1 Schauder fixed point

Warning: Brouwer’s Thm is false in infinite dimensions. Example: $\ell_2(\mathbb{N})$, with unit closed ball $B$. Then

$$f : B \to \partial B, \quad f(x) = (\|x\|^2 - 1, x_1, x_2, \ldots)$$

is continuous, and if it had a fixed point, the fixed point equations would be $x_1 = 0, x_2 = x_1, \ldots, x_{n+1} = x_n$, so the fixed point would be 0, but it had to have norm equal to 1.

**Definition 1.** A continuous function $F : S \subset X \to X$, where $X$ is a Banach space, is compact if it maps bounded closed sets to relatively compact sets (sets whose closure is compact).

**Theorem 1.** Let $f : S \to X$ where $S$ is closed and bounded in the Banach space $X$. Then $f$ is compact iff it is a uniform limit of continuous finite range maps.

**Proof.** If $f$ is compact then $K = \overline{f(S)}$ is compact. Given $\epsilon > 0$ there exist $x_1, \ldots, x_{j(\epsilon)} \in K$ such that the balls $B_i$ of centers $x_i$ and radii $\epsilon$ cover $K$. Let $\psi_i$ be a partition of unity for $K$ subordinated to the cover, i.e $\psi_i \geq 0$ is supported in $B_i$ and $\sum_i \psi_i = 1$ on $K$. Let

$$f_\epsilon(x) = \sum_{i=1}^{j(\epsilon)} \psi_i(f(x))x_i$$

Then $f_\epsilon(x)$ belongs to the convex hull of $x_i$ and

$$\|f(x) - f_\epsilon(x)\| \leq \sum_{i=1}^{j(\epsilon)} \psi_i(f(x))\|f(x) - x_i\| \leq \epsilon$$

The argument in the other direction is an exercise.
Theorem 2. (Schauder fixed point). Let \( S \) be a closed, convex, bounded subset of a Banach space \( X \), and let \( f : S \to S \) be a compact map. Then \( f \) has a fixed point.

Proof. Consider \( f_\epsilon(x) \) defined above, and let \( X_\epsilon \) be the finite dimensional linear spaced spanned by \( x_i, i = 1, \ldots, j(\epsilon) \). Since \( S \) is convex and \( f_\epsilon(S) \) is contained in the convex hull of \( f(S) \) we have \( f_\epsilon : S \to S \cap X_\epsilon \). Therefore \( f_\epsilon \) maps the closed bounded set \( S \cap X_\epsilon \) to itself. This is a subset of \( X_\epsilon \) so we may apply the finite dimensional Brouwer fixed point theorem, and find \( x_\epsilon \in S \cap X_\epsilon \) such that \( x_\epsilon = f_\epsilon(x_\epsilon) \). Now \( f_\epsilon(x_\epsilon) \) has a convergent subsequence by the relative compactness of \( f(S) \). Passing to the limit and using \( x_\epsilon - f(x_\epsilon) = f_\epsilon(x_\epsilon) - f(x_\epsilon) \), we finish the proof.

2 Leray-Schauder Degree

If \( X \) is a Banach space and \( \phi = I - K \) where \( K : \overline{\Omega} \to X \) is a compact transformation, then we the image under \( \phi(S) \) of a closed bounded set is closed. Indeed, if \( y_n = \phi(x_n) \) with \( x_n \in S \) converges to \( y \in X \) then, because \( S \) is bounded and \( K \) is compact we may extract a subsequence, relabeled \( x_n \), such that \( Kx_n \to z \), and then \( x_n = \phi(x_n) + Kx_n \) converges to \( x = y + Kz \). By continuity, \( y = x - Kz \).

If \( y_0 \notin \phi(\partial\Omega) \), then it is at positive distance \( \delta \) from \( \partial\Omega \). We take an \( \epsilon \)-approximation \( K_\epsilon \) of \( K \) with range in \( X_\epsilon \), a finite dimensional subspace of \( X \) such that \( y_0 \in X_\epsilon \). If \( \epsilon \leq \frac{\delta}{2} \) then \( y_0 \notin \phi_\epsilon(\partial\Omega) \) where \( \phi_\epsilon = I - K_\epsilon \). We consider

\[
\phi_\epsilon|_{X_\epsilon \cap \overline{\Omega}} : X_\epsilon \cap \overline{\Omega} \to X_\epsilon
\]

Definition 2.

\[
\deg(\phi, \Omega, y_0) = \deg(\phi_\epsilon|_{X_\epsilon \cap \overline{\Omega}}, \Omega \cap X_\epsilon, y_0)
\]

This is well defined by the last proposition in the chapter on finite dimensional degree. That means that we may change the finite dimensional space \( X_\epsilon \), and we may also change the finite range approximation \( K_\epsilon \). This follows by first placing both approximation ranges in a common (larger) finite dimensional space, and the using homotopy.

We note that if \( y_0 \notin \phi(\overline{\Omega}) \) then \( \deg(\phi, \Omega, y_0) = 0 \). All results in the chapter on finite dimensional degree are valid. In particular \( \deg(\phi, \Omega, y_0) \)
depends only on the homotopy class of \( \phi : \partial \Omega \to X \setminus \{ y_0 \} \), where the homotopy is of the form \( \phi_t = I - K_t \), with \( K_t \) continuous in \( t \in [0, 1] \) and compact for each \( t \). In particular, the image of an open set under a one-to-one map \( \phi = I - K \) is open.

3 First elementary applications

First, an application of Schauder’s fixed point theorem. Let \( K(s, t) \) be a continuous function and let

\[
Ku(s) = \int_0^1 K(s, t)f(t, u(t))dt
\]

where \( f : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is continuous and bounded. Taking \( X = C([0, 1]) \) we have that \( K \) is a compact map on any ball \( \|u\| \leq R \). By the Schauder fixed point, there exists \( u \) continuous, such that

\[
u(s) = Ku(s).
\]

Indeed we want to find \( R \) such that \( K \) maps the ball of radius \( R \) into itself. Now, let \( M = \sup |f| \) and \( L = \sup |K| \). The range of \( K \) obeys \( \|Ku\| \leq ML \), so that if we take \( R \geq ML \) we are done.

We recall from functional analysis that if \( K \) is a linear compact operator then \( I - K \) is Fredholm of index zero. That is, range is closed, of finite codimension, kernel is finite dimensional, and

\[
dim \ker(I - K) = \text{codim} \text{Range} \ (I - K).
\]

We recall here also \( P(x, D) \) linear elliptic operators in Sobolev spaces and Hölder spaces, and embedding theorems.

Now an application involving elliptic operators. Let \( P = P(x, \partial) \) be an elliptic operator of order \( m \)

\[
P(x, D)u = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha u
\]

with principal symbol

\[
p_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha \xi^\alpha
\]
that does not vanish for \( x \in \overline{\Omega} \) and \( \xi \in \mathbb{R}^n \setminus \{0\} \). We consider boundary conditions on \( \partial \Omega \) that are good: \( Bu = 0 \) on \( \partial \Omega \) imply that the \( P : X \to Y \) is a Fredholm operator (kernel finite dimensional, closed range with finite dimensional codimension. In many cases the index of \( P \) is zero, i.e. the dimension of the kernel equals the dimension of the coimage. Examples are the Laplacian with Neumann or Dirichlet BC.

Now we consider a sublinear function \( g(x, \partial^\alpha u) \) with \( |\alpha| \leq m - 1 \), satisfying

\[
|g(x, \partial^\alpha u)| \leq C(1 + \sum_{|\alpha|\leq 1} |\partial^\alpha u|)^r
\]

with \( r < 1 \), uniformly for \( x \in \overline{\Omega} \) and arbitrary entries \( \partial^\alpha u \in \mathbb{R}^M \) where \( M \) is the number of such things. We consider the equation

\[
P(x, D)u = g(x, \partial^\alpha u)
\]

with boundary conditions \( Bu = 0 \). We assume that the index of \( P \) is zero and \( P \) is injective. Then there exists a \( C^\infty(\overline{\Omega}) \) solution. (Assuming the boundary, and all coefficients are smooth all the way to the boundary).

The idea of the proof is to take \( I - P^{-1}g(x, \partial^\alpha u) \) and apply degree theory. We may choose the space \( X = C^{m-1}(\overline{\Omega}) \cap \{ Bu = 0 \} \).

The steps of the proof are instructive. First we establish a priori estimates. For example, we can look at \( W^{m,p}(\Omega) \), \( p > n \), and assuming a solution, obtain uniform bounds

\[
\|u\|_{m,p} \leq C_{m,p}
\]

with constant independent of anything. This comes from \( r < 1 \) and ellipticity. We could have had a fully nonlinear equation here (right hand side depending on all \( m \) derivatives). Then we show that this means that solutions have to belong to a fixed ball of \( X \). This uses Sobolev embedding and \( p > n \) and the fact that the right hand side sees \( m - 1 \) derivatives only. Then we take a strictly larger ball \( B \subset X \). There are no solution on the boundary of this ball. Also, by embeddings, \( K(u) = P^{-1}g(x, \partial^\alpha u) \) is compact (because its range is bounded in the Hölder space \( C^{m-1,\gamma}(\Omega) \), with \( \gamma = 1 - \frac{n}{p} \). By homotopy to \( I \) vis \( I - tK \), the degree \( \deg(I - K, B, 0) = 1 \), and therefore there is a solution. Smoothness follows by bootstrapping.

This was sublinear, but set the stage. Here is a semilinear example that is not trivial: the existence of steady solutions of Navier-Stokes equations with arbitrary forcing in both 2 and 3 dimensions.
The equation
\[ Au + B(u, u) = f \]
where \( A \) is the Stokes operator and \( B(u, v) = \mathbb{P}(u \cdot \nabla v) \) has solutions \( u \in V \) for any \( f \in L^2(\Omega)^d \) with \( \mathbb{P}f = f \).

Here \( \Omega \) is an open bounded set with smooth boundary, \( d = 2, 3 \) and \( \mathbb{P} \) is the projector on divergence-free functions in \( L^2 \). We recall notations: \( V \) is the closure of the space of divergence-free \( C^\infty_0(\Omega) \) vectors in the topology of \( H^1(\Omega)^d \), \( d = 2, 3 \). The Stokes operator is \( A = -\mathbb{P}\Delta \) with domain \( \mathcal{D}(A) = V \cap H^2(\Omega)^d \).

The function
\[ K(u) = A^{-1}B(u, u) : V \to V \]
is compact. This follows because \( A^{-\frac{3}{2}}B(u, u) \) is continuous
\[ \|A^{-\frac{3}{2}}B(u, v)\|_V \leq C\|u\|_V\|v\|_V \]
(see [2]). For any \( t \in [0, 1] \), the equation
\[ u + tK(u) = tA^{-1}f \]
has no solutions on the boundary of the ball \( B_R = \{ u \mid \|u\|_V < R \} \) for \( R > \|A^{-1}f\|_V \). Indeed, any solution in \( V \) obeys
\[ \|u\|^2_V = t\langle A^{-1}f, u \rangle_V. \]
Therefore, \( \phi(u) = u + K(u) - A^{-1}f \) obeys \( \text{deg}(\phi, B_R, 0) = 1 \) and the equation has solution in \( B_R \).

Finally, for a quasilinear example: Damped and driven Euler equations in 2D.

Consider a bounded domain \( \Omega \subset \mathbb{R}^2 \). Consider a time independent force \( F \in H^1(\Omega) \) and a positive constant \( \gamma > 0 \). Then there exist \( H^1(\Omega) \) solutions of the damped Euler equations
\[ \gamma u + u \cdot \nabla u + \nabla p = F, \quad \text{div} \, u = 0 \]
in \( \Omega \) with \( u \cdot n = 0 \) on \( \partial\Omega \).

The proof starts by adding artificial viscosity, thus producing a semilinear equation. We take the vorticity-stream formulation of the equation, \( \omega = \Delta \psi \), \( u = \nabla^\perp \psi \). The vorticity equation is
\[ \gamma \omega + u \cdot \nabla \omega = f \]
with $f = \nabla^\perp \cdot F$. This we want to solve in $L^2$. We take first $\nu > 0$ and seek solutions of

\[- \nu \Delta \omega + \gamma \omega + u \cdot \nabla \omega = f\]

with the artificial boundary condition $\omega = 0$ at $\partial \Omega$. We should think of this as being

\[\nu \Delta^2 \psi + \gamma (-\Delta \psi) + J(\psi, \Delta \psi) = f\]

where $J(f, g) = \partial_1 f \partial_2 g - \partial_2 f \partial_1 g$ is the Poisson bracket. The boundary conditions are $\psi = \Delta \psi = 0$ at $\partial \Omega$. (These are “good”).

We start by showing there exist solutions at fixed $\nu$. Then we pass to the limit as $\nu \to 0$. At fixed $\nu$.

References
