Homework 1

Functional Analysis

1. (a) Let $\{e_n\}$ be an infinite orthonormal sequence in a Hilbert space H. Show that this gives an example of bounded and closed set that is not compact. Let $\{\delta_n\}$ be a sequence of positive numbers and consider the set

$$Q_\delta = \{f = \sum_{n=1}^{\infty} c_n e_n \mid |c_n| \leq \delta_n\}.$$

Show that Q_δ is compact if and only if $\sum_{n=1}^{\infty} \delta_n^2 < \infty$.

(b) Give an example of a nonempty closed set in a Hilbert space that contains no element of smallest norm.

2. Let τ_h denote the translation operator $\tau_h(f)(x) = f(x-h)$ in $L^p(dx)$ where dx is Lebesgue measure in $\mathbb{R}^n$. For $1 \leq p \leq \infty$ define

$$\omega_p(f;h) = \|\tau_h(f) - f\|_{L^p(dx)}.$$

Take $1 \leq p < \infty$ and $f \in L^p(dx)$. Prove that $\lim_{h \to 0} \omega_p(f;h) = 0$. Is the same true for $p = \infty$?

3. Prove that the set of nowhere differentiable functions in $C^\alpha(\mathbb{R})$ is generic (contains a dense G_δ), if $0 < \alpha < 1$.

(Note however that by Rademacher's thm, Lipschitz continuous functions are a.e. differentiable).

4. Let X be a Banach space and let X^* be its dual, i.e. the space of all linear continuous functionals

$$L : X \to \mathbb{C}$$

endowed with the norm

$$\|L\| = \sup_{\|x\| \leq 1} |L(x)|.$$

(a) Prove that X^* is a Banach space.

(b) Prove that $x \mapsto \delta_x$ defined by

$$\delta_x(L) = L(x)$$

provides a linear continuous (isometric ($\|\delta_x\| = \|x\|$)) embedding of X into the bidual X^{**}.

(c) Let $\{x_n\}$ be a sequence with the property that $L(x_n)$ is bounded for every $L \in X^*$. Prove that the sequence $\{x_n\}$ is bounded i.e., $\sup_n \|x_n\| < \infty$.

5. Let X be a Banach space. (a) We say that a set $U \subset X$ is open in the weak topology on X if $\forall x_0 \in U \; \exists \epsilon > 0$, and $\exists \; T_1, \dots T_n \in X^*$ such that,

$$B_{T_1,\dots T_n,\epsilon}(x_0) = \{x \in X \mid \max_{1 \le i \le n} |T_i(x_0 - x)| < \epsilon\} \subset U$$

Prove that this topology is Hausdorff. More generally, prove that any convex, closed (in the usual topology of X) set is closed in the weak topology.

(b) Give an example of a set in a Banach space that is closed but not weakly closed.