Transport in Rotating Fluids

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Dedicated to Professor M. I. Vishik with deep admiration.

Abstract. We consider uniformly rotating incompressible Euler and Navier-Stokes equations. We study the suppression of vertical gradients of Lagrangian displacement ("vertical" refers to the direction of the rotation axis). We employ a formalism that relates the total vorticity to the gradient of the back-to-labels map (the inverse Lagrangian map, for inviscid flows, a diffusive analogue for viscous flows). We obtain bounds for the vertical gradients of the Lagrangian displacement that vanish linearly with the maximal local Rossby number. Consequently, the change in vertical separation between fluid masses carried by the flow vanishes linearly with the maximal local Rossby number.

1 Introduction

Consider a container filled with water and rotated at a constant angular velocity Ω. The equations ([2]) are

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla \pi + 2\Omega \widehat{z} \times u = 0 \tag{1}$$

with

$$\nabla \cdot u = 0 \tag{2}$$

$\Delta = \nabla^2$ is the Laplacian. We use Cartesian coordinates, the unit vector $\widehat{z} = e_3$ is the rotation axis. The vector u is the relative velocity, a function of three space variables and time, representing the velocity of the fluid recorded by an ideal observer attached to the container and rotating thus with it, at uniform angular velocity. The function π contains the physical pressure and the centrifugal force. The Taylor-Proudman theorem ([2]) states that, if one neglects viscosity and inertia, then the only time independent solutions are two-dimensional.

Neglecting viscosity brings us to the three dimensional incompressible Euler equations

$$\partial_t u + u \cdot \nabla u + \nabla \pi + 2\Omega \widehat{z} \times u = 0. \tag{3}$$

Neglecting inertia brings about the linear equation

$$\partial_t u + 2\Omega \widehat{z} \times u + \nabla \pi = 0, \quad \nabla \cdot u = 0, \tag{4}$$

and time independent solutions of (4) obey $\partial_3 \pi = 0$, $u_1 = -(2\Omega)^{-1}\partial_2 \pi$, $u_2 = (2\Omega)^{-1}\partial_1 \pi$. Incompressibility demands then $\partial_3 u_3 = 0$ and standard boundary conditions require then $u_3 = 0$, so indeed, the solutions are two dimensional. G.I. Taylor showed experimentally that, when motions are slow and steady, the motion of the fluid is organized in sheets that remain parallel to the axis of rotation. Two fluid elements which are initially on a line parallel to the rotation axis, will remain so. If these elements are a certain distance apart, they will remain at the same distance apart ([2]). In Taylor's and in more recent [1] experiments the rotation axis coincides with the direction of gravity, so we refer to the direction of the axis as "vertical". Thus, for slow, steady inviscid motions, there is no vertical transport.

The main objective of this paper is the quantitative study of the suppression of vertical transport in the presence of inertia.

Extensive mathematical studies, ([5], [6] and more recent works), based on the averaging of the interaction of the fast waves of (4) with inertia show that two dimensional structures emerge in the limit $\Omega \to \infty$.

In the present work we employ a formalism that relates the total vorticity to the gradient of the back-to-labels map (the inverse Lagrangian map, for inviscid flows, a diffusive analogue for viscous flows ([3], [4])). In the presence of rotation the total vorticity decomposes in local vorticity (curl of relative velocity) and Ωe_3. The ratio between the magnitude of the relative vorticity and Ω is the local Rossby number ρ. Rotation dominated flows have small Rossby numbers. For solutions of the inviscid equations (3), the Lagrangian displacement $\lambda(a,t) = X(a,t) - a$ obeys the time independent differential equation (see 89):

$$\left(\partial_{a_3} + \frac{1}{2}\rho_0(a)\xi_0(a)\cdot\nabla_a\right)\lambda(a,t) =$$

$$= \frac{1}{2}\left(\rho_t(a)\xi(a,t) - \rho_0(a)\xi_0(a)\right)$$

where

$$\rho_0(a) = \frac{|\omega_0(a)|}{\Omega}, \quad \xi_0(a) = \frac{\omega_0(a)}{|\omega_0(a)|},$$

$$\rho_t(a) = \frac{|\omega(X(a,t),t)|}{\Omega}, \quad \xi(a,t) = \frac{\omega(X(a,t),t)}{|\omega(X(a,t),t)|}$$

and ω is the relative vorticity. Note that, in the limit $\rho \to 0$ we recover

$$\partial_{a_3}\lambda_3(a,t) = 0,$$

that is, the Taylor-Proudman theorem. For small local Rossby numbers we obtain bounds for vertical gradients of the Lagrangian displacements that vanish linearly with ρ.

2 Maps, Velocity and Vorticity

We consider flow in an open domain R with smooth boundary ∂R and we impose natural boundary conditions: the relative velocity is tangent to the boundary: $u(x,t)\cdot n(x) = 0$, for $x \in \partial R$. We denote by $x = (x_1, x_2, x_3)$ or $a = (a_1, a_2, a_3)$ Cartesian coordinates of independent variables. The third Cartesian direction $\widehat{z} = e_3$ is singled out. The fluid can be described employing Lagrangian or Eulerian descriptions. While doing this, we always keep the Cartesian coordinates for independent variables. The relative Lagrangian path map $X(a,t)$, defined for $a \in R$, and associated to the relative velocity $u(x,t)$ is the solution of the familiar ordinary differential equation

$$\partial_t(X(a,t)) = u(X(a,t),t) \tag{5}$$

with initial data

$$X(a,0) = a. \tag{6}$$

The rotating Euler equations (3) are thus

$$\frac{d}{dt}(\partial_t X(a,t)) + (\nabla_x \pi)(X(a,t),t) + 2\Omega\widehat{z} \times u(X(a,t),t) = 0 \tag{7}$$

There is no difference between $\frac{d}{dt}$ and ∂_t: the labels a are held fixed. We use both notations to emphasize the operation at hand. We take one of the directions $a = (a_1, a_2, a_3)$, say a_α and differentiate the Lagrangian map in the direction a_α to obtain the vector $\partial_\alpha(X(a,t))$. We take the scalar product of the equation (7) with this vector, and using the chain rule, we get:

$$\frac{d}{dt}\left(\partial_t X \cdot \partial_\alpha X\right) + \partial_\alpha(\pi - \frac{1}{2}\left|\partial_t X\right|^2) + 2\Omega\left(\widehat{z}; \partial_t X; \partial_\alpha X\right) = 0. \tag{8}$$

where $(u; v; w)$ denotes the determinant of the matrix whose columns are u, v and respectively, w. We compute $(\widehat{z}; \partial_t X; \partial_\alpha X) = \partial_t\left(\widehat{z}; X; \partial_\alpha X\right) - (\widehat{z}; X; \partial_\alpha \partial_t X) = \partial_t\left(\widehat{z}; X; \partial_\alpha X\right) - \partial_\alpha\left(\widehat{z}; X; \partial_t X\right) + \left(\widehat{z}; \partial_\alpha X; \partial_t X\right)$, and thus, using the antisymmetry of the determinant, we get

$$2\left(\widehat{z}; \partial_t X; \partial_\alpha X\right) = \partial_t\left(\widehat{z}; X; \partial_\alpha X\right) - \partial_\alpha\left(\widehat{z}; X; \partial_t X\right). \tag{9}$$

Inserting (9) in (8) we get

$$\frac{d}{dt}\left\{\left(\partial_t X \cdot \partial_\alpha X\right) + \Omega\left(\widehat{z}; X; \partial_\alpha X\right)\right\} + \partial_\alpha \rho = 0 \tag{10}$$

where

$$\rho = \pi - \frac{1}{2}|\partial_t X|^2 - \Omega\left(\widehat{z}; X; \partial_t X\right). \tag{11}$$

Integrating in time, from time zero to time t, and remembering that time integration commutes with label derivative, we obtain

$$\left\{\left(\partial_t X \cdot \partial_\alpha X\right) + \Omega\left(\widehat{z}; X; \partial_\alpha X\right)\right\}$$

$$= \left\{\left(\partial_t X \cdot \partial_\alpha X\right) + \Omega\left(\widehat{z}; X; \partial_\alpha X\right)\right\}_{|t=0} - \partial_\alpha \int_0^t \rho. \tag{12}$$

The initial data area $X(a,0) = 0$ and $\partial_t X(a,0) = u_0(a)$. The equation (12) is therefore

$$\left\{\partial_t X + \Omega\widehat{z} \times X\right\} \cdot \partial_\alpha X = \left\{u_0(a) + \Omega\widehat{z} \times a\right\} \cdot e_\alpha - \partial_\alpha q \tag{13}$$

where

$$q = \int_0^t \rho \tag{14}$$

and $e_\alpha = (\delta_{i\alpha})$ is the unit vector associated with the direction α. We consider the "back-to-labels" map, (the inverse Lagrangian map)

$$A(x,t) = X^{-1}(x,t) \tag{15}$$

We read (14) at $a = A(x,t)$, we multiply it by $(\partial_i A^\alpha)(x,t)$ and sum. We use the chain rule, and obtain:

$$u^i(x,t) + \Omega\left(\widehat{z}; x; e_i\right) =$$

$$\partial_i A^\alpha(x,t)\left\{u_0^\alpha(A(x,t),t) + \Omega\left(\widehat{z}; A(x,t); e_\alpha\right)\right\} - \partial_i r(x,t),$$

where

$$r(x,t) = q(A(x,t),t). \tag{16}$$

Re-arranging the expressions a bit, we write

$$u^i(x,t) = (\partial_i A^\alpha(x,t)) u_0^\alpha(A(x,t),t) - \partial_i r(x,t) +$$

$$\Omega\left\{(\widehat{z}; A(x,t), \partial_i A(x,t) - (\widehat{z}; x; e_i)\right\}. \tag{17}$$

This is the Weber formula for rotating Euler equations. Obviously, the "back-to-labels" map A obeys the equation

$$\partial_t A(x,t) + u(x,t)\cdot\nabla A(x,t) = 0 \tag{18}$$

which follows by time differentiation from the statement that $A(x,t)$ is the label corresponding to x, $a = A(X(a,t),t)$. One can verify, by direct calculation, that if $A(x,t)$ solves the equation (18) with initial data

$$A(x,0) = x \tag{19}$$

and with velocity computed from (17), then the velocity satisfies the rotating Euler equations. Indeed, writing $\Gamma_0(u,\nabla) = \partial_t + u\cdot\nabla$, and applying $\Gamma_0(u,\nabla)$ to (17) we obtain, using (18)

$$\Gamma_0(u,\nabla)u^i = -\Omega\left(\widehat{z}; u; e_i\right) - (\partial_i u^j)(\partial_j A^\alpha(x,t))\left\{u_0^\alpha(A) + \Omega\left(\widehat{z}; A(x,t); e_\alpha\right)\right\}$$

$$-\partial_i(\Gamma_0(u,\nabla)r) + (\partial_i u^j)\partial_j r. \tag{20}$$

Now insering the definition (17) in the equation above, we have

$$\Gamma_0(u,\nabla)u^i = -\Omega\,(\widehat{z};u;e_i) - (\partial_i u^j)u^j - \partial_i(\Gamma_0(u,\nabla)r) - \Omega(\partial_i u^j)\,(\widehat{z};x;e_j) \tag{21}$$

We notice that

$$(\partial_i u^j)\,(\widehat{z};x;e_j) = \partial_i\,(\widehat{z};x;u) - (\widehat{z};e_i;u) \tag{22}$$

so that the equation (21) reads

$$\Gamma_0(u,\nabla)u^i = -2\Omega\,(\widehat{z};u,e_i) - \partial_i\pi \tag{23}$$

with

$$\pi = \Gamma_0(u,\nabla)r + \frac{1}{2}|u|^2 + \Omega\,(\widehat{z};x;u) \tag{24}$$

The obtained equation (23) is just (3), so we have proved therefore:

Theorem 1. *Any twice continuously differentiable solution of the active vector system (17, 18) solves the rotating Euler equations (3) with pressure π given in (24). Viceversa, if u is a twice differentiable solution of the rotating Euler equations, then it obeys (17) with A determined by (18).*

The Navier-Stokes equations (1) in a rotating frame admit a somewhat similar treatment. One considers a map $A(x,t)$ that obeys

$$(\partial_t + u\cdot\nabla - \nu\Delta)\,A(x,t) = 0 \tag{25}$$

together with the initial condition (19). It is convenient to denote

$$\Gamma_\nu(u,\nabla) = \partial_t + u\cdot\nabla - \nu\Delta \tag{26}$$

the operator of advection by u and diffusion with diffusivity ν. One starts with an ansatz like (17),

$$u^i = (\partial_i A^\alpha)v^\alpha + \Omega\,\{(\widehat{z};A;\partial_i A) - (\widehat{z};x,e_i)\} - \partial_i r \tag{27}$$

which is exactly (17) except that instaed of $u_0(A(x,t))$ we have now $v(x,t)$. Note that, in the inviscid case, the function $v_0(x,t) = u_0(A(x,t))$ obeys

$\Gamma_0(u,\nabla)v_0 = 0$. The viscosity introduces corrections to this equation. We apply $\Gamma_\nu(u,\nabla)$ to (27) and compute the various pieces separately. First, we get

$$\Gamma_\nu(u,\nabla)\left\{(\partial_i A^\alpha)\, v^\alpha\right\} =$$

$$(\partial_i A^\alpha)\Gamma_\nu(u,\nabla)v^\alpha - 2\nu(\partial_j\partial_i A^\alpha)\partial_j v^\alpha - (\partial_i u^j)\,(\partial_j A^\alpha)\, v^\alpha. \tag{28}$$

Secondly, we have

$$\Gamma_\nu(u,\nabla)(\partial_i r) = \partial_i\left(\Gamma_\nu(u,\nabla)r\right) - (\partial_i u^j)(\partial_j r) \tag{29}$$

Subtracting (29) from (28) we get

$$\Gamma_\nu(u,\nabla)\left\{(\partial_i A^\alpha)\, v^\alpha - \partial_i r\right\} = -\partial_i\left(\Gamma_\nu(u,\nabla)r\right) +$$

$$(\partial_i A^\alpha)\Gamma_\nu(u,\nabla)v^\alpha - 2\nu(\partial_j\partial_i A^\alpha)\partial_j v^\alpha - (\partial_i u^j)\left\{(\partial_j A^\alpha)\, v^\alpha - \partial_j r\right\}. \tag{30}$$

Let us give a temporary name to the term involving Ω in (27):

$$U^i = \Omega\left\{(\widehat{z}; A; \partial_i A) - (\widehat{z}; x; e_i)\right\}. \tag{31}$$

Then, using (27) we write (30) as

$$\Gamma_\nu(u,\nabla)u = (\partial_i A^\alpha)\Gamma_\nu(u,\nabla)v^\alpha - 2\nu(\partial_j\partial_i A^\alpha)\partial_j v^\alpha - \partial_i\left(\Gamma_\nu(u,\nabla)r\right)$$

$$-(\partial_i u^j)u^j + (\partial_i u^j)U^j + \Gamma_\nu(u,\nabla)U^i \tag{32}$$

Now we compute $\Gamma_\nu(u,\nabla)U^i$:

$$\Gamma_\nu(u,\nabla)U^i = -\Omega\left\{(\partial_i u^j)(\widehat{z}; A; \partial_j A) + (\widehat{z}; u; e_i) + 2\nu(\widehat{z}; \partial_j A, \partial_j\partial_i A)\right\} \tag{33}$$

Using

$$\partial_i\left(-\Omega(\widehat{z}; x; u)\right) = -\Omega(\widehat{z}; x; (\partial_i u^j)e_j) - \Omega(\widehat{z}; e_i; u)$$

and noticing that two terms $\Omega(\partial_i u^j)(\widehat{z}; A; \partial_j A)$ cancel, we deduce that

$$(\partial_i u^j)U^j + \Gamma_\nu(u,\nabla)U^i = -\Omega\partial_i(\widehat{z}; x; u) - 2\Omega(\widehat{z}; u; e_i) - 2\Omega\nu(\widehat{z}; \partial_j A; \partial_i\partial_j A). \tag{34}$$

Therefore, we have arrived at

$$\Gamma_\nu(u,\nabla)u = -\partial_i\left\{\Gamma_\nu(u,\nabla)r + \frac{1}{2}|u|^2 + \Omega(\widehat{z};x;u)\right\} - 2\Omega(\widehat{z};u;e_i)+$$

$$(\partial_i A^\alpha)\Gamma_\nu(u,\nabla)v^\alpha - 2\nu(\partial_j\partial_i A^\alpha)(\partial_j v^\alpha) - 2\Omega\nu(\widehat{z};\partial_j A;\partial_i\partial_j A). \tag{35}$$

Now the last piece in (35) vanishes if v solves the equation

$$\Gamma_\nu(u,\nabla v) = 2\nu C^\alpha_{j;\beta}v^\alpha_j + 2\Omega\nu(\widehat{z};\partial_j A;C_{j;\beta}) \tag{36}$$

Here

$$C^\alpha_{j;\beta} = (\nabla A)^{-1}_{k\beta}\partial_j\partial_k A^\alpha = -\Gamma^\alpha_{\beta\gamma}(\partial_j A^\gamma) \tag{37}$$

is related to the Riemann-Christoffel symbol $\Gamma^\alpha_{\beta\gamma}$. Thus we have proved

Theorem 2. *Assume that A solves (25) with u given by (27) and with v evolving according to (36). Then u solves the rotating Navier-Stokes equation (1). Vice-versa, if u solves the rotating Navier-Stokes equations and if we solve the linear equation (25) and then the linear inhomogeneous equation (36), then (27) holds.*

Let us derive the analogue of (27) for the vorticity. We will take the curl of (27) and express the vorticity $\omega = \nabla\times u$ in terms of the diffusive back-to-labels map A associated to the relative velocity, and the virtual vorticity $\zeta = \nabla_A \times v$. This object is a generalization of the Cauchy invariant $\zeta_0(x,t) = \omega_0(A(x,t))$, to which it reduces if $\nu = 0$. Thus, ζ is defined by

$$\zeta^\alpha(x,t) = \frac{1}{2}\epsilon_{\alpha\beta\gamma}(\nabla A)^{-1}_{j\beta}(\partial_j v^\gamma). \tag{38}$$

We compute the curl of (27); we start with the curl of $(\partial_i A^\alpha)v^\alpha$:

$$\epsilon_{ijk}\partial_j\left\{(\partial_k A^\alpha)v^\alpha\right\} = \epsilon_{ijk}(\partial_k A^\alpha)\partial_j v^\alpha.$$

Using the convenient notation $v^\alpha_{;\beta} = (\nabla A)^{-1}_{j\beta}\partial_j v^\alpha$, we obtain

$$\epsilon_{ijk}\partial_j\left\{(\partial_k A^\alpha)v^\alpha\right\} = \epsilon_{ijk}(\partial_k A^\alpha)(\partial_j A^\beta)v^\alpha_{;\beta} = \frac{1}{2}\epsilon_{ijk}(\partial_k A^\alpha)(\partial_j A^\beta)(v^\alpha_{;\beta} - v^\beta_{;\alpha})$$

The last equality holds because the contribution of $\frac{1}{2}(v^{\alpha}_{;\beta}+v^{\beta}_{;\alpha})$ vanishes. Thus

$$\epsilon_{ijk}\partial_j\left\{(\partial_k A^{\alpha})v^{\alpha}\right\} = \frac{1}{2}\epsilon_{ijk}(\partial_k A^{\alpha})(\partial_j A^{\beta})\epsilon_{\beta\alpha\gamma}\zeta^{\gamma}$$

and consequently

$$\epsilon_{ijk}\partial_j\left\{(\partial_k A^{\alpha})v^{\alpha}\right\} = \frac{1}{2}\epsilon_{ijk}(\partial_j A;\partial_k A;\zeta) \tag{39}$$

Now we compute the curl of (31):

$$\epsilon_{ijk}\partial_j U^k = \Omega\epsilon_{ijk}\left\{(\widehat{z};\partial_j A;\partial_k A;) + (\widehat{z};A;\partial_j\partial_k A) - (\widehat{z};e_j;e_k)\right\}$$

which gives

$$\epsilon_{ijk}\partial_j U^k = \Omega\epsilon_{ijk}\left\{(\widehat{z};\partial_j A;\partial_k A;) - (\widehat{z};e_j;e_k)\right\}. \tag{40}$$

Adding (39) to (40 we deduce

$$\omega^i = \frac{1}{2}\epsilon_{ijk}\left\{(\partial_j A;\partial_k A;\zeta) + 2\Omega\left[(\partial_j A;\partial_k A;\widehat{z}) - (e_j;e_k;\widehat{z})\right]\right\}. \tag{41}$$

This states that the total vorticity $\omega + 2\Omega\widehat{z}$ obeys the Cauchy invariance ([3], [4])

$$\omega + 2\Omega\widehat{z} = \mathcal{C}(\nabla A;\zeta + 2\Omega\widehat{z}) \tag{42}$$

with respect to the trajectories of the relative velocity. Here we used the notation $\mathcal{C}(M;v) = \frac{1}{2}\epsilon_{ijk}(M_{.j};M_{.k};v)$.

3 Lagrangian Transport

We will look more closely now at the term involving Ω in the expression (41). We start by writing

$$\omega = \mathcal{C}(\nabla A;\zeta) + 2\Omega\mathcal{R}(\ell), \tag{43}$$

Here

$$\ell(x,t) = A(x,t) - x \tag{44}$$

is the Eulerian-Lagrangian displacement, and

$$(\mathcal{R}(\ell))_i = \frac{1}{2}\epsilon_{ijk}\left\{(\widehat{z};\partial_j\ell;\partial_k\ell) + (\widehat{z};e_j;\partial_k\ell) + (\widehat{z};\partial_j\ell;e_k)\right\} \tag{45}$$

Using $\widehat{z} = e_3$, the term depending linearly on ℓ is

$$\frac{1}{2}\epsilon_{ijk}\left\{(e_j;\partial_k\ell;e_3) + (\partial_j\ell;e_k;e_3)\right\} = \epsilon_{ijk}(e_j;\partial_k\ell;e_3) =$$

$$\{-\delta_{k1}\delta_{j2}\delta_{i3}(\partial_2\ell;e_1;e_3) + \delta_{k1}\delta_{j3}\delta_{i2}(\partial_3\ell;e_1;e_3) + \delta_{k2}\delta_{j1}\delta_{i3}(\partial_1\ell;e_2;e_3)$$
$$-\delta_{k2}\delta_{j3}\delta_{i1}(\partial_3;e_2;e_3)\} = \{(\partial_2\ell_2)e_3 - (\partial_3\ell_2)e_2\} +$$

$$\{(\partial_1\ell_1)e_3 - (\partial_3\ell_1)e_1\} = \{-\partial_z\ell + (\nabla\cdot\ell)e_3\} \tag{46}$$

The quadratic term is

$$\frac{1}{2}\epsilon_{ijk}(\partial_j\ell;\partial_k\ell;e_3) = \frac{1}{2}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}(\partial_j\ell_\alpha)(\partial_k\ell_\beta)\delta_{\gamma 3} =$$

$$= \epsilon_{ijk}(\partial_j\ell_1)(\partial_k\ell_2) = \nabla\ell_1 \times \nabla\ell_2. \tag{47}$$

Putting together (46) and (47) we have thus

$$\mathcal{R}(\ell) = -\partial_z\ell + (\nabla\cdot\ell)\widehat{z} + (\nabla\ell_1 \times \nabla\ell_2) \tag{48}$$

The detailed expression is:

$$\mathcal{R}(\ell) = \begin{pmatrix} -\partial_3\ell_1 + (\partial_2\ell_1)(\partial_3\ell_2) - (\partial_3\ell_1)(\partial_2\ell_2) \\ -\partial_3\ell_2 + (\partial_3\ell_1)(\partial_1\ell_2) - (\partial_1\ell_1)(\partial_3\ell_2) \\ -\partial_3\ell_3 + (\nabla\cdot\ell) + (\partial_1\ell_1)(\partial_2\ell_2) - (\partial_2\ell_1)(\partial_1\ell_2) \end{pmatrix} \tag{49}$$

Let us consider the case of the rotating three dimensional Euler equations. In this case the back-to-labels map is the inverse Lagrangian path map. The Eulerian-Lagrangian displacement $\ell(x,t)$ is related to the Lagrangian displacement

$$\lambda(a,t) = X(a,t) - a \tag{50}$$

by

$$\lambda(a,t) + \ell(x,t) = 0, \quad a = A(x,t). \tag{51}$$

As long as the evolution is smooth, the incompressibility constraint ($\nabla \cdot u = 0$) implies

$$Det(\nabla A) = 1 \tag{52}$$

In view of $\nabla A = \mathbf{1} + (\nabla \ell)$, one has therefore

$$\nabla \cdot \ell + (\partial_2 \ell_2)(\partial_3 \ell_3) - (\partial_3 \ell_2)(\partial_2 \ell_3) + (\partial_1 \ell_1)(\partial_3 \ell_3) - (\partial_3 \ell_1)(\partial_1 \ell_3) +$$

$$(\partial_1 \ell_1)(\partial_2 \ell_2) - (\partial_2 \ell_1)(\partial_1 \ell_2) = 0 \tag{53}$$

Now comes the main observation concerning $\mathcal{R}(\ell)$: The only terms in (49) that are not explicitly multiples of some component of $\partial_z \ell$ are found in the third component; but, in view of (53), $(\nabla \cdot \ell) + (\partial_1 \ell_1)(\partial_2 \ell_2) - (\partial_2 \ell_1)(\partial_1 \ell_2)$ is a quadratic expression involving exclusively multiples of $\partial_z \ell$. Therefore one can factor

$$\mathcal{R}(\ell) = -\mathcal{M}(\nabla \ell)\partial_z \ell, \tag{54}$$

where the matrix $\mathcal{M}(\nabla \ell)$ is:

$$\mathcal{M}(\nabla \ell) = \begin{pmatrix} 1 + \partial_2 \ell_2 & -\partial_2 \ell_1 & 0 \\ -\partial_1 \ell_2 & 1 + \partial_1 \ell_1 & 0 \\ -\partial_1 \ell_3 & -\partial_2 \ell_3 & 1 + \partial_1 \ell_1 + \partial_2 \ell_2 \end{pmatrix} \tag{55}$$

Returning to (43) we obtain the relation

$$\mathcal{M}(\nabla \ell)\partial_z \ell = \frac{1}{2\Omega}\left\{\mathcal{C}(\nabla A;\zeta) - \omega\right\} \tag{56}$$

Let us consider the matrix

$$\mathcal{N}(\nabla \ell) = \begin{pmatrix} 1 + \partial_1 \ell_1 & \partial_2 \ell_1 & 0 \\ \partial_1 \ell_2 & 1 + \partial_2 \ell_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{57}$$

and multiply both sides of (56) by (57) from the left. Let us denote the right hand side by the single letter s (for small).

$$s = \frac{1}{2\Omega}\mathcal{N}(\nabla\ell)\left\{\mathcal{C}(\nabla A;\zeta) - \omega\right\} \tag{58}$$

We obtain thus

$$D_2(\nabla\ell)\partial_z\ell_1 = s_1, \tag{59}$$

$$D_2(\nabla\ell)\partial_z\ell_2 = s_2, \tag{60}$$

and

$$(1 + t_2(\nabla\ell))\partial_z\ell_3 - (\partial_1\ell_3)(\partial_3\ell_1) - (\partial_2\ell_3)(\partial_3\ell_2) = s_3. \tag{61}$$

We use the notations

$$D_2(\nabla\ell) = (1 + \partial_1\ell_1)(1 + \partial_2\ell_2) - (\partial_1\ell_2)(\partial_2\ell_1), \tag{62}$$

$$d_2(\nabla\ell) = (\partial_1\ell_1)(\partial_2\ell_2) - (\partial_1\ell_2)(\partial_2\ell_1), \tag{63}$$

$$t_2(\nabla\ell) = \partial_1\ell_1 + \partial_2\ell_2. \tag{64}$$

From definitions obviously

$$D_2(\nabla\ell) = 1 + t_2(\nabla\ell) + d_2(\nabla\ell). \tag{65}$$

Also, the information contained in (53) can be written as

$$t_2(\nabla\ell) + d_2(\nabla\ell) + (1 + t_2(\nabla\ell))\partial_z\ell_3 = (\partial_1\ell_3)(\partial_3\ell_1) + (\partial_2\ell_3)(\partial_3\ell_2) \tag{66}$$

Substituting (66) in (61) we obtain

$$t_2(\nabla\ell) + d_2(\nabla\ell) = -s_3 \tag{67}$$

Using (65) we obtain

Proposition 1. *Let ω be the relative vorticity $\omega = \nabla \times u$ of a solution of the rotating three dimensional Euler equations (3), let A be the inverse Lagrangian map associated to the relative velocity u, let ζ be the Cauchy invariant ($\zeta(x,t) = \omega_0(A(x,t))$, with $\omega_0 = \omega_{|t=0}$), and let $D_2(\nabla\ell)$ be the determinant of the block $(1,2)$ of ∇A, i.e.*

$$D_2(\nabla\ell) = (\partial_1 A_1)(\partial_2 A_2) - (\partial_1 A_2)(\partial_2 A_1). \tag{68}$$

Then one has

$$D_2(\nabla\ell) = 1 + \frac{1}{2\Omega}\left\{\omega_3 - (\partial_1 A; \partial_2 A; \zeta)\right\}, \tag{69}$$

and the equations (59, 60, 61).

Let us assume now that

$$\sup_{t\in[0,T]} \|\omega(\cdot,t)\|_{L^\infty(dx)} = M < \infty. \tag{70}$$

Let us assume also that on a time interval I we have

$$\sup_{t\in I} \|\nabla\ell(\cdot,t)\|_{L^\infty(dx)} \le g. \tag{71}$$

Clearly then

$$|s_3| \le (1 + 2g + 3g^2)\frac{M}{\Omega} \tag{72}$$

and

$$|s_j| \le (1+2g)(1 + 2g + 3g^2)\frac{M}{\Omega} \tag{73}$$

$j = 1, 2$ hold on this time interval. We deduce from (69) that

$$D_2(\nabla\ell) \ge 1 - (1 + 2g + 3g^2)\frac{M}{\Omega} \tag{74}$$

holds on the time interval I. Using (59, 60) we deduce that

$$|\partial_z \ell_j| \le (1+2g)(1 + 2g + 3g^2)\left(1 - (1 + 2g + 3g^2)\frac{M}{\Omega}\right)^{-1}\frac{M}{\Omega} \tag{75}$$

holds for $j = 1, 2$ on the time interval I_i. From (65, 74) we deduce that

$$1 + t_2(\nabla \ell) \geq 1 - 2g^2 - (1 + 2g + 3g^2)\frac{M}{\Omega} \tag{76}$$

holds on the same interval. From (61, 76) and from (75) we deduce now that

$$|\partial_z \ell_3| \leq C_g \frac{M}{\Omega} \tag{77}$$

with

$$C_g = \left(1 - 2g^2 - (1 + 2g + 3g^2)\frac{M}{\Omega}\right)^{-1} \times$$

$$\left[1 + 2g(1 + 2g)\left(1 - (1 + 2g + 3g^2)\frac{M}{\Omega}\right)^{-1}\right](1 + 2g + 3g^2)\frac{M}{\Omega} \tag{78}$$

In order to simplify the exposition, let us assume for example, that

$$g \leq \frac{1}{4} \tag{79}$$

Let us introduce the maximal local Rossby number

$$\rho = \frac{M}{\Omega} = \frac{\sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_{L^\infty(dx)}}{\Omega}. \tag{80}$$

The most stringent condition of invertibility, $1 > 2g^2 + (1 + 2g + 3g^2)\rho$ is satisfied if the Rossby number is small enough. Let us assume that the Rossby number satisfies

$$\rho \leq \frac{1}{4} \tag{81}$$

Then we deduce

Proposition 2. *Let a solution of the rotating Euler equation (3) satisfy*

$$\sup_{t \in I} \|\nabla \ell(\cdot, t)\|_{L^\infty} \leq \frac{1}{4} \tag{82}$$

on a time interval I and assume that the local Rossby number ρ (80) does not exceed $1/4$ (81). Then

$$\sup_{t \in I} \|\partial_z \ell\|_{L^\infty(dx)} \leq 14\rho \tag{83}$$

holds on the interval I.

In order to bound the vertical derivative of the direct Lagrangian map $X(a,t)$ we recall that

$$(\nabla_a X)(A(x,t),t) = (\nabla A(x,t))^{-1} \tag{84}$$

and in particular

$$\frac{\partial X}{\partial a_3}(A(x,t),t) = (\nabla A_1(x,t)) \times (\nabla A_2(x,t)). \tag{85}$$

Expressing this in terms of the displacement ℓ we arrive at

$$\frac{\partial X}{\partial a_3}(A(x,t),t) = e_3 + \begin{pmatrix} (\partial_2\ell_1)(\partial_3\ell_2) - (1+\partial_2\ell_2)(\partial_3\ell_1) \\ (\partial_1\ell_2)(\partial_3\ell_1) - (1+\partial_1\ell_1)(\partial_3\ell_2) \\ t_2(\nabla\ell) + d_2(\nabla\ell) \end{pmatrix} \tag{86}$$

In view of (65, 67, 69, 72, 75) we deduce

Proposition 3. *Let a solution of the rotating Euler equation (3). Consider the Lagrangian map $X(a,t)$ associated to the relative velocity. Then*

$$\frac{\partial X_3}{\partial a_3}(a,t) = 1 + \frac{1}{2\Omega}\left(\omega_3(X(a,t),t) - \zeta(a)\cdot\frac{\partial X_3}{\partial a}(a,t)\right) \tag{87}$$

holds. If the displacement satisfies (82) on a time interval I and the local Rossby number ρ does not exceed $1/4$ then

$$\sup_{t\in I}\|\partial_{a_3}\left(X(a,t)-a\right)\|_{L^\infty(da)} \le 9\rho \tag{88}$$

holds on the interval I.

Comparing (86) to (49) employing (66), and then recalling the definition (50) of the Lagrangian displacement we have

Theorem 3. *The Lagrangian displacement $\lambda(a,t)$ of a solution of the rotating Euler equations (3) for local Rossby number less than two obeys the differential equation*

$$\partial_{a_3}\lambda(a,t) + \frac{1}{2}\rho_0(a)\xi_0(a)\cdot\nabla_a\lambda(a,t) =$$

$$= \frac{1}{2}\left(\rho_t(a)\xi(a,t) - \rho_0(a)\xi(a,0)\right) \tag{89}$$

where

$$\rho_t(a) = \frac{|\omega(X(a,t),t)|}{\Omega}. \tag{90}$$

and

$$\xi(a,t) = \frac{\omega(X(a,t),t)}{|\omega(X(a,t),t)|} \tag{91}$$

Let us consider the motion of a rotating incompressible ideal fluid for a duration T, and assume that in this period of time the flow is smooth, and so,

$$\int_0^T \|\nabla u(\cdot,t)\|_{L^\infty} < \infty \tag{92}$$

Notice that, from the equation

$$\frac{\partial_t(\nabla_a X)}{\partial t}(a,t) = (\nabla u(X(a,t),t))(\nabla_a X)(a,t) \tag{93}$$

it follows immediately that

$$\left\| \frac{\partial X}{\partial a}(a,t) \right\|_{L^\infty(da)} \leq e^{\int_{t_0}^t \|\nabla u(\cdot,t)\|_{L^\infty(dx)} dt} \tag{94}$$

holds.

Theorem 4. *Consider a smooth (92) solution of the rotating Euler equations (3), defined on a time interval* $[0,T]$. *Consider, at time* t_0, *two particles* $P = (a,b,c)$ *and* $Q = (a,b,c+d)$ *separated by a vertical segment of length* d. *(Vertical refers to the direction of the rotation axis* $e_3 = \widehat{z}$*). Consider the Lagrangian evolution of the particles* $X(P,t)$, $X(Q,t)$. *Then, the separation between the two particles obeys*

$$|X(P,t) - X(Q,t) + d| \leq \frac{\rho d}{2}\left(1 + e^{\int_{t_0}^t \|\nabla u(\cdot,t)\|_{L^\infty(dx)} dt}\right) \tag{95}$$

for all $t_0 \leq t \leq T$.

Ideed, (89) can be written as

$$\partial_{a_3}\lambda(a,t) = \frac{1}{2\Omega}\left(\omega(X(a,t),t) - \omega_0(a)\cdot\nabla_a X(a,t)\right) \tag{96}$$

and the statement follows by integrating $\int_c^{c+d} da_3$ both sides of (96) and holding $a_1 = a, a_2 = b$. The right hand side is the bounded using the definition of ρ and the inequality (94).

For a lower bound we need to restrict the duration of time. Let us denote

$$g(t) = \|\nabla \ell(\cdot,t)\|_{L^\infty}. \tag{97}$$

In view of the equation

$$(\partial_t + u\cdot\nabla)(\nabla\ell) + \nabla u + (\nabla\ell)\nabla u = 0 \tag{98}$$

obeyed by the gradient of the displacement, we deduce that

$$g(t) \le \left(e^{\int_{t_0}^t \|\nabla u(\cdot,t)\|_{L^\infty(dx)}dt} - 1\right) \tag{99}$$

holds on any interval $I = [t_0, t_0+\tau]$ where $\ell(x,t_0) = 0$.

If wish to ensure that the Lagrangian displacement obeys (82) on the interval of time $I = [t_0, t_0+\tau]$, then, if (92) holds, it is enough to require

$$\int_{t_0}^{t_0+\tau} \|\nabla u(\cdot,t)\|_{L^\infty(dx)}dt \le \log\frac{5}{4}. \tag{100}$$

Now we are in position to prove a lower bound for the vertical separation of Lagrangian trajectories in terms of the maximal local Rossby number:

We consider two fluid masses (sets) $\Sigma_j(t)$, $j = 1,2$. We define their vertical separation as

$$\delta(t) = \inf\{|z_1 - z_2|\ ;\ (x,y,z_j)\in\Sigma_j(t)\} \tag{101}$$

and distance $d(t)$

$$d(t) = \inf\{|(x_1,y_1,z_1) - (x_2,y_2,z_2)\ ;\ (x_j,y_j,z_j)\in\Sigma_j(t)\} \tag{102}$$

Obviously, $d(t)\le\delta(t)$.

Theorem 5. *Let $\Sigma_j(t)$ be two sets carried by the flow of a smooth (92) solution of the rotating Euler equations (3). Assume that at some time t_0, the distance between the two sets is positive, $d(t_0) > 0$. Consider a time interval $I = [t_0, t_0 + \tau]$ in which (82) holds. Then*

$$\delta(t) \geq (1 + 14\rho)^{-1} d(t_0) \tag{103}$$

holds for $t \in [t_0, t_0 + \tau]$

Indeed, if two points are in the respective sets, $(x, y, z_j) \in \Sigma_j(t)$, then, considering the back-to-labels map starting from time t_0, $\ell(x, t_0) = 0$ we have by assumption $A(x, y, z_j, t) \in \Sigma_j(t_0)$, and, in view of (83) we have

$$d(t_0) \leq |A(x, y, z_1, t) - A(x, y, z_2, t)| = |z_1 - z_2 + \ell(x, y, z_1, t) - \ell(x, y, z_2, t)| \leq$$

$$|z_1 - z_2|(1 + 14\rho).$$

Taking the infimum produces (103). The bounds above hold in great generality; they become most effective when $\rho \to 0$. It is quite easy to exhibit at least some situations in which this limit is well defined mathematically.

Proposition 4. *Consider the rotating Euler system (3) with spatially periodic boundary conditions, $u(x + L_1, y + L_2, z + L_3, t) = u(x, y, z, t)$. Assume that the initial data $u(x, y, z, 0) = u_0(x, y, z)$ belongs to the space H^s of divergence-free square integrable functions having $s > \frac{5}{2}$ square integrable derivatives. Then, there exists a time T and a constant M_1, depending only on the norm of the initial data in H^s, such that, for any $\Omega > 0$, the solution of (3) with initial data u_0 exists on the time interval $[0, T]$ and obeys*

$$\sup_{0 \leq t \leq T} \|\nabla u(\cdot, t)\|_{L^\infty(dx)} \leq M_1. \tag{104}$$

By restricing the duration of time further (see 100),

$$M_1 T \leq \log \frac{5}{4} \tag{105}$$

we guarantee (82) on the time interval $[0, T]$. Fixing thus u_0 and T, we may let $\Omega \to \infty$ and consequently $\rho \leq \frac{M_1}{\Omega} \to 0$. Assuming that

$$\Omega \geq 4M_1 \tag{106}$$

we have the conditions for the bounds (83, 88) to hold.

Theorem 6. *Let $u_0 \in H^s$, $s > \frac{5}{2}$ and $T > 0$ be fixed as above satisfying (104, 105). Then, for each Ω satisfying (106), consider the inverse and direct Lagrangian displacements*

$$\ell(x, y, z, t) = A(x, y, z, t) - (x, y, z)$$

and

$$\lambda(a_1, a_2, a_3, t) = X(a_1, a_2, a_3, t) - (a_1, a_2, a_3).$$

They obey

$$\|\partial_z \ell(\cdot, t)\|_{L^\infty(dx)} \leq 14\rho \tag{107}$$

and

$$\|\partial_{a_3} \lambda(\cdot, t)\|_{L^\infty(da)} \leq 9\rho \tag{108}$$

with $\rho = \Omega^{-1} \sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_{L^\infty(dx)} \leq \Omega^{-1} M_1$.

Let $\Omega_j \to \infty$ be an arbitrary sequence and let $X^j(a_1, a_2, a_3, t)$ denote the Lagrangian paths associated to Ω_j. Then, there exists a subsequence (denoted for convenience by the same letter j) an invertible map $X(a_1, a_2, a_3, t)$, and a periodic function of two variables $\lambda(a_1, a_2, t)$ such that

$$\lim_{j \to \infty} X^j(a_1, a_2, a_3, t) = X(a_1, a_2, a_3, t)$$

holds uniformly in a, t and

$$X(a_1, a_2, a_3) = (a_1, a_2, a_3) + \lambda(a_1, a_2, t)$$

The proof of the second part follows easily from the Arzeli-Ascola theorem and inequalities of the form

$$\left|X^{(j)}(a_1, a_2, b_3, t) - X^{(j)}(a_1, a_2, c_3, t) + c_3 - b_3\right| \leq 9\rho_j |b_3 - c_3|$$

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