) + 2nP = \int_{S^{n-1}} |\nabla_g V|^2\psi\,\sigma(d\phi) \tag{10}$$

where

$$|\nabla_g V|^2 = g^{ij}\partial_i V\partial_j V$$

On the other hand, considering the (negative) Gibbs-Boltzmann entropy

$$E(t) = \int_{S^{n-1}} \psi(\phi,t)\log(\psi(\phi,t))\sigma(d\phi) \tag{11}$$

one has, using (1) and one integration by parts

$$\frac{d}{dt}E = -\int_{S^{n-1}} \left|\nabla_g\psi\right|^2\psi^{-1}\sigma(d\phi) - \int_{S^{n-1}} g^{ij}\partial_j V\partial_i\psi\,\sigma(d\phi),$$

and using one more integration by parts and (8) one obtains

$$\frac{d}{dt}E = -\int_{S^{n-1}} \left|\nabla_g\psi\right|^2\psi^{-1}\sigma(d\phi) + 2nP(t). \tag{12}$$

Note that if the initial data ψ_0 is strictly positive, then ψ remains strictly positive for all time. Actually, in view of the inequality

$$-b\left(1-\frac{1}{n}\right) \le V(x,t) \le \frac{b}{n} \tag{13}$$

one can show, using the equations (1), (8) and the maximum principle that

$$\min_\phi \psi(\phi,t) \ge e^{-2bt}\min_\phi \psi_0(\phi) \tag{14}$$

and thus (12) is meaningful. Also, if ψ_0 is bounded above one has

$$\sup_{\phi} \psi(\phi, t) \leq e^{2b(n-1)t} \sup_{\phi} \psi_0(\phi) \tag{15}$$

Theorem 2.2. *Let ψ_0 be a continuous, strictly positive function. Let the minimum and maximum of ψ_0 be denoted, respectively, $m_0 = \min_\phi \psi_0(\phi)$ and $M_0 = \max_\phi \psi_0(\phi)$. Then the unique solution $\psi(\phi, t)$ of (1) with initial datum ψ_0 obeys the inequality*

$$\frac{1}{t}\int_0^t \left(\int_{S^{n-1}} |\nabla_g \psi(\phi, s)|\sigma(d\phi)\right)^2 ds \leq 2nb + \frac{1}{t}\log\left(\frac{M_0}{m_0}\right) \tag{16}$$

for all $t \geq 0$. Let now $t > 0$ be fixed and consider a sequence of solutions of (1) coresponding to a sequence of parameters $b \to \infty$. Then

$$\lim_{b\to\infty} \frac{1}{b^2 t}\int_0^t \int_{S^{n-1}} |\nabla_g V|^2 \psi\, \sigma(d\phi)ds = 0 \tag{17}$$

Proof. We start by noting, from (14) and (15) and (5) that

$$-2bt + \log m_0 \leq E(t) \leq 2b(n-1)t + \log M_0 \tag{18}$$

holds for all $t \geq 0$. We take the time average of (12):

$$\frac{1}{t}\int_0^t \int_{S^{n-1}} |\nabla_g \psi|^2 \psi^{-1} \sigma(d\phi)ds = 2n\frac{1}{t}\int_0^t P(s)ds + \frac{1}{t}\left(E(0) - E(t)\right).$$

Using (13), (5) and (18) we deduce

$$\frac{1}{t}\int_0^t \int_{S^{n-1}} |\nabla_g \psi|^2 \psi^{-1} \sigma(d\phi)ds \leq 2nb + \frac{1}{t}\log\left(\frac{M_0}{m_0}\right) \tag{19}$$

The inequality

$$\int_{S^{n-1}} |\nabla_g \psi|\sigma(d\phi) \leq \sqrt{\int_{S^{n-1}} |\nabla_g \psi|^2 \psi^{-1}\sigma(d\phi)}$$

follows from the normalization (5) and the Schwartz inequality by dividing and multiplying inside the the left hand side integral by $\sqrt{\psi}$. The above inequality and (19) prove (16). In order to prove (17) we integrate (10) in time from 0 to t and divide by $b^2 t$. We use (13) to deduce that

$$\frac{1}{b^2 t}\int_0^t \int_{S^{n-1}} |\nabla_g V|^2 \psi\, \sigma(d\phi)ds = O\left(\frac{1}{b}\right)$$

and thus prove (17). This ends the proof of the theorem. The relationship (17) implies a strong constraint on moments of the solutions, in the limit of large b (for $n = 2$ see (43) below).

Let us study the evolution of the L^2 norm. Using the equation (1), integrating by parts twice and using (8) we deduce

$$\frac{d}{2dt}\int_{S^{n-1}} |\psi(\phi, t)|^2 \sigma(d\phi) + \int_{S^{n-1}} |\nabla_g \psi(\phi, t)|^2 \sigma(d\phi)$$
$$= -n\int_{S^{n-1}} V(x(\phi), t)\, |\psi(\phi, t)|^2 \sigma(d\phi)$$

Now we are going to specialize to the case $n = 3$. The case $n = 2$ will be treated separately in the section concerning dynamics. The Gagliardo-Nirenberg-Sobolev inequality ([3]) for $W^{1,1}$ functions in two dimensions, together with straightforward localization implies that there exists a constant C_3 such that

$$\int_{S^2} |\psi(\phi)|^2 \sigma(d\phi) \leq C_3 \left\{ 1 + \left(\int_{S^2} |\nabla_g \psi(\phi)| \sigma(d\phi) \right)^2 \right\} \tag{20}$$

holds for any $W^{1,1}(S^2)$ function with unit L^1 norm. Using this inequality we deduce

Theorem 2.3. *Let $n = 3$, and let ψ_0 be a continuous, strictly positive function. Let the minimum and maximum of ψ_0 be denoted as above, respectively, $m_0 = \min_\phi \psi_0(\phi)$ and $M_0 = \max_\phi \psi_0(\phi)$. Then the unique solution $\psi(\phi, t)$ of (1) with initial datum ψ_0 obeys the inequalities*

$$\frac{1}{t} \int_0^t \int_{S^2} |\psi(\phi, s)|^2 \sigma(d\phi) ds \leq C_3 \left[(6b+1) + \frac{1}{t} \log\left(\frac{M_0}{m_0} \right) \right] \tag{21}$$

and

$$\frac{1}{t} \int_0^t \int_{S^2} |\nabla_g \psi(\phi, s)|^2 \sigma(d\phi) ds \leq 2C_3 b(6b+1)$$

$$+ \frac{1}{t} \left[\frac{1}{2} \int_{S^2} |\psi_0(\phi)|^2 \sigma(d\phi) + 2bC_3 \log\left(\frac{M_0}{m_0} \right) \right] \tag{22}$$

for all $t \geq 0$ with C_3 the embedding constant of (20).

Proof. The inequality (21) follows directly from (16) and (20). The inequality (13) and the evolution equation for the L^2 norm imply that

$$\frac{d}{2dt} \int_{S^2} |\psi(\phi, t)|^2 \sigma(d\phi) + \int_{S^2} |\nabla_g \psi(\phi, t)|^2 \sigma(d\phi)$$

$$\leq 2b \int_{S^2} |\psi(\phi, t)|^2 \sigma(d\phi) \tag{23}$$

Using (20) in the right hand side of (23), integrating and applying (21) we obtain

$$\frac{1}{2} \int_{S^2} |\psi(\phi, t)|^2 \sigma(d\phi) + \int_0^t \int_{S^2} |\nabla_g \psi(\phi, s)|^2 \sigma(d\phi) ds$$

$$\leq \frac{1}{2} \int_{S^2} |\psi_0(\phi)|^2 \sigma(d\phi) + 2bC_3 \left[(6b+1)t + \log\left(\frac{M_0}{m_0} \right) \right] \tag{24}$$

Dividing by t and gives (22) and ends the proof.

3. **Steady States.** Let us consider steady states of (1). Because the matrix $g^{ij}(\phi)$ is nonnegative, diagonal and pointwise invertible, one can prove that any time independent solution of (1) must satisfy

$$e^V \psi = c$$

for an appropriate constant c. Thus

$$\psi(\phi) = Z^{-1} e^{bS^{ij} x_i(\phi) x_j(\phi)} \tag{25}$$

We have constraints for the coefficients. Z fixes the normalization $\int \psi d\sigma = 1$. The matrix S is symmetric and traceless. Its eigenvalues must lie between $-1/n$ and $(n-1)/n$. Indeed, for any unit vector v, the number

$$S^{ij} v_i v_j = \int (v \cdot x(\phi))^2 \psi(\phi) d\sigma(\phi) - \frac{1}{n}$$

is in the interval above. The uniform distribution is the special solution for which the matrix S vanishes, Z is the area of S^{n-1} and $\psi = Z^{-1}$. In order to parametrize all steady solutions let us consider the real valued map

$$(S,b) \mapsto Z(S,b) \tag{26}$$

defined for any real, symmetric, traceless matrix S and any positive b by the formula

$$Z(S,b) = \int_{S^{n-1}} e^{bS^{ij}x_i(\phi)x_j(\phi)} d\sigma(\phi). \tag{27}$$

Let us also consider the function

$$\psi_{S,b}(\phi) = (Z(S,b))^{-1} e^{b(S^{ij}x_i(\phi)x_j(\phi))} \tag{28}$$

associated to any real, traceless, symmetric S and $b > 0$. Finally, for any real, traceless symmetric S and $b > 0$, denote

$$\left(\widehat{S}(S,b)\right)^{ij} = \int_{S^{n-1}} x_i(\phi)x_j(\phi)\psi_{S,b}(\phi)d\sigma(\phi). \tag{29}$$

Obviously $\widehat{S}$ is a function of S and b. Actually, one can check that $Z(S,b)$ depends only on the conjugacy class OSO^{-1}, $O \in O(n)$. More specifically, if $S_1 = OSO^{-1}$ then the rotation invariance of the measure on the unit sphere implies that $Z(S,b) = Z(S_1,b)$ and therefore $\psi_{S,b}(\phi) = \psi_{S_1,b}(T\phi)$ where $T\phi$ is the angle translation associated to the rotation O, $Ox(\phi) = x(T\phi)$. The rotation invariance implies then that $\widehat{S}(S_1,b) = O\left(\widehat{S}(S,b)\right)O^{-1}$.

Theorem 3.1. *The steady solutions of (1) are in one-to-one correspondence with the solutions of the implicit trancendental matrix equation*

$$\widehat{S}(S,b) = S + \frac{1}{n}\mathbf{I} \tag{30}$$

where $\widehat{S}(S,b)$ is associated to S and b by the formalism (27), (28), (29) above.

Note that

$$\frac{\partial(\log Z(S,b))}{\partial b} = Tr\left(S\widehat{S}\right) \tag{31}$$

and that

$$\frac{\partial \log Z(S,b)}{\partial S} = b\widehat{S}. \tag{32}$$

Let us consider this problem for $n = 2$. We look for a symmetric, traceless matrix

$$S = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}.$$

To it we associate $\psi_{S,b}$

$$\psi_{S,b}(\phi) = Z^{-1}e^{b(Sx\cdot x)}$$

with $x_1 = \cos\phi$, $x_2 = \sin\phi$ and $(x \cdot y)$ the usual Euclidean scalar product in $\mathbf{R}^2$. Putting

$$bA = \alpha, \;\; bB = \beta$$

we get

$$\psi_{S,b} = Z^{-1}e^{\alpha\cos(2\phi)+\beta\sin(2\phi)}.$$

The steady state equation

$$\widehat{S} = S + \frac{1}{2}\mathbf{I}$$

together with the normalization

$$\int_0^{2\pi} \psi_{S,b}(\phi)d\phi = 1$$

translate into the four transcendental equations

$$\frac{\alpha}{b} + \frac{1}{2} = Z^{-1}\int_0^{2\pi} (\cos\phi)^2 e^{\alpha\cos(2\phi)+\beta\sin(2\phi)}d\phi$$

$$-\frac{\alpha}{b} + \frac{1}{2} = Z^{-1}\int_0^{2\pi} (\sin\phi)^2 e^{\alpha\cos(2\phi)+\beta\sin(2\phi)}d\phi,$$

$$\frac{\beta}{b} = Z^{-1}\int_0^{2\pi} (\sin\phi\cos\phi) e^{\alpha\cos(2\phi)+\beta\sin(2\phi)}d\phi$$

and

$$Z = \int_0^{2\pi} e^{\alpha\cos(2\phi)+\beta\sin(2\phi)}d\phi$$

for the unknowns α, β, Z depending on the parameter b. There are only three independent equations, the second equation follows from the first and the last. Elementary trigonometry brings us to

$$\frac{2\alpha}{b} = Z^{-1}\int_0^{2\pi} \cos\phi\, e^{\alpha\cos\phi+\beta\sin\phi}d\phi,$$

$$\frac{2\beta}{b} = Z^{-1}\int_0^{2\pi} \sin\phi\, e^{\alpha\cos\phi+\beta\sin\phi}d\phi,$$

and

$$Z = \int_0^{2\pi} e^{\alpha\cos\phi+\beta\sin\phi}d\phi.$$

Introducing now

$$\alpha = r\cos\phi_0, \;\; \beta = r\sin\phi_0$$

and using the rotation invariance of the measure on the circle (translation invariance of $d\phi$), we identify ϕ_0 as the free $O(2)$ conjugacy parameter. The two remaining equations are

$$Z(r) = \int_0^{2\pi} e^{r\cos\phi}d\phi \tag{33}$$

and

$$\frac{2r}{b} = (Z(r))^{-1}\int_0^{2\pi} \cos\phi e^{r\cos\phi}d\phi. \tag{34}$$

The first of these equations, (33), can be thought of as defining the function $Z(r)$ for all values $r \geq 0$, and the second (34) of the equations is then thought of as selecting the appropriate number (or numbers) r corresponding to the parameter b. These, in turn, together with the free parameter ϕ_0, determine the matrices S, and steady states ψ. The eigenvalues of the matrix S are

$$\lambda_{\pm} = \pm\frac{r}{b}. \tag{35}$$

The constraint that the eigenvalues of S lie in the interval $[-1/n, 1-1/n]$ implies that

$$0 \leq r \leq \frac{b}{2} \tag{36}$$

is the admissible range of solutions r. At $r = 0$, $Z = 2\pi$ is the trivial (uniform state) solution. In order to determine whether we have nontrivial solutions we note that $Z(r)$ is a positive function, defined for all $r \in [0, \infty)$.

$$Z(r) = 2\pi \sum_{k=0}^{\infty} r^{2k} 2^{-2k} \frac{1}{(k!)^2}$$

and so $\log(Z(r))$ makes sense for $r \geq 0$. Now we define, for any continuous periodic function $f(\phi)$, a transformed function

$$[f](r) = (Z(r))^{-1} \int_0^{2\pi} f(\phi) e^{r\cos\phi} d\phi \tag{37}$$

which is a continuous function of $r \geq 0$.

Lemma 3.1. *For any analytic periodic function $f(\phi)$ defined for $\phi \in [0, 2\pi]$, the function $[f](r)$ defined by (37) obeys*

$$\lim_{r\to\infty} [f](r) = f(0).$$

Proof. We divide both numerator and denominator of

$$\frac{\int_0^{2\pi} f(\phi) e^{r\cos\phi} d\phi}{\int_0^{2\pi} e^{r\cos\phi} d\phi}$$

by e^r, and then we use steepest descent: splitting

$$\int_{-\pi}^{\pi} f(\phi) e^{-r(1-\cos\phi)} d\phi = \int_{-\epsilon}^{\epsilon} f(\phi) e^{-r(1-\cos\phi)} d\phi + O(e^{-r/2})$$

we get

$$\int_{-\pi}^{\pi} f(\phi) e^{-r(1-\cos\phi)} d\phi = f(0)\sqrt{\frac{\pi}{r}} + O(\frac{1}{r})$$

and dividing by the integral corresponding to $f = 1$ we finish the proof of the lemma.

The object of our interest is the function

$$\frac{d}{dr} \log Z = [\cos](r).$$

Note that

$$\frac{d^2}{dr^2} \log Z = [(\cos - [\cos](r))^2](r) > 0,$$

so $r \mapsto [\cos](r)$ is increasing. Because $[\cos](0) = 0$ and $\cos 0 = 1$, it follows from the lemma above that

$$0 \leq [\cos](r) \leq 1$$

holds for all $r \geq 0$. Introducing

$$H(r) = \frac{d}{dr} \log Z - \frac{2r}{b} = [\cos](r) - \frac{2r}{b} \tag{38}$$

the equation we wish to study (34) can be written as

$$Z'(r) - \frac{2r}{b} Z(r) = 0 \tag{39}$$

or, equivalently, as

$$H(r) = 0.$$

Note that, because the range of [cos] is included in the interval $[0,1]$, it follows that all solutions of (39) are automatically in the required range (36). The explicit formula

$$Z'(r) - \frac{2r}{b}Z(r) = -\frac{8\pi}{b}\sum_{k=1}^{\infty}\left(k - \frac{b}{4}\right)\frac{k}{(k!)^2}\left(\frac{r}{2}\right)^{2k-1} \tag{40}$$

shows that there are no zeros other than $r = 0$ if $b \le 4$. At $r = 0$ we have

$$H'(0) = \frac{1}{2} - \frac{2}{b}$$

so, for $b > 4$ we have $H'(0) > 0$. On the other hand, using the lemma

$$\lim_{r\to\infty} H'(r) = -\frac{2}{b}$$

so we have at least one nontrivial solution, for $b > 4$.

Theorem 3.2. *Let $n = 2$. Let $N(b)$ denote the number of distinct steady solutions of (1) modulo the $O(2)$ conjugacy. Then, if $b \le 4$ then $N(b) = 1$. If $b > 4$ then $N(b) \ge 2$. For $b > 4$*

$$N(b) \le 2\left[\frac{b}{4}\right]$$

where $[x]$ denotes the largest integer not exceeding x.

Proof. We just proved that there are at least two solutions for $b > 4$ and only the trivial solution for $b \le 4$. Now we need to estimate the number of zeros of $Z'(r) - \frac{2r}{b}Z(r)$ for $b > 4$. The expression (40) shows that

$$Z'(r) - \frac{2r}{b}Z(r) = P(r) - Q(r)$$

where $P(r)$ is a polynomial of order $2\left[\frac{b}{4}\right] - 1$ and

$$Q(r) = \frac{8\pi}{b}\sum_{k=[\frac{b}{4}]+1}^{\infty}\left(k - \frac{b}{4}\right)\frac{k}{(k!)^2}\left(\frac{r}{2}\right)^{2k-1}$$

Differentiating $m = 2\left[\frac{b}{4}\right]$ times we obtain

$$\frac{d^m}{dr^m}\left(Z'(r) - \frac{2r}{b}Z(r)\right) < 0$$

for all $r > 0$. This, and Rolle's theorem, finish the proof.

4. **Dynamics.** The two-dimensional (n=2) time dependent problem is

$$\partial_t\psi = \partial_\phi\left(e^{-V}\partial_\phi(e^V\psi)\right)$$

with

$$V(\phi, t) = -\frac{b}{2}\left[\cos(2(\phi - \cdot))\right]$$

and

$$[f(\cdot)] = \int_0^{2\pi} f(\phi)\psi(\phi, t)d\phi.$$

Thus,

$$V(\phi, t) = -\frac{b}{2}\int_0^{2\pi}\cos(2(\phi - y))\psi(y, t)dy$$

and the equation is

$$\partial_t\psi = \partial_\phi^2\psi + \partial_\phi(V_\phi\psi)$$

with 2π periodic boundary conditions. Denote

$$\widehat{\psi}(j,t) = \int_0^{2\pi} e^{-ij\phi}\psi(\phi,t)d\phi$$

Clearly $V_\phi\psi$ is a simple quadratic nonlinearity; in Fourier representation

$$\partial_\phi \widehat{(V_\phi\psi)}(j,t) = \frac{bj}{2}\left(\widehat{\psi}(j-2,t)\widehat{\psi}(2,t) - \widehat{\psi}(j+2,t)\widehat{\psi}(-2,t)\right)$$

Thus, the system becomes

$$\widehat{\psi}(0,t) = 1$$

and

$$\partial_t\widehat{\psi}(j,t) = -j^2\widehat{\psi}(j,t) + \frac{bj}{2}\left(\widehat{\psi}(j-2,t)\widehat{\psi}(2,t) - \widehat{\psi}(j+2,t)\widehat{\psi}(-2,t)\right)$$

Because we are interested in real solutions we have

$$\widehat{\psi}(-j,t) = \widehat{\psi}(j,t)^*.$$

Certain symmetries are preserved. Even functions satisfy

$$\widehat{\psi}(-j,t) = \widehat{\psi}(j,t).$$

This is preserved by the flow. Also, $\widehat{\psi}(2k+1,t) = 0$ for all $k \in \mathbf{Z}$ if initially it is zero. Returning to the physical space representation, we have $\psi(\phi,t) > 0$ if initially, ψ is probability density,

$$V_{\phi\phi} = -4V$$

and V and V_ϕ are bounded in absolute value pointwise by $\frac{b}{2}$ and, respectively, by b. No finite time blow up can occur, ψ is bounded pointwise by an exponential function of time. In order to show that the system is dissipative (i.e. the phase space explored is bounded), we denote

$$y_k = y_k(t) = \widehat{\psi}(2k,t)$$

From the PDE information we will use

$$|y_k(t)| \le 1, \;\; y_0(t) = 1.$$

We restrict to the symmetric case (so we take only even Fourier coefficients). This means we are dealing with a cosine series

$$\psi(\phi,t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} y_k(t)\cos(2k\phi).$$

The numbers y_k are real and obey

$$y_1' = (b - 4 - by_2)y_1 \tag{41}$$

and

$$y_k' = -4k^2 y_k + bky_1\left(y_{k-1} - y_{k+1}\right) \tag{42}$$

for $k \ge 1$ where $y' = \frac{dy}{dt}$. Note that the sign of y_1 does not change in the evolution. Note also, with these symmetries in place, the steady state calculation for $b > 4$ has explicit coefficients y_k computed inductively starting from $y_1 = c \ne 0$ and $y_2 = 1 - \frac{4}{b}$: $y_{k+1} = y_{k-1} - 4k\frac{y_k}{by_1}$. The parameter c is then determined by the condition that the resulting series converge. That becomes a nontrivial calculation in this setting, and the fact that there are at most $2\left[\frac{b}{4}\right]$ solutions is not easily seen. In the present example the statement (17) is: for every fixed positive t,

$$\lim_{b\to\infty}\frac{1}{t}\int_0^t y_1^2(s)\,(1-y_2(s))\,ds = 0. \tag{43}$$

In order to prove dissipativity of the ODE for y_k, we multiply (42) by $k^{-1}y_k$ to get

$$\frac{y_k}{k}y_k' = -4ky_k^2 + by_1(y_{k-1}-y_{k+1})y_k, \quad k\geq 1,$$

and summing from $k=1$ we deduce

$$\frac{d}{2dt}\left(\sum_{k=1}^{\infty}\frac{1}{k}y_k^2\right) = -4\sum_{k=1}^{\infty}ky_k^2 + by_1^2.$$

Using $|y_1|\leq 1$ we obtain:

Theorem 4.1. *Consider the evolution equation (1) for n=2, with positive continuous initial data*

$$\psi_0(\phi) = \frac{1}{2\pi} + \sum_{k=1}^{\infty} y_k(0)\cos(2k\phi)$$

Then

$$\|\psi(\cdot,t)\|^2_{H^{-\frac{1}{2}}(\mathbf{S}^1)} \leq \frac{b}{4} + e^{-8t}\|\psi_0\|^2_{H^{-\frac{1}{2}}(\mathbf{S}^1)} \tag{44}$$

and

$$\frac{1}{t}\int_0^t \|\psi(\cdot,s)\|^2_{H^{\frac{1}{2}}(\mathbf{S}^1)}ds \leq \frac{b}{4} + \frac{1}{2t}\|\psi_0\|^2_{H^{-\frac{1}{2}}(\mathbf{S}^1)} \tag{45}$$

hold for all $t\geq 0$. Here

$$\|\psi\|^2_{H^s(S^1)} = \sum_{k=1}^{\infty} k^{2s}y_k^2.$$

The theorem shows that the ball in $H^{-\frac{1}{2}}(S^1)$ of radius $\sqrt{b}$ centered at the uniform state $\psi = \frac{1}{2\pi}$ absorbs all trajectories in finite time.

Let us consider now two solutions $\psi^{(1)}(\phi,t)$ and $\psi^{(2)}(\phi,t)$ which are given by cosine series

$$\psi^{(j)}(\phi,t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} y_k^{(j)}(t)\cos(2k\phi)$$

Let us consider their difference

$$\psi(\phi,t) = \psi^{(1)}(\phi,t) - \psi^{(2)}(\phi,t),$$

and semi-sum

$$\overline{\psi}(\phi,t) = \frac{1}{2}(\psi^{(1)}(\phi,t) + \psi^{(2)}(\phi,t)).$$

The Fourier coefficients are denoted accordingly,

$$y_k(t) = y_k^{(1)}(t) - y_k^{(2)}(t)$$

and

$$\overline{y}_k(t) = \frac{1}{2}\left(y_k^{(1)}(t) + y_k^{(2)}(t)\right)$$

The equation for the difference is

$$y_k' = -4k^2y_k + bk\overline{y}_1(y_{k-1}-y_{k+1}) + bky_1(\overline{y}_{k-1}-\overline{y}_{k+1}) \tag{46}$$

valid for all $k \geq 1$. In the equation for $k = 1$ it is understood that $y_0 = 0$ and $\overline{y}_0 = 1$. Multiplying by $\frac{1}{k}y_k$ and adding we obtain

$$\frac{d}{2dt}\|\psi(\cdot,t)\|^2_{H^{-\frac{1}{2}}(S^1)} + 4\|\psi(\cdot,t)\|^2_{H^{\frac{1}{2}}(S^1)} = by_1\sum_{k=1}^{\infty}(\overline{y}_{k-1} - \overline{y}_{k+1})y_k$$

Using a Schwartz inequality we deduce

$$\frac{d}{2dt}\|\psi(\cdot,t)\|^2_{H^{-\frac{1}{2}}(S^1)} + 4\|\psi(\cdot,t)\|^2_{H^{\frac{1}{2}}(S^1)}$$

$$\leq b|y_1|\|\psi(\cdot,t)\|_{H^{\frac{1}{2}}(S^1)}\sqrt{\sum_{k=1}^{\infty}\frac{1}{k}(\overline{y}_{k-1} - \overline{y}_{k+1})^2}$$

Therefore, using Young's inequality

$$\frac{d}{dt}\|\psi(\cdot,t)\|^2_{H^{-\frac{1}{2}}(S^1)} + \|\psi(\cdot,t)\|^2_{H^{\frac{1}{2}}(S^1)}$$

$$\leq 2b^2y_1^2\left(1 + \|\psi^{(1)}(\cdot,t)\|^2_{H^{-\frac{1}{2}}(S^1)} + \|\psi^{(2)}(\cdot,t)\|^2_{H^{-\frac{1}{2}}(S^1)}\right)$$

Using the inequality (44) we obtain

$$\frac{d}{dt}\|\psi(\cdot,t)\|^2_{H^{-\frac{1}{2}}(S^1)} + \|\psi(\cdot,t)\|^2_{H^{\frac{1}{2}}(S^1)}$$

$$\leq 2b^2y_1^2\left(1 + \frac{b}{2} + e^{-8t}\|\psi_0^{(1)}\|^2_{H^{-\frac{1}{2}}(S^1)} + e^{-8t}\|\psi_0^{(2)}\|^2_{H^{-\frac{1}{2}}(S^1)}\right). \tag{47}$$

Theorem 4.2. *Let $\psi^{(j)}(\phi,t)$, $j = 1, 2$, be two solutions of (1) for $n = 2$. Assume that the corresponding potentials $V^{(1)}(\phi,t)$ and $V^{(2)}(\phi,t)$ become close asymptotically at $\phi = 0$:*

$$\lim_{t\to\infty}|V^{(1)}(0,t) - V^{(2)}(0,t)| = 0.$$

Then the two solutions approach each other:

$$\lim_{t\to\infty}\|\psi^{(1)}(\cdot,t) - \psi^{(2)}(\cdot,t)\|_{H^{-\frac{1}{2}}(S^1)} = 0$$

Proof. Note that

$$V^{(j)}(\phi,t) = -\frac{b}{2}y_1^{(j)}(t)\cos(2\phi)$$

and therefore the asumption implies

$$\lim_{t\to\infty}|y_1(t)| = 0$$

where $y_1(t)$ is the Fourier coefficient of the difference of solutions. The conclusion of the theorem follows then from (47).

This shows that the system has one determining mode. Using standard tools ([1]) one can prove that this system has a finite dimensional global attractor.

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