Take Home Final, due Friday May 11 at 12 p.m. in my mail box on the second floor

Functional Analysis

1.

(i) $U : \mathbb{C}^n \to \mathbb{C}$ is the Fourier transform of a distribution $u \in \mathcal{E}'(\mathbb{R}^n)$ with $supp u \subset B(0,R) = \{x \in \mathbb{R}^n \mid \ |x| \le R\}$ if and only if there exist N and C such that

$$|U(\zeta)| \le C(1+|\zeta|)^N e^{R|Im\,\zeta|}$$

holds for all $\zeta \in \mathbb{C}^n$.

(ii) U is the Fourier transform of a function $u \in \mathcal{D}(\mathbb{R}^n)$ supported in $B(0,R)$ if and only if, for every N there exists C_N such that

$$|U(\zeta)| \le C_N(1+|\zeta|)^{-N} e^{R|Im\,\zeta|}$$

holds for all $\zeta \in \mathbb{C}^n$.

Hint. For the direct ("only if") inplication of part (i) consider the function $e(x,\zeta) = \psi(|\zeta|(|x|-R))e^{-i(x\cdot\zeta)}$ where ψ is a smooth function of a positive variable r, supported in $0 \le r \le 1$ and identically one on $0 \le r \le \frac{1}{2}$. Show that for x in the support of e we have $|x| \le R + \frac{1}{|\zeta|}$, $|\partial_x^\alpha e| \le C(1+|\zeta|)^{|\alpha|}e^{(R+\frac{1}{|\zeta|})|Im\,\zeta|} \le C_1(1+|\zeta|)^{|\alpha|}e^{R|Im\,\zeta|}$. For the converse ("if") implication of part (ii) use a contour integral $\xi + i\eta$ for fixed η in the inverse Fourier transform formula to obtain

$$|u(x)| \le Ce^{-x\cdot\eta}e^{R|Im\,\eta|}$$

with a constant that is independent of η. Let $\eta = \lambda x$ and send $\lambda \to \infty$. For the converse implication ("if") of part (i), mollify and use the converse part of (ii).

2. We start by recalling some definitions. A finite Borel regular measure in $\mathbb{R}^n$ is a positive measure such that the Borel sets are measurable, $\mu(\mathbb{R}^n) < \infty$, and for any measurable set A $\mu(A) = inf\{\mu(U) \mid A \subset U, U \text{ open}\}$. The Fourier transform of a finite Borel measure is

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-i(x\cdot\xi)}d\mu(x)$$

We say that a finite regular Borel measure μ in $\mathbb{R}^n$ has an atom at x if $\mu(\{x\}) > 0$. A measure without atoms is said to be *continuous*.

(i) Prove that the set of atoms of a finite regular Borel measure μ is at most countable and that the measure can be written as

$$\mu = \sum_j a_j \delta_{x_j} + \mu_C$$

where x_j is the sequence of distinct atoms, μ_C is Borel regular, $a_j \geq 0$ and and $\sum_j a_j = A < \infty$.

(ii) Let $f \in L^\infty(\mathbb{R})$. Assume that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt = 0.$$

Then, for any $g \in L^\infty(\mathbb{R})$,

$$\lim_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} |(f+g)(t)|^2 dt - \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 dt \right) = 0.$$

(iii) If $\mu = \sum_j a_j \delta_{x_j}$ is a countable sum of point masses with $x_j \in \mathbb{R}$, $a_j > 0$, $A = \sum a_j < \infty$ then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\hat{\mu}(\xi)|^2 d\xi = \sum_j a_j^2.$$

(iv) If μ is a continuous finite regular Borel measure then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\hat{\mu}(\xi)|^2 d\xi = 0.$$

(v) If $\mu = \sum_j a_j \delta_{x_j} + \mu_C$ is a finite regular Borel measure with atoms at x_j, $a_j \geq 0$, and with μ_C continuous, then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\hat{\mu}(\xi)|^2 = \sum_j a_j^2$$

Hints. For (iii): use the fact that

$$\sum_{j\neq k} a_j a_k \frac{sin((x_j - x_k)T)}{(x_j - x_k)T}$$

is uniformly (in T) small if j or k are large enough, and then take T large enough to show that the rest of this sum is small as well. For (iv) prove that if μ is a continuous finite regular Borel measure, then for any $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(B(x,\delta)) \leq \epsilon$ holds for all x. Use this to dominate the contribution near the diagonal $|x - y| \leq \delta$ in

$$\int\int \frac{\sin((x-y)T)}{(x-y)T} d\mu(x) d\mu(y)$$

For (v) use (ii), (iii), (iv).

3.(a) Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $1 \leq p < \infty$ and $m \geq 0$. Show that the dual $W_0^{m,p}(\Omega)'$ of $W_0^{m,p}(\Omega)$ (the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$) is a space of distributions. i.e. for any $L \in W_0^{m,p}(\Omega)'$ there exists a unique $u \in \mathcal{D}'(\Omega)$ so that $u(\phi) = L(\phi)$ for all $\phi \in \mathcal{D}(\Omega)$. The dual is denoted $W^{-m,q}(\Omega)$, with $q^{-1} + p^{-1} = 1$. Prove that $f \in W^{-1,2}(\mathbb{R}^n)$ if and only if $(1 + |\xi|^2)^{-\frac{1}{2}} \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$ where $\widehat{f}$ is the Fourier transform. (You need to justify that the Fourier transform is defined.)

(b) Let μ be a Radon measure on $K \subset\subset \Omega \subset \mathbb{R}^n$. ($K$ is compact, Ω is open). Radon measure means that $f \mapsto \int_K f d\mu$ is a continuous positive linear functional on $C(K)$, the Banach space of continuous functions. Show that μ defines a distribution in $\mathcal{D}'(\Omega)$ naturally, by $\mu(\phi) = \int_K \phi_{|K} d\mu$. (Here $\phi_{|K}$ is the restriction of $\phi \in \mathcal{D}(\Omega)$ to K.)

Prove that $K = \{0\}$, $\mu = \delta$, $\Omega = \mathbb{R}^3$ is an example of μ that does not belong to $W^{-1,2}(\Omega)$.

(c) Let $K = S^2$ be the unit sphere and $\mu = dS$ be the surface measure. Show that $\mu \in W^{-1,2}(\mathbb{R}^3)$.

4. (a) Let f_j be a sequence of functions in $L^1(\mathbb{R}^n)$ that is uniformly bounded, i.e. $\exists\, C > 0, \|f_j\|_{L^1} \leq C$, and is uniformly absolutely continuous i.e. $\forall \epsilon > 0,\ \exists \delta > 0,\ |A| \leq \delta \Rightarrow \int_A |f_j| dx \leq \epsilon$. Prove that there exists a subsequence f_{j_k} and a function $f \in L^1(\mathbb{R}^n)$ such that $\lim_{k\to\infty} \int \phi(x) f_{j_k}(x) dx = \int \phi(x) f(x) dx$ holds for any continuous, compactly supported function ϕ.

(b) Assume that $f \in L^1(\mathbb{R}^n)$. Consider the finite differences

$$(\delta_{he_1} f)(x) = f(x_1 - h, x_2, \ldots, x_n) - f(x)$$

and assume that there exists a constant $C > 0$, such that

$$\|\delta_{he_1} f\|_{L^1} \leq C|h|$$

for all $|h| \leq 1$. Assume also that the family $h^{-1}\delta_{he_1} f$ is uniformly absolutely continuous: $\forall \epsilon > 0,\ \exists \delta > 0,\ |A| \leq \delta \Rightarrow \int_A |\delta_{he_1} f| dx \leq \epsilon |h|$. Show that $\partial_1 f \in L^1(\mathbb{R}^n)$.

(c) Verify that the condition $(1+|\xi|)\widehat{f}(\xi) \in L^1(\mathbb{R}^n)$ is a sufficient condition for the uniform absolute continuity of $h^{-1}\delta_h f$ in $L^1(\mathbb{R}^n)$.

5. Let s be a real number, $s > \frac{n}{2}$, and consider the space $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ of functions with s derivatives in L^2, with norm $\|f\|_s$ defined by Fourier transform.

(a) Prove that $H^s(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap \mathcal{F}^{-1}(L^1(\mathbb{R}^n)) \subset L^\infty(\mathbb{R}^n)$ and the embeddings are continuous. Here $\mathcal{F}(f) = \widehat{f}$ is the Fourier transform and

$$\mathcal{F}^{-1}(L^1(\mathbb{R}^n)) = \{f \in \mathcal{S}'(\mathbb{R}^n) \mid \ \widehat{f} \in L^1(\mathbb{R}^n)\}$$

(Not to be confused it with the image of $L^1(\mathbb{R}^n)$ under the inverse Fourier transform!).

(b) Prove that there exists a constant so that

$$\|fg\|_s \leq C\left\{\|\widehat{f}\|_{L^1(\mathbb{R}^n)}\|g\|_s + \|\widehat{g}\|_{L^1(\mathbb{R}^n)}\|f\|_s\right\}$$

(a) Prove that $H^s(\mathbf{R}^n)$ is a Banach algebra, i.e. there exists a constant C so that $\|fg\|_s \leq C\|f\|_s\|g\|_s$ holds for all $f, g \in H^s(\mathbb{R}^n)$.

(d) Let $K \in \mathcal{S}(\mathbb{R}^n)$ be a smooth, rapidly decaying function. Show that there exists $\epsilon > 0$ such that, if $\|f\|_s \leq \epsilon$, then the equation

$$u(x) + \int K(x-y)u^2(y)dy = f(x)$$

has always a solution in $H^s(\mathbb{R}^n)$.
Hint: Consider successive approximations,

$$u_{n+1}(x) = f(x) - \int K(x-y)u_n^2(y)dy.$$

6. Let the real valued functions $V_j : \mathbb{R}^n \to \mathbb{R}$ be smooth ($\mathcal{C}^1$) and uniformly bounded,

$$\sup_{x \in \mathbb{R}^n} \sum_{j=1}^{n} |V_j(x)| \le C < \infty.$$

Assume that $V(x) = (V_1(x), \dots V_n(x))$ is divergence-free,

$$0 = \nabla \cdot V = \sum_{j=1}^{n} \frac{\partial V_j}{\partial x_j}.$$

Consider the operator

$$A = -\Delta + \frac{1}{i} V \cdot \nabla$$

i.e.

$$Au(x) = -(\Delta u)(x) + \frac{1}{i} \sum_{j=1}^{n} V_j(x) \frac{\partial u}{\partial x_j}(x)$$

defined for $u \in C_0^\infty(\mathbb{R}^n)$. Show that A is essentially selfadjoint as operator $A : C_0^\infty(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

7. Let $F : B \to \mathbb{R}$ be a function defined on the reflexive Banach space B. We say that F is weakly lower semicontinuous if from $u = w\lim_{n\to\infty} u_n$ it follows that

$$F(u) \le \lim \inf_{n\to\infty} F(u_n)$$

Here $w\lim_{n\to\infty} u_n$ means the weak limit.

(a) Consider $B = L^4([0, 2\pi])$. Take $u_n = \sin(nx)$ and

$$F(u) = \int_0^{2\pi} (1 - |u(x)|^2)^2 dx$$

Does u_n converge weakly in B? Is F continuous (strongly)? Is F weakly lower semicontinuous?

(b) Is the norm in a reflexive Banach space weakly lower semicontinuous? Are continuous functions on reflexive Banach spaces weakly lower semicontinuous?

(c) Let $B = H_0^1(0, 2\pi)$. Consider the same functional F as above,

$$F(u) = \int_0^{2\pi} (1 - |u(x)|^2)^2 dx.$$

Is F weakly lower semicontinuous?