Notes on Distributions

Functional Analysis

1 Locally Convex Spaces

Definition 1. *A vector space (over $\mathbb{R}$ or $\mathbb{C}$) is said to be a topological vector space (TVS) if it is a Hausdorff topological space and the operations $+$ and $\cdot$ are continuous.*

For a subset A in a vector space E and $\lambda \in \mathbb{C}$ we denote

$$\lambda A = \{y = \lambda a \mid \ a \in A\}$$

and for two subsets, A, B we denote

$$A + B = \{a + b \mid \ a \in A,\ b \in B\}.$$

Definition 2. *$A \subset E$, E vector space, is*
balanced *if $\alpha A \subset A$, for all $|\alpha| \le 1$,*
convex *if $(1-t)A + tA \subset A$ for all $t \in [0,1]$,*
absorbing *if $\forall x \in E$, $\exists t > 0$, $x \in tA$.*

Definition 3. *We denote by $\mathcal{U}_0$ the collection of neighborhoods of 0 in a TVS E. A subset A of a TVS E is* bounded *if for any neighborhood of zero $V \in \mathcal{U}_0$ there exists $t > 0$ such that $A \subset tV$.*

Definition 4. *E TVS is locally convex (l.c.) if there exists a basis of $\mathcal{U}_0$ formed with convex sets.*

We recall that a basis $\mathcal{B} \subset \mathcal{U}_0$ is a collection of neighborhoods that has the property that for any $V \in \mathcal{U}_0$, there exists $B \in \mathcal{B}$ such that $B \subset V$.

Definition 5. *A TVS is said to be metrizable if its topology is given by a translation invariant metric.*

Definition 6. *If X is a vector space, a seminorm on X is a function*

$$p : X \to \mathbb{R}$$

obeying

$$\begin{aligned} p(\lambda x) = |\lambda| p(x), \quad \forall \lambda \in \mathbb{C},\, x \in X, \\ p(x+y) \le p(x) + p(y), \quad \forall x, y \in X. \end{aligned}$$

Note that $p(0) = 0$ and $p(-x) = p(x)$ and then $p(x) \ge 0$ follow from the definition.

Proposition 1. *(Minkwoski seminorm). Let $A \subset E$ be a convex, balanced and absorbing subset of a vector space E. Let*

$$p_A(x) = \inf\{t > 0 \mid \; x \in tA\}.$$

Then p_A is a seminorm and

$$\{x \mid \; p_A(x) < 1\} \subset A \subset \{x \mid \; p_A(x) \le 1\}$$

Proof. If $t^{-1}x \in A$ and $s^{-1}y \in A$ then, by convexity

$$\frac{t}{t+s} t^{-1} x + \frac{s}{t+s} s^{-1} y \in A$$

which proves $p_A(x+y) \le p_A(x) + p_A(y)$. Also

$$\lambda t^{-1} x = |\lambda| \frac{\lambda}{|\lambda|} t^{-1} x$$

shows, in view of the fact that A is balanced, that $p_A(\lambda x) = |\lambda| p_A(x)$. The first inclusion $\{x \mid \; p_A(x) < 1\} \subset A$ follows from convexity and the fact that any absorbing set contains 0. The second inclusion follows from definition.

Definition 7. *A family $\mathcal{P}$ of seminorms on X is said to be* sufficient *if for any $x \in X$ there exists $p \in \mathcal{P}$ such that $p(x) \neq 0$.*

Let $\mathcal{P}$ be a sufficient family of seminorms on X. The l.c. topology generated by them is the coarsest l.c. topology on X such that all $p \in \mathcal{P}$ are continuous. A basis of $\mathcal{U}_0$ in it is formed with the sets

$$V = \{x \mid \; p_i(x) < \epsilon_i, \; i = 1, \dots n\}$$

where n is arbitrary, $p_i \in \mathcal{P}$ and $\epsilon_i > 0$.

Proposition 2. *A TVS E is l.c. if and only if it has a basis of $\mathcal{U}_0$ formed with convex, balanced and absorbing sets.*

Proof. "Only if": if U is a convex neighborhood of zero in E, there exists a balanced neighborhood of zero $W \in \mathcal{U}_0$ such that $W \subset U$. Indeed, because of the continuity of the map $(\beta, x) \mapsto \beta x$ at $(0,0)$, there exists $\delta > 0$ and $V \in \mathcal{U}_0$ such that $\beta V \subset U$ for all $|\beta| < \delta$. Clearly $W = \cup_{|\beta|<\delta} \beta V$ is a balanced neighborhood of zero included in U. Now let $A = \cap_{|\alpha|=1} \alpha U$. Because $\alpha W = W$ it follows that A is nonempty and it is easy to see that it is balanced, convex and absorbing (all neighborhoods of zero are). The "if" part of the proof follows from definition.

Note that if $\mathcal{B}$ is a basis of $\mathcal{U}_0$ formed with absorbing, convex and balanced neighborhoods then $\mathcal{P} = \{p_V \mid \ V \in \mathcal{B}\}$ is sufficient.

2 Examples

Let $\Omega \subset \mathbb{R}^n$ be an open set and let us consider, for $m \in \mathbb{N} \cup \{0\}$

$$C^m(\Omega) = \{f : \Omega \to \mathbb{C} \mid \ \partial^\alpha f \ \textit{continuous}, \ \forall \ |\alpha| \le m\}$$

Here $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Definition 8. *Let $K \subset\subset \mathbb{R}^n$ be a compact set. (The notation $A \subset\subset B$ will be used always to mean A compact subset of B). Let $m \ge 0$ be an integer or zero. We define the seminorm $p_{K,m}$ on $C^m(\Omega)$ by*

$$p_{K,m}(f) = \sup_{x \in K} \sum_{|\alpha| \le m} |\partial^\alpha f(x)|$$

Definition 9. *The l.c. topology on $C^m(\Omega)$ is generated by the seminorms $p_{K,m}$ as $K \subset\subset \Omega$ runs through all compact subsets of Ω. This is the topology of uniform convergence together with m derivatives.*

Definition 10. *When $m = \infty$ we denote by $\mathcal{E}(\Omega)$ the vector space $C^\infty(\Omega)$ endowed with the l.c. topology generated by the seminorms $p_{K,m}$ as $K \subset\subset \Omega$ runs through all compact subsets of Ω and $m \in \mathbb{N}$ runs through all nonnegative integers.*

Definition 11. *If $K \subset\subset \Omega$ is a compact set in Ω we denote by $\mathcal{D}_K$ the set*

$$\mathcal{D}_K(\Omega) = \{f \in C_0^\infty(\Omega) \mid \; supp(f) \subset K\}$$

endowed with the l.c. topology generated by $p_{K,m}$ as $m \in \mathbb{N}$.

Proposition 3. *Let X be a l.c. space with topology generated by a counatble family of seminorms. Then X is metrizable.*

Proof. If p_n are the seminorms in question, set

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x-y)}{1 + p_n(x-y)}.$$

A metrizable l.c. space which is complete is called a Fréchet space.

Proposition 4. *$C^m(\Omega)$, $\mathcal{E}(\Omega)$, $\mathcal{D}_K(\Omega)$ are Fréchet.*

The idea of the proof here is based on the existence of an exhaustion of Ω by compacts

$$Int(K_n) \subset K_n \subset\subset Int(K_{n+1}) \subset \Omega$$

with $Int(K)$ meaning the interior of K and with

$$\Omega = \cup K_n$$

Then clearly $p_{K_n,m}$ is a cofinal countable set of seminorms in $C^m(\Omega)$ and $\mathcal{E}(\Omega)$, meaning simply that for each $K \subset\subset \Omega$, there exists n such that $p_{K,m} \leq p_{K_n,m}$. Then the topology generated by $p_{K_n,m}$ is the same as the topology generated by the whole family $p_{K,m}$. Cauchy sequences in the metric are sequences of functions such that for every ϵ and for every seminorm p in the sequence of sufficient seminorms, there exists a rank N such that $p(f_k - f_l) \leq \epsilon$ if $l, k \geq N$.

3 Strict inductive limits and the space $\mathcal{D}(\Omega)$

The space $C_0^\infty(\Omega)$ has a a more complicated topology. As a point set we see that

$$C_0^\infty(\Omega) = \cup_{K \subset\subset \Omega} \mathcal{D}_K(\Omega).$$

Moreover, if $K \subset L \subset\subset \Omega$ it is clear that the inclusions

$$i_{K,L} : \mathcal{D}_K(\Omega) \to \mathcal{D}_L(\Omega)$$

are injective and continuous and the restriction of the topology of $\mathcal{D}_L(\Omega)$ to $\mathcal{D}_K(\Omega)$ (i.e. the collection of preimages of open sets) gives the topology of $\mathcal{D}_K(\Omega)$. Moreover, an exhaustion with compacts K_n gives a countable cofinal family,

$$C_0^\infty(\Omega) = \cup_n \mathcal{D}_{K_n}(\Omega).$$

Definition 12. *The topology on $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ is the finest l.c. topology that makes all the inclusions*

$$i_K : \mathcal{D}_K(\Omega) \to \mathcal{D}(\Omega)$$

continuous.

In general, if we have a nondecreasing sequence of l.c. spaces X_n (or a partially ordered family (under inclusion) with a countable cofinal chain) with the property that if $X_i \subset X_j$ then the topology on X_i is exactly the topology induced from X_j we can put the strict inductive limit topology on the union $X = \cup X_n$ by demanding that all the inclusions $i_n : X_n \to X$ be continuous.

Lemma 1. *If X is l.c. and Y is a closed linear subspace and if $B \in \mathcal{U}_{0,Y}$, B convex, (B is a convex neighborhood of zero in the induced topology in Y) and if $x \in X \setminus B$ then there exists $A \in \mathcal{U}_{0,X}$ a convex neighborhood of zero in X such that $B = A \cap Y$ and $x_0 \notin A$.*

Proof. Let $B = W \cap Y$ with some $W \in \mathcal{U}_{0,X}$. For each $V \in \mathcal{U}_{0,X}$ consider

$$A_V = co(B \cup (W \cap V))$$

the convex hull of the union of B with the intersection $V \cap W$. Clearly A_V is convex, $A_V \cap Y = B$. (Indeed, if $y = (1-t)b + tw$ if $t = 0$ then $y \in B$ but if $t \neq 0$ then $w \in W \cap Y = B$ so $y \in B$.) Now $A_V \in \mathcal{U}_{0,X}$. If $x_0 \in A_V$ for all V then $x_0 \in \overline{Y} = Y$, which is absurd because $x_0 \notin B$.

Now if $X = \varinjlim X_n$ is a strict inductive limit of l.c. spaces then the family

$$\{co\,(\cup U_n)\,|\ U_n \subset X_n, \text{ convex, balanced nbhood of } 0\}$$

is the basis of zero in a locally convex topology. Clearly, $i_n : X_n \to X$ are continuous with this topology on X. Conversely, if all i_n are continuous then for any convex neighborhood of zero in X there exists $U_n \subset X_n$ a balanced convex neighborhood of zero in X_n such that $i_n(U_n) \subset V$, i.e. $U_n \subset V$ and then $co(\cup U_n) \subset V$. The lemma above shows that the topology is Hausdorff and furthermore we can take a basis of topology on X formed with

$$\mathcal{B} = \{U \subset X \mid U \text{ convex, balanced, } U \cap X_n \text{ nbhood of } 0 \text{ in } X_n\}$$

Proposition 5. *(a) If $X = \varinjlim X_n$ is the strict inductive limit of the l.c. spaces X_n then a linear map $T : X \to Y$ with Y l.c. is continuous if and olny if $T \circ i_n : X_n \to Y$ are all continuous.*
(b) If a set F is bounded in X, there exists n such that $F \subset X_n$ and F is bounded in X_n.
(c) A sequence $x_n \in X$ converges in X if and only if there exists k such that the sequence belongs to X_k and converges there.

In particular, a sequence of $C_0^\infty(\Omega)$ functions converges in $\mathcal{D}(\Omega)$ if, and only if there exists a compact $K \subset\subset \Omega$ such that all the functions have compact support included in K and the convergence of the sequence takes place in $\mathcal{D}_K(\Omega)$.

4 Distributions

Definition 13. *A distribution u in the open subset Ω of $\mathbb{R}^n$ is a linear continuous functional*

$$u : \mathcal{D}(\Omega) \to \mathbb{C}.$$

In other words, a distribution is an element of the dual of $\mathcal{D}(\Omega)$, $u \in \mathcal{D}'(\Omega)$.

Because it is continuous, a distribution obeys: $\forall K \subset\subset \Omega$, $\exists m$, $\exists C$ depending on K, m, such that

$$|u(\phi)| \le C p_{K,m}(\phi), \quad \forall\, \phi \in \mathcal{D}_K(\Omega).$$

We recall now some properties of the convolution

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

defined for instance for $f, g \in \mathcal{D}(\mathbb{R}^n)$.

Proposition 6. *(i)* $f * g = g * f$
(ii) $(f * g) * h = f * (g * h)$
(iii) $\partial^\alpha (f * g) = \partial^\alpha f * g = f * \partial^\alpha g$
(iv) $supp(f * g) \subset supp\ f + supp\ g$
(v) $\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$

If $\phi \in \mathcal{D}(\mathbb{R}^n)$, $\phi \ge 0$, $\int_{\mathbb{R}^n} \phi(x)dx = 1$ then, if $f \in C_0^0(\mathbb{R}^n)$ (compactly supported, continuous functions) then $\phi_\epsilon * f \to f$ uniformly, as $\epsilon \to 0$, and if $f \in L^p(\mathbb{R}^n)$, $\phi_\epsilon * f \to f$ in norm in L^p. Here $\phi_\epsilon(x) = \epsilon^{-n}\phi(\frac{x}{\epsilon})$.

Proposition 7. *Let $K \subset\subset \Omega$, with Ω open in $\mathbb{R}^n$. Then there exists $\chi \in \mathcal{D}(\Omega)$ such that $\chi(x) = 1$ for all $x \in K$.*

Proof. Consider

$$K_\epsilon = \{x \mid \ dist(x, K) < \epsilon\},$$

take $\epsilon < \frac{dist(K, \partial\Omega)}{3}$ and take

$$\chi = \phi_\epsilon * 1_{K_\epsilon}$$

where 1_{K_ϵ} is the indicator function of K_ϵ, equal to 1 on K_ϵ and equal to 0 outside it. We take ϕ_ϵ to be a standard mollifier with ϕ supported in $|x| < 1$.

Proposition 8. *(Partition of unity subordinated to a cover). Let*

$$\Omega = \cup_{a \in A} \Omega_a$$

a union of open sets $\Omega_a \subset \mathbb{R}^n$. There exists a sequence $\psi_n \in \mathcal{D}(\Omega)$ such that
(i) $\sum_n \psi_n(x) = 1$, with the sum being locally finite, and
(ii) for each n there exists a such that $supp\ \psi_n \subset \Omega_a$.

Proof. There exist two sequences of open balls B_n^1 and B_n^2 such that

$$B_n^1 \subset \overline{B_n^1} \subset\subset B_n^2, \quad \cup_n B_n^1 = \Omega$$

and such that for any n there exists a such that $B_n^2 \subset \Omega_a$. Indeed, consider all the balls with centers with rational coordinates and with rational radii included in some Ω_a. Let these be B_n^2 and let B_n^1 have the same centers and half the radius. Then let any point $x \in \Omega$. there exists a such that $x \in \Omega_a$. There exists a positive rational r such that $B(x, 3r) \subset \Omega_a$, and there exists q with rational coordinates such that $|x - q| < r$. Then $x \in B(q, r)$ and

$B(q,2r) \subset \Omega_a$. Now take $\phi_j \in \mathcal{D}(B_j^2)$ identically equal to 1 on $\overline{B_j^1}$ and set $\psi_1 = \phi_1$,

$$\psi_{n+1} = \phi_{n+1}\Pi_{j=1}^n(1-\phi_j)$$

Note by induction that

$$\psi_1 + \cdots + \psi_n = 1 - \Pi_{j=1}^n(1-\phi_j).$$

Note also that the sum is locally finite: if $K \subset\subset \Omega$ a finite number of B_k^1 cover K, and if n is larger than the largest k in the finite cover, then ψ_n and the product in the right hand side of the above identity vanish.

Definition 14. *If $U \subset \Omega$ are open sets, the restriction of $u \in \mathcal{D}'(\Omega)$ to $\mathcal{D}(U)$, $u_{|U}$, is defined naturally, as the restiction of the linear operator, because $\mathcal{D}(U) \subset \mathcal{D}(\Omega)$. We say that two distributions u_1 and u_2 in $\mathcal{D}'(\Omega)$ agree on U if*

$$(u_1 - u_2)_{|U} = 0$$

Proposition 9. *If $\Omega = \cup_a \Omega_a$ and if $u, v \in \mathcal{D}'(\Omega)$ obey $u_{|\Omega_a} = v_{|\Omega_a}$ for all a, then $u = v$.*

Proof. Taking a partition of unity subordinated to the cover we have

$$u(\phi) = \sum_n u(\psi_n\phi) = \sum_n v(\psi_n\phi) = v(\phi)$$

for any $\phi \in \mathcal{D}(\Omega)$.

Definition 15. *A distribution $u \in \mathcal{D}'(\Omega)$ is said to be of finite order if $\exists m$ such that $\forall K \subset\subset \Omega$ there exists C such that*

$$|u(\phi)| \le C p_{K,m}(\phi), \quad \forall\, \phi \in \mathcal{D}(\Omega)$$

Examples of distributions:

$$PV\left(\frac{1}{x}\right)(\phi) = \lim_{\epsilon\to 0}\int_{|x|>\epsilon}\frac{\phi(x)}{x}dx = \int_0^\infty(\phi(x)-\phi(-x))\frac{dx}{x},$$

$$\frac{1}{x \pm i0} = PV\left(\frac{1}{x}\right) \mp i\delta$$

Definition 16. *The support of a distribution $u \in \mathcal{D}'(\Omega)$ is the complement of the largest open set $U \subset \Omega$ such that $u_{|U} = 0$.*

Proposition 10. *Denoting $\mathcal{E}'(\Omega)$ the set of linear continuous functionals on $\mathcal{E}(\Omega)$ we have that $u \in \mathcal{D}'(\Omega)$ has compact support if and only if there exists an extension $v \in \mathcal{E}'(\Omega)$ such that $v_{|\mathcal{D}(\Omega)} = u$.*

Definition 17. *The derivatives of $u \in \mathcal{D}'(\Omega)$, $\partial^\alpha u$, are defined by*

$$\partial^\alpha u(\phi) = (-1)^{|\alpha|} u(\partial^\alpha \phi)$$

Proposition 11. *(Local structure). Let ω be an open set relatively compact in Ω, $\omega \subset \overline{\omega} \subset\subset \Omega$. Let $u \in \mathcal{D}'(\Omega)$. There exists $f \in L^\infty(\omega)$ and m such that*

$$u_{|\omega} = \partial_1^m \dots \partial_n^m f$$

Proof. Clearly, if the proposition is to be true, an inequality

$$\left|u_{|\omega}(\phi)\right| \le C \int_\omega |\partial_1^m \dots \partial_n^m \phi(x)| dx \quad (*)$$

must hold for all $\phi \in \mathcal{D}(\omega)$ (with $C = \|f\|_{L^\infty(\omega)}$). Let us assume this inequality for a moment. Then we consider the linear subspace F of $L^1(\omega)$ formed with ψ of the form

$$\psi = \partial_1^m \dots \partial_n^m \phi$$

for some $\phi \in \mathcal{D}(\omega)$. We define a linear operator L on F by

$$L(\psi) = u(\phi)$$

The operator is well defined because of the assumed inequality (*). By Hahn-Banach L has an extension to $L^1(\omega)$ which still obeys

$$|L(\psi)| \le C\|\psi\|_{L^1(\Omega)}.$$

Because ${L^1}^* = L^\infty$ we can represent

$$L(\psi) = (-1)^{nm} \int_\omega \psi(x) f(x) dx$$

with $f \in L^\infty(\omega)$. This proves the proposition, modulo the proof of $(*)$. For that, let us take the compact $K = \overline{\omega}$. There exists m_0 and a constant C_0 such that

$$|u_{|\omega}(\phi)| = C_0 p_{K,m_0}(\phi)$$

holds for all $\phi \in \mathcal{D}_K(\Omega)$, and in particular for all $\phi \in \mathcal{D}(\omega)$. Now consider $m = m_0 + 1$ and let M be a large enough number so that $K \subset [-M, M]^n$. Then

$$\partial^\alpha \phi(x_1, \ldots x_n) =$$
$$\int_{-M}^{x_1} dz_1^{(1)} \int_{-M}^{z_1^{(1)}} dz_1^{(2)} \ldots \int_{-M}^{x_n} dz_n^{(1)} \ldots (\partial_1^m \ldots \partial_n^m \phi)(z) dz_1^{(p_1)} \ldots dz_n^{(p_n)}$$

where there are $p_1 = m - \alpha_1$ integrals with respect the first variable, $p_2 = m - \alpha_2$ integrals with respect the second variable, etc. This shows that

$$p_{K,m_0}(\phi) \leq C(M) \|\partial_1^m \ldots \partial_1^m \phi\|_{L^1(\omega)}$$

for all $\phi \in \mathcal{D}(\omega)$.

Proposition 12. *Let m be the order of $u \in \mathcal{E}'(\Omega)$. Assume that $\psi \in \mathcal{E}(\Omega)$ vanishes of order m on supp u. (i.e. $\partial^\alpha \psi(x) = 0$, for all $|\alpha| \leq m$ and $x \in$ supp u.) Then $u(\psi) = 0$.*

Proof. Let $K = \text{supp} u$ and let $\epsilon > 0$, $\epsilon < \frac{dist(K, \partial\Omega)}{3}$. Take

$$\chi_\epsilon = \phi_\epsilon * 1_{K_{2\epsilon}}$$

Note that $\chi_\epsilon(x) = 1$ on K_ϵ. Then note that

$$u(\psi) = u(\chi_\epsilon \psi).$$

On the other hand

$$|u(\phi)| \leq C p_{K,m}(\phi)$$

holds for all $\phi \in \mathcal{E}(\Omega)$. Note that

$$\partial^\beta \chi_\epsilon = O(\epsilon^{-|\beta|})$$

and

$$\partial^\gamma \psi = O(\epsilon^{m+1-|\gamma|})$$

at $K_{3\epsilon}$. So, for any α with $|\alpha| \leq m$ we have by Leibniz

$$\partial^\alpha (\chi_\epsilon \psi) = O(\epsilon)$$

everywhere and thus

$$|u(\psi)| = O(\epsilon),$$

for any smallenough ϵ.

Corollary 1. *Let $u \in \mathcal{D}'(\Omega)$ have singleton support, $\operatorname{supp} u = \{x_0\}$. Then there exists m and constants c_α, $|\alpha| \le m$ such that*

$$u = \sum_{|\alpha| \le m} c_\alpha \partial^\alpha \delta_{x_0}$$

Proof. Let m be the order of u. For any $\phi \in \mathcal{D}(\Omega)$,

$$\psi = \phi - \sum_{|\alpha| \le m} (\alpha!)^{-1} \partial^\alpha \phi(x_0)(x - x_0)^\alpha$$

is a function in $\mathcal{E}(\Omega)$ which vanishes of order m at the support of u. Thus,

$$u(\phi) = \sum_{|\alpha| \le m} b_\alpha \partial^\alpha \phi(x_0) = \left(\sum_{|\alpha| \le m} (-1)^{|\alpha|} b_\alpha \partial^\alpha \delta_{x_0} \right) (\phi)$$

with

$$b_\alpha = \frac{u((x - x_0)^\alpha)}{\alpha!} = (-1)^{|\alpha|} c_\alpha.$$