Dissipativity and Gevrey Regularity of a Smoluchowski Equation

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December 13, 2004

Abstract

We investigate a Smoluchowski equation (a nonlinear Fokker-Planck equation on the unit sphere), which arises in modeling of colloidal suspensions. We prove the dissipativity of the equation in 2D and 3D, in certain Gevrey classes of analytic functions.

MSC2000: 35Kxx, 70Kxx

1 Introduction

The Smoluchowski equation is an equation describing the temporal evolution of the distribution ψ of directions of rod-like particles in a suspension. The equation has the form of a Fokker-Planck equation

$$\partial_t \psi = \Delta\psi + \mathrm{div}(\psi \,\mathrm{grad}\, V),$$

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except that it is nonlinear and it is phrased on the unit sphere (so the Laplacian, divergence and gradient are suitably modified). One thinks of $\psi d\sigma$ as the proportion of particles whose directions belong to the area element $d\sigma$ on the unit sphere. The equation is nonlinear because the mean field potential V depends on ψ. If this dependence is linear then the equation has an energy functional, and its steady solutions are solutions of nonlinear (and typically non-local) equations. Historically, the steady equation arose first, in the work of Onsager ([14]) concerning the effect of the shape of particles in a suspension on their distribution. The time dependent kinetic theory ([7]), and the particular type of potential (Maier-Saupe) we study in this paper are a further development. There are relatively few rigorous mathematical papers concerning this equation. In two previous works ([4] and [5]) mostly questions regarding the steady states were discussed. The Smoluchowski equation is dissipative. This means that the solutions, viewed as trajectories in a phase space, after a transient time, enter and remain in a bounded region of phase space. The dissipativity of the Smoluchowski equation is however a subtle matter. The energy functional is not positive definite in general, and it cannot be used directly. Instead, the conservation law associated to the equation, namely the fact that $\int \psi$ does not change in time, needs to be used in order to prove dissipativity. In [4] dissipativity was proved in $2D$ in a weak phase space, (a phase space in which it is not clear that the equation is well posed), using a cancellation special to $2D$. The dissipativity in three dimensions was until now an open problem. In this paper we prove among other things dissipativity in very strong analytic spaces both in two and three dimensions. The proof of Gevrey regularity and dissipativity in three dimensions uses a slightly different approach than the classical method of [9] (see also [1], [2], [8], [10] and [12]) making use of the special nature of the Fokker-Planck nonlinearity.

2 Preliminaries

We consider the Smoluchowski equation written in local coordinates $\phi = (\phi_1, \phi_2, \dots, \phi_{n-1})$ on the unit sphere S^{n-1} in $\mathbb{R}^n$ as:

$$\partial_t \psi = \frac{1}{\sqrt{g}} \partial_i \left(e^{-V} \sqrt{g} g^{ij} \partial_j (e^V \psi) \right). \tag{2.1}$$

The potential V is given by

$$V(x,t) = -b x_i x_j S^{ij}(t),\\ S^{ij}(t) := \int_{S^{n-1}} x_i(\phi) x_j(\phi) \psi(\phi,t) \sigma(d\phi) - \tfrac{1}{n}\delta_{ij}, \tag{2.2}$$

where x_i are Cartesian coordinates in $\mathbb{R}^n$, $\sigma(d\phi) = \sqrt{g} d\phi$ the surface area, and $b > 0$ is a given parameter representing the intensity of the potential. As a result of applying the product rule, (2.1) can be written in the form of a Fokker-Planck equation

$$\partial_t \psi + A\psi = B(\psi, V), \tag{2.3}$$

where

$$A = -\Delta_g = -\frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j)$$

is the Laplace-Beltrami operator, and

$$B(\psi, V) := \mathrm{div}_g(\psi \nabla_g V) = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} (\partial_j V) \psi).$$

Because of the dependence of V on ψ, the Smoluchowski equation is nonlinear (quadratic) in ψ.

Regarding the existence, uniqueness and regularity of solutions of (2.3), it is easy to prove the following theorem (see [4], [5] for the same claim)

Theorem 1 *Let ψ_0 be a nonnegative continuous function on S^{n-1}. The solutions of (2.3) with initial data $\psi(\cdot, 0) = \psi_0$ exist for all nonnegative times, are smooth, nonnegative and normalized*

$$\int_{S^{n-1}} \psi(\phi, t) \sigma(d\phi) = \int_{S^{n-1}} \psi_0(\phi) \sigma(d\phi).$$

In addition, they are analytic for all positive times.

From now on we will choose the normalization

$$\int_{S^{n-1}} \psi(\phi, t) \sigma(d\phi) = 1.$$

The normalization yields that the matrix S is trace-free ($Tr(S) = 0$), which implies that the homogeneous quadratic polynomial $V(x,t)$ is harmonic.

This, in turn, implies that V, restricted to the sphere, is an eigenvector of A corresponding to the eigenvalue $2n$:

$$AV = 2nV.$$

Moreover, one has the following inequality:

$$-b\left(1-\frac{1}{n}\right) \leq V(x,t) \leq \frac{b}{n}.$$

In particular

$$|V(x,t)| \leq b.$$

The following nontrivial property of the Fokker-Planck bilinear form B will be crucial in the sequel:

Lemma 1 *For $\psi,\chi,V \in D(A)$*

$$(B(\psi,V),\chi)_g = \frac{1}{2}\int_{S^{n-1}} [V(\chi A\psi - \psi A\chi) - \psi\chi AV]\,\sigma(d\phi), \tag{2.4}$$

where

$$(u,v)_g = \int_{S^{n-1}} uv\sigma(d\phi)$$

is the scalar product on $L^2(S^{n-1})$.

Proof : Assuming first that $\psi,\chi,V \in \mathcal{C}^\infty(S^{n-1})$ and applying integration by parts one has

$$\begin{aligned}
(B(\psi),\chi) &= \int_{S^{n-1}} \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j V\psi)\chi\ \sigma(d\phi)\\
&= \int \partial_i(\sqrt{g}g^{ij}\partial_j V\psi)\chi\ d\phi^{n-1}\\
&= -\int \sqrt{g}g^{ij}\partial_j V\psi\partial_i\chi\ d\phi^{n-1}\left(= -\int_{S^{n-1}} g^{ij}\partial_j V\psi\partial_i\chi\ \sigma(d\phi)\right)\\
&= \int V\partial_j\psi\sqrt{g}g^{ij}\partial_i\chi\ d\phi^{n-1} + \int V\psi\partial_j(\sqrt{g}g^{ij}\partial_i\chi)\ d\phi^{n-1}\\
&= \int_{S^{n-1}} Vg^{ij}\partial_j\psi\partial_i\chi\ \sigma(d\phi) + \int_{S^{n-1}} V\psi\Delta_g\chi\ \sigma(d\phi)\\
&= -\int_{S^{n-1}} g^{ij}\partial_i V\partial_j\psi\chi\ \sigma(d\phi) - \int_{S^{n-1}} V\chi\Delta_g\psi\ \sigma(d\phi) + \int_{S^{n-1}} V\psi\Delta_g\chi\ \sigma(d\phi)\\
&= \int_{S^{n-1}} g^{ij}\partial_i V\psi\partial_j\chi\ \sigma(d\phi) + \int_{S^{n-1}} \Delta_g V\psi\chi\sigma(d\phi)\\
&\quad - \int_{S^{n-1}} V\chi\Delta_g\psi\ \sigma(d\phi) + \int_{S^{n-1}} V\psi\Delta_g\chi\ \sigma(d\phi)\\
&= -(B(\psi),\chi) + \int_{S^{n-1}} V(\psi\Delta_g\chi - \chi\Delta_g\psi - 2n\psi\chi)\ \sigma(d\phi),
\end{aligned}$$

and the statement of the Lemma follows by the above and the density of $\mathcal{C}^\infty(S^{n-1})$ in $D(A)$. A similar proof is obtained using $\nabla V \cdot_g \nabla\omega = 1/2(\Delta_g(V\omega) - \omega\Delta_g V - V\Delta_g\omega)$ and integration by parts. □

3 The 2D Case

When $n=2$, the unit circle has one local coordinate $\phi\in[0,\pi]$, and $x_1(\phi)=\cos\phi$, $x_2(\phi)=\sin\phi$, and $g^{11}=g=1$. Thus, in two dimensions, the equation can be rewritten as

$$\partial_t\psi - \partial_\phi^2\psi = \partial_\phi(\partial_\phi V\psi). \tag{3.5}$$

The potential V can be written as a function of the local coordinate ϕ as:

$$V(\phi) = -\frac{b}{2}\int_0^{2\pi}\cos(2(\phi-\tilde{\phi}))\psi(\tilde{\phi},t)\; d\tilde{\phi}. \tag{3.6}$$

In the following sections, we will use the Fourier Transform to rewrite (3.5) as a system of ODEs for which we will prove that the solutions belong to certain Gevrey classes, in which they dissipate, and are real-entire.

3.1 2D Smoluchowski as an infinite system of ODEs

We expand ψ in Fourier series as

$$\psi(\phi,t) = \frac{1}{2\pi}\sum_{j\in\mathbb{Z}}\widehat{\psi}(j,t)e^{ij\phi},$$

where

$$\widehat{\psi}(j,t) = \int_0^{2\pi} e^{-ij\phi}\psi(\phi,t)d\phi$$

are the Fourier coefficients. Requiring $\widehat{\psi}(-j,t)=\widehat{\psi}(j,t)^*$ will insure that ψ is a real-valued function. The system (3.5) becomes a system of ODEs

$$\frac{d\widehat{\psi}}{dt}(j,t) + j^2\widehat{\psi}(j,t) = \frac{bj}{2}\left(\widehat{\psi}(j-2,t)\widehat{\psi}(2,t) - \widehat{\psi}(j+2,t)\widehat{\psi}(-2,t)\right),$$

and the normalization is equivalent to $\widehat{\psi}(0,t)=1$ in this setting.

One can easily verify that the evenness of the initial datum will be preserved by the flow. In terms of Fourier coefficients, this means $\widehat{\psi}(-j,t)=$

$\widehat{\psi}(j,t)$ for $j \in \mathbb{Z}$. Moreover, $\widehat{\psi}(2j+1,t)=0$, for $j \in \mathbb{Z}$ is preserved by the flow, as well. Therefore, we can restrict our study to solutions that have the above symmetries, i.e., solutions of the form

$$\psi(\phi,t)=\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty} y_k(t)\cos(2k\phi),$$

where

$$y_k(t)=\widehat{\psi}(2k,t)=\int_0^{2\pi}\cos(2k\phi)\psi(\phi,t)d\phi.$$

The normalization implies $y_0=1$ and $|y_k|\le 1$. Notice that for such ψ the potential becomes

$$V(\phi,t)=-\frac{b}{2}y_1(t)\cos(2\phi).$$

In this new setting, the 2D Smoluchowski equation can be written in terms of the Fourier coefficients as an infinite system of ODEs:

$$\begin{aligned} &y_0=1\\ &y_k'+4k^2y_k=bky_1(y_{k-1}-y_{k+1}), \quad k=1,2,\dots \end{aligned} \tag{3.7}$$

In [4] the authors have proven that the solutions of the 2D Smoluchowski equation with nonnegative continuous initial data of the form

$$\psi_0(\phi)=\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty} y_k(0)\cos(2k\phi) \tag{3.8}$$

dissipate in the space $H^{-1/2}(S^1)$ according to the inequality

$$\|\psi(t)\|^2_{H^{-1/2}}\le \frac{b}{4}+e^{-8t}\|\psi_0\|^2_{H^{-1/2}}.$$

Also, the existence of one determining mode was proven: If for two solutions

$$\lim_{t\to\infty}|V^{(1)}(0,t)-V^{(2)}(0,t)|=0,$$

then

$$\lim_{t\to\infty}\|\psi^{(1)}(t)-\psi^{(2)}(t)\|_{H^{-1/2}}=0.$$

By $S(t)$ we will denote the semi-group of solution operators, i.e. $\psi(t)=S(t)\psi_0$. The 2D Smoluchowski equation has a compact global attractor $\mathcal{A}$, the maximal bounded set which satisfies $S(t)\mathcal{A}=\mathcal{A}$ for all $t\in\mathbb{R}$. Thanks to the existence of one determining mode, or the Gevrey regularity which we will prove in the next section, one can easily show that the global attractor $\mathcal{A}$ is finite dimensional.

3.2 Gevrey regularity and dissipativity in 2D

Let us denote the uniform state by $\psi_u = 1/2\pi$. Also denote

$$\|\psi\|_{L^2}^2 := \pi \|\psi - \psi_u\|_{L^2(S^1)}^2 = \sum_{k=1}^{\infty} y_k^2,$$

$$\|\psi\|_{H^s}^2 := 2^{-2s}\pi \|\psi - \psi_u\|_{H^s(S^1)}^2 = \sum_{k=1}^{\infty} k^{2s} y_k^2.$$

For a positive function f defined on positive integers let us define the following classes of functions:

$$H_f := \left\{ \psi(\phi) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} y_k \cos(2k\phi) :\ \sum_{k=1}^{\infty} \frac{f(k)}{k} y_k^2 < \infty \right\}$$

and

$$V_f := \left\{ \psi(\phi) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} y_k \cos(2k\phi) :\ \sum_{k=1}^{\infty} k f(k) y_k^2 < \infty \right\},$$

endowed with the 'norms'

$$|\psi|_f = \left(\sum_{k=1}^{\infty} \frac{f(k)}{k} y_k^2 \right)^{1/2} \quad \text{and} \quad \|\psi\|_f = \left(\sum_{k=1}^{\infty} k f(k) y_k^2 \right)^{1/2},$$

respectively. For f that grows at least exponentially with k it is well known that H_f and V_f are subsets of the set of real analytic functions. Also, for each $n \in \mathbb{N}$ there exists a combinatorial constant $M_n \in (0,\infty)$ depending on f, such that

$$\|\partial_\phi^n \psi\|_{L^\infty} \le M_n |\psi|_f, \quad \psi \in H_f.$$

Theorem 2 *Consider the equation (3.5) for $b > 4$ with nonnegative continuous initial data of the form (3.8). Let $h(t) = \min\{t,1\}$, and let $f(k,t) = a^{2kh(t)}$, $1 < a^2 \le 1 + b^{-1}$, or alternatively $f(k,t) = [(k-1)!]^{2h(t)}/b^{2(k-1)}$. In either case, a solution ψ dissipates according to the inequality*

$$|\psi(t)|_f^2 \le \frac{b+1}{2} + e^{-4t}\|\psi_0\|_{H^{-1/2}}^2,\ t \ge 0, \tag{3.9}$$

and is real-entire for $t > 0$. In particular, the ball of radius $\sqrt{b}$ in H_f centered at the uniform state ψ_u absorbs all trajectories in finite time.

Proof : Multiplying (3.7) by $f(k,t)y_k/k$ and summing over $k=1,2,3,\ldots$, we obtain the following a priori estimate. The computations are formal, and can be made rigorous by considering Galerkin approximations (see [3]).

$$\begin{aligned}
\frac{d}{2dt}\sum_{k=1}^{\infty}\frac{f(k,t)}{k}y_k^2 - \frac{1}{2}\sum_{k=1}^{\infty}\frac{f'(k,t)}{k}y_k^2 + 4\sum_{k=1}^{\infty}kf(k,t)y_k^2 \\
= f(1)by_1^2 + by_1\sum_{k=1}^{\infty}(f(k+1)-f(k))y_ky_{k+1} \\
\leq f(1)by_1^2 + b|y_1|\sqrt{\sum_{k=1}^{\infty}kf(k+1)y_{k+1}^2}\sqrt{\sum_{k=1}^{\infty}\frac{f(k+1)}{k}y_k^2}.
\end{aligned}$$

For $f(k,t)=a^{2kh(t)}$, $1<a^2\leq 1+b^{-1}$, we have $f(k+1)-f(k)=(a^{2h(t)}-1)f(k)$, and

$$\frac{d}{2dt}|\psi|_f^2 + 2\|\psi\|_f^2 \leq b+1.$$

For

$$f(k,t)=\left[\frac{(k-1)!^{h(t)}}{b^{k-1}}\right]^2,$$

one has $b^2f(k+1)=k^{2h(t)}f(k)\leq k^2f(k)$, and therefore

$$\frac{d}{2dt}|\psi|_f^2 + 2\|\psi\|_f^2 \leq b.$$

In both cases, (3.9) follows. □

Remark 1 *Observe that from $y_k^2\leq 1$, $k=1,2,\ldots$ the dissipativity follows in H_f for any f for which*

$$f(k)\leq[(k-1)!/b^{k-1}]^2,\quad k=k_0,k_0+1,\ldots,$$

for some $k_0\in\mathbb{Z}$. In particular, this is true for $f(k)=a^{2k}$ for any $a>1$. Moreover, the dissipativity in Gevrey classes implies the dissipativity of the solution and all its derivatives in L^∞:

$$\|\partial_\phi^n\psi(t)\|_{L^\infty}^2 \leq M_n^2(b+e^{-4t}\|\psi_0\|_{H^{-1/2}}^2),\ t>0.$$

In particular

$$\sup_{\psi\in\mathcal{A}}\|\partial_\phi^n\psi\|_{L^\infty}\leq M_n\sqrt{b},$$

and

$$\lim_{t\to\infty}\inf_{\psi\in\mathcal{A}}\|\partial_\phi^n S(t)\psi_0-\partial_\phi^n\psi\|_{L^\infty}=0.$$

The Fourier coefficients of the elements of the global attractor $\mathcal{A}$ decay according to:

$$y_k^2\le\min\left\{1,bk(1+b^{-1})^{-k},bk\left[\frac{b^{k-1}}{(k-1)!}\right]^2\right\}.$$

□

Remark 2 *The quotient $z_k=y_k/y_1$ satisfies the following ODE:*

$$z_k'+4(k^2-1)z_k=bky_1(z_{k-1}-z_{k+1})-bz_k(1-y_2),\quad k=2,3,4,\dots$$

and therefore

$$\frac{d}{2dt}(z_k^2)+4(k^2-1)z_k^2=bky_1(z_{k-1}z_k-z_kz_{k+1})-bz_k^2(1-y_2),\quad k=2,3,4,\dots$$

As before, for the same choice of f as in Theorem 2, multiplying by $f(k)z_k/k$ and summing over $k=2,3,\dots$ gives the following inequality:

$$\frac{d}{2dt}\sum_{k=2}^\infty\frac{f(k)}{k}z_k^2+2\sum_{k=2}^\infty kf(k)z_k^2\le b|y_2|.$$

In particular, $|\psi(t)|_f^2/|V(t,0)|^2$ is dissipated in time, until eventually

$$|\psi(t)|_f\le\frac{2}{\sqrt{b}}|V(0,t)|,\quad t\ge T.$$

for some T. □

In [4] the authors proved the existence of one determining mode. Here we improve the result leading to the convergence in stronger norms.

Theorem 3 *Let $\psi^{(j)}(\phi,t)$, $j=1,2$, be two solutions of (3.7) corresponding to nonnegative continuous initial data*

$$\psi_0^{(j)}(\phi)=\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^\infty y_k^{(j)}(0)\cos(2k\phi)$$

respectively. Let $V^{(j)}(\phi,t)$ *be the corresponding potential to the solution* $\psi^{(j)}(\phi,t)$. *Assume that*

$$\lim_{t\to\infty} |V^{(1)}(0,t) - V^{(2)}(0,t)| = 0,$$

i.e.

$$\lim_{t\to\infty} |y_1^{(1)}(t) - y_1^{(2)}(t)| = 0,$$

then for $f(k) = a^{2k}$, $1 < a^2 \le 1 + b^{-1}$

$$\lim_{t\to\infty} |\psi^{(1)}(t) - \psi^{(2)}(t)|_f = 0,$$

and for every $n = 0,1,2,\ldots$

$$\lim_{t\to\infty} \|\partial_\phi^n \psi^{(1)}(t) - \partial_\phi^n \psi^{(2)}(t)\|_{L^\infty} = 0.$$

Proof : Let $\psi = \psi^{(1)} - \psi^{(2)}$ and $\bar{\psi} = \frac{1}{2}(\psi^{(1)} + \psi^{(2)})$. The Fourier coefficients are defined accordingly by $y_k = y_k^{(1)} - y_k^{(2)}$, and $\bar{y}_k = \frac{1}{2}(y_k^{(1)} + y_k^{(2)})$, $k = 0,1,2,\ldots$ The equation for the difference in terms of the Fourier coefficients reads

$$\begin{aligned} &y_0 = 0,\\ &y_k' + 4k^2 y_k = bk\bar{y}_1(y_{k-1} - y_{k+1}) + bky_1(\bar{y}_{k-1} - \bar{y}_{k+1}), \quad k = 1,2,\ldots \end{aligned} \tag{3.10}$$

Multiplying (3.10) by $f(k)y_k/k$ and summing over $k = 1,2,\ldots$, we obtain

$$\frac{d}{2dt}|\psi|_f^2 + 4\|\psi\|_f^2 = b\bar{y}_1 \sum_{k=1}^{\infty} (f(k+1) - f(k)) y_k y_{k+1} + by_1 \sum_{k=1}^{\infty} f(k)(\bar{y}_{k-1} - \bar{y}_{k+1}) y_k.$$

Similarly as before, and using a Schwartz inequality we obtain

$$\frac{d}{2dt}|\psi|_f^2 + 3\|\psi\|_f^2 \le b|y_1|\|\psi\|_f \sqrt{\sum_{k=1}^{\infty} \frac{f(k)}{k}(\bar{y}_{k-1} - \bar{y}_{k+1})^2}$$

Using Young's inequality,

$$\frac{d}{dt}|\psi|_f^2 + \|\psi\|_f^2 \le 16b^2 y_1^2 \left(1 + |\psi^{(1)}|_f^2 + |\psi^{(2)}|_f^2\right).$$

There exists $T > 0$ so that for $t \ge T$

$$\frac{d}{dt}|\psi(t)|_f^2 + \|\psi(t)\|_f^2 \le 16b^2 y_1^2 (1 + 2b).$$

Therefore $y_1(t)=-\frac{2}{b}V(0,t)\to 0$ when $t\to\infty$ will imply $\lim_{t\to\infty}|\psi(t)|_f=0$, and $\lim_{t\to\infty}\|\partial_\phi^n\psi(t)\|_{L^\infty}=0$. This completes the proof. $\square$

The following Theorem shows that there are finite number of determining nodes for the 2D Smoluchowski equation.

Theorem 4 *There exists $n=n(b)$, so that for any n equidistant points $\phi_0<\phi_2<\cdots<\phi_n=\phi_0$, if*

$$\psi^{(1)}(t,\phi_j)-\psi^{(2)}(t,\phi_j)\to 0,\ j=1,2,\dots,n,$$

then for every $l=0,1,2,\dots$

$$\lim_{t\to\infty}\|\partial_\phi^l\psi^{(1)}(t)-\partial_\phi^l\psi^{(2)}(t)\|_\infty=0.$$

Proof : Let us write the 2D Smoluchowski equation in the following form:

$$\partial_t\psi-\partial_\phi^2\psi=by_1\partial_\phi(\sin(2\phi)\psi),$$

and the equation for the difference of two solutions as

$$\partial_t\psi-\partial_\phi^2\psi=b\bar{y}_1\partial_\phi(\sin(2\phi)\psi)+by_1\partial_\phi(\sin(2\phi)\bar{\psi}).$$

Let $0=\phi_0<\phi_2<\cdots<\phi_n=2\pi$, such that $\phi_{i+1}-\phi_i=d$. Multiplying the above equation by ψ and integrating over $[\phi_i,\phi_i+d]$, we obtain

$$\begin{aligned}
&\frac{d}{2dt}\int_{\phi_i}^{\phi_i+d}|\psi|^2+\int_{\phi_i}^{\phi_i+d}|\psi_\phi|^2-[\psi_\phi\psi]_{\phi_i}^{\phi_i+d}\\
&=b\bar{y}_1[\sin(2\phi)\psi^2]_{\phi_i}^{\phi_i+d}-b\bar{y}_1\int_{\phi_i}^{\phi_i+d}\sin(2\phi)\psi\psi_\phi+by_1\int_{\phi_i}^{\phi_i+d}\partial_\phi(\sin(2\phi)\bar{\psi})\psi\\
&\le b\bar{y}_1[\sin(2\phi)\psi^2]_{\phi_i}^{\phi_i+d}+b\int_{\phi_i}^{\phi_i+d}|\psi\psi_\phi|+b(\|\bar{\psi}_\phi\|_{L^\infty}+2\|\bar{\psi}\|_{L^\infty})\int_{\phi_i}^{\phi_i+d}|y_1\psi|\\
&\le b\bar{y}_1[\sin(2\phi)\psi^2]_{\phi_i}^{\phi_i+d}+\frac{1}{2}\int_{\phi_i}^{\phi_i+d}|\psi_\phi|^2+\frac{b^2}{2}\int_{\phi_i}^{\phi_i+d}|\psi|^2+b^3(4M_0^2+M_1^2)\int_{\phi_i}^{\phi_i+d}|\psi|^2+\frac{d}{2}y_1^2,
\end{aligned}$$

where the constants M_0 and M_1 are as in Remark 1. Observe that

$$\int_{\phi_i}^{\phi_i+d}|\psi|^2\le 2d^2\int_{\phi_i}^{\phi_i+d}|\psi_\phi|^2+2d|\psi(\phi_i)|^2.$$

Now

$$\begin{aligned}
&\frac{d}{2dt}\int_{\phi_i}^{\phi_i+d}|\psi|^2+\left(\frac{1}{4d^2}-\frac{b^2(1+2M_1b+8M_0b)}{2}\right)\int_{\phi_i}^{\phi_i+d}|\psi|^2\\
&\quad\le[\psi_\phi\psi]_{\phi_i}^{\phi_i+d}+b\bar{y}_1[\sin(2\phi)\psi^2]_{\phi_i}^{\phi_i+d}+\frac{1}{2d}|\psi(\phi_i)|^2+\frac{d}{2}y_1^2
\end{aligned}$$

Choosing d small enough that $\frac{1}{4d^2} - \frac{b^2(1+2M_1^2 b+8M_0^2 b)}{2} > 2\pi^2$, we obtain

$$\frac{d}{2dt}\int_{\phi_i}^{\phi_i+d}|\psi|^2 + 2\pi^2\int_{\phi_i}^{\phi_i+d}|\psi|^2 \le [\psi_\phi\psi]_{\phi_i}^{\phi_i+d} + b\bar{y}_1[\sin(2\phi)\psi^2]_{\phi_i}^{\phi_i+d}$$
$$+\frac{1}{2d}|\psi(\phi_i)|^2 + \frac{d}{2}\pi\int_0^{2\pi}|\psi|^2.$$

Summing the above equations for $i=0,1,2,\dots,n-1$, we obtain

$$\frac{d}{dt}\int_0^{2\pi}|\psi|^2 + 2\pi^2\int_0^{2\pi}|\psi|^2 \le \frac{n}{2\pi}\sum_{i=0}^{n-1}|\psi(\phi_i)|^2.$$

Therefore

$$\lim_{t\to\infty}\|\psi^{(1)}(t)-\psi^{(2)}(t)\|_{L^2} = 0.$$

In particular $y_1 \to 0$, and the Theorem follows. □

4 The 3D Case

When $n=3$, the local coordinates on S^2 are $\phi=(\theta,\varphi)$, and one has $x_1(\theta,\varphi) = \sin\theta\cos\varphi$, $x_2(\theta,\varphi)=\sin\theta\sin\varphi$, and $x_2(\theta,\varphi)=\cos\theta$. Also, $g^{11}=1$, $g^{22}=\sin^{-2}\theta$, $g^{12}=g^{21}=0$, and $\sqrt{g}=\sin\theta$. In terms of the local coordinates,

$$A\psi = -\Delta_g\psi = -\left(\frac{1}{\sin\theta}\partial_\theta(\sin\theta\ \partial_\theta\psi) + \frac{1}{\sin^2\theta}\partial_\varphi^2\psi\right),$$

$$B(\psi) = \frac{1}{\sin\theta}\ \partial_\theta\ (\sin\theta\ (\partial_\theta V)\psi) + \frac{1}{\sin^2\theta}\partial_\varphi((\partial_\varphi V)\psi),$$

and

$$V(\varphi,\theta,t) = \int_0^\pi\int_0^{2\pi}(\sin\theta\sin\tilde{\theta}\cos(\varphi-\tilde{\varphi})+\cos\theta\cos\tilde{\theta})^2\psi(\tilde{\varphi},\tilde{\theta},t)\ d\tilde{\varphi}\ d\tilde{\theta} - \frac{1}{3}.$$

In the following section, we will use the expansion of solutions in spherical harmonics in order to prove the regularity and dissipativity of solutions in certain Gevrey classes.

4.1 Spherical Harmonics

Let P_k denote the Legendre polynomial of degree k. For $k=0,1,2,\ldots$ and $j=0,\pm1,\pm2,\ldots,\pm k$ let us define

$$Y_k^j(\theta,\varphi)=C_k^j e^{ij\varphi}P_k^j(\cos\theta),$$

where

$$C_k^j=\left[\frac{2k+1}{4\pi}\frac{(k-|j|)!}{(k+|j|)!}\right]^{1/2},$$

$$P_k^j(x)=(1-x^2)^{j/2}\frac{d^jP_k}{dx^j}(x),\quad j=0,1,2,\ldots,k,$$

and

$$P_k^j=P_k^{-j},\quad j=-1,-2,\ldots,\ -k.$$

The following are well known facts about the operator $A=-\Delta_g$ (see [13]):

1. Each Y_k^j is an eigenvector of A corresponding to the eigenvalue $\lambda_k=k^2+k$:
$$AY_k^j=\lambda_k Y_k^j.$$

2. The set $\{Y_k^j:\ k=0,1,2,\ldots;j=0,\pm1,\pm2,\ldots,\pm k\}$ forms an orthonormal basis in $L^2(S^2)$; in particular, for each $\psi\in L^2(S^2)$ there is a representation
$$\psi=\sum_{k=0}^{\infty}\sum_{j=-k}^{k}\psi_k^j Y_k^j,$$
where
$$\psi_k^j=\int_{S^2}\psi Y_k^{-j}\ \sigma(d\phi).$$
Observe that ψ is a real-valued function if and only if $\psi_k^{-j}=\bar{\psi}_k^j$, and ψ is an even function in variable φ, if and only if $\psi_k^{-j}=\psi_k^j$. For the simplicity of notation, let us also denote $Y_k^j=0$ and $\psi_k^j=0$ for $|j|>k$.

3. For each $k=0,1,2,\ldots$, we have the point-wise identity
$$\sum_{j=-k}^{k}|Y_k^j(\theta,\varphi)|^2=\frac{2k+1}{4\pi}. \tag{4.11}$$

4. If
$$\int_{S^2} Y_n^m Y_k^j Y_\alpha^{-\beta} \neq 0,$$
then all of the following must hold:
   - $\beta = m + j$
   - $\alpha \leq n + k$
   - $k \leq n + \alpha$
   - $n \leq \alpha + k$
   - $\alpha + n + k$ is even.

4.2 Gevrey Regularity

Let (ψ, V) be a solution of (2.3) for $n = 3$. Let $\psi = \sum_{k=0}^{\infty} \sum_{j=-k}^{k} \psi_k^j Y_k^j$ be the expansion of ψ in spherical harmonics. The normalization yields

$$\psi_0^0 = \frac{1}{\sqrt{4\pi}}$$

and

$$|\psi_k^j| \leq \int_{S^2} \psi |Y_k^{-j}| \sigma(d\phi) \leq \sqrt{\frac{2k+1}{4\pi}}. \tag{4.12}$$

Since V is an eigenvector corresponding to the eigenvalue $\lambda_2 = 6$,

$$V = \sum_{m=-2}^{2} V^m Y_2^m,$$

where $V^m = \int_{S^2} V Y_2^{-m} \ \sigma(d\phi)$. Also

$$|V^m| \leq b \int_{S^2} |Y_2^{-m}| \ \sigma(d\phi) \leq b\sqrt{20\pi}. \tag{4.13}$$

Observe also that the equation (2.3) for $n = 3$ preserves the evenness in φ. We will only consider solutions with this symmetry, i.e. solutions for which $\psi_k^{-j} = \psi_k^j$, and $V^{-j} = V^j$.

Lemma 2 *Let* $\mathcal{F}=f(A)=f(-\Delta_g)$ *and* $\mathcal{G}=g(A)=g(-\Delta_g)$ *be two spectral operators defined by*

$$\mathcal{F}\psi=\sum_{k=0}^{\infty} f(\lambda_k)\sum_{j=-k}^{k}\psi_k^j Y_k^j,$$

and

$$\mathcal{G}\psi=\sum_{k=0}^{\infty} g(\lambda_k)\sum_{j=-k}^{k}\psi_k^j Y_k^j,$$

where f *and* g *are positive functions defined on the set of eigenvalues of* A. *Then for* $\psi\in D(\mathcal{F})\cap D(\mathcal{G})$

$$\int_{S^2} V\mathcal{F}\psi\mathcal{G}\psi\sigma(d\phi)=$$

$$=\sum_{m=-2}^{2}\sum_{k=0}^{\infty}\sum_{j=-k}^{k} f(\lambda_k)g(\lambda_k)V^m\psi_k^j\psi_k^{-(m+j)}\int_{S^2} Y_2^m Y_k^j Y_k^{-(m+j)}\sigma(d\phi)$$

$$+\sum_{m=-2}^{2}\sum_{k=0}^{\infty}\sum_{j=-k}^{k}[f(\lambda_k)g(\lambda_{k+2})+f(\lambda_{k+2})g(\lambda_k)]V^m\psi_k^j\psi_{k+2}^{-(m+j)}\int_{S^2} Y_2^m Y_k^j Y_{k+2}^{-(m+j)}\sigma(d\phi).$$

Proof : Since

$$\int_{S^2} V\mathcal{F}\psi\mathcal{G}\psi\sigma(d\phi)=\sum_{m=-2}^{2}\sum_{k=0}^{\infty}\sum_{j=-k}^{k}\sum_{\alpha=0}^{\infty}\sum_{\beta=-\alpha}^{\alpha} V^m f(\lambda_k)\psi_k^j g(\lambda_\alpha)\psi_\alpha^{-\beta}\int_{S^2} Y_2^m Y_k^j Y_\alpha^{-\beta}\sigma(d\phi),$$

and since $\int_{S^2} Y_2^m Y_k^j Y_\alpha^{-\beta}\sigma(d\phi)\neq 0$ implies $\beta=m+j$, and $\alpha=k+2$, or $k=\alpha+2$, or $\alpha=k$, we have

$$\begin{aligned}\int_{S^2} V\mathcal{F}\psi\mathcal{G}\psi\sigma(d\phi) &= \sum_{m=-2}^{2}\sum_{k=0}^{\infty}\sum_{j=-k}^{k} V^m f(\lambda_k)\psi_k^j g(\lambda_k)\psi_k^{-(m+j)}\int_{S^2} Y_2^m Y_k^j Y_k^{-(m+j)}\sigma(d\phi)\\ &+\sum_{m=-2}^{2}\sum_{k=0}^{\infty}\sum_{j=-k}^{k} V^m f(\lambda_k)\psi_k^j g(\lambda_{k+2})\psi_{k+2}^{-(m+j)}\int_{S^2} Y_2^m Y_k^j Y_{k+2}^{-(m+j)}\sigma(d\phi)\\ &+\sum_{m=-2}^{2}\sum_{\alpha=0}^{\infty}\sum_{\beta=-\alpha}^{\alpha} V^m f(\lambda_{\alpha+2})\psi_{\alpha+2}^{\beta-m} g(\lambda_\alpha)\psi_\alpha^{-\beta}\int_{S^2} Y_2^m Y_{\alpha+2}^{\beta-m} Y_\alpha^{-\beta}\sigma(d\phi).\end{aligned}$$

Since we assume that $V^{-m}=V^m$, and $\psi_k^{-j}=\psi_k^j$,

$$\sum_{m=-2}^{2}\sum_{\alpha=0}^{\infty}\sum_{\beta=-\alpha}^{\alpha} V^m f(\lambda_{\alpha+2})\psi_{\alpha+2}^{\beta-m} g(\lambda_\alpha)\psi_\alpha^{-\beta}\int_{S^2} Y_2^m Y_{\alpha+2}^{\beta-m} Y_\alpha^{-\beta}\sigma(d\phi)=$$

$$\sum_{m=-2}^{2}\sum_{\alpha=0}^{\infty}\sum_{\beta=-\alpha}^{\alpha} V^{-m} f(\lambda_{\alpha+2})\psi_{\alpha+2}^{\beta+m} g(\lambda_\alpha)\psi_\alpha^{-\beta}\int_{S^2} Y_2^{-m} Y_{\alpha+2}^{\beta+m} Y_\alpha^{-\beta}\sigma(d\phi) =$$
$$\sum_{m=-2}^{2}\sum_{\alpha=0}^{\infty}\sum_{\beta=-\alpha}^{\alpha} V^{m} f(\lambda_{\alpha+2})\psi_{\alpha+2}^{-(\beta+m)} g(\lambda_\alpha)\psi_\alpha^{\beta}\int_{S^2} Y_2^{m} Y_{\alpha+2}^{-(\beta+m)} Y_\alpha^{\beta}\sigma(d\phi),$$

and the proof follows. □

The following Lemma establishes important estimates regarding the nonlinear term, and it will be used to prove the Gevrey regularity and dissipativity of solutions that are even in the φ variable.

Lemma 3 *Let $f(\lambda_k) = a^{2k}$ for $a \ge 1$, and $\mathcal{F}\psi = \sum_{k=0}^{\infty} f(\lambda_k)\sum_{j=-k}^{k}\psi_k^j Y_k^j$. There exists $C > 0$, independent of a and b, and $C_b > 0$ depending on b only, such that for any ψ even in φ, for which $\sum_{k=1}^{\infty} k^2 a^{2k}\sum_{j=-k}^{k}|\psi_k^j|^2 < \infty$ we have*

$$|(B(\psi,V),\mathcal{F}\psi)_g| \le C a^4 b\left(1 + \sum_{k=1}^{\infty} k a^{2k}\sum_{j=-k}^{k}|\psi_k^j|^2\right) + C(a^4-1)b\sum_{k=0}^{\infty} k^2 a^{2k}\sum_{j=-k}^{k}|\psi_k^j|^2, \tag{4.14}$$

and if $1 \le a^4 \le 1 + (4Cb)^{-1}$, then also

$$|(B(\psi,V),\mathcal{F}\psi)_g| \le C_b + \frac{1}{2}(A\psi,\mathcal{F}\psi).$$

Proof : Due to Lemma 1,

$$(B(\psi,V),\mathcal{F}\psi)_g = \frac{1}{2}\int_{S^2} V\left(\mathcal{F}\psi A\psi - \psi A\mathcal{F}\psi - 6\psi\mathcal{F}\psi\right)\sigma(d\phi)$$

Therefore, by Lemma 2 and the fact that $\lambda_{k+2} - \lambda_k = 4k + 6$

$$\begin{aligned}
&(B(\psi,V),\mathcal{F}\psi)_g \\
&= -3\sum_{m=-2}^{2}\sum_{k=0}^{\infty}\sum_{j=-k}^{k} f(\lambda_k)V^m\psi_k^j\psi_k^{-(m+j)}\int_{S^2} Y_2^m Y_k^j Y_k^{-(m+j)}\sigma(d\phi) \\
&\quad -3\sum_{m=-2}^{2}\sum_{k=0}^{\infty}\sum_{j=-k}^{k} (f(\lambda_k)+f(\lambda_{k+2}))V^m\psi_k^j\psi_{k+2}^{-(m+j)}\int_{S^2} Y_2^m Y_k^j Y_{k+2}^{-(m+j)}\sigma(d\phi) \\
&\quad -\frac{1}{2}\sum_{m=-2}^{2}\sum_{k=0}^{\infty}\sum_{j=-k}^{k} (\lambda_{k+2}-\lambda_k)(f(\lambda_{k+2})-f(\lambda_k))V^m\psi_k^j\psi_{k+2}^{-(m+j)}\int_{S^2} Y_2^m Y_k^j Y_{k+2}^{-(m+j)}\sigma(d\phi) \\
&= -3\sum_{m=-2}^{2}\sum_{k=0}^{\infty}\sum_{j=-k}^{k} f(\lambda_k)V^m\psi_k^j\psi_k^{-(m+j)}\int_{S^2} Y_2^m Y_k^j Y_k^{-(m+j)}\sigma(d\phi)
\end{aligned}$$

$$-6\sum_{m=-2}^{2}\sum_{k=0}^{\infty}\sum_{j=-k}^{k} f(\lambda_{k+2})V^m\psi_k^j\psi_{k+2}^{-(m+j)}\int_{S^2} Y_2^m Y_k^j Y_{k+2}^{-(m+j)}\sigma(d\phi)$$
$$-2\sum_{m=-2}^{2}\sum_{k=0}^{\infty}\sum_{j=-k}^{k} k(f(\lambda_{k+2})-f(\lambda_k))V^m\psi_k^j\psi_{k+2}^{-(m+j)}\int_{S^2} Y_2^m Y_k^j Y_{k+2}^{-(m+j)}\sigma(d\phi)$$
$$=-\frac{3a^4}{2\pi}\sum_{m=-2}^{2} V^m\psi_2^{-m}$$
$$-3\sum_{m=-2}^{2}\sum_{k=1}^{\infty}\sum_{j=-k}^{k} a^{2k}V^m\psi_k^j\psi_k^{m+j}\int_{S^2} Y_2^m Y_k^j Y_k^{-(m+j)}\ \sigma(d\phi)$$
$$-6\sum_{m=-2}^{2}\sum_{k=1}^{\infty}\sum_{j=-k}^{k} a^{2k+4}V^m\psi_k^j\psi_{k+2}^{-(m+j)}\int_{S^2} Y_2^m Y_k^j Y_{k+2}^{-(m+j)}\sigma(d\phi)$$
$$-2(a^4-1)\sum_{m=-2}^{2}\sum_{k=1}^{\infty}\sum_{j=-k}^{k} ka^{2k}V^m\psi_k^j\psi_{k+2}^{m+j}\int_{S^2} Y_2^m Y_k^j Y_{k+2}^{-(m+j)}\ \sigma(d\phi).$$

The following estimates are obtained using (4.12), (4.11), and (4.13). We have

$$\left|-3\sum_{m=-2}^{2}\sum_{k=1}^{\infty}\sum_{j=-k}^{k} a^{2k}V^m\psi_k^j\psi_k^{m+j}\int_{S^2} Y_2^m Y_k^j Y_k^{-(m+j)}\ \sigma(d\phi)\right|$$
$$\le 15b\sum_{m=-2}^{2}\sum_{k=1}^{\infty}\sum_{j=-k}^{k} a^{2k}|\psi_k^j\psi_k^{m+j}|\int_{S^2}|Y_k^j Y_k^{-(m+j)}|\sigma(d\phi)$$
$$\le 15b\sum_{k=0}^{\infty}(2k+1)a^{2k}\sum_{j=-k}^{k}|\psi_k^j|^2 \le 60b\sum_{k=0}^{\infty}ka^{2k}\sum_{j=-k}^{k}|\psi_k^j|^2,$$

and

$$\left|-6\sum_{m=-2}^{2}\sum_{k=1}^{\infty}\sum_{j=-k}^{k} a^{2k+4}V^m\psi_k^j\psi_{k+2}^{-(m+j)}\int_{S^2} Y_2^m Y_k^j Y_{k+2}^{-(m+j)}\sigma(d\phi)\right|$$
$$\le 30b\sum_{m=-2}^{2}\sum_{k=1}^{\infty}\sum_{j=-k}^{k} a^{2k+4}|\psi_k^j\psi_{k+2}^{m+j}|\int_{S^2}|Y_k^j Y_{k+2}^{-(m+j)}|\sigma(d\phi)$$
$$\le 60b\left(\sum_{k=1}^{\infty}(k+2)a^{2k+4}\sum_{j=-k}^{k}|\psi_k^j|^2+\sum_{k=1}^{\infty}(k+2)a^{2k+4}\sum_{j=-k}^{k}|\psi_{k+2}^j|^2\right)$$
$$\le 60b(3a^4+1)\sum_{k=1}^{\infty}ka^{2k}\sum_{j=-k}^{k}|\psi_k^j|^2,$$

and also

$$
\begin{aligned}
&\left|-2(a^4-1)\sum_{m=-2}^{2}\sum_{k=1}^{\infty}\sum_{j=-k}^{k} ka^{2k}V^m\psi_k^j\psi_{k+2}^{m+j}\int_{S^2}Y_2^mY_k^jY_{k+2}^{-(m+j)}\ \sigma(d\phi)\right| \\
&\le 10b(a^4-1)\sum_{m=-2}^{2}\sum_{k=1}^{\infty}\sum_{j=-k}^{k} ka^{2k}|\psi_k^j\psi_{k+2}^{m+j}|\int_{S^2}|Y_k^jY_{k+2}^{-(m+j)}|\sigma(d\phi) \\
&\le 20b(a^4-1)\left(\sum_{k=1}^{\infty}k(k+2)a^{2k}\sum_{j=-k}^{k}|\psi_k^j|^2+\sum_{k=0}^{\infty}k(k+2)a^{2k}\sum_{j=-k}^{k}|\psi_{k+2}^j|^2\right) \\
&\le 20b(a^4-1)\left(3\sum_{k=1}^{\infty}k^2a^{2k}\sum_{j=-k}^{k}|\psi_k^j|^2+\frac{1}{a^4}\sum_{k=0}^{\infty}(k+2)^2a^{2k+4}\sum_{j=-k}^{k}|\psi_{k+2}^j|^2\right) \\
&\le 80b(a^4-1)\sum_{k=1}^{\infty}k^2a^{2k}\sum_{j=-k}^{k}|\psi_k^j|^2.
\end{aligned}
$$

The estimate (4.14) follows. For any a, which satisfies $1\le a^4\le 1+(4Cb)^{-1}$, and an integer k_0 such that $4Ca^4b\le k_0<4Ca^4b+1$, and by virtue of (4.12) one has

$$
\begin{aligned}
|(B(\psi),\mathcal{F}\psi)_g|-\frac{1}{2}(A\psi,\mathcal{F}\psi)_g &\le Ca^4b+Ca^4b\sum_{k=1}^{\infty}ka^{2k}\sum_{j=-k}^{k}|\psi_k^j|^2-\frac{1}{4}\sum_{k=1}^{\infty}k^2a^{2k}\sum_{j=-k}^{k}|\psi_k^j|^2 \\
&\le Ca^4b+Ca^4b\sum_{k=1}^{\infty}ka^{2k}\sum_{j=-k}^{k}|\psi_k^j|^2-Ca^4b\sum_{k=k_0}^{\infty}ka^{2k}\sum_{j=-k}^{k}|\psi_k^j|^2 \\
&= Ca^4b+Ca^4b\sum_{k=1}^{k_0-1}ka^{2k}\frac{(2k+1)^2}{4\pi}\le C_b.
\end{aligned}
$$

□

The next Theorem is an application of Lemma 3 for the choice of $a=1$, and establishes the dissipation of solutions in $L^2(S^2)$.

Theorem 5 *Let ψ_0 be a nonnegative continuous function on S^2. Then the unique solution $\psi(\phi,t)$ of (2.3) for $n=3$ with initial datum ψ_0 dissipates in $L^2(S^2)$ according to the inequality*

$$\|\psi(t)\|_{L^2}^2\le C_1b^5+e^{-t}\|\psi_0\|_{L^2},\quad t>0,$$

where C_1 is a constant independent of b.

Proof : Applying Lemma 3 for $a=1$ one obtains

$$\frac{d}{2dt}(\psi,\psi)_g+(A\psi,\psi)_g=(B(\psi),\psi)_g\le C_b+\frac{1}{2}(A\psi,\psi)_g,$$

thus

$$\frac{d}{2dt}\|\psi\|_{L^2}^2+\frac{1}{2}\|\psi\|_{L^2}\le C_b.$$

One can easily see that $C_b=C_1b^5$ for a constant C_1 independent of b, and the Theorem follows. □

The following Theorem establishes the regularity and the dissipativity of solutions in a Gevrey class. The idea of the proof is inspired by the work of [9] and its generalization in [1], [2] and [8]. The proof presented here is formal and can be easily made rigorous by applying the Galerkin procedure.

Theorem 6 *Let ψ_0 be a nonnegative continuous function on S^2, and $\psi(\phi,t)$ the unique solution of (2.3) $(n=3)$ corresponding to that initial datum. Let a be such that $1<a^4\le\min\{e,1+(4Cb)^{-1}\}$, and let $h(t)=\min\{t,1\}$. Then*

$$\sum_{k=1}^{\infty}a^{2kh(t)}\sum_{j=-k}^{k}|\psi_k^j|^2\le 4C_b+e^{-t/2}\|\psi_0\|_{L^2},\quad t\ge 0.$$

Proof : Let

$$\mathcal{F}(t)\psi(t)=\sum_{k=0}^{\infty}a^{2kh(t)}\sum_{j=-k}^{k}\psi_k^j(t)Y_k^j,$$

and

$$\mathcal{F}'(t)\psi(t)=2\ln a\;h'(t)\sum_{k=0}^{\infty}ka^{2kh(t)}\sum_{j=-k}^{k}\psi_k^j(t)Y_k^j.$$

Multiplying the equation (2.3) by $\mathcal{F}(t)\psi$ and integrating over S^2 one obtains

$$\frac{d}{2dt}(\psi,\mathcal{F}(\cdot)\psi)_g-\frac{1}{2}(\psi,\mathcal{F}'(\cdot)\psi)_g+(A\psi,\mathcal{F}(\cdot)\psi)_g=(B(\psi),\mathcal{F}(\cdot)\psi)_g$$

which together with Lemma 3 yields

$$\frac{d}{2dt}\sum_{k=0}^{\infty}a^{2kh(t)}\sum_{j=-k}^{k}|\psi_k^j|^2-\ln a\,h'(t)\sum_{k=0}^{\infty}ka^{2kh(t)}\sum_{j=-k}^{k}|\psi_k^j|^2+\frac{1}{2}\sum_{k=0}^{\infty}k^2a^{2kh(t)}\sum_{j=-k}^{k}|\psi_k^j|^2\le C_b,$$

thus

$$\frac{d}{2dt}\sum_{k=0}^{\infty}a^{2kh(t)}\sum_{j=-k}^{k}|\psi_k^j|^2+\frac{1}{4}\sum_{k=0}^{\infty}k^2a^{2kh(t)}\sum_{j=-k}^{k}|\psi_k^j|^2\le C_b,$$

and the Theorem follows. □

Remark 3 *As in the 2D case, the dissipativity in Gevrey classes implies the dissipativity of ψ and its partial derivatives in $L^\infty(S^2)$. In particular, the global attractor $\mathcal{A}$ exists in this case as well, it is finite-dimensional, and there are constants $\tilde{M}(n,b)$, depending on n and b only, such that*

$$\sup_{\psi\in\mathcal{A}} \|\nabla_g^n \psi\|_{L^\infty} \le \tilde{M}(n,b),$$

and

$$\lim_{t\to\infty} \inf_{\psi\in\mathcal{A}} \|\nabla_g^n S(t)\psi_0 - \nabla_g^n \psi\|_{L^\infty} = 0.$$

Remark 4 *As a result of the Gevrey regularity one can easily prove that the Galerkin scheme, based on the eigenfunctions of the Laplacian (in the 2D case) and the Laplace-Beltrami operator (in the 3D case) , converges exponentially fast to the exact solution of the underlying equation (see, e.g., [6] and [11]).*

5 Acknowledgments

We would like to thank I.G. Kevrekidis for stimulating and inspiring discussions. The work of P. C. was supported in part by the NSF grant DMS-0202531. The work of E. S. T. was supported in part by the NSF grant DMS-0204794, the MAOF Fellowship of the Israeli Council of Higher Education, and the US Civilian Research and Development Foundation, grant RM1-2343-MO-02.

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