Decay for Schrödinger and related equations.

Introduction to PDE


The system we study (Schrödinger-Davey-Stewartson-Zakharov) is

\begin{align*}
  i\partial_t u + L_1 u &= a|u|^2u + vu \\
  L_2 v &= L_3(|u|^2)
\end{align*}

(1)

where \( a \in \mathbb{R} \), and \( L_i \) are second order constant coefficient differential operators

\[ L_i = g^{ij} \frac{\partial^2}{\partial x^i \partial x^j}, \quad i = 1, 2, 3. \]

(2)

where the constant real matrices \( g^{ij} \) are symmetric and invertible. We assume \( L_2 \) to be elliptic, and we write then the system as a single equation

\[ i\partial_t u + P(D)u = L(|u|^2)u \]

(3)

where \( P(D) = L_1 \) and we drop the index:

\[ P(D) = g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} \]

(4)

and

\[ L = aI + L_2^{-1}L_3 \]

(5)

The properties we will use for the linear operator \( L \) are: \( L : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n) \) is bounded for any \( 1 < p < \infty \), \( L \) commutes with translations (and hence with differentiation, \( L \) is real (i.e. it commutes with complex conjugation).
1 Sobolev lemma and the linear equation

We denote by $g_{jk}$ the inverse matrix $(g_{jk}) = (g^{jk})^{-1}$. We do not assume that $P(D)$ is elliptic. When $P(D) = \Delta$ then

$$iu_t + P(D)u = 0$$

is the free Schrödinger equation. We introduce the differential operators

$$Q_j((x,t),D) = Q_j = 2t\partial_j - ig_{jk}x^k$$

They commute with the free equation:

$$[i\partial_t + P(D), Q_j] = 0$$

This can be checked by hand. We can also easily check using the Fourier transform that

$$Q_j = e^{itP(D)}(-ig_{jk}x^k)e^{-itP(D)}$$

Indeed,

$$\mathcal{F}(e^{itP(D)}(-ig_{jk}x^k)e^{-itP(D)}u)(\xi) = e^{-itg^{ab}\xi_a\xi_b}(g_{jk}\partial_{\xi_k}(e^{itg^{cd}\xi_c\xi_d}\mathcal{F}(u)(\xi)))
= g_{jk}\partial_{\xi_k}\mathcal{F}(u)(\xi) + g_{jk}(itg^{cd}(\xi_d\delta_{jk} + \xi_k\delta_{dc})\mathcal{F}(u)(\xi)
= \mathcal{F}[(-ig_{jk}x^k + 2t\partial_j)u](\xi)$$

where we used

$$\mathcal{F}(u)(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)dx$$

and $\delta_{jk}$ the Kronecker delta.

We have as well that

$$Q_j = 2te^{i\psi}\partial_j e^{-i\psi}$$

where

$$\psi(x,t) = \frac{g_{jk}x^jx^k}{4t}$$

and where the right hand side of (9) is considered as a product of operators (multiplication by $e^{-i\psi}$ followed by differentiation, followed by multiplication by $2te^{i\psi}$). The operators $Q_j$ commute. They generate a Lie algebra denoted $\mathcal{A}$. The Lie algebra generated by the collection $Q_j, \partial_j$ is denoted $\mathcal{B}$. For any
Lie algebra \( \mathcal{A} \) of differential operators with generators \( A_1, \ldots A_N \) we use the notation

\[
|u(x,t)|_{\mathcal{A},m} = \sum_{j=0}^{m} \left( \sum_{|\alpha| = j} (A_\alpha u(x,t))^2 \right)^{\frac{1}{2}}
\]

(11)

where

\[ A_\alpha = A_1^{\alpha_1} \cdots A_N^{\alpha_N}, \quad \text{and} \quad |\alpha| = \alpha_1 + \cdots + \alpha_N. \]

We define generalized \( W^{m,p} \) norms via

\[
\|u(\cdot, t)\|_{\mathcal{A},m,p} = \left( \int_{\mathbb{R}^n} |u(x,t)|_{\mathcal{A},m}^p dx \right)^{\frac{1}{p}}.
\]

(12)

**Lemma 1.** There exists a constant \( C = C(n) \) such that

\[
|u(x,t)| \leq C |t|^{-\frac{n}{2}} \|u(\cdot, t)\|_{\mathcal{A},[\frac{n}{2}]+1,2}
\]

(13)

holds for all \((x,t)\) and all \( u \).

Proof. Let us consider the function

\[
v(x,t) = e^{-i\psi} u(x,t)
\]

and apply the local Sobolev Lemma to it

\[
|v(x,t)| \leq C \sum_{j=0}^{1+[\frac{n}{2}]} R^{j-\frac{n}{2}} \left( \sum_{|\alpha| = j} \int_{|x-y| \leq R} |\partial_\alpha y v(y,t)|^2 dy \right)^{\frac{1}{2}}
\]

This holds for any \( R \), and the constant \( C \) is independent of \( R \). Now we observe that

\[
|v(x,t)| = |u(x,t)|
\]

and, in view of (9), by induction, it holds that

\[
|\partial_\alpha y v(y,t)| = (2|t|)^{-|\alpha|} Q^\alpha u(y,t).
\]

The inequality (13) follows by choosing \( R = 2t \).

We remove the singularity at \( t = 0 \) by augmenting to \( \mathcal{B} \):
Lemma 2. There exists a constant $C = C(n)$ such that

$$|u(x,t)| \leq C(1 + |t|)^{-\frac{n}{2}} \|u(\cdot, t)\|_{\mathcal{B}_0, \frac{n}{2} + 1, 2}$$

(14)

holds for all $(x, t)$ and all $u$.

This is trivial because for $|t| \leq 1$ we use the usual Sobolev inequality. A direct application is:

Theorem 1. Let $u(x, t)$ be a solution of

$$iu_t + P(D)u = 0$$

(15)

with initial datum $u(x, 0) = u_0(x)$. Then

$$|u(x, t)| \leq C|t|^{-\frac{n}{2}} \|u_0\|_{X, 1 + \frac{n}{2}, 2}$$

(16)

where $X$ is the Lie algebra generated by the operators of multiplication by $x^j$, $j = 1, \ldots, n$. More generally,

$$|u(x, t)|_{X,k} \leq C|t|^{-\frac{n}{2}} \|u_0\|_{X, k + 1 + \frac{n}{2}, 2}$$

(17)

and

$$|u(x, t)|_{\mathcal{B}_0, k} \leq C(1 + |t|)^{-\frac{n}{2}} \|u_0\|_{\mathcal{B}_0, k + 1 + \frac{n}{2}, 2}$$

(18)

where $\mathcal{B}_0$ is the Lie algebra generated by the operators of multiplication by 1, $x^j$ and by $\partial_j$, $j = 1, \ldots, n$.

2 The nonlinear equation

Lemma 3. Let $0 < j < m$. There exists a constant $C$ depending on $j, m$ and $n$ such that

$$\sum_{|\beta| = j} \|Q^\beta u(\cdot, t)\|_{L^{2m}(\mathbb{R}^n)} \leq C\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)}^{1 - \frac{1}{m}} \left( \sum_{|\alpha| = m} \|Q^\alpha u(\cdot, t)\|_{L^2(\mathbb{R}^n)} \right)^{\frac{j}{m}}$$

(19)

The inequality (19) in which the operators $Q_j$ are replaced by $\partial_j$ is a well known Gagliardo-Nirenberg inequality. Applying it to $v = e^{-i\psi}u$ and using (9) and the scale invariance of the inequality, we immediately obtain (19).

Now we state a Leibniz rule:
Lemma 4. For any multi-index $\alpha$ it holds that

$$Q^\alpha(L(|u|^2 v) = \sum_{\beta+\gamma+\delta=\alpha} \frac{\alpha!}{\beta!\gamma!\delta!} L((Q^\beta u)\overline{Q^\gamma u})Q^\delta v$$  \hspace{1cm} (20)$$

Proof. We used the notation $\alpha! = \alpha_1! \cdots \alpha_n!$. The proof is done by induction on $|\alpha|$ and it follows from the observations that

$$Q_j(ab) = (2t\partial_j a) b + aQ_j(b)$$

and that

$$2t\partial_j(a\overline{b}) = (Q_j(a))\overline{b} + a(Q_j(b))$$

The first equality is used to write

$$Q_j(L((Q^\beta u)\overline{Q^\gamma u})Q^\delta v) = L((Q^\beta u)\overline{Q^\gamma u}))Q^\delta v + L((Q^\beta u)\overline{Q^\gamma u})Q_jQ^\delta v$$

and the second one to finish

$$Q_j(L((Q^\beta u)\overline{Q^\gamma u})Q^\delta v) = L((Q_jQ^\beta u)\overline{Q^\gamma u} + (Q^\beta u)\overline{Q_jQ^\gamma u})Q^\delta v + L((Q^\beta u)\overline{Q^\gamma u})Q_jQ^\delta v$$

So, $Q_j$ distributes just like a derivative in the product. The fact that the complex conjugate is inside the operator $L$ is used crucially. Let us start by denoting

$$I_m(w)(t) = I_m = \left( \sum_{|\alpha|=m} \int_{\mathbb{R}^n} |Q^\alpha w|^2 dx \right)^{\frac{1}{2}}$$  \hspace{1cm} (21)$$

Let us assume that $w$ solves the equation

$$iw_t + P(D)w = L(|u|^2)w$$  \hspace{1cm} (22)$$

for some given (smooth) function $u$. Note that this is a linear Schrödinger equation if $P(D)$ is elliptic. Then, in view of (7) and the fact that $P(D)$ is real, we have

$$\frac{1}{2} \frac{d}{dt} I_m^2 = \text{Im} \sum_{|\alpha|=m} \int_{\mathbb{R}^n} Q^\alpha(L(|u|^2)w)\overline{Q^\alpha w} dx$$  \hspace{1cm} (23)$$
Now, in view of our Leibniz formula (20), the right-hand side of (23) is a sum for $|\alpha| = m$ and $\alpha = \beta + \gamma + \delta$ of terms
\[
\frac{\alpha!}{\beta!\gamma!\delta!} \text{Im} \int_{\mathbb{R}^n} L((Q^\beta u)(\overline{Q^\gamma u}))(Q^\delta w) \overline{Q^\delta w} \, dx
\] (24)

The term in (24) corresponding to $\beta = \gamma = 0$ vanishes. This is very important, because $L$ is not bounded in $L^\infty$. If $\beta = \delta = 0$ or $\gamma = \delta = 0$, then we estimate (24) using
\[
\left| \int_{\mathbb{R}^n} L(uQ^\alpha u)wQ^\alpha w \, dx \right| \leq C \|uQ^\alpha u\|_{L^2(\mathbb{R}^n)} \|wQ^\alpha w\|_{L^2(\mathbb{R}^n)}
\] (25)

The rest of the terms have $0 < |\delta| < m$. In these terms we apply a Hölder inequality, raising the last term to the second power, the term involving $Q^\delta w$ to the power $2m|\delta|$, and the term involving $L$ to the power $q = 2(1 - |\delta|/m)^{-1}$. Using the boundedness of $L$ in $L^q$ spaces and our Gagliardo-Nirenberg inequality (19) we majorize (24) by
\[
C \|w(\cdot, t)\|_{L^\infty(\mathbb{R}^n)}^{1 - |\delta|/m} I_m(w)^{1 + |\delta|/m} \left( \int_{\mathbb{R}^n} |Q^\beta u|^q |Q^\gamma u|^q \, dx \right)^{1/q}
\]

In the last integral we use a Hölder inequality with powers $2m/|\beta|$ and $2m/|\gamma|$ (their inverses do add up to 1) and again our Gagliardo-Nirenberg inequality (19). The result is that all these terms in the (24) can be bound by
\[
C \|u\|_{L^\infty(\mathbb{R}^n)}^{1 - |\delta|/m} I_m(u)^{1 - |\delta|/m} \|w\|_{L^\infty(\mathbb{R}^n)}^{1 - |\delta|/m} I_m(w)^{1 + |\delta|/m}
\] (26)

We note that (25) has the form of (26) with $|\delta| = 0$. Dividing by $I_m(w)$ we obtained
\[
\frac{d}{dt} I_m(w) \leq C \sum_{j=0}^{m-1} \|u\|_{L^\infty(\mathbb{R}^n)}^{1 + \frac{j}{m}} I_m(u)^{1 - \frac{j}{m}} \|w\|_{L^\infty(\mathbb{R}^n)}^{1 - \frac{j}{m}} I_m(w)^{\frac{j}{m}}
\] (27)

Now the exact same calculation applies to integrals
\[
J_m(w) = \left( \sum_{|\alpha| = m} \int_{\mathbb{R}^n} |\partial^\alpha w|^2 \, dx \right)^{1/2}
\] (28)
using the usual Leibniz formula and Gagliardo-Nirenberg inequalities. Denoting
\[ K_m(w) = I_m(w) + J_m(w) \]
we obtain by adding the two similar inequalities and using \( \max \{I, J\} \leq K \)
\[
\frac{d}{dt} K_m(w) \leq C \sum_{j=0}^{m-1} \|u\|_{L^\infty}^{1+\frac{j}{m}} K_m(u)^{1-\frac{j}{m}} \|w\|_{L^\infty}^{1-\frac{j}{m}} K_m(w)^{\frac{1}{m}}
\]  
(30)

Let us introduce now
\[
E_N(w) = \sum_{m=0}^{N} K_m(w)
\]
and take \( N \geq 1 + \lfloor \frac{n}{2} \rfloor \). Note that (14) implies
\[
\|w\|_{L^\infty} \leq C(1 + |t|)^{-\frac{n}{2}} E_N(w)
\]
(32)

Using the same inequality for \( u \) and majorizing each \( K_m \) by \( E_N \), we obtain from (30)
\[
\frac{d}{dt} K_m(w) \leq (1 + |t|)^{-n} E_N(u) E_N(w)
\]
(33)
for \( m = 0, 1, \ldots, N \). (Note that \( K_0 \) is conserved.) Adding in \( m \) we obtain

**Theorem 2.** Let \( w \) solve the linear equation (22). For \( N \geq 1 + \lfloor \frac{n}{2} \rfloor \) there exists a constant \( C = C(n, N) \) such the norm \( E_N(w) \) satisfies
\[
\frac{d}{dt} E_N(w) \leq C(1 + |t|)^{-n} E_N(u) E_N(w)
\]
(34)

**Theorem 3.** Let \( N \geq 1 + \lfloor \frac{n}{2} \rfloor \). Then there exists \( \epsilon = \epsilon(N) \) and \( C = C(n, N) \) such that, if \( u_0 \) satisfies
\[
\sum_{|\alpha| \leq N} \int_{\mathbb{R}^n} \left[ |Q^\alpha(x,0,D)u_0(x)|^2 + |\partial^\alpha u_0(x)|^2 \right] dx \leq \epsilon
\]
then the solution of (3) exists for all time and satisfies
\[
\sum_{|\alpha| \leq N} \int_{\mathbb{R}^n} \left[ |Q^\alpha(x,t,D)u(x,t)|^2 + |\partial^\alpha u(x,t)|^2 \right] dx \leq C\epsilon
\]
and
\[
|u(x,t)| \leq C\epsilon^{\frac{1}{2}}(1 + |t|)^{-\frac{n}{2}}
\]
Proof. We prove first local existence (short time existence) and uniqueness in $H^N$. This is done by considering the map

$$u(t) \mapsto e^{itP(D)}u_0 - i \int_0^t e^{i(t-s)P(D)}L(|u(s)|^2)u(s)ds$$

for $u \in C(0, T; H^N)$ with $u(0) = u_0$. We obtain unique solutions on time intervals depending on the norm of $u_0$ in $H^N$. Then we use (34) with $u = w$ to deduce that

$$E_N(t)(1 - C_NE(0)) \leq E_N(0).$$