Homework 7 = Higher order elliptic regularity

Functional Analysis

We say that the operator $P = P(x, \partial) = \sum_{|\alpha| \le m} a_\alpha(x)\partial^\alpha$ is (uniformly) elliptic in $\Omega \subset \mathbb{R}^n$ if there exists $\gamma > 0$ such that

$$\left| \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \right| \ge \gamma |\xi|^m$$

holds for all $\xi \in \mathbb{R}^n$ and all $x \in \Omega$. Here $m \ge 2$ is an integer. You are going to prove interior regularity for solutions of $Pu = f$. We use the notation $\|u\|_s$ for the norm in H^s. We use Sobolev spaces $H_0^s(\Omega)$. We recall that these can be defined for all s as the completion of $\mathcal{D}(\Omega)$ in the $\|\cdot\|_s$ norm. The norm is defined via Fourier transform.

1. Consider the constant coefficient case $P = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha$. Assume the operator is elliptic. Prove that there exists a constant C_s (depending on s and γ) such that

$$\|u\|_s \le C_s \left(\|u\|_0 + \|Pu\|_{s-m} \right)$$

holds for any $s \ge m$ and any $u \in H^s(\mathbb{R}^n)$.

Hint: Use Fourier transform and ellipticity.

2. Assume that the coefficients a_α are C^∞ and P is elliptic in Ω. Let $x_0 \in \Omega$ and let $s \ge m$. There exists $\delta > 0$ and a constant C_s such that if $u \in H_0^s(B(x_0, \delta))$ then

$$\|u\|_s \le C_s \left(\|u\|_{s-1} + \|Pu\|_{s-m} \right).$$

Hint: Use the previous result and the fact $\|(P(x, \partial) - P(x_0, \partial))u\|_{s-m}$ can be bound by $\epsilon \|u\|_s + C\|u\|_{s-1}$.

3. Let $s \ge m$. There exists a constant C_s such that

$$\|u\|_s \le C_s \left(\|u\|_{s-1} + \|Pu\|_{s-m} \right)$$

holds for all $u \in H_0^s(\Omega)$.

Hint: Partition of unity $u = \sum \phi_j u$ to achieve small supports, and commutators

$$[P, \phi_j]u = P(\phi_j u) - \phi_j P u \in H_{s-m}$$

controlled by $\|u\|_{s-1}$.

4. Let $s \geq m$, $t \leq s-1$. There exists a constant C_t such that

$$\|u\|_s \leq C_t \left(\|u\|_t + \|Pu\|_{s-m}\right)$$

holds for all $u \in H_0^s(\Omega)$.

Hint:

$$\|u\|_{s-1} \leq \epsilon \|u\|_s + C_\epsilon \|u\|_t$$

5. Let Ω be open, P elliptic of order m with C^∞ coefficients and $s \geq m$. If $u \in H^s_{loc}(\Omega)$ and $Pu \in H^{s-m+1}_{loc}(\Omega)$ then $u \in H^{s+1}_{loc}(\Omega)$.

Hint: Let $\psi \in \mathcal{D}(\Omega)$. Then $\psi u \in H_0^s(\Omega)$ and $\psi P u \in H_0^{s-m+1}(\Omega)$. Because also the commutator $P(\psi u) - \psi P(u) \in H_0^{s-m+1}(\Omega)$ it follows that $P(\psi u) \in H_0^{s-m+1}(\Omega)$. Take finite difference quotients and show, using commutators that

$$\|\frac{1}{h}\delta_h(\psi u)\|_s \leq C\left[\|P(\psi u)\|_{s-m+1} + \|\psi u\|_s\right]$$

6. Let P be an uniformly elliptic operator of order m with C^∞ coefficients in the open set $\Omega \subset \mathbb{R}^n$. Let $f \in H^t_{loc}(\Omega)$, with $t \geq 0$. If $u \in L^2_{loc}(\Omega)$ solves $Pu = f$ in $\mathcal{D}'(\Omega)$ then $u \in H^{t+m}_{loc}(\Omega)$.