Homework 6 = Mazur separation thm script and applications

Functional Analysis

We recall the real Hahn-Banach theorem: If X a real vector space and $p : X \to \mathbb{R}_+$ satisfies

$$(i) \quad p(x+y) \leq p(x) + p(y) \quad \forall\, x, y \in X,$$
$$(ii) \quad p(tx) = tp(x) \quad \forall\, t > 0, x \in X,$$

then any real linear map $L : F \to \mathbb{R}$, defined on a linear subspace F of X and satisfying

$$L(f) \leq p(f), \quad \forall\, f \in F$$

has a linear extension $\widetilde{L} : X \to \mathbb{R}$ satisfying

$$\widetilde{L}(x) \leq p(x) \quad \forall\, x \in X.$$

1. The Mazur separation theorem is the following: Let X be a complex locally convex space. Let C be a closed convex subset $C \subset X$, and let $x_0 \notin C$. Then there exists a continuous linear functional $F : X \to \mathbb{C}$ such that

$$\sup_{x \in C} Re(F(x)) < Re(F(x_0)).$$

You are going to prove this by proving the following statements.

(a) WLOG $0 \in C$.

(b) Let V be an absorbing, balanced, convex neighborhood of 0 in X such that $V + x_0 \cap C = \emptyset$. Let $A = C + \frac{V}{2}$ and let p_A be the Minkowski functional

$$p_A(x) = \inf\{t > 0 \,|\, x \in tA\}$$

associated to A. Prove that p_A exists, and satisfies the conditions of the real Hahn-Banach thm.

(c) Show that $p(x_0+z) \geq 1$ for all $z \in \frac{V}{2}$. (Use the fact that A is starshaped i.e. $tA \subset A$ for any $t \in [0,1]$, and consequently that $p_A^{-1}([0,1)) \subset A$.)

(d) Let $\widetilde{L}$ be a real linear extension to X (which obviously is a real vector space as well) of the real linear map

$$L : \mathbb{R}x_0 \to \mathbb{R}, \quad L(tx_0) = tp_A(x_0)$$

which obeys

$$\widetilde{L}(x) \leq p_A(x) \quad \forall\, x \in X.$$

Define $F : X \to \mathbb{C}$ by

$$F(x) = \widetilde{L}(x) - i\widetilde{L}(ix)$$

Show that F is (complex) linear, that

$$Re(F(x)) \leq p_A(x) \quad \forall\, x \in X,$$

and

$$sup_{x\in C} Re(F(x)) \leq 1.$$

(e) Prove that F is continuous. Use, for instance, $K = co\left(\cup_{|z|\leq 1} zC\right)$, the convex hull of $\cup_{|z|\leq 1} zC$ and the associated Minkowski functional p_B of the convex, balanced, absorbing open set

$$B = K + \frac{V}{2}.$$

(f) Prove that $Re(F(x_0)) > 1$.

2. Let $C \subset X$, X locally convex space. Assume that C is convex and closed. Then it is weakly closed (i.e. it is closed in the locally convex topology on X generated by X^*, the linear continuous functionals on X.)

3. Let X be a Banach space and let x_n be a sequence which converges weakly to x. Prove that there exists a squence of convex combinations of the x_n, $y_n = \sum_{j=1}^n \alpha_j^{(n)} x_j$, $\sum_{j=1}^n \alpha_j^{(n)} = 1$, $\alpha_j^{(n)} \geq 0$ that converges in norm to x.