Boundedness, Harnack inequality and Hölder continuity for weak solutions

Introduction to PDE

We describe results for weak solutions of elliptic equations with bounded coefficients in divergence-form. The ideas of proofs come from DeGiorgi, Nash and Moser. References abound; we mostly use Gilbarg and Trudinger. We discuss equations

\[ Lu = g + \partial_i f_i \]  

where

\[ Lu = -\partial_i (a_{ij} \partial_j u + b_i u) + c_j \partial_j u + du \]  

We assume uniform ellipticity

\[ a_{ij}(x)\xi_i \xi_j \geq \lambda|\xi|^2 \]  

with \( \lambda > 0 \) good for all \( x \) considered. We assume that \( a_{ij} = a_{ji} \) are measurable and bounded,

\[ \sum |a_{ij}(x)|^2 \leq \Lambda^2. \]  

We assume that the coefficients \( b, c, d \) are bounded

\[ \lambda^{-2}(\sum |b_i(x)|^2 + |c^j(x)|^2) + \lambda^{-1}|d(x)| \leq \Gamma \]  

and that the right hand side \( g \in L^q_\Omega, f \in (L^q)^n \), for some \( q > n \). Denoting by

\[ A_i(x, z, p) = a_{ij}(x)p_j + b_i(x)z + f_i, \]
\[ B(x, z, p) = c_j(x)p_j + d(x)z - g(x), \]

we say that \( u \) is a \( W^{1,2}(\Omega) \) weak subsolution in \( \Omega \) if

\[ \int_{\Omega} (\partial_i v)A_i(x, u, \nabla u) + vB(x, u, \nabla u)dx \leq 0 \]
holds for all $v \in C^1_0(\Omega)$, $v \geq 0$. We say that $u$ is a supersolution if the inequality is reversed

$$\int_{\Omega} (\partial_i v) A_i(x,u,\nabla u) + v B(x,u,\nabla u) dx \geq 0. \quad (7)$$

We will quote theorems in full generality but we will present the ideas of proofs only, and for that purpose we will take $b = c = d = f = g = 0$.

### 1 Local boundedness, Harnack inequality

We denote $k(R) = \lambda^{-1} \left( R^{1-\frac{n}{p}} \| f \|_{L^q} + R^{2(1-\frac{n}{q})} \| g \|_{L^2} \right)$

**Theorem 1.** If $u$ is a $W^{1,2}(\Omega)$ subsolution of (1) then there exists a constant $C = C(n, \lambda, q, p, \Gamma R)$ such that, for all balls $B(y, 2R) \subset \Omega$, and $p > 1$ we have

$$\sup_{B(y,R)} u \leq C \left( R^{-\frac{n}{p}} \| u^+ \|_{L^p(B(y,2R))} + k(R) \right). \quad (8)$$

If $u$ is a $W^{1,2}(\Omega)$ supersolution then

$$\sup_{B(y,R)} (-u) \leq C \left( R^{-\frac{n}{p}} \| u^- \|_{L^p(B(y,2R))} + k(R) \right). \quad (9)$$

The idea of proof is to use $v = \eta^2 u^\beta$ as test function, where $\eta$ is a cutoff function, $\beta$ is arbitrary, positive, and deduce bounds of the type

$$\int |\nabla u|^{2} u^{\beta-1} \eta^2 dx \leq C \int |\nabla \eta|^{2} u^{\beta+1} dx$$

This, together with a Sobolev embedding produces bounds for higher $L^p$ norms on the left hand side, depending on lower $L^p$ norms on the right hand side on larger domains. An iteration, due to Moser, finishes the proof. Unfortunately, because $u^\beta$ is not an admissible test function we have to trim it first. We consider $k > 0$ a small positive constant, $M$ a large positive constant. We take $\overline{u} = u^+ + k$ and set $\overline{u}_M = \overline{u}$ if $u < M$, $\overline{u}_M = M + k$ if $u \geq M$. We take now

$$v = \eta^2 (\overline{u}_M^\beta - \overline{u} - k^\beta).$$
where \( \eta \) is a nonnegative smooth cutoff function and \( \beta > 0 \). We take without loss of generality \( y = 0, \ R = 4 \) and \( \eta \in C^1_0(B_4) \). We denote \( B_R = B(0, R) \). Now \( \eta \) is a legitimate test function and
\[
\nabla v = \eta^2\bar{u}_M^{\beta - 1}[(\beta - 1)\nabla \bar{u}_M + \nabla \bar{u}] + 2\eta\nabla \eta(\bar{u}_M^{\beta - 1}\bar{u} - k^\beta)
\]
Here we used the fact that \( \bar{u}_M^{\beta - 1}\bar{u} - k^\beta \geq 0 \) to drop the \( k^\beta \) term in the right hand side. After hiding the term involving \( \nabla \bar{u} \) from the right hand side in the left hand side, we have (new constant \( C! \))
\[
\int \eta^2\bar{u}_M^{\beta - 1}[(\beta - 1)|\nabla \bar{u}_M|^2 + |\nabla \bar{u}|^2] \leq C \int |\nabla \eta|\eta(|\nabla \bar{u}|\bar{u}_M^{\beta - 1}\bar{u})dx.
\]
We let now \( w = \bar{u}_M^{\beta - 1}\bar{u} \). The inequality above implies
\[
\int \eta^2|\nabla w|^2dx \leq C(\beta + 1) \int |\nabla \eta|^2w^2dx
\]
In view of the fact that \( \nabla(\eta w) = \eta \nabla w + w \nabla \eta \) and the estimate above we have, using the Sobolev embedding,
\[
\left( \int (\eta w)^{\frac{2n}{n-2}}dx \right)^{\frac{n-2}{n}} \leq C(\beta + 1) \int |\nabla \eta|^2w^2dx
\]
if \( n > 2 \). (If \( n = 2 \) we replace \( \frac{2n}{n-2} \) in the left hand side by any \( q < \infty \)). We choose appropriately now the test function \( \eta \) so that, for balls \( B_r \subset B_R \) we obtain
\[
\left( \int_{B_r} w^{\frac{2n}{n-2}}dx \right)^{\frac{n-2}{n}} \leq C(\beta + 1) \left( \frac{1}{R - r} \right)^2 \int_{B_R} \bar{u}_M^{\beta + 1}dx
\]
Writing \( q = \beta + 1 \), we have
\[
\left( \int_{B_r} \bar{w}_M^{\frac{nq}{n-2}}dx \right)^{\frac{n-2}{n}} \leq Cq \left( \frac{1}{R - r} \right)^2 \int_{B_R} \bar{u}_M^{\beta}dx
\]
for any $q > 1$ and $r < R$. Letting $M \to \infty$ and then $k \to 0$ we obtain
\[
\|u^+\|_{L^{\frac{nq}{n-2}}(B_r)} \leq \left( C \frac{q}{(R - r)^2} \right)^{\frac{1}{q}} \|u^+\|_{L^n(B_R)}
\]
Denoting the amplification factor by $a = \frac{n}{n-2} > 1$, and choosing for $i = 0, 1, \ldots, q_i = pa^i, r_i = 2 + 2^{1-i}$, we obtain
\[
\|u^+\|_{L^{q_i+1}(B_{r_i+1})} \leq C_a \|u^+\|_{L^{q_i}(B_{r_i})}
\]
and so
\[
\|u^+\|_{L^{q_i}(B_{r_i})} \leq C \sum_i \frac{1}{a^i} \|u^+\|_{L^{p_i}(B_{r_i})}
\]
proving the theorem for subsolutions. The same idea of proof works for supersolutions. Now we prove a lemma, a weak Harnack inequality.

**Theorem 2.** If $u$ is a supersolution of (1) that is nonnegative in a ball $B(y, 4R) \subset \Omega$ then
\[
R^{-\frac{\alpha}{p}} \|u\|_{L^p(B(y, 2R))} \leq C \left( \inf_{B(y, R)} u + k(R) \right)
\]
holds for $1 \leq p < \frac{n}{n-2}$ with $C = C(n, \frac{\alpha}{\lambda}, q, p, R\Gamma)$.

The idea of proof is similar to the one for $L^\infty$ bounds except we are going to use negative powers, and a result of John-Nirenberg, of independent interest. We start by assuming without loss of generality that $y = 0, R = 1$, and so on. Also, we may assume $u$ is bounded (either by the previous result, or by a trimming procedure like in the proof above). We set
\[
v = \eta^2 \bar{u}^\beta
\]
with $\eta \in C^1_0(B_4)$ a nonnegative cutoff function,
\[
\bar{u} = u + k
\]
with $k > 0$ and $\beta \neq 0$. Using the fact that $u$ is a supersolution we have, after hiding one term,
\[
\int \eta^2 \bar{u}^{\beta - 1} |\nabla u|^2 dx \leq C(|\beta|) \int |\nabla \eta|^2 \bar{u}^{\beta + 1} dx
\]
The constant $C(\beta)$ is bounded when $|\beta| > \epsilon > 0$. We set $w = \bar{u}^{\beta+1 \over 2}$ if $\beta \neq -1$ and $w = \log \bar{u}$ if $\beta = -1$. Letting $\gamma = \beta + 1$ we have

$$
\int |\eta \nabla w|^2 \leq C(|\beta|) \gamma^2 \int |\nabla \eta|^2 w^2 dx
$$

if $\beta \neq -1$, and

$$
\int |\eta \nabla w|^2 \leq C \int |\nabla \eta|^2 dx \quad (*)
$$

if $\beta = -1$. We will refer to this inequality at the end of the proof. Thus, for $n > 2$, from the Sobolev inequality we obtain the inequality

$$
\|\eta w\|_{L^{2n \over n-2}} \leq C (1 + |\gamma|) \|\nabla \eta w\|_{L^2}
$$

Choosing $\eta$ like before, we have

$$
\|w\|_{L^{2n \over n-2}(B_{r_1})} \leq C (1 + |\gamma|)(r_2 - r_1)^{-1} \|w\|_{L^2(B_{r_2})}
$$

with $a = n \over n-2$ as before, and $r_2 > r_1$. Denote

$$
\Phi(p, r) = \left( \int_{B_r} |\bar{u}|^p dx \right)^{1 \over p}
$$

We have the inequalities

$$
\Phi(a \gamma, r_1) \leq \left( \frac{C(1 + |\gamma|)}{r_2 - r_1} \right)^{2 \over 2m} \Phi(\gamma, r_2), \quad \text{if } \gamma > 0
$$

$$
\Phi(\gamma, r_2) \leq \left( \frac{C(1 + |\gamma|)}{r_2 - r_1} \right)^{2 \over 2m} \Phi(a \gamma, r_1), \quad \text{if } \gamma < 0.
$$

We take any $0 < p_0 < p < a$ and we have, with $r_m = 1 + 2^{-m}$, $m = 0, \ldots$, and appropriate $\gamma_m < 0$,

$$
\Phi(-p_0, 3) \leq C \Phi(-\infty, 1)
$$

We can check that $\Phi(-\infty, r) = \inf_{B_r} \bar{u}$. We also can prove one step, using $\gamma > 0$,

$$
\Phi(p, 2) \leq C \Phi(p_0, 3)
$$

We would be therefore done if we could find some $p_0 > 0$ such that

$$
\Phi(p_0, 3) \leq C \Phi(-p_0, 3)
$$

This follows from a result of F. John and Nirenberg.
Theorem 3. (F. John-L. Nirenberg) Let \( u \in W^{1,1}(U) \) where \( U \) is convex. Suppose that there exists a constant \( K \) so that

\[
\int_{U \cap B_r} |\nabla u| \, dx \leq Kr^{n-1}
\]

holds for all balls \( B_r \). Then there exists \( \sigma_0 > 0 \) and \( C \) depending only on \( n \) such that

\[
\int_{\Omega} \exp \left( \frac{\sigma}{K} |u - u_U| \right) \, dx \leq C(diam \, U)^n
\]

holds with \( \sigma = \sigma_0 |U| (diam \, U)^{-n} \) and \( u_U = \frac{1}{|U|} \int_U u \).

Let us note that, from the inequality (*), we obtain, via Schwartz

\[
\int_{B_r} |\nabla w| \, dx \leq C r^{\frac{n}{2}} \left( \int_{B_r} |\nabla w|^2 \, dx \right)^{\frac{1}{2}} \leq Cr^{n-1}
\]

Therefore, by the theorem of John-Nirenberg, there exists a constant \( p_0 > 0 \) such that

\[
\int_{B_3} \exp (p_0 |w - w_3|) \, dx \leq C
\]

where \( w_3 = \frac{1}{|B_3|} \int_{B_3} w \, dx \). Therefore

\[
(\int_{B_3} e^{p_0 w} \, dx)(\int_{B_3} e^{-p_0 w} \, dx) \leq Ce^{p_0 w_3} e^{-p_0 w_3} = C
\]

This concludes the proof of the weak Harnack inequality, modulo the John-Nirenberg result. Putting together Theorem 1 and Theorem 2 we have the full Harnack inequality for weak solutions:

Theorem 4. If \( u \) is a weak \( W^{1,2}(\Omega) \) solution with \( u \geq 0 \) then there exists a constant \( C \) so that

\[
\sup_{B(y,R)} u \leq C \inf_{B(y,R)} u
\]

holds for any \( y \in \Omega \) so that \( B(y,4R) \subset \Omega \). Moreover, for any \( U \subset \subset \Omega \) there exists a constant, depending on \( U \) and \( \Omega \) so that

\[
\sup_U u \leq C \inf_U u
\]
2 Hölder continuity

The weak Harnack inequality is sufficient to prove the Hölder continuity of weak solutions.

Theorem 5. Let \( u \) be a weak \( W^{1,2}(\Omega) \) solution of (1). Then \( u \) is locally Hölder continuous in \( \Omega \). For every ball \( B_0 = B(y,R_0) \subset \Omega \) there exists a constant \( C = C(n, \frac{A}{X}, \Gamma, q, R_0) \) such that

\[
\text{osc}_{B(y,R)} u \leq CR^{-\alpha}(R_0^{-\alpha} \sup_{B_0}|u| + k)
\]

where \( \alpha = \alpha(n, \frac{A}{X}, \Gamma R_0, q) > 0 \) and \( k = \lambda^{-1}(\|f\|_{L^q} + \|g\|_{L^\frac{q}{2}}) \).

Moreover, for every \( U \subset\subset \Omega \), there exists \( \alpha = \alpha(n, \frac{A}{X}, \Gamma d) > 0 \) where \( d = \text{dist}(U, \partial \Omega) \), so that

\[
\|u\|_{C^{\alpha}(U)} \leq C(\|u\|_{L^2(\Omega)} + k)
\]

We provide the proof for the case \( b = c = d = f = g = 0 \). Let \( M_0 = \sup_{B_0}|u| \), \( M_4 = \sup_{B(y,4R)} u \), \( m_4 = \inf_{B(y,4R)} u \), \( M_1 = \sup_{B(y,R)} u \), \( m_1 = \inf_{B(y,R)} u \). We have

\[
L(M_4 - u) = 0, \quad L(u - m_4) = 0
\]

so, we can apply the weak Harnack inequality of Theorem 2 with \( p = 1 \). We obtain

\[
R^{-n} \int_{B(y,2R)} (M_4 - u)dx \leq C(M_4 - M_1)
\]

and

\[
R^{-n} \int_{B(y,2R)} (u - m_4) \leq C(m_1 - m_4)
\]

Adding, we obtain

\[
(M_4 - m_4) \leq C[M_4 - m_4 - (M_1 - m_1)]
\]

and so

\[
M_1 - m_1 \leq \gamma(M_4 - m_4)
\]

with \( \gamma = \frac{C^{-1}}{C} < 1 \). We have thus, for \( \omega(R) = \text{osc}_{B(y,R)} u \)

\[
\omega(R) \leq \gamma \omega(4R)
\]

It follows by iteration (exercise!) that

\[
\omega(R) \leq CR^\alpha \omega(R_0)
\]
3 Riesz potentials and the John-Nirenberg inequality

We use the notation of Gilbarg and Trudinger

\[(V_\mu f)(x) = \int_\Omega |x-y|^{n(\mu-1)} f(y) dy\]

with \(\mu \in (0,1]\). This is the same as the Riesz potential of order \(n\mu\) of \(f\chi_\Omega\), where \(\Omega\) is a bounded open set and \(\chi_\Omega\) its characteristic (indicator) function.

Note that

\[V_\mu 1 \leq \mu^{-1}\omega_n^{1-n} |\Omega|^{\mu}\]

Indeed, choosing \(R\) so that \(|\Omega| = \omega_n R^n\), we have

\[(V_\mu 1)(x) = \int_\Omega |x-y|^{n(\mu-1)} dy \leq \int_{B(x,R)} |x-y|^{n(\mu-1)} dy\]

because \(|\Omega \setminus B(x,R)| = |B(x,R) \setminus \Omega|\) and the points in \(\Omega \setminus B(x,R)\) are farther away and hence have smaller potential the points in \(B(x,R) \setminus \Omega\).

**Lemma 1.** The operator \(V_\mu\) maps \(L^p(\Omega)\) continuously into \(L^q(\Omega)\), \(1 \leq q \leq \infty\)

\[0 \leq \delta = \delta(p,q) = p^{-1} - q^{-1} < \mu\]

Moreover

\[\|V_\mu f\|_{L^q} \leq \left(\frac{1 - \delta}{\mu - \delta}\right)^{1-\delta} \omega_n^{1-\mu} |\Omega|^{\mu-\delta} \|f\|_{L^p}\]

The proof: choose \(r\) so that

\[r^{-1} = 1 + q^{-1} - p^{-1} = 1 - \delta\]

Then \(h(x-y) = |x-y|^{n(\mu-1)}\) is in \(L^r(\Omega)\) and, using the same trick as above

\[\|h\|_{L^r} \leq \left(\frac{1 - \delta}{\mu - \delta}\right)^{1-\delta} \omega_n^{1-\mu} |\Omega|^{\mu-\delta}\]

Now we write

\[h|f| = h^{\frac{\delta}{\mu}} h^{(1-p^{-1})} |f|^{\frac{\mu}{\delta}} |f|^{p\delta}\]
and using Hölder we get
\[
|V_\mu(x)| \leq \left[ \int_\Omega h^r(x-y)|f(y)|^p dy \right]^\frac{1}{p} \left[ \int_\Omega h^r(x-y) dy \right]^{1-p^{-1}} \|f\|_{L^p}^p
\]
But sup_{x \in \Omega} \left[ \int_\Omega h^r(x-y) dy \right]^\frac{1}{r} is finite and raising the inequality above to power q and integrating we obtain the desired result.

**Lemma 2.** Let \( \Omega \) be convex and \( u \in W^{1,1}(\Omega) \). Let \( S \) be any measurable set. Then
\[
|u(x) - u_S| \leq \frac{d_n}{n|S|} \int_\Omega |x-y|^{1-n} |\nabla u(y)| dy
\]
a.e. in \( \Omega \), where
\[
u_S = \frac{1}{|S|} \int_S u dy
\]
and \( d = \text{diam } \Omega \).

It is enough to establish the inequality for \( C^1(\Omega) \) functions. Then
\[
u(x) - u(y) = - \int_0^{||x-y||} \partial_r u(x + r\omega) dr
\]
where \( \omega = \frac{y-x}{||y-x||} \) and \( \partial_r = \omega \cdot \nabla \). Integrating in \( y \) over \( S \):
\[
|S|(u(x) - u_S) = - \int_S dy \int_0^{||x-y||} \partial_r u(x + r\omega) dr
\]
We define
\[
V(x) = \begin{cases} 
|\partial_r u(x)|, & \text{if } x \in \Omega \\
0, & \text{if } x \notin \Omega
\end{cases}
\]
and thus we have
\[
|u(x) - u_S| \leq \frac{1}{|S|} \int_{|x-y| \leq d} dy \int_0^\infty V(x + r\omega) dr
\]
\[
= \frac{1}{|S|} \int_0^\infty dr \int_{|\omega| = 1} d\omega \int_0^\infty V(x + r\omega) r^{n-1} d\rho
\]
\[
= \frac{d^n}{n|S|} \int_{|\omega| = 1} V(x + r\omega) dr d\omega
\]
\[
= \frac{d^n}{n|S|} \int_\Omega |x-y|^{1-n} |\partial_r u(y)| dy
\]
We introduce now Morrey spaces: We say that \( f \in M^p(\Omega) \) if there exists a constant \( K \) so that
\[
\int_{B_r \cap \Omega} |f| dx \leq Kr^{n(1-\frac{1}{p})}
\]
holds for all balls $B_r = B(x_0, r)$. The norm $\|f\|_{MP(\Omega)}$ is the smallest such constant $K$.

**Lemma 3.** Let $f \in M^p(\Omega)$, and $\delta = p^{-1} < \mu$. Then

$$|V_\mu f(x)| \leq \frac{1}{\mu - \delta} (\text{diam } \Omega)^{\frac{n}{n-\delta}} \|f\|_{MP(\Omega)}$$

**Proof.** We extend $f$ by zero outside $\Omega$ and denote

$$m(r) = \int_{B(x, r)} |f| dy$$

Then,

$$|V_\mu f(x)| \leq \int_{\Omega} r^{n(\mu-1)} |f(y)| dy, \quad r = |x - y|,$$

$$= \int_0^d r^{n(\mu-1)} m'(r) dr, \quad d = \text{diam } \Omega$$

$$= d^{n(\mu-1)} m(d) + n(1 - \mu) \int_0^d r^{n(\mu-1)-1} m(r) dr$$

$$\leq C \frac{1}{\mu - \delta} d^{n(\mu-\delta)}$$

We note here a generalization of Morrey’s inequality

**Proposition 1.** Let $u \in W^{1,1}(\Omega)$ and assume that there exist $K > 0$ and $0 < \alpha \leq 1$ so that

$$\int_{B_r} |\nabla u| dx \leq K r^{n-1+\alpha}$$

for all balls $B_r \subset \Omega$. Then $u \in C^\alpha(\Omega)$ and

$$\text{osc}_{B_r} u \leq C K r^\alpha$$

The proof is a direct application of Lemma 2 with $S = \Omega = B_r$ and Lemma 3 with $\Omega = B_r$.

**Lemma 4.** Let $f \in M^p(\Omega)$ with $p > 1$ and let $g = V_\mu f$ with $\mu = p^{-1}$. Then there exist constants $c_1, c_2$ depending only on $n$ and $p$ so that

$$\int_\Omega \exp \left( \frac{|g|}{c_1 K} \right) dx \leq c_2 (\text{diam } \Omega)^n$$

where $K = \|f\|_{MP(\Omega)}$. 

Proof: we write, for $q \geq 1$:

$$|x - y|^n(\mu - 1) = |x - y|^n\left(\frac{\mu}{q} - 1\right)\frac{\mu}{q}$$

and by Hölder

$$|g(x)| \leq \left(\frac{V_\mu}{q}|f|\right)^{\frac{1}{q}} \left(\frac{V_{\mu + \frac{\mu}{q}}}{\frac{\mu}{q}}|f|\right)^{1 - \frac{1}{q}}$$

By Lemma 3

$$V_{\mu + \frac{\mu}{q}}|f| \leq \frac{(1 - \mu)q}{\mu} d\frac{n}{m} K, \quad d = \text{diam } \Omega$$

and by Lemma 1

$$\int_\Omega V_\mu |f| dx \leq pq\omega_{n - 1}^{\frac{1}{m}} |\Omega|^{\frac{1}{m}} \|f\|_{L^1}$$

$$\leq pq\omega_{n} K d\frac{n}{m} (1^{\frac{1}{m}} + \frac{1}{m})$$

Therefore

$$\int_\Omega |g|^q dx \leq p(p - 1)^{q - 1} \omega_{n} q^d n^q K^q$$

$$\leq p' \omega_{n} \{(p - 1)qK\}^q d^n$$

where $p' = \frac{p}{p - 1}$. Choosing $c_1 > e(p - 1)$ and summing, we have

$$\int_\Omega \sum \frac{|g|^m}{m!c_1 Kn} dx \leq p' \omega_{n} d^n \sum \left(\frac{p - 1}{c_1}\right)^m \frac{m^n}{m!}$$

$$\leq c_2 d^n$$

Combining Lemma 2 and Lemma 4 we proved Theorem 3.