Homework 4– due March 27

Functional Analysis

1. Let Ω be an open set with compact orientable boundary of class C^m, $m \geq 1$. Let $1 \leq p < \infty$. Prove that there exists a linear continuous map

$$E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^n)$$

with the property that

$$E(f_{|\Omega})(x) = f(x), \quad \forall x \in \Omega$$

holds for all $f \in C^\infty(\mathbb{R}^n) \cap W^{m,p}(\Omega)$.
Hint: Localize by a partition of unity, and use the formula from class for $\mathbb{R}^n_+$.
2. Let Ω be open, with compact orientable boundary of class C^m, $m \geq 1$. Let $1 \leq p < \infty$. Prove that there exists a bounded linear map

$$\gamma : W^{m,p}(\Omega) \to W^{m-1,p}(\partial\Omega)$$

with the property that

$$\gamma(f) = f_{|\partial\Omega}$$

for any $f \in C^\infty(\mathbb{R}^n) \cap W^{m,p}(\Omega)$.
Hint: Reduce to the case of $\mathbb{R}^n_+$, and smooth functions with compact support in $\overline{\mathbb{R}^n_+} \cap \mathbb{R}^{n-1} \times [0, h)$ for some $h > 0$. Then consider the map

$$f \mapsto (R_\delta f)(x') = -\int_\delta^h \frac{\partial f(x', t)}{\partial x_n} dt$$

Prove that

$$\|R_\delta f\|_{W^{m-1,p}(\mathbb{R}^{n-1})} \leq C h^{1-\frac{1}{p}} \|f\|_{W^{m,p}(\mathbb{R}^n_+)}$$

and that $R_\delta f$ is a Cauchy sequence as $\delta \to 0$.
3. Read the first 50 pages of Stein, singular Integrals and differentiability properties of functions, PUP, 1970. (This should involve just reviewing your notes, and filling in all missing details).
4. Let Ω be a smooth function on $\mathbb{S}^{n-1}$ with the property that

$$\int_{\mathbb{S}^{n-1}} \Omega(x) d\sigma = 0$$

where $d\sigma$ is surface measure. Let $0 < r < R$ and let

$$K_{r,R}(x) = \begin{cases} \frac{\Omega(x)}{|x|^n}, & \text{if } r \le |x| \le R \\ 0, & \text{otherwise} \end{cases}$$

Prove:

(a) There exists a constant A independent of r, R such that

$$\left|\widehat{K_{r,R}}(\xi)\right| \le A$$

holds for any $\xi \in \mathbb{R}^n$.

(b) The limit

$$\lim_{r\to 0, R\to\infty} \widehat{K_{r,R}}(\xi) = m(\xi)$$

exists, and compute it.

Hint: Stein, singular Integrals and differentiability properties of functions, PUP, 1970, Ch. II, Thm 4.2.