Homework 5 = unbounded operators script

Functional Analysis

Definitions Let $T : \mathcal{D}(T) \subset H \to H$ be a linear operator defined on a dense linear subset $\mathcal{D}(T)$ of a Hilbert space H. We say that T is closed if the graph $G(T) = \{(x, y) \mid x \in \mathcal{D}(T), y = Tx\}$ is closed in $H \times H$. We say that $S : \mathcal{D}(S) \to H$ is an extension of T and we write $T \subset S$ if $G(T) \subset G(S)$. We say that T is closeable if $\exists\, S$, S closed such that $T \subset S$. If T is closeable we define $\overline{T}$ to be the smallest closed extension of T. We define the domain $\mathcal{D}(T^*)$ of the adjoint T^* of a densely defined operator T to be the set of $y \in H$ such that the map

$$z \in \mathcal{D}(T) \mapsto \langle Tz, y \rangle$$

is continuous. By Riesz representation (and uniqueness of the extension of linear continuous maps from dense subspaces) T^*y is defined by the relation

$$\langle Tz, y \rangle = \langle z, T^*y \rangle, \quad \forall\, z \in \mathcal{D}(T).$$

1. (i) The adjoint T^* of the densely defined $T : \mathcal{D}(T) \to H$ is closed.

(ii) T is closeable if and only if T^* is densely defined, in which case $\overline{T} = T^{**}$

Hint. Let $H \times H$ be the product Hilbert space with natural structure, let $G(T)$ denote the graph of an operator. Let $\mathcal{U}$ be the unitary transformation $\mathcal{U} : H \times H \to H \times H$ given by $\mathcal{U}(x, y) = (-y, x)$. Prove that $\mathcal{U}(G(T)^{\perp}) = G(T^*)$ holds for any densely defined operator T.

Definition We say that the densely defined operator T is symmetric if $T \subset T^*$.

2. (i) T symmetric implies T closeable and $T \subset T^{**} \subset T^*$.

(ii) If T is closed and symmetric then $T = T^{**}$.

Definition We say that the symmetric operator T is essentially selfadjoint if $\overline{T}$ is selfadjoint. If T is closed, then a linear space D is called a core of T if the closure of the restriction of T to D is T:

$$\overline{T_{|D}} = T$$

3. If T is essentially selfadjoint then it has a unique selfadjoint extension.

4. Let T be symmetric. The following are equivalent (TFAE):

(i) T is selfadjoint.

(ii) T is closed and both $\ker(T^* \pm i\mathbb{I}) = \{0\}$. (Both refers to the two signs).

(iii) $\mathrm{Ran}(T \pm i\mathbb{I}) = H$ (both signs).

Hints. For $(iii) \Rightarrow (i)$: for given ϕ, solve $(T - i)u = (T^* - i)\phi$, and use that $T + i$ is onto to deduce $T^* - i$ is one to one. (We will ommit $\mathbb{I}$ from now on, so $T + i$ means $T + i\mathbb{I}$.)

5. Let T be symmetric. TFAE:

(i) T is essentially selfadjoint.

(ii) $\ker(T^* \pm i\mathbb{I}) = \{0\}$. (Both signs).

(iii) $\overline{\mathrm{Ran}(T \pm i\mathbb{I})} = H$ (both signs).

6. Let $T = -i\frac{d}{dx}$ and let $\mathcal{D}(T) = \{\phi \mid \phi \in AC([0,1]),\ \phi(0) = \phi(1)\}$. ($AC([0,1])$ is the space of absolutely continuous functions on $[0,1]$. We take $H = L^2([0,1])$. Compute the adjoint T^* and all selfadjoint extensions of T.

7. Let T be selfadjoint in H and let B be symmetric, defined on the same domain $\mathcal{D}(T)$ as T and obey the bound

$$\|Bx\| \leq a\|Tx\| + b\|x\|$$

with $0 \leq a < 1$ and $0 \leq b < \infty$. Then $T + B$ is selfadjoint.
Hint: Take M large enough so that $a + \frac{b}{M} < 1$ and use the fact that $T + iM$ has continuous inverse, and $\|(T + iM)x\|^2 = \|Tx\|^2 + M^2\|x\|^2$ so that if $y = (T + iM)x$ then $\|Tx\| \leq \|y\|$ and $\|x\| \leq M^{-1}\|y\|$.

8. Let $T = -\Delta$ be the Laplacian in $\mathbb{R}^n$ with $\mathcal{D}(T) = H^2(\mathbb{R}^n)$.

(i) Prove that $-\mathcal{D}$ is selfadjoint in $H = L^2(\mathbb{R}^n)$.

(ii) Let V be a real function $V \in L^2(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$. (That means V is a sum of a bounded and a square integrable function.) Let B be the multiplication operator $Bf = Vf$. Prove that $-\Delta + V$ is selfadjoint. (We write with obvious abuse of notation $-\Delta + V$ for $T + B$.)